Uniform Elgot Iteration in Foundations

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Abstract

Category theory is famous for its innovative way of thinking of concepts by their descriptions, in particular by establishing *universal properties*. Concepts that can be characterized in a universal way receive a certain quality seal, which makes them easily transferable across application domains. The notion of partiality is however notoriously difficult to characterize in this way, although the importance of it is certain, especially for computer science where entire research areas, such as *synthetic* and *axiomatic domain theory* revolve around it. More recently, this issue resurfaced in the context of (constructive) *intensional type theory*. Here, we provide a generic categorical iteration-based notion of partiality, which is arguably the most basic one. We show that the emerging free structures, which we dub *uniform-iteration algebras* enjoy various desirable properties, in particular, yield an *equational lifting monad*. We then study the impact of classicality assumptions and choice principles on this monad, in particular, we establish a suitable categorial formulation of the *axiom of countable* choice entailing that the monad is an *Elgot monad*.

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1 Introduction

Natural numbers form a prototypical domain for programming and reasoning. Both in *category theory* and in *type theory* they are characterized by a universal property, which consists of two parts: a definitional principle – *(structural) primitive recursion* and a reasoning principle – *induction*. Dualization yields respectively *co-natural numbers*, *co-recursion* and *co-induction*. Amid these two structuralist extremes, here, we analyse the challenging case of non-structural recursion in the form of iteration, which arises as follows. A map

$$h\colon S \to X + S$$

presents the simplest possible model of a computation process: with S regarded as a state space, h sends any state either to a successor *state* or to a terminal *value* in X. We wish to be able to form an object KX of *denotations* potentially reachable via such processes. Besides the values of X reachable in a finite number of steps, KX must also contain a designated value for divergence, generated by the right injection $h = \mathsf{inr}$. We then ask: what would be the generic universal characterization of KX and what properties it would imply? Somewhat surprisingly, this question has not been addressed yet on a level of generality, sufficiently close to the settings where the question can be posed, although many similar closely related questions have been addressed, mostly couched in type-theoretic terms.

The question trivializes whenever one of the two following perspectives is adopted.

- *intensional perspective:* the domain KX keeps track not only of results, but also of the number of steps needed to reach them. This leads to the identification of KX as the final coalgebra $DX = \nu\gamma.\ X + \gamma$, known as *Capretta's monad* or the *delay monad* [10].
- *non-constructive perspective:* assuming non-constructive principles, such as the *law of excluded middle*, leads to the identification of KX as the *maybe-monad* $X + 1$.

Here, we generally keep aloof from these interpretations of KX and work both extensionally and generically, using the language of the category theory to analyse the issue in the abstract, and keeping the potential class of models possibly large.

We introduce KX as a certain free structure, equipped with an iteration operator, which sends any $f\colon S \to KX + S$ to $f^\sharp\colon S \to KX$, and satisfies the following two basic and uncontroversial principles:

- *fixpoint:* $f^\sharp$ is in an obvious sense a fixpoint of f;
- *uniformity:* the structure of the state space S is ineffective (i.e. merging or adding new states done coherently does not influence the result).

We dub such structures *uniform-iteration algebras* and show that on a high level of generality (in any extensive category with finite limits and a stable natural number object) if KX exists then it satisfies a number of other properties: K extends to a monad $\mathbf{K}$, which is an equational lifting monad [9], the Kleisli category of K is enriched over partial orders and monotone maps, and the iteration operator is a least fixpoint operator w.r.t. this order; moreover, the iteration operator satisfies an additional principle, previously dubbed *compositionality* [2].

In some environments, such as *homotopy type theory (HoTT)*, $\mathbf{K}$ can be constructed directly, by using *higher inductive types.* One can then define a universal map from the delay monad $\mathbf{D}$ to $\mathbf{K}$ and regard it is a form of *extensional collapse.* However, proving $\mathbf{K}$ to be a quotient of $\mathbf{D}$ seems to be impossible without using (weak) choice principles [11, 5, 16]. We interpret this categorically, first by introducing a categorical *limited principle of omniscience (LPO)* under which $\mathbf{K}$ turns out to be isomorphic to the *maybe-monad* $(\text{–} +1)$ and also turns out to be an *Elgot monad.* This generalizes slightly previous results [17] obtained for *hyper-extensive categories* [3]. Second, we identify other cases of $\mathbf{K}$ being a quotient of $\mathbf{D}$ and additionally being an (initial) Elgot monad, by introducing certain coequalizer preservation conditions, abstractly capturing the corresponding instances of the axiom of countable choice.

From the type-theoretic perspective, in our work we revisit the familiar waymarks of using/avoiding principles of classical/constructive mathematics in view of the tradoffs in expressive power of the corresponding constructions. Our present approach of uniform-iteration algebras as a fundamental primitive is entirely new, though. Moreover, we would like to emphasize that our results, being generic, apply to a wide range of categories, whose objects need not be like sets, or types in any conventional sense. This has a massive impact on the underlying proof methods. In topos theory, calculations are facilitated by existence of the subobject classifier Ω, which is used as a global parent space for propositions. Predicative theories, such as HoTT make do without Ω, but it is still possible to form predicative types of propositions per universe, implying that the style of proofs can to a significant extent be maintained, with Ω intuitively regarded as "scattered" over the cumulative universe hierarchy. Contrastingly, here we do not assume any kind of general reference spaces for propositions, which results in completely different proof methods. Nevertheless, we conjecture that our results can be implemented in HoTT. This is clear for the universe of sets, which in HoTT form a *pretopos* [30], and hence directly satisfy our assumptions. For types of higher homotopy levels this should be possible by using existing recipes of formalizing *precategories of types* [33].

Previous related work. We relate to the work on iteration theories, starting from a seminal paper of Elgot [15], who identified iteration as a fundamental unifying notion. Equational properties of Elgot iteration were extensively explored by Bloom and Ésik [8] with the initial iteration structure playing a prominent role, however, since the whole setup therein is inherently classical, most of our present agenda is essentially moot there. The uniformity property occurred under the name *functorial dagger implication* in Bloom and Ésik's monograph, and is an established and powerful principle, thus notably recognized in Simpson and Plotkin's work [32], in the context of generic recursion (as opposed to the present dual case of generic iteration). Adámek et al [2] introduced axioms of *(guarded) Elgot algebras*, and it follows from their results that these axioms are complete w.r.t. the algebras of the delay monad. Uniform-iteration algebras are generally a proper weakening of Elgot algebras, but we show that KX as a free uniform-iteration algebra over X is in fact also an Elgot algebra.

Another line of research we relate to is concerned with notions of partiality, via *dominances*, in particular the *Rosolini dominance* in *synthetic domain* theory [31], via *equational lifting monads* [9], and via *restriction categories* [13]. We remark that these approaches are rather concerned with specifying a notion of partiality than with defining it. This distinction is particularly significant in the context of constructive type theories, such as HoTT, which revitalized the interest to defining a notion of partiality both predicatively and constructively and to understanding the impact of (restricted) choice principles. Chapman et al [11] provided a construction of a partiality monad as a quotient of the delay monad assuming countable choice. Also, Uustalu and Veltri [35] explored universal properties of the obtained quotient as an initial *ω-complete pointed classifying monad*. Altenkirch et al [5] directly based on ω-complete partial orders to obtain a partiality monad in HoTT as a certain *quotient inductive-inductive type* without using any choice whatsoever, but established an equivalence with the delay monad quotient under countable choice. Chapman et al [12] subsequently used more basic *quotient inductive types* for the same purpose.

Recently, Escardó and Knapp [16] reinforced the issue of discrepancy between the quotient of the delay monad and partiality monads, by showing that the quotient precisely captures extensions of Turing computable values, whereas in the absence of any choice, the reasonable partiality monads seem to yield proply larger carriers. The latter view is particularly fine grained, and involves a monad, which is essentially our monad $\mathbf{K}$. According to them, showing the desired connection between $\mathbf{K}$ and the delay monad still amounts to (very weak) choice principles (albeit still not natively available in HoTT), while equivalence to more expressive monads would again require countable choice. Further relevant details of type-theoretic analysis of partiality can be found in recent theses [37, 25]. A comparison of various lifting monads in type theory using a unifying notion of *container* was recently provided by Uustalu and Veltri [36].

2 Categories and Monads

We assume familiarity with standard categorical concepts [28, 6]. In what follows, we generally work in an ambient extensive category $\mathbf{C}$ with finite products, a stable natural number object $\mathbb{N}$ and exponentials $X^{\mathbb{N}}$. By $|\mathbf{C}|$ we refer to the objects of $\mathbf{C}$. We often drop indices of natural transformations to avoid clutter. For the same purpose, we juxtaposition of morphisms as composition. Let us clarify this and fix some conventions.

Extensive categories and pointful reasoning. Extensiveness means existence of disjoint finite coproducts and stability of them under pullbacks (which must exist). Every extensive category is *distributive*, that is, every morphism $[\mathsf{id}\times\mathsf{inl},\mathsf{id}\times\mathsf{inr}]\colon X\times Y+X\times Z\to X\times(Y+Z)$ is an isomorphism whose inverse we denote $\mathsf{dstr}\colon X\times(Y+Z)\to X\times Y+X\times Z$. Let $\mathsf{dstl}\colon (X+Y)\times Z\to X\times Z+Y\times Z$ be the obvious dual to dstr.

In order to simplify reasoning, we occasionally use a rudimentary pointful notation for stating equalities in $\mathbf{C}$, most notably we use the case distinction operator case, e.g. we write

$$f(x) = \mathsf{case}\, g(x)\,\mathsf{of}\,\mathsf{inl}\,y\mapsto h(y);\ \mathsf{inr}\,z\mapsto u(z)$$

meaning $f=[h,u]\ g$ where $f\colon X\to W$, $g\colon X\to Y+Z$, $h\colon Y\to W$ and $u\colon Z\to W$.

Natural numbers and primitive recursion. A *stable natural number object* (NNO) in a Cartesian category $\mathbf{C}$, is an object $\mathbb{N}$ equipped with two morphisms $\mathsf{o}\colon 1\to\mathbb{N}$ (*zero*) and $\mathsf{s}\colon\mathbb{N}\to\mathbb{N}$ (*successor*) such that for any $X,Y\in|\mathbf{C}|$ and any $f\colon X\to Y$ and $g\colon Y\to Y$ there is unique $\mathsf{init}[f,g]\colon X\times\mathbb{N}\to Y$ such that

$$\begin{tikzcd}
X \arrow[r, "{\langle \mathsf{id}, \mathsf{o}\,!\rangle}"] \arrow[dr, "f"'] & X\times\mathbb{N} \arrow[r, "\mathsf{id}\times\mathsf{s}"] \arrow[d, dotted, "{\mathsf{init}[f,g]}"] & X\times\mathbb{N} \arrow[d, dotted, "{\mathsf{init}[f,g]}"] \\
& Y \arrow[r, "g"] & Y
\end{tikzcd}$$

commutes. This combines two separate properties: there exists an initial $(1+\text{–})$-algebra $(\mathbb{N},[\mathsf{o},\mathsf{s}]\colon 1+\mathbb{N}\to\mathbb{N})$, and $(X\times\mathbb{N},[\langle\mathsf{id},\mathsf{o}\,!\rangle,\mathsf{id}\times\mathsf{s}]\colon X+X\times\mathbb{N}\to X\times\mathbb{N})$ is an initial $(X+\text{–})$-algebra. The latter property follows from the former in Cartesian closed categories.

More generally, we need the derivable Lawvere's internalization of *primitive recursion* [27]: Given $f\colon X\to Y$ and $g\colon Y\times X\times\mathbb{N}\to Y$ there is unique $\mathsf{p\text{-}rec}(f,g)\colon X\times\mathbb{N}\to Y$ such that

$$\mathsf{p\text{-}rec}(f,g)(x,\mathsf{o}) = f(x), \qquad\qquad \mathsf{p\text{-}rec}(f,g)(x,\mathsf{s}\,n) = g(\mathsf{p\text{-}rec}(f,g)(x,n),x,n).$$

We thus say that $\mathsf{p\text{-}rec}(f,g)$ is *defined* by (primitive) recursion, whereas *induction* is a *proof principle*, stating that $\mathsf{p\text{-}rec}(f,g)=w$ for any $w\colon X\times\mathbb{N}\to Y$ satisfying the same equations.

Exponentials $X^{\mathbb{N}}$ are adjoint to products $X\times\mathbb{N}$, meaning that there is an isomorphism $\mathsf{curry}\colon \mathbf{C}(X\times\mathbb{N},Y)\to\mathbf{C}(X,Y^{\mathbb{N}})$ natural in X. This induces an evaluation morphism $\mathsf{ev}=\mathsf{curry}^{-1}\,\mathsf{id}\colon X^{\mathbb{N}}\times\mathbb{N}\to X$ with the standard properties.

Strong functors and monads. A functor T is strong if it is equipped with a natural transformation *strength* $\tau_{X,Y}: X\times TY\to T(X\times Y)$, satisfying standard coherence conditions w.r.t. the monoidal structure $(1,\times)$ of $\mathbf{C}$ [26]. This induces the obvious dual $\hat\tau_{X,Y}: TX\times Y\to T(X\times Y)$. A natural transformation $\alpha\colon F\to G$ between two strong functors is itself *strong* if it preserves strength in the obvious sense, i.e. $\alpha\ \tau=\tau\ (\mathsf{id}\times\alpha)$.

A monad $\mathbf{T}$ (in the form of a Kleisli triple) consists of an endomap $T\colon|\mathbf{C}|\to|\mathbf{C}|$, a family of morphisms $(\eta_X\in\mathbf{C}(X,TX))_{X\in|\mathbf{C}|}$ and a lifting operation $(\text{–})^\star\colon\mathbf{C}(X,TY)\to\mathbf{C}(TX,TY)$, satisfying standard laws [29]. It then follows that T is an endofunctor with $Tf=(\eta\ f)^\star$, η extends to a natural transformation, and the *multiplication* transformation $\mu\colon TT\to T$ is definable as $\mathsf{id}^\star$. For every monad $\mathbf{T}$, whose underlying functor T is strong, η and μ are strong (with id being a strength of Id and $(T\tau)\ \tau$ being a strength of μ). Such monad $\mathbf{T}$ is then called *strong* if both η and μ are strong. A strong monad is *commutative* if $\tau^\star\ \hat\tau=\hat\tau^\star\ \tau$.

We adopt Moggi's perspective [29] to strong monads as carriers of computational effects, and thus say that a morphism $f\colon X\to TY$ *computes a value in* Y. Since, the only effect we deal with here is divergence, f can either produce a value or diverge (modulo the inherent linguistic inaccuracy of the excluded middle law baked into the natural language).

Functor algebras and monad algebras. For an endofunctor T, we distinguish T-algebras, which are pairs $(A, a\colon TA \to A)$, from $\mathbf{T}$-algebras, which can only be formed for monads $\mathbf{T}$ on T: a $\mathbf{T}$-algebra is a T-algebra (A, a), which additionally satisfies $a\,\eta = \mathsf{id}$ and $a\,\mu = a\,Ta$. Both T- and $\mathbf{T}$-algebras form categories under the standard structure preserving morphisms, the latter fully embeds into the former.

With our assumptions on $\mathbf{C}$, we mean to cover the following (classes of) categories.

1. Zermello-Fraenkel set theory with choice (ZFC) and further variants of set theory: ETCS, ZF, CZF, etc.
2. Toposes satisfying countable choice, e.g. the *topological topos* [23].
3. Toposes not satisfying countable choice, e.g. *nominal sets*.
4. Pretoposes, e.g. ΠW-pretoposes, compact Hausdoff spaces.
5. The category of topological spaces **Top**, and its subcategories, such as the category of directed complete sets **dCpo**.

3 Basic Properties of the Delay Monad

The final coalgebras $DX = \nu\gamma.\, X + \gamma$ jointly yield a monad $\mathbf{D}$, called the *delay monad* [10]. Capretta [10] showed that $\mathbf{D}$ is strong, which remains valid in our setting. By Lambek's lemma, the final coalgebra structure $\mathsf{out}\colon DX \to X + DX$ is an isomorphism. Its inverse $\mathsf{out}^{-1} = [\mathsf{now}, \mathsf{later}]\colon X + DX \to DX$ is composed of the morphisms, conventionally called now and later, of which the first one is the monad unit, and the effect of the second one is intuitively to postpone the argument computation by one time unit. In what follows, we will write $\triangleright$ instead of later for the sake of succinctness. As a final coalgebra, DX comes together with a *coiteration operator*: for any $f\colon Y \to X + Y$, $\mathsf{coit}\, f\colon Y \to DX$ is the unique morphism, such that $\mathsf{out}\,(\mathsf{coit}\, f) = (\mathsf{id} + \mathsf{coit}\, f)\; f$.

We denote $D1$ by $\bar{\mathbb{N}}$, and think of it as an object of co-natural or possibly infinite natural numbers. Note that the initial algebra structure $[\langle \mathsf{id}, \mathsf{o}\,!\rangle, \mathsf{id} \times \mathsf{s}]\colon X + X \times \mathbb{N} \to X \times \mathbb{N}$, is an isomorphism, and thus yields a D-coalgebra structure on $X \times \mathbb{N}$. This induces a unique D-coalgebra morphism $\iota_X\colon X \times \mathbb{N} \to DX$. Alternatively, we can regard ι_X as defined by primitive recursion:

$$\iota_X(x, \mathsf{o}) = \mathsf{now}(x) \qquad\qquad \iota_X(x, \mathsf{s}(n)) = \triangleright(\iota_X(x, n))$$

Let $\hat{\iota}\colon \mathbb{N} \to \bar{\mathbb{N}}$ be ι_1 modulo the obvious isomorphism.

In our setting, DX need not be postulated, for it is in fact definable as a retract of the object $(X + 1)^{\mathbb{N}}$ of *infinite streams*, which is elaborated in detail by Chapman et al [11]. This also entails that ι is a componentwise monic. Intuitively, DX consists of precisely those streams, which contain at most one element of the form $\mathsf{inl}\, x$. This intuition becomes precise in (possibly non-classical) set theory, where

$$\mathsf{now}\, x = (\mathsf{inl}\, x, \mathsf{inr}\,\star, \mathsf{inr}\,\star, \ldots) \qquad\qquad \triangleright\,(e_1, e_2, \ldots) = (\mathsf{inr}\,\star, e_1, e_2, \ldots)$$

This explains why classically, more precisely, under the *law of excluded*, DX is isomorphic to $X \times \mathbb{N} + 1$. We provide a stronger result to this effect further below. Let us record some general facts about $\mathbf{D}$ first.

▶ **Proposition 1.** *The monad* $\mathbf{D}$ *admits the following characterization:*

1. *unit* $\mathsf{now}\colon X \to DX$ *of* $\mathbf{D}$ *satisfies* $\mathsf{out}\ \mathsf{now} = \mathsf{inl}$*;*
2. *Kleisli lifting of* $f\colon X \to DY$ *is the unique morphism* $f^\star\colon DX \to DY$*, for which the diagram*

$$\begin{tikzcd} DX \arrow[r, "f^\star"] \arrow[d, "\mathsf{out}"'] & DY \arrow[d, "\mathsf{out}"] \\ X + DX \arrow[r, "{[\mathsf{out}\, f,\ \mathsf{inr}\, f^\star]}"] & Y + DY \end{tikzcd}$$

commutes;

3. *strength* $\tau\colon X \times DY \to D(X \times Y)$ *is a unique such morphism that the diagram*

$$\begin{tikzcd} X \times DY \arrow[rr, "\tau"] \arrow[d, "\mathsf{id}\times\mathsf{out}"'] & & D(X \times Y) \arrow[d, "\mathsf{out}"] \\ X \times (Y + DY) \arrow[r, "\mathsf{dstr}"] & X \times Y + X \times DY \arrow[r, "\mathsf{id}+\tau"] & Y + DY \end{tikzcd}$$

commutes.

Proof. (1) and (2) follow from a more general characterization by Uustalu [34]; (3) is established in [19]. ◀

▶ **Proposition 2.** $\mathbf{D}$ *is commutative.*

Let us proceed with a characterization of the situations when $DX \cong X \times \mathbb{N} + 1$. Recall that a monic σ is called *complemented* if there exists $\sigma'\colon X' \hookrightarrow Y$, such that Y is a coproduct of X and X' with σ and σ' as coproduct injections. The *law of excluded middle* states that any monic is complemented. We involve a rather more specific property.

▶ **Proposition 3.** *The monic* $\hat\iota\colon \mathbb{N} \hookrightarrow \bar{\mathbb{N}}$ *is complemented iff* $DX \cong X \times \mathbb{N} + 1$*.*

Proof (Sketch). The necessity is obvious. Let us proceed with the sufficiency. Using extensiveness of $\mathbf{C}$ one can obtain the following pullback:

$$\begin{tikzcd} X \times \mathbb{N} \arrow[r, "\mathsf{snd}"] \arrow[d, hook, "\iota"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{N} \arrow[d, hook, "\hat\iota"] \\ DX \arrow[r, "D!"] & \bar{\mathbb{N}} \end{tikzcd}$$

By assumption, $\hat\iota$ is complemented, and since $\mathbf{C}$ is extensive, so is ι. We obtain that $DX \cong \mathbb{N} \times X + R$ for some R, and then it follows from finality of DX that $R \cong 1$. ◀

The property of $\hat\iota\colon \mathbb{N} \hookrightarrow \bar{\mathbb{N}}$ to be complemented is a categorical formulation of the *limited principle of omniscience (LPO)*, which is rejected in constructive mathematics. Informally, LPO states that every infinite bit-stream either contains 1 at some position or contains only 0 everywhere (the constraint that the stream contains *at most one* 1, does not make a difference). We say that $\mathbf{C}$ is an LPO category if $\hat\iota\colon \mathbb{N} \hookrightarrow \bar{\mathbb{N}}$ is complemented.

▶ **Corollary 4.** *Suppose that (i)* $\mathbf{C}$ *has countable products and (ii) given a family* $(\sigma_i\colon A_i \to A)_{i\in\omega}$ *of complemented pairwise disjoint monos, the induced universal morphism* $\coprod_i A_i \to A$ *is complemented. Then* $\mathbf{C}$ *satisfies LPO and hence* $DX \cong X \times \mathbb{N} + 1$*.*

Proof. It is folklore that in categories with countable products $\mathbb{N}$ is isomorphic to the sum of ω copies of 1. Thus $\hat\iota\colon \mathbb{N} \to \bar{\mathbb{N}}$ is the induced universal map, which is complemented by (ii). ◀

▶ **Example 5.** As expected, Proposition 3 does not apply to models, designed with constructivist principles in mind, such as intensional type theories, or realizability toposes, although, it is technically possible to design a realizability topos, satisfying LPO [7], in which thus $DX \cong X \times \mathbb{N} + 1$. Another class of examples to which Proposition 3 does not apply stems from topology. In **Top**, $\bar{\mathbb{N}}$ is a subspace of the *Cantor space* $2^{\mathbb{N}}$ whose topology is generated by the base of opens of the form $\{sr \mid r \in \{0,1\}^{\omega}\}$ with $s \in 2^{\star}$. Then $\bar{\mathbb{N}}$ is isomorphic to a *one-point compactification* of $\mathbb{N}$, i.e. it is the set $\mathbb{N} \cup \{\infty\}$, whose opens are all subsets of $\mathbb{N}$ and additionally all complements of finite subsets of $\mathbb{N}$ in $\mathbb{N} \cup \{\infty\}$. Clearly, $\bar{\mathbb{N}} \ncong \mathbb{N} + 1$. This kind of arguments is inherited by higher order topology-based models, such as Johnstone's *topological topos* [23], which is a Grothendieck topos not satisfying LPO.

▶ **Example 6.** Proposition 3 and Corollary 4 cover quite a few models constructed in the scope of classical mathematics. Every set theory satisfying the law of excluded middle satisfies LPO. Every presheaf topos (w.r.t. a classical set theory) inherits countable coproducts from **Set** and those satisfy (ii) of Corollary 4. As we indicated in Example 5, a Grothendieck topos generally need not satisfy LPO, but, e.g. *Schanuel topos* (aka the topos of nominal sets) does satisfy it, because this topos is Boolean. As we indicated in Example 5, **Top** does not satisfy LPO, but curiously the full subcategory of directed complete partial orders **dCpo** (under Scott topology) does. Both **Top** and **dCpo** have countable coproducts, but **Top** fails to satisfy condition (ii), of Corollary 4, while **dCpo** does satisfy it. This can be read as a manifestation of (undesirable) effects, which motivated synthetic domain theory [21].

Conditions (i) and (ii) in Corollary 4 are essentially the axioms of *hyper-extensive categories* by Adámek et al [3] (modulo our background extensiveness assumption). An example of an LPO category that fails (i) is Lawvere's ETCS. Another example of a Grothendieck topos that fails (ii) can be rendered as a certain category of *Jónsson-Tarski algebras* [3].

The above examples indicate that in models developed w.r.t. constructive foundations LPO fails by design, while in models developed w.r.t. classical foundations, depending on the purposes, constructively questioned principles may leak in from the metalogic level inside of the category, possibly in a weakened form, resulting in an explicit expression for DX.

4 Unguarded Elgot Algebras

Recall the following notion from [2] where the term *complete Elgot algebra over H* is used.

▶ **Definition 7** (Guarded Elgot Algebras). *Given an endofunctor H, an* (H-)guarded Elgot algebra *is a tuple $(A, a\colon HA \to A, (-)^\sharp)$ where the* iteration $f^\sharp\colon X \to A$ *for every given $f\colon X \to A + HX$, satisfies the following axioms:*

- (**Fixpoint**) *for every $f\colon X \to A + HX$, $f^\sharp = [\mathsf{id}, a\ Hf^\sharp]\ f$;*
- (**Uniformity**) *for every $f\colon X \to A + HX$ every $g\colon Y \to A + HY$ and every $h\colon X \to Y$, $(\mathsf{id} + Hh)\ f = g\ h$ implies $f^\sharp = g^\sharp\ h$;*
- (**Compositionality**) *for every $h\colon Y \to X + HY$ and $f\colon X \to A + HX$, $((f^\sharp + \mathsf{id})\ h)^\sharp = \big([(\mathsf{id} + H\,\mathsf{inl})\ f, \mathsf{inr}\ (H\,\mathsf{inr})]\ [\mathsf{inl}, h]\colon X + Y \to A + H(X+Y)\big)^\sharp\ \mathsf{inr}$.*

H-guarded Elgot algebras form a category together with iteration preserving *morphisms defined as follows: a morphism h from $(A, a, (-)^\sharp)$ to $(B, b, (-)^\sharp)$ is a morphism $h\colon A \to B$ between carriers, such that $h\ f^\sharp = ((h + \mathsf{id})\ f)^\sharp$ for every $f\colon X \to A + HX$ (this entails $h\ a = b\ (Hh)$ [2, Lemma 5.2]).*

The **Compositionality** axiom is the most sophisticated one. It intuitively states that running h in a loop over Y as the state space, and subsequently running f in a loop over X as the state space, equivalently corresponds to running a certain term constructed from f and g in a single loop over the combined state $X + Y$.

The axioms of guarded Elgot algebras are complete in the following sense.

▶ **Theorem 8** ([2, Theorem 5.4, Corollary 5.7, Theorem 5.8]). *For every X, a final coalgebra $\nu\gamma.\, X + H\gamma$ is a free H-guarded algebra over X, in particular, existence of final coalgebras is equivalent to existence of free H-guarded Elgot algebras. The categories of H-guarded Elgot algebras and algebras of the monad $\nu\gamma.\, X + H\gamma$ are isomorphic.*

By Theorem 8, free algebras of the delay monad are thus precisely the free Id-guarded Elgot algebras. We then introduce *un-guarded Elgot algebras* as a certain subcategory of Id-guarded ones.

▶ **Definition 9** (Unguarded Elgot Algebras). *We call* Id*-guarded Elgot algebras of the form* $(A, \mathsf{id}\colon A \to A, (-)^\sharp)$ unguarded Elgot algebras*, or simply* Elgot algebras *if no confusion arises. Given two Elgot algebras A and B, we call $f\colon X \times A \to B$ right iteration preserving if*

$$f(\mathsf{id} \times h^\sharp) = \big(X \times Z \xrightarrow{\mathsf{id}\times h} X \times (A + Z) \xrightarrow{\mathsf{dstr}} X \times A + X \times Z \xrightarrow{f+\mathsf{id}} B + X \times Z\big)^\sharp$$

for any $h\colon Z \to A + Z$. This generalizes Elgot algebra morphisms under $X = 1$.

We write simply "iteration preserving" instead of "right iteration preserving" in the sequel if the decomposition of $X \times A$ into the Elgot algebra part A and the parameter part X is clear from the context. Parametrization will be needed later for characterizing stability of free algebras (Lemma 18).

The unguarded Elgot algebras thus differ from the Id-guarded ones in that the Id-algebra structures $a\colon A \to A$ in the former case are forced to be trivial. This has an impact on forming the corresponding free structures: in the guarded case, the Id-algebra structures must be maximally unrestricted, which is the reason why we obtain a free Id-guarded Elgot algebra DX with the Id-algebra structure playing the role of delays. Intuitively, a free unguarded Elgot algebra must be a quotient of a free guarded one under removing delays, which is indeed what happens for LPO categories, as we show later. Otherwise, the situation is much more subtle, and it is one of our goals to demonstrate that free unguarded Elgot algebras are exactly the semantic carriers generated by unguarded iteration.

In the unguarded case **Compositionality** can be replaced by a simpler looking new law that we dub **Folding**:

▶ **Proposition 10.** *Given $A \in |\mathbf{C}|$, $(A, (-)^\sharp)$ is an Elgot algebra iff $(-)^\sharp$ satisfies*

- (**Fixpoint**) *for every $f\colon X \to A + X$, $f^\sharp = [\mathsf{id}, f^\sharp]\; f$;*
- (**Uniformity**) *for every $f\colon X \to A + X$ every $g\colon Y \to A + Y$ and every $h\colon X \to Y$, $(\mathsf{id} + h)\; f = g\; h$ implies $f^\sharp = g^\sharp\; h$;*
- (**Folding**) *for every $h\colon Y \to X + Y$ and $f\colon X \to A + X$, $(f^\sharp + h)^\sharp = [(\mathsf{id} + \mathsf{inl})\; f, \mathsf{inr}\;\; h]^\sharp$.*

The laws of Elgot algebras are summarised in Fig. 1 in the style of string diagrams, akin to those, which are used for axiomatizing traced symmetric monoidal categories [24]. In contrast to the latter, here we essentially can only form traces of morphisms of the form $X + Y \to A + Y$ where A is an Elgot algebra. Merging wires is to be interpreted as calling codiagonal morphisms $\nabla\colon X + X \to X$.

As expected, products and exponents of Elgot algebras can be formed in a canonical way.

▶ **Lemma 11.** *Given two Elgot algebras $(A, (-)^\sharp)$ and $(B, (-)^\sharp)$ and an object $X \in |\mathbf{C}|$,*

1. *$(A \times B, (-)^{\hat\sharp})$ is an Elgot algebra with $h^{\hat\sharp} = \langle((\mathsf{fst} + \mathsf{id})\; h)^\sharp, ((\mathsf{snd} + \mathsf{id})\; h)^\sharp\rangle$ for any $h\colon Z \to A \times B + Z$.*
2. *If A^X exists then $(A^X, (-)^{\hat\sharp})$ is an Elgot algebra with $h^{\hat\sharp} = \mathsf{curry}((\mathsf{ev} + \mathsf{id})\;\; \mathsf{dstl}\; (h \times \mathsf{id}))^\sharp$ for any $h\colon Z \to A^X + Z$.*

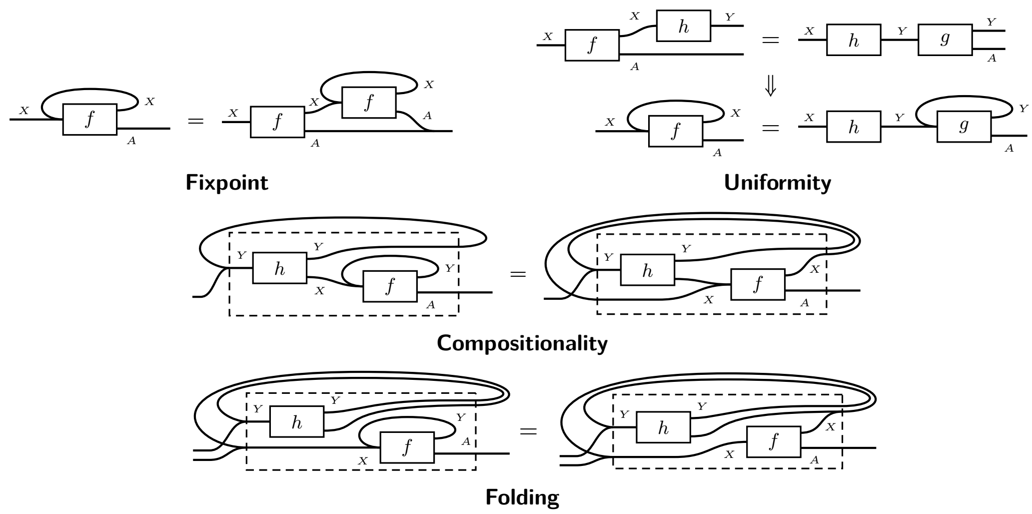

Figure 1 Laws of (unguarded) Elgot algebras.

Every Elgot algebra $(A, (-)^\sharp)$ comes together with a divergence constant $\bot\colon 1 \to A = (\mathsf{inr}\colon 1 \to A + 1)^\sharp$. Note that $\bot$ is automatically preserved by Elgot algebra morphisms.

By omitting the not entirely self-motivating **Compositionality** (or **Folding**) law, we obtain what we dub *uniform-iteration algebras.* As we see later, this law is automatic for free uniform-iteration algebras.

▶ **Definition 12** (Uniform-Iteration Algebras). *A* uniform-iteration algebra *is a tuple* $(A, (-)^\sharp)$ *as in Definition 9 but* $(-)^\sharp$ *is only required to satisfy* **Fixpoint** *and* **Uniformity**. *Morphisms of uniform-iteration algebras are defined in the same way.*

5 The Initial Pre-Elgot Monad

The goal of this section is to show that free uniform-iteration algebras coincide with free Elgot algebras (Theorem 29), and enjoy a number of other characteristic properties. In particular, we characterize the functor sending any X to a free uniform-iteration algebra on X as an initial pre-Elgot monad. We define pre-Elgot monads as follows.

▶ **Definition 13** (Pre-Elgot Monads). *We call a monad* $\mathbf{T}$ pre-Elgot *if every* TX *is equipped with an Elgot algebra structure, in such a way that* $h^\star\, f^\sharp = ((h^\star + \mathsf{id})\ f)^\sharp$ *for any* $f\colon Z \to TX + Z$ *and any* $h\colon X \to TY$. *A pre-Elgot monad* $\mathbf{T}$ *is* strong pre-Elgot *if* $\mathbf{T}$ *is strong as a monad and strength is iteration preserving.*

Pre-Elgot monads are to be compared with Elgot monads, which support a stronger type profile for the iteration operator, and satisfy more sophisticated axioms.

▶ **Definition 14** (Elgot Monads [15, 4]). *A monad* $\mathbf{T}$ *is an* Elgot monad *if it is equipped with an iteration operator sending each* $f\colon X \to T(Y + X)$ *to* $f^\dagger\colon X \to TY$ *and satisfying:*

- (**Fixpoint**) $f^\dagger = [\eta, f^\dagger]^\star\ f$;
- (**Naturality**) $g^\star\ f^\dagger = ([(T\,\mathsf{inl})\ g,\ \eta\ \mathsf{inr}]^\star\ f)^\dagger$ *for* $f\colon X \to T(Y + X)$, $g\colon Y \to TZ$;
- (**Codiagonal**) $(T[\mathsf{id}, \mathsf{inr}]\ f)^\dagger = f^{\dagger\dagger}$ *for* $f\colon X \to T((Y + X) + X)$;
- (**Uniformity**) $f\ h = T(\mathsf{id}{+}h)\ g$ *implies* $f^\dagger\ h = g^\dagger$ *for* $f\colon X \to T(Y{+}X)$, $g\colon Z \to T(Y{+}Z)$ *and* $h\colon Z \to X$.

If $\mathbf{T}$ *is additionally strong then* $\mathbf{T}$ *is strong Elgot if moreover:*

- (**Strength**) $\tau\ (\mathsf{id} \times f^\dagger) = ((T\,\mathsf{dstr})\ \tau\ (\mathsf{id} \times f))^\dagger$ *for any* $f\colon X \to T(Y + X)$.

▶ **Proposition 15.** *(Strong) Elgot monads are (strong) pre-Elgot under* $f^\sharp = ([T\,\mathsf{inl}, \eta\,\mathsf{inr}]\ f)^\dagger$.

It has been argued [17, 20] that strong Elgot monads are minimal semantic structures for interpreting effectful while-languages. In that sense, we acknowledge an expressivity gap between Elgot and pre-Elgot monads, which generally happen to be too weak. We will consider approaches to close this gap, in particular by drawing on some versions of the axiom of countable choice. Even though, in general, the gap presumably cannot be closed, we regard the initial pre-Elgot monad to be an important notion, which arises from first principles and carries a very clear operational intuition. The discrepancy between pre-Elgot monads and Elgot monads seems to represent a very basic form of discrepancy between operational and denotational semantics. We thus find it important to conceptually delineate between Elgot monads and pre-Elgot monads, no matter how desirable it is to have them to be equivalent.

▶ **Lemma 16.** *If for every* $X \in |\mathbf{C}|$ *a free uniform-iteration algebra* KX *exists then* K *extends to a monad* $\mathbf{K}$ *whose algebras are precisely uniform-iteration algebras.*

As in the case of natural numbers, one cannot make much progress without stability.

▶ **Definition 17** (Stable Free Uniform-Iteration Algebras)**.** *A free uniform-iteration algebra* KY *over* Y *is* stable *if for every* $X \in |\mathbf{C}|$, $\mathsf{fst}\colon X \times KY \to X$ *is a free uniform-iteration algebra in the slice category* $\mathbf{C}/X$.

▶ **Lemma 18.** *For* $Y \in |\mathbf{C}|$, KY *is stable iff for every uniform-iteration* A *and every* $f\colon X \times Y \to A$, *there is unique iteration preserving* $f^\#\colon X \times KY \to A$ *such that* $f = f^\#\ (\mathsf{id} \times \eta)$.

Using Lemma 11, it is easy to show that in Cartesian closed categories every KX is stable. For the rest of the section, we assume that all free uniform-iteration algebras KX exist and are stable.

▶ **Proposition 19.** *The monad* $\mathbf{K}$ *is strong, with the components of strength* $\tau\colon X \times KY \to K(X \times Y)$ *uniquely identified by the conditions:*

$$\tau\ (\mathsf{id} \times \eta) = \eta, \qquad \tau\ (\mathsf{id} \times h^\sharp) = ((\tau + \mathsf{id})\ \mathsf{dstr}\ (\mathsf{id} \times h))^\sharp \qquad (h\colon Z \to KY + Z)$$

Proof. In the notation of Lemma 18 we define strength of $\mathbf{K}$ as $(\eta\colon X \times Y \to K(X \times Y))^\#$. The axioms of strength are easy to verify. ◀

As a next step, we show that $\mathbf{K}$ is an equational lifting monad in the sense of Bucalo et al [9]. This means precisely that $\mathbf{K}$ is commutative and satisfies the equational law:

$$\tau\ \Delta = K\langle \eta, \mathsf{id}\rangle. \tag{1}$$

This law is rather restrictive, and roughly means that some form of non-termination is the only possible effect of the monad. Proving (1) is nontrivial. The key step is the following property, which allows for splitting a loop involving a product of algebras into two loops.

▶ **Lemma 20.** *Given uniform-iteration algebras* A *and* B, $f\colon Z \to A \times B + Z$ *and* $h\colon A \times B \to C$, $((h + \mathsf{id})\ f)^\sharp = ((h + \mathsf{id})\ \mathsf{dstr}\ (\mathsf{id} \times (\mathsf{snd} + \mathsf{id})\ f))^\sharp\ \langle ((\mathsf{fst} + \mathsf{id})\ f)^\sharp, \mathsf{id}\rangle$.

▶ **Lemma 21.** *Given* $X, Z \in |\mathbf{C}|$, *and* $h\colon Z \to KX + Z$, *then* $\tau\ \langle h^\sharp, h^\sharp\rangle = ((\tau\ \Delta + \mathsf{id})\ h)^\sharp$.

Proof. It follows from Lemma 20 that $((\tau + \mathsf{id})\ \mathsf{dstr}\ (\mathsf{id} \times h))^\sharp\ \langle h^\sharp, \mathsf{id}\rangle = ((\tau\ \Delta + \mathsf{id})\ h)^\sharp$. On the other hand, by Proposition 19, $((\tau + \mathsf{id})\ \mathsf{dstr}\ (\mathsf{id} \times h))^\sharp\ \langle h^\sharp, \mathsf{id}\rangle = \tau\ \langle h^\sharp, h^\sharp\rangle$. By combining the last two identities, we obtain the goal. ◀

▶ **Theorem 22.** $\mathbf{K}$ *is an equational lifting monad.*

Proof. Let us sketch the proof of (1). Since $K\langle\eta,\mathsf{id}\rangle = (\eta\ \langle\eta,\mathsf{id}\rangle)^\star$, using the definition of Kleisli star for K, it suffices to show that $\tau\ \Delta$ is a unique iteration preserving morphism for which $\eta\ \langle\eta,\mathsf{id}\rangle = \tau\ \Delta\ \eta$. Indeed, $\tau\ \Delta\ \eta = \tau\ (\mathsf{id}\times\eta)\ \langle\eta,\mathsf{id}\rangle = \eta\ \langle\eta,\mathsf{id}\rangle$, and $\tau\ \Delta$ is iteration preserving by Lemma 21. ◀

The fact that $\mathbf{K}$ is an equational lifting monad has a number of implications, in particular, the Kleisli category of $\mathbf{K}$ is a *restriction category* [13]. That is, we can calculate the *domain (of definiteness)*, represented by an idempotent Kleisli morphism as follows: given $f\colon X\to KY$,

$$\mathsf{dom}\, f = (K\,\mathsf{fst})\ \tau\ \langle\mathsf{id}, f\rangle\colon X\to KX,$$

We additionally use the notation $f\downharpoonleft g = \mathsf{fst}^\star\ \tau\ \langle f,g\rangle$, meaning: restrict f to the domain of g. It is easy to see that $\mathsf{dom}\, f = \eta\downharpoonleft f$ and $f\downharpoonleft g = f^\star\ (\mathsf{dom}\, g)$. Let $f\sqsubseteq g$ abbreviate $f = g\downharpoonleft f$. Under this definition, every $\mathbf{C}(X,KY)$ is partially ordered, which is a general fact about restriction categories. In our case, moreover, this partial order additionally has a bottom element $\bot = \mathsf{inr}^\sharp$; $\mathsf{dom}(\eta\ f) = \eta$ for any $f\colon X\to KY$, and $\mathsf{dom}\, f\sqsubseteq\eta$ for any f.

▶ **Proposition 23.** *The Kleisli category of* $\mathbf{K}$ *is enriched over pointed partial orders and strict monotone maps. Moreover, strength preserves* $\bot$ *and* $\sqsubseteq$ *as follows:*

$$\tau\ (\mathsf{id}\times\bot) = \bot \qquad\qquad f\sqsubseteq g \quad \textit{implies} \quad \tau\ (\mathsf{id}\times f)\sqsubseteq\tau\ (\mathsf{id}\times g)$$

▶ **Corollary 24.** $K\emptyset\cong 1$.

Proof. Since $!\,\bot = \mathsf{id}\colon 1\to 1$ and $\bot\,! = \mathsf{id}\colon K\emptyset\to K\emptyset$, we obtain an isomorphism $K\emptyset\cong 1$. ◀

▶ **Proposition 25.** *The monad* $\mathbf{K}$ *is* copyable *and* weakly discardable *[18], i.e.:* $\hat\tau^\star\ \tau\ \Delta = K\Delta$ *and* $(K\,\mathsf{fst})\ \hat\tau^\star\ \tau\ \langle f,g\rangle\sqsubseteq f$ *for* $f\colon X\to KY$ *and* $g\colon X\to KZ$.

▶ **Definition 26** (Bounded Iteration)**.** *Let A be a* pointed object*, i.e. an object with a canonical map* $\bot\colon 1\to A$*. Then we define* bounded iteration $(\text{–})^{(\sharp}\colon \mathbf{C}(X,A+X)\to\mathbf{C}(X\times\mathbb{N},A)$ *by primitive recursion as follows:*

$$f^{(\sharp}(x,\mathsf{o}) = \bot \qquad\qquad f^{(\sharp}(x,\mathsf{s}\,n) = \mathsf{case}\ f(x)\ \mathsf{of}\ \mathsf{inl}\, a\mapsto a;\ \mathsf{inr}\, y\mapsto f^{(\sharp}(y,n).$$

Intuitively, $f^{(\sharp}(x,n)$ behaves as $f^\sharp(x)$ except that at each iteration the counter n is decreased, and $\bot$ is returned once $n = \mathsf{o}$. We next show that $f^\sharp(x)$ is in a suitable sense a limit of the $f^{(\sharp}(x,n)$ as n tends to infinity. This is, of course, a form of *Kleene fixpoint theorem.*

▶ **Theorem 27** (Kleene Fixpoint Theorem)**.** *Given* $f\colon X\to KY+X$*, and* $g\colon X\to KY$*, (i)* $f^{(\sharp}\sqsubseteq f^\sharp\ \mathsf{fst}$*, and (ii)* $f^{(\sharp}\sqsubseteq g\ \mathsf{fst}$ *implies* $f^\sharp\sqsubseteq g$.

▶ **Corollary 28.** *Given* $f\colon X\to KY+X$*,* $f^\sharp\colon X\to KY$ *is the least pre-fixpoint of the map* $[\mathsf{id},\text{–}]\ f\colon\mathbf{C}(X,KY)\to\mathbf{C}(X,KY)$.

Finally, we obtain

▶ **Theorem 29.** $\mathbf{K}$ *is an initial pre-Elgot monad and an initial strong pre-Elgot monad.*

$$\begin{tikzcd}
& D\text{-algebras} & \\
\begin{array}{c}\text{uniform-iteration algebras/}\\ \text{search-algebras}\end{array} \arrow[ur, bend left] & & \mathbf{D}\text{-algebras} \arrow[ul, bend right] \\
& \text{Elgot algebras} \arrow[ul, bend left] \arrow[ur, bend right] &
\end{tikzcd}$$

Figure 2 Connections between classes of D-algebras.

6 Quotienting the Delay Monad

By Theorem 8, Id-guarded Elgot algebras are precisely the $\mathbf{D}$-algebras. We proceed to characterize uniform-iteration and Elgot algebras as certain D-algebras, which we dub *search-algebras*. Intuitively, modulo identification of DA with a set of streams from $(A+1)^{\mathbb{N}}$, a search-algebra structure $a\colon DA \to A$ is guaranteed to find the first element in the stream of the form $\mathsf{inl}\, a$ if it exists. We expect that this notion can be formulated more generally, but we do not pursue it here.

▶ **Definition 30** (Search-Algebra). *We call a D-algebra $(A, a\colon DA \to A)$ a* search-algebra *if it satisfies the conditions: $a\ \mathsf{now} = \mathsf{id}$, $a\ \triangleright = a$. Search-algebras form a full subcategory of the category of all D-algebras.*

Uniform-iteration algebras capture the structure of search-algebras independently of the assumption that D exists. This and further connections between categories of D-algebras illustrated in Fig. 2 (arrows indicate full embeddings of categories) are formalized as follows.

▶ **Proposition 31.**

1. *The categories of uniform-iteration algebras and search-algebras are isomorphic under:*

$$\begin{aligned}(A, (-)^{\sharp}) &\mapsto (A, \mathsf{out}^{\sharp}\colon DA \to A),\\ (A, a\colon DA \to A) &\mapsto (A, a\ \ \mathsf{coit}(-)\colon \mathbf{C}(X, A+X) \to \mathbf{C}(X, A)).\end{aligned}$$

2. *Elgot algebras are precisely those D-algebras, which are search-algebras and $\mathbf{D}$-algebras.*

▶ **Lemma 32.** *Every Elgot algebra $(DA, a\colon DA \to A)$ satisfies $a\ \iota^{\star} = a\ (D\,\mathsf{fst})$.*

We proceed to model the construction of quotienting $\mathbf{D}$ by weak bisimilarity $\approx$, previously described in type-theoretic terms [11]. Modulo identification of DX with the object of those streams $\sigma\colon \mathbb{N} \to X+1$ for which $\sigma(n) \neq \mathsf{inr}\,\star$ for at most one n, $\approx$ can be described as follows: $\sigma \approx \sigma'$ if for every a, $\sigma(n) = a$ for some n iff $\sigma'(n) = a$ for some n.

Recall the embedding $\iota\colon X \times \mathbb{N} \hookrightarrow DX$, and define the quotient of DX by the coequalizer

$$D(X \times \mathbb{N}) \overset{\iota^{\star}}{\underset{D\,\mathsf{fst}}{\rightrightarrows}} DX \xrightarrow{\ \rho_X\ } \widetilde{D}X \tag{2}$$

which we assume to exist and be preserved by products. It is then straightforward that $\widetilde{D}$ is a functor and ρ_X is natural in X. It also follows that $X \xrightarrow{\mathsf{now}} DX \xrightarrow{\rho} \widetilde{D}X$ is strong. Following tradition, we denote $\widetilde{D}1$ as Σ.

▶ **Lemma 33.** $\rho\ \triangleright = \rho$.

Defining ρ as a coequalizer of $\triangleright$ and id in the first place does not seem to be sufficient, though, in particular, for showing the following property. We leave open the question of identifying conditions under which it is possible.

▶ **Proposition 34.** *The following is a coequalizer:*

$$D(X+(X\times\mathbb{N}+X\times\mathbb{N})) \underset{[\eta,[\eta\ \mathsf{fst},\iota\ (\mathsf{id}\times\mathsf{s})]]^\star}{\overset{[\eta,[\iota\ (\mathsf{id}\times\mathsf{s}),\eta\ \mathsf{fst}]]^\star}{\rightrightarrows}} DX \xrightarrow{\rho_X} \widetilde{D}X \tag{3}$$

The last proposition brings the definition of ρ in accordance with the intuition; the coproduct $X+(X\times\mathbb{N}+X\times\mathbb{N})$ covers three alternatives for $\sigma\approx\sigma'$: either $\sigma=\sigma'$, or σ terminates earlier than σ' by a specified number, or the other way around. It can be verified that the embedding $D(X+(X\times\mathbb{N}+X\times\mathbb{N}))\hookrightarrow DX\times DX$ is an internal equivalence relation.

▶ **Theorem 35.** *The following conditions are equivalent:*
1. *for every X, coequalizer (2) is preserved by D;*
2. *every $\widetilde{D}X$ extends to a search-algebra, so that each ρ_X is a D-algebra morphism;*
3. *for every X, $(\widetilde{D}X,\rho\ \mathsf{now}\colon X\to\widetilde{D}X)$ is a stable free Elgot algebra on X, ρ_X is a D-algebra morphism and $\rho_X=((\rho_X\ \mathsf{now}+\mathsf{id})\ \mathsf{out})^\sharp$;*
4. *$\widetilde{D}$ extends to a strong monad, so that ρ is a strong monad morphism.*

If the equivalent conditions of Theorem 35 are satisfied, we obtain an explicit construction of the initial pre-Elgot monad K, which we explored previously. Let us consider concrete examples.

▶ **Example 36** (Maybe-Monad)**.** Suppose that $\mathbf{C}$ is an LPO category, and recall that DX is isomorphic to $X\times\mathbb{N}+1$. It is then easy to check that (2) exists, it is preserved by products, $\widetilde{D}X\cong X+1$ and $\rho=\mathsf{fst}+\mathsf{id}\colon X+1\to X\times\mathbb{N}+1$. Since D is the composition of $(-\times\mathbb{N})$ and $(-+1)$, and both these functors preserve coequalizers (first as a left adjoint, and second by extensiveness of $\mathbf{C}$), D preserves (2). We thus obtain that the maybe-monad is an initial pre-Elgot monad. This covers instances of LPO categories from Example 6. Moreover, the initial pre-Elgot monad is in fact an initial Elgot monad in this case: the profiles of the iteration operators $(-)^\sharp$ and $(-)^\dagger$ agree up to rearrangement of summands, and the axioms of Definition 14 become the axioms of Definition 13, except for **Codiagonal**, which can be checked directly.

Note that Example 36 entails that the maybe-monad is the initial Elgot monad in $\mathbf{dCpo}$. This is a result of our assumption that $\mathbf{dCpo}$ is developed w.r.t. a classical set theory, which entails that $\mathbf{dCpo}$ is an LPO category. This would not be the case if we defined $\mathbf{dCpo}$ internally to a non-classical environment, which is indeed the core idea of synthetic domain theory.

Another direction for obtaining an Elgot monad from (2) is by using a suitable instance of the *axiom of countable choice*. In our setting this takes the following form.

▶ **Theorem 37.** *Suppose that the coequalizers (2) are preserved by the exponentiation $(-)^{\mathbb{N}}$.*
1. *The equivalent conditions of Theorem 35 hold, in particular, K is an initial (strong) pre-Elgot monad.*
2. *If every (3) is an effective quotient, i.e. $D(X+(X\times\mathbb{N}+X\times\mathbb{N}))$ is a kernel pair of ρ_X, then K is a strong Elgot monad with $f^\dagger$ being the least fixpoint of $[\eta,-]^\star\ f\colon\mathbf{C}(X,TY)\to\mathbf{C}(X,TY)$ for any $f\colon X\to T(Y+X)$.*

The effectiveness assumption in clause 2. is satisfied in any exact category (e.g. in any pretopos) – by definition, every internal equivalence relation there is effective.

▶ **Example 38.** Theorem 37 applies to **Top**, yielding a concrete description for **K**. Recall that in **Top**, coequalizers are computed as in **Set** and are equipped with the quotient topology. Note that DX is the set $X \times \mathbb{N} \cup \{\infty\}$ whose base opens are $\{(x, n) \mid x \in O\}$ and $\{(x, k) \mid x \in X, k \geqslant n\} \cup \{\infty\}$ with $n \in \mathbb{N}$ and O ranging over the opens of X. The collapse $\widetilde{D}X$ computed with (2) is thus the set $X \cup \{\infty\}$, whose opens are those of X and additionally the entire space $X \cup \{\infty\}$, in particular, $\widetilde{D}1$ is the *Sierpiński space.*

To obtain that (2) is preserved by $(-)^{\mathbb{N}}$, it suffices to show that the opens of $(X \cup \{\infty\})^{\mathbb{N}}$ are precisely those, whose inverse images under $\rho^{\mathbb{N}}$ are open. This is in fact true for any regular epi in **Top**. The effectiveness condition in 2. is not vacuous for **Top**, which is not an exact category (and not even regular), but it can be checked manually.

In every pretopos, preservation of (2) by $(-)^{\mathbb{N}}$ is a proper instance of the *internal axiom of countable choice*, or *internal projectivity of* $\mathbb{N}$, which means preservation of epis by $(-)^{\mathbb{N}}$, roughly because every pretopos is *exact* and our quotienting morphism ρ is associated with an internal equivalence relation by Proposition 34. Theorem 37 can thus be related to the existing result in synthetic domain theory, that Rosolini dominance, i.e. our Σ, is indeed a dominance [31], which applies to Hyland's *effective topos* [22], as it satisfies countable choice. Contrastingly, we cannot apply Theorem 37 to nominal sets, which falsify countable choice, however, as a Boolean topos, nominal sets fall into the scope of Example 36.

We currently do not have a concrete example of **K** being definable, but not being an Elgot monad. Theorem 37 and Example 36 indicate that a category to witness this must neither support excluded middle nor the axiom of countable choice.

7 Conclusions and Further Work

Iteration and iteration theories emerged as unifying concepts for computer science semantics and reasoning. By interpreting iteration suitably, one obtains a basic extensible equational logic of programs, shown to be sound and complete across various models [8]. Elgot monads implement this inherently algebraic view in the general categorical realm of abstract data types and effects. The class of Elgot monads (over a fixed category) is stable under various categorical constructions (monad transformers), and thus one can build new Elgot monads from old, but the simplest Elgot monad, the initial one, does not arise in this way.

Here, we proposed an approach to defining an initial iteration structure from first principles, characterized it in various ways, analysed conditions, under which it can be concretely described, and to yield an Elgot monad. Unsurprisingly, these conditions generally cannot be lifted, as the previous research in type theory indicates. We consider broadening the scope in which results about notions of partiality apply, and unifying both classical and non-classical models, as an important part of our contribution. Universal properties play a central role in category theory, but many important concepts are not covered by them. One example is Sierpiński space, which is fundamental in topology, duality theory and domain theory. It follows from our results, that it is in fact a free uniform-iteration algebra on one generator. We believe that the structure of our results can be reused in more sophisticated setting, such as semantics of *hybrid systems*, which require a notion of partiality, combined with continuous evolution, and rise semantic issues, structurally similar to those, we considered here [14]. Another potential for taking further the present work is to consider more general shapes of the basic functor (instead of the current $(X + -)$), prospectively leading to more sophisticated (non-)structural recursion scenarios (see e.g. [1]).

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