Powerset-Like Monads Weakly Distribute over Themselves in Toposes and Compact Hausdorff Spaces

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Abstract

The powerset monad on the category of sets does not distribute over itself. Nevertheless a weaker form of distributive law of the powerset monad over itself exists and it essentially stems from the canonical Egli-Milner extension of the powerset to the category of relations. On the other hand, any regular category yields a category of relations, and some regular categories also possess a powerset-like monad, as is the Vietoris monad on compact Hausdorff spaces. We derive the Egli-Milner extension in three different frameworks : sets, toposes, and compact Hausdorff spaces. We prove that it corresponds to a monotone weak distributive law in each case by showing that the multiplication extends to relations but the unit does not. We provide an application to coalgebraic determinization of alternating automata.

2012 ACM Subject Classification Theory of computation → Formal languages and automata theory

Keywords and phrases Egli-Milner relation, weak extension, weak distributive law, weak lifting, powerset monad, Vietoris monad, topos, alternating automaton, generalized determinization

Digital Object Identifier 10.4230/LIPIcs.ICALP.2021.132

Category Track B: Automata, Logic, Semantics, and Theory of Programming

1 Introduction

Composing monads is usually achieved using distributive laws. Unfortunately, sometimes these do not exist. It is known since the work of Varacca [24] that there is no distributive law between the monad **D** of probability distributions and the powerset monad **P**. The proof, attributed to Plotkin, relies on a manipulation of the naturality squares of the unit of **D**. More recently, [14] showed that there is no distributive law of the powerset **P** over itself, and even more nor over its iterations. More negative results for other Set-based algebraic theories are presented in [26].

One way to circumvent such negative results is to compose monads using weaker forms of distributive laws. In the definition of a distributive law between monads we have four axioms specifying the interactions of the law with the units, respectively with the multiplications of the two monads. In a weak distributive law, an axiom involving the unit of one of the monads is dropped. In our previous paper [11] we exhibited a canonical weak distributive law between the monads **D** and **P**. It comes as no surprise that the axiom that is dropped from the definition of such a law is the one involving the unit of **D** – on which the argument of Plotkin relied. Our work in turn, was based on Garner's results [10]. In loc. cit. he exhibited a weak distributive law between the powerset monad and the ultrafilter monad β. This leads to a weak lifting of the powerset to the category of Eilenberg-Moore algebras of β, that is, to the category of compact Hausdorff spaces KHaus. This weak lifting is the Vietoris monad and is indeed the closest there is to a powerset-like monad on KHaus. In [10] it is also shown that the powerset monad on Set weakly distributes over itself.

The weak distributive laws in [10] and [11] were all of the form $\mathbf{TP} \to \mathbf{PT}$, where $\mathbf{P}$ is the powerset monad on Set. The recipe for obtaining these laws was based on the monotone weak extension of the monad $\mathbf{T}$ to the category of relations, this is, the Kleisli category of $\mathbf{P}$.

The motivation of the present paper is to understand composition of monads via weak distributive laws in other settings than for Set-based monads. In particular our aim here is to build the technology necessary for combining various forms of nondeterminism (ordinary and probabilistic) in a continuous setting.

An obvious starting point is to consider the category of compact Hausdorff spaces KHaus, where we have the Vietoris monad $\mathbf{V}$ whose Kleisli category $\mathsf{Kl}(\mathbf{V})$ can be seen as a category of relations satisfying additional continuity constraints. The first question we have asked ourselves is whether the Vietoris monad $\mathbf{V}$ weakly distributes over itself. Can we extend Garner's result from Set to KHaus? It turns out the answer is positive, see Theorem 24. In the process we have used various results from the literature, in particular, the work of [7] for extending functors on a regular category C to the category of relations $\mathsf{Rel}(\mathsf{C})$. Nevertheless, the results are far from immediate since $\mathsf{Kl}(\mathbf{V})$ is only a subcategory of $\mathsf{Rel}(\mathsf{KHaus})$, hence some additional work is needed to obtain the canonical extension of $\mathbf{V}$ to $\mathsf{Kl}(\mathbf{V})$. We also notice that despite the fact that the Vietoris functor does not preserve pullbacks (see also [6]), it nearly preserves pullbacks, and this is exactly the condition needed in [7] to provide the relational extension. Once the monotone extension of the monad $\mathbf{V}$ to $\mathsf{Kl}(\mathbf{V})$ is found, we can provide the weak distributive law via the same mechanism as in [10].

Another extension we provide is for the powerset-like monad on a topos C. This monad is defined on objects as Ω^X where Ω is the subobject classifier of the topos. Notice that we are not considering the contravariant powerset-like functor usually appearing in topos theory, but rather the monad $\exists$ of [19]. Its Kleisli category is simply $\mathsf{Rel}(\mathsf{C})$ and, as far as the underlying functor is concerned, a monotone extension is obtained again by leveraging the work in [7]. To obtain the extension of the *monad* $\exists$, we need to investigate the properties of its unit and of its multiplication. As far as the multiplication is concerned, we can internalize the proof from the Set case, using the internal logic of the topos. As far as the unit is concerned, we show that it has the required property for obtaining a strong extension of the monad, only when the topos is degenerate. In all other meaningful cases, we thus only obtain a weak extension of $\exists$ to $\mathsf{Kl}(\exists)$, and hence a weak distributive law of $\exists$ over itself, see Theorem 13.

The paper is structured as follows. In Section 2 we recall the necessary preliminaries on weak distributive laws and relations in regular categories. In Section 3 we recall the weak distributive law of the powerset over itself and provide an application to coalgebraic determinization of alternating automata. We find it instructive to understand the proofs first in Set, as they will serve as a basis for the generalization to toposes, performed in Section 4. In Section 5 we provide the weak distributive law of the Vietoris functor over itself and we conclude with a summary and directions for future work in Section 6.

2 Preliminaries

Notations

For a relation $R \subseteq X \times Y$ between two sets and $A \subseteq X$, $B \subseteq Y$, we write $R[A] = \{y \in Y \mid (x, y) \in R \text{ and } x \in A\}$ and $R^{-1}[B] = \{x \in X \mid (x, y) \in R \text{ and } y \in B\}$. The complement of $A \subseteq X$ is denoted by A^c. In the whole paper, C denotes a generic category, $F, G : \mathsf{C} \to \mathsf{C}$ denote functors and $\alpha : F \to G$ denotes a natural transformation. Identity morphisms, functors and natural transformations will all be denoted by 1.

2.1 (Weak) Extensions, (Weak) Distributive Laws, (Weak) Liftings

We assume the reader is familiar with the basic theory of monads, and fix here some notations. A *monad* is a triple $\mathbf{T} = (T, \eta^{\mathbf{T}}, \mu^{\mathbf{T}})$ where $T : \mathsf{C} \to \mathsf{C}$ is a functor, $\eta^{\mathbf{T}} : 1 \to T$ and $\mu^{\mathbf{T}} : TT \to T$ are natural transformations called respectively unit and multiplication and equations $\mu^{\mathbf{T}} \circ T\eta^{\mathbf{T}} = 1 = \mu^{\mathbf{T}} \circ \eta^{\mathbf{T}}T$, $\mu^{\mathbf{T}} \circ T\mu^{\mathbf{T}} = \mu^{\mathbf{T}} \circ \mu^{\mathbf{T}}T$ hold. In the following we fix two monads $\mathbf{T}$ and $\mathbf{S}$ on C. The *Kleisli category* of $\mathbf{S}$ is denoted by $\mathsf{Kl}(\mathbf{S})$. A morphism in $\mathsf{Kl}(\mathbf{S})$ will be denoted by $X \nrightarrow Y$ and corresponds to a morphism $X \to SY$ in C. The Kleisli free and forgetful functor are denoted respectively by $F_{\mathbf{S}} : \mathsf{C} \to \mathsf{Kl}(\mathbf{S})$ and $U_{\mathbf{S}} : \mathsf{Kl}(\mathbf{S}) \to \mathsf{C}$. The *Eilenberg-Moore category* of $\mathbf{T}$ is denoted by $\mathsf{EM}(\mathbf{T})$, objects are algebras (X, x) where X is an object of C and $x : TX \to X$ satisfies $x \circ \eta^{\mathbf{T}}_X = 1_X$ and $x \circ \mu^{\mathbf{T}}_X = x \circ Tx$. The Eilenberg-Moore free and forgetful functor are denoted respectively by $F^{\mathbf{T}} : \mathsf{C} \to \mathsf{EM}(\mathbf{T})$ and $U^{\mathbf{T}} : \mathsf{EM}(\mathbf{T}) \to \mathsf{C}$.

▶ **Example 1.** The *powerset monad* $\mathbf{P}$ on the category of sets and functions Set is defined as follows. The functor P maps a set X to the set of its subsets and acts on functions by taking direct images. Unit is given by the singleton operation $\eta^{\mathbf{P}}_X(x) = \{x\}$ and multiplication by union $\mu^{\mathbf{P}}_X(\mathcal{A}) = \bigcup \mathcal{A}$.

Monads are not stable under composition. However, Beck introduced the framework of distributive laws [1] as a tool to generate composite monads. Distributive laws are actually one face of a three-sided coin comprising also extensions and liftings. Interestingly enough, each component of this triptych can be weakened in such a way that the correspondence still stands between weak distributive laws, weak extensions and weak liftings [10]. The rest of this section aims at jointly recalling both the usual and the weakened framework.

An extension of F to $\mathsf{Kl}(\mathbf{S})$ is a functor $\overline{F} : \mathsf{Kl}(\mathbf{S}) \to \mathsf{Kl}(\mathbf{S})$ such that $\overline{F}F_{\mathbf{S}} = F_{\mathbf{S}}F$. Similarly, an extension of $\alpha : F \to G$ is a natural transformation $\overline{\alpha} : \overline{F} \to \overline{G}$ such that the equation $\overline{\alpha}F_{\mathbf{S}} = F_{\mathbf{S}}\alpha$ holds.

▶ **Definition 2** (Extension). *An* extension *of* $\mathbf{T}$ *to* $\mathsf{Kl}(\mathbf{S})$ *is a monad* $\overline{\mathbf{T}}$ *on* $\mathsf{Kl}(\mathbf{S})$ *whose functor, unit and multiplication are extensions of those of* $\mathbf{T}$*. A* weak extension *only requires the extension of the functor and of the multiplication of* $\mathbf{T}$*.*

▶ **Definition 3** (Distributive law). *A* distributive law *of type* $\mathbf{TS} \to \mathbf{ST}$ *is a natural transformation* $\delta : TS \to ST$ *such that the four following diagrams commute*

$$\begin{tikzcd}
TTS \arrow[r, "T\delta"] \arrow[d, "\mu^{\mathbf{T}}S"'] & TST \arrow[r, "\delta T"] \arrow[d, phantom, "(\mu^{\mathbf{T}})"] & STT \arrow[d, "S\mu^{\mathbf{T}}"] \\
TS \arrow[rr, "\delta"'] & & ST
\end{tikzcd}
\qquad
\begin{tikzcd}
TSS \arrow[r, "\delta S"] \arrow[d, "T\mu^{\mathbf{S}}"'] & STS \arrow[r, "S\delta"] \arrow[d, phantom, "(\mu^{\mathbf{S}})"] & SST \arrow[d, "\mu^{\mathbf{S}}T"] \\
TS \arrow[rr, "\delta"'] & & ST
\end{tikzcd}$$

$$\begin{tikzcd}
TS \arrow[rr, "\delta"] & (\eta^{\mathbf{T}}) & ST \\
& S \arrow[ul, "\eta^{\mathbf{T}}S"] \arrow[ur, "S\eta^{\mathbf{T}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
TS \arrow[rr, "\delta"] & (\eta^{\mathbf{S}}) & ST \\
& T \arrow[ul, "T\eta^{\mathbf{S}}"] \arrow[ur, "\eta^{\mathbf{S}}T"'] &
\end{tikzcd}$$

A weak distributive law *only requires the* $(\eta^{\mathbf{S}})$*,* $(\mu^{\mathbf{S}})$ *and* $(\mu^{\mathbf{T}})$ *diagrams.*

▶ **Definition 4** (Lifting). *A* lifting *of* S *to* $\mathsf{EM}(\mathsf{T})$ *is a monad* $\widehat{\mathsf{S}} : \mathsf{EM}(\mathsf{T}) \to \mathsf{EM}(\mathsf{T})$ *such that* $U^{\mathsf{T}}\widehat{S} = SU^{\mathsf{T}}$, $U^{\mathsf{T}}\widehat{\eta^{\mathsf{S}}} = \eta^{\mathsf{S}}U^{\mathsf{T}}$ *and* $U^{\mathsf{T}}\widehat{\mu^{\mathsf{S}}} = \mu^{\mathsf{S}}U^{\mathsf{T}}$. *A* weak lifting *of* S *on* T *is a monad* $\widehat{\mathsf{S}} : \mathsf{EM}(\mathsf{T}) \to \mathsf{EM}(\mathsf{T})$ *along with two natural transformations* $\pi : SU^{\mathsf{T}} \to U^{\mathsf{T}}\widehat{S}$, $\iota : U^{\mathsf{T}}\widehat{S} \to SU^{\mathsf{T}}$ *such that* $\pi \circ \iota = 1$ *and the following diagrams commute:*

$$\begin{tikzcd}
U^{\mathsf{T}}\widehat{S}\widehat{S} \arrow[r, "\iota\widehat{S}"] \arrow[d, "U^{\mathsf{T}}\widehat{\mu^{\mathsf{S}}}"'] & SU^{\mathsf{T}}\widehat{S} \arrow[r, "S\iota"] \arrow[d, phantom, "(\iota\mu)"] & SSU^{\mathsf{T}} \arrow[d, "\mu^{\mathsf{S}}U^{\mathsf{T}}"] \\
U^{\mathsf{T}}\widehat{S} \arrow[rr, "\iota"'] & & SU^{\mathsf{T}}
\end{tikzcd}
\qquad
\begin{tikzcd}
SSU^{\mathsf{T}} \arrow[r, "S\pi"] \arrow[d, "\mu^{\mathsf{S}}U^{\mathsf{T}}"'] & SU^{\mathsf{T}}\widehat{S} \arrow[r, "\pi\widehat{S}"] \arrow[d, phantom, "(\pi\mu)"] & U^{\mathsf{T}}\widehat{S}\widehat{S} \arrow[d, "U^{\mathsf{T}}\widehat{\mu^{\mathsf{S}}}"] \\
SU^{\mathsf{T}} \arrow[rr, "\pi"'] & & U^{\mathsf{T}}\widehat{S}
\end{tikzcd}$$

$$\begin{tikzcd}
U^{\mathsf{T}}\widehat{S} \arrow[rr, "\iota"] & (\iota\eta) & SU^{\mathsf{T}} \\
& U^{\mathsf{T}} \arrow[ul, "U^{\mathsf{T}}\widehat{\eta^{\mathsf{S}}}"] \arrow[ur, "\eta^{\mathsf{S}}U^{\mathsf{T}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
SU^{\mathsf{T}} \arrow[rr, "\pi"] & (\pi\eta) & U^{\mathsf{T}}\widehat{S} \\
& U^{\mathsf{T}} \arrow[ul, "\eta^{\mathsf{S}}U^{\mathsf{T}}"] \arrow[ur, "U^{\mathsf{T}}\widehat{\eta^{\mathsf{S}}}"'] &
\end{tikzcd}$$

Recall that an idempotent morphism $e : X \to X$ *splits* if there is an object Y and morphisms $f : X \to Y$, $g : Y \to X$ such that $g \circ f = e$ and $f \circ g = 1_Y$.

▶ **Theorem 5** ([1, 10]). *There is a bijective correspondence between extensions of* T *to* $\mathsf{Kl}(\mathsf{S})$, *distributive laws of type* $\mathsf{TS} \to \mathsf{ST}$, *and liftings of* S *to* $\mathsf{EM}(\mathsf{T})$. *This extends to a bijective correspondence between weak extensions, weak distributive laws, and (if all idempotents split in* C*) weak liftings.*

2.2 Relations in Regular Categories

From now on, we make the assumption that C is a *regular* category, this is, finitely complete with pullback-stable image factorizations. (Note that in a regular category, regular epis and strong epis coincide; notions will be phrased in terms of strong epis, as in [7].) In particular, C has all pullbacks: in this context, a *weak pullback* (resp. *near pullback*) is a commutative square such that the mediating morphism into the pullback is a split epi (resp. strong epi). The functor F is *weakly cartesian* (resp. *nearly cartesian*) if it maps pullbacks into weak pullbacks (resp. near pullbacks), and α is weakly cartesian (resp. nearly cartesian) if its naturality squares are weak pullbacks (resp. near pullbacks). Note that in the literature, strong epis are sometimes called *covers*, and within this terminology a nearly cartesian functor is a functor that *covers pullbacks* [22].

Regular categories have a (strong epi, mono) factorization system. In such a factorization $f = m \circ e$ we recall that the subobject defined by the mono m is the *image* of the morphism f. Note that these factorizations imply that idempotents split in C, so that Theorem 5 can be fully applied. As explained in [7], one can build a category $\mathsf{Rel}(\mathsf{C})$ whose morphisms will stand for relations. The objects of $\mathsf{Rel}(\mathsf{C})$ are the objects of C. A morphism $r : X \rightsquigarrow Y$ in $\mathsf{Rel}(\mathsf{C})$ is a subobject of $X \times Y$ in C and is called a relation – the notation $\rightsquigarrow$ tells relations apart from morphisms in C. We will not distinguish monomorphisms from their equivalence classes, hence a relation is equivalently a jointly monic span

$$\begin{tikzcd}
& R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
X & & Y
\end{tikzcd}$$

The composition of relations $r = \langle r_1, r_2\rangle : R \to X \times Y$ and $s = \langle s_1, s_2\rangle : S \to Y \times Z$ is obtained using a pullback Θ, by taking the image of the morphism $\Theta \to X \times Z$ below:

$$\begin{tikzcd}
 & & \Theta \arrow[dl, "\theta_1"'] \arrow[dr, "\theta_2"] \arrow[dd, phantom, very near start, "\vee"] & & \\
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
X & & Y & & Z
\end{tikzcd}$$

This relational composition of r and s will be denoted by $s.r$. Identities are obtained via the diagonal monomorphism $\langle 1_X, 1_X\rangle : X \to X \times X$. There is a contravariant involution $-^\circ : \mathsf{Rel}(\mathsf{C})^{\mathsf{op}} \to \mathsf{Rel}(\mathsf{C})$ given by $X^\circ = X$ and $\langle r_1, r_2\rangle^\circ = \langle r_2, r_1\rangle$. The graph functor $\mathcal{G} : \mathsf{C} \to \mathsf{Rel}(\mathsf{C})$ is defined by $\mathcal{G}X = X$ and $\mathcal{G}f = \langle 1_X, f\rangle$ for any $f : X \to Y$. These two fundamental functors have a nice interplay, as for every $r : X \rightsquigarrow Y$ we have $r = \mathcal{G}r_2.(\mathcal{G}r_1)^\circ$. Most of the time, the mention of $\mathcal{G}$ will be omitted, e.g. the previous equation writes $r = r_2.r_1^\circ$ and functoriality of $\mathcal{G}$ writes $g.f = g \circ f$. Given two relations $r : R \to X \times Y$ and $s : S \to X \times Y$, the subobject order is defined by:

$$r \leq s \iff \exists h : R \to S, r = s \circ h \tag{1}$$

Accordingly, a functor $H : \mathsf{Rel}(\mathsf{C}) \to \mathsf{Rel}(\mathsf{C})$ is called *monotone* if $r \leq s \Rightarrow Hr \leq Hs$.

A *relational extension* of F is a monotone functor $\mathsf{Rel}(F) : \mathsf{Rel}(\mathsf{C}) \to \mathsf{Rel}(\mathsf{C})$ such that $\mathsf{Rel}(F)\mathcal{G} = \mathcal{G}F$. This actually forces $\mathsf{Rel}(F)X = FX$ and

$$\mathsf{Rel}(F)r = (Fr_2).(Fr_1)^\circ \tag{2}$$

so that F has at most one relational extension. Similarly, a relational extension of $\alpha : F \to G$ is a natural transformation $\mathsf{Rel}(\alpha) : \mathsf{Rel}(F) \to \mathsf{Rel}(G)$ such that $\mathsf{Rel}(\alpha)\mathcal{G} = \mathcal{G}\alpha$, and α has at most one such extension. Collecting results from the literature ([7, §4.3], [21, Corollary 1.5.7]) we get the following existence theorem

▶ **Theorem 6** (Existence of relational extensions). *A functor $F : \mathsf{C} \to \mathsf{C}$ on a regular category C has a (unique) relational extension if and only if F preserves strong epis and F is nearly cartesian. Provided these conditions hold for both F and G, a natural transformation $\alpha : F \to G$ has a (unique) relational extension if and only if α is nearly cartesian.*

Note that whenever $\mathsf{Rel}(F)$ and $\mathsf{Rel}(G)$ exist, then $\mathsf{Rel}(G)\mathsf{Rel}(F) = \mathsf{Rel}(GF)$ (see also [7, §4.4]).

3 Sets

There is no distributive law of type $\mathbf{PP} \to \mathbf{PP}$ on Set [14]. However, there is a weak distributive law recently described by Garner [10]. In this section, we detail how this law is obtained using Theorem 6. Our aim is twofold. First, we lay the ground for Sections 4 and 5 where we will generalize this reasoning in two different directions. Second, as an application we present how this weak distributive law allows to retrieve a known procedure that transforms alternating automata into non-deterministic automata [13].

In this section, C is the (regular) category Set of sets and functions and $\mathbf{S}$ is the powerset monad $\mathbf{P}$ defined in Example 1. It turns out that both $\mathsf{Kl}(\mathbf{P})$ and $\mathsf{Rel}(\mathsf{Set})$ can be identified to the category Rel of sets and relations. Under this identification we have $F_{\mathbf{P}} = \mathcal{G}$. In this context a relation $R : X \nrightarrow Y$ is just a subset of the product $R \subseteq X \times Y$ and composition is defined by the usual formula $S \circ R = \{(x, z) \in X \times Z \mid (x, y) \in R \text{ and } (y, z) \in S\}$. A relational extension of a functor is nothing but a (Kleisli) extension that is monotone with respect to relation inclusion. The axiom of choice yields that all epis are split, henceforth any functor automatically preserves strong epis, and near pullbacks coincide with weak pullbacks. Theorem 6 therefore boils down to saying that F (resp. α) extends to Rel iff F (resp. α) is weakly cartesian – this is [10, Proposition 15].

Let $\mathbf{T}$ also be the powerset monad $\mathbf{P}$. The following example is essentially in [10] – however the proofs there are done with $\mathbf{T}$ being the *finite* powerset monad. An obvious consequence of the above paragraph is that $\mathbf{P}$ has a weak extension to Rel iff P and $\mu^{\mathbf{P}}$ are weakly cartesian, which both are known results, see e.g. [23] for P and [10] for $\mu^{\mathbf{P}}$. Further, the unit $\eta^{\mathbf{P}}$ is not nearly cartesian [10], so that the weak extension is not an extension. We recall these proofs here because they will be used in the next sections.

▶ **Proposition 7.** *The powerset functor P is weakly cartesian.*

Proof. Equivalently, P being nearly cartesian amounts to showing that for every $f : X \to Z$, $g : Y \to Z$ and $(A, B) \in PX \times PY$ such that $f[A] = g[B]$, there is $C \subseteq P := \{(x, y) \in X \times Y \mid f(x) = g(y)\}$ such that $\pi_1[C] = A$ and $\pi_2[C] = B$, where $\pi_1 : P \to X$, $\pi_2 : P \to Y$ are the projections from the pullback. One can easily check that $C = \pi_1^{-1}(A) \cap \pi_2^{-1}(B)$ completes the proof. ◀

▶ **Proposition 8.** *The unit $\eta^{\mathbf{P}}$ is not weakly cartesian.*

Proof. Consider the naturality square for the unique map $! : \{0, 1\} \to \{0\}$. The corresponding pullback $\{(0, \{0\}), (0, \{1\}), (0, \{0, 1\})\}$ has cardinality 3, so there cannot be a surjective map from $\{0, 1\}$ into it. ◀

▶ **Proposition 9.** *The multiplication $\mu^{\mathbf{P}}$ is weakly cartesian.*

Proof. For any $f : X \to Y$ and $(A, \mathcal{B}) \in PX \times PPY$ such that $f[A] = \bigcup \mathcal{B}$, we must find $\mathcal{A} \in PPX$ such that $\bigcup \mathcal{A} = A$ and $(Pf)[\mathcal{A}] = \mathcal{B}$. Take $\mathcal{A} = \{A \cap f^{-1}(B) \mid B \in \mathcal{B}\}$ and check

$$(Pf)[\mathcal{A}] = \{f[A \cap f^{-1}(B)] \mid B \in \mathcal{B}\} = \{f[A] \cap B \mid B \in \mathcal{B}\} = \{B \mid B \in \mathcal{B}\} = \mathcal{B}$$

$$\bigcup \mathcal{A} = A \cap f^{-1}\left(\bigcup \mathcal{B}\right) = A \cap f^{-1}(f[A]) = A$$ ◀

Computing the weak extension of $\mathbf{P}$ to Rel using equation (2) yields the well-known Egli-Milner relation

$$\overline{P}R = \{(A, B) \in PX \times PY \mid \forall x \in A, \exists y \in B, (x, y) \in R \text{ and } \forall y \in B, \exists x \in A, (x, y) \in R\} \quad (3)$$

The corresponding weak distributive law $\lambda : \mathbf{PP} \to \mathbf{PP}$ is given by

$$\lambda_X(\mathcal{A}) = \left\{B \in PX \mid B \subseteq \bigcup \mathcal{A} \text{ and } \forall A \in \mathcal{A}, A \cap B \neq \emptyset\right\} \quad (4)$$

The corresponding weak lifting of $\mathbf{P}$ to the category of complete join semi-lattices $\mathsf{EM}(\mathbf{P})$ is as follows: $\widehat{P}(X, x) = (S, s)$ has underlying set $S = \{A \in PX \mid \forall B \subseteq A, B \neq \emptyset \Rightarrow x(B) \in A\}$ with join given for every $\mathcal{A} \in PS$ by $s(\mathcal{A}) = \{x(\{a_A \mid A \in \mathcal{A}\}) \mid \forall A \in \mathcal{A}, a_A \in A\}$. The morphism $\pi_{(X,x)} : PX \to S$ sends a subset $A \subseteq X$ to its closure under non-empty join $\{x(B) \mid B \in P(A) \setminus \{\emptyset\}\}$, whereas $\iota_{(X,x)} : S \to PX$ is just the inclusion. Practically speaking, disposing of such a weak lifting allows to perform generalized determinization of PP-coalgebras as in [11, Section 5].

An F-*coalgebra* (X, c) is a morphism $c : X \to FX$, and a morphism of F-coalgebras $f : (X, c) \to (Y, d)$ is a morphism $f : X \to Y$ such that $Ff \circ c = d \circ f$. Let $\mathsf{Coalg}(F)$ be the category of F-coalgebras. Generalized determinization of PP-coalgebras is a transformation into a P-coalgebra via a process that factors through $\widehat{P}$-coalgebras as follows:

$$\begin{tikzcd} \mathsf{Coalg}(PP) \arrow[r] \arrow[d] & \mathsf{Coalg}(\widehat{P}) \arrow[r] \arrow[d] & \mathsf{Coalg}(P) \arrow[d] \\ \mathsf{Set} \arrow[r, "F^{\mathbf{P}}"] & \mathsf{EM}(\mathbf{P}) \arrow[r, "U^{\mathbf{P}}"] & \mathsf{Set} \end{tikzcd} \tag{5}$$

More precisely, this construction maps a coalgebra $X \xrightarrow{c} PPX$ to

$$c^+ = PX \xrightarrow{Pc} PPPX \xrightarrow{\lambda_{PX}} PPPX \xrightarrow{P\mu^{\mathbf{P}}_X} PPX \tag{6}$$

so $c^+(A) = \left\{ \bigcup_{x \in A} B_x \mid \forall x \in A, B_x \in \overline{c(x)} \right\}$ where $\overline{c(x)}$ is the closure of $c(x)$ under non-empty unions. See Figure 1 for a concrete example. An interesting view is to interpret P-coalgebras as non-deterministic automata and PP-coalgebras as alternating automata, meaning a transition from a state $x \in X$ consists in choosing non-deterministically a set $U \in c(x)$, then going simultaneously into every state $y \in U$. Alternating automata have been introduced in [8] and have known some difficulties to be properly modelled as PP-coalgebras ([2, 13]). In particular, this non-standard transformation of alternating automata was already described in [13], but did not fit very well into any general framework. There, λ was only identified as a non-distributive law, because the $(\eta^{\mathbf{T}})$ diagram does not commute – which in this case is equivalent to $\eta^{\mathbf{P}}$ not being weakly cartesian. We hereby pinpoint that this automata transformation is canonical in the sense that it comes from the unique monotone weak extension of $\mathbf{P}$ to Rel.

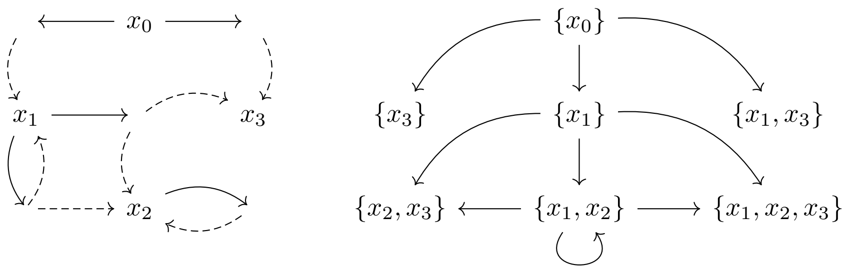

Figure 1 On the left, an alternating automaton: a transition consists in going in one of the solid lines, then in all of the available dashed lines. On the right, a portion of the non-deterministic automaton obtained after the process described in (5).

4 Toposes

The category Set is a special case of the more general notion of topos. Our contribution in this section is to generalize the results of Section 3 to arbitrary toposes. Some standard references about the theory of toposes are [12], [16], and our approach will be close to the one of [19]. Our presentation of toposes will be self-contained, with the exception of internal logic. In short, toposes are sufficiently set-behaved to internalize a logic in which one may reason as if they were picking elements in sets, and accommodate internally constructive proofs, i.e., not using either the law of excluded middle or the axiom of choice. For more details about the internal logic, see [12, Part D], [16, Section VI.5] or the more accessible [20], [18, Chapters 14-16].

A *topos* is a finitely complete cartesian closed category with a subobject classifier Ω. Having a subobject classifier means that there is a morphism out of the terminal object $\mathsf{true} : 1 \to \Omega$ such that for every subobject $m : A \hookrightarrow X$, there is a unique morphism $\chi_m : X \to \Omega$ (called the characteristic morphism of m) such that the following diagram is a pullback:

$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d, hook, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\mathsf{true}"] \\ X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

Note that $!_A$ denotes the unique morphism of type $A \to 1$. Toposes are finitely cocomplete. Every topos is regular, and conversely a regular category C is a topos if and only if the graph functor $\mathcal{G} : \mathsf{C} \to \mathsf{Rel}(\mathsf{C})$ has a right adjoint ([19, §6.1.1], [9, §1.911]). This adjunction yields a monad $\boldsymbol{\exists} : \mathsf{C} \to \mathsf{C}$ which is the generalization of the powerset monad on the topos Set – as hinted by the equality $\exists X = \Omega^X$ in C similar to $PX = 2^X$ in Set. In the internal logic of the topos, the data of $\boldsymbol{\exists}$ can be expressed as

$$\begin{aligned} \exists f(a) &= \{y : Y \mid \exists x : X, x \in a \wedge f(x) = y\} && \text{[20, Proposition 4.9]} \\ \eta^{\boldsymbol{\exists}}_X(x) &= \{x' : X \mid x = x'\} && \text{[20, Proposition 4.17]} \\ \mu^{\boldsymbol{\exists}}_X(t) &= \{x : X \mid \exists s : \Omega^X, x \in s \wedge s \in t\} && \text{[20, Proposition 4.19]} \end{aligned}$$

Another view on $\eta^{\boldsymbol{\exists}}_X : X \to \Omega^X$ is that it is the exponential transpose of the characteristic morphism $X \times X \to \Omega$ of the diagonal monomorphism $\langle 1_X, 1_X\rangle : X \hookrightarrow X \times X$ ([12, page 86]).

The Kleisli category of $\boldsymbol{\exists}$ is nothing but the category $\mathsf{Rel}(\mathsf{C})$ ([19, §6.1.10]), with again the identification $F_{\boldsymbol{\exists}} = \mathcal{G}$ and relational extensions being exactly monotone (Kleisli) extensions. Given the similarities with Set, a natural question is whether the results of the previous section extend to any topos: is there a weak distributive law of type $\boldsymbol{\exists\exists} \to \boldsymbol{\exists\exists}$?

Some of the ingredients required for obtaining such a law are already in the literature:

▶ **Proposition 10** ([19, Proposition 6.5.1])**.** *The functor* $\exists$ *is weakly cartesian and preserves strong epis.*

De Moor [19] deduces that the *functor* $\exists$ is a *relator*, i.e., it has a monotone extension to $\mathsf{Rel}(\mathsf{C})$. One can compute this generalized Egli-Milner formula using equation (2) and the internal logic notations, although it is not relevant for the subsequent developments.

The extension $\overline{\exists}$ corresponds to a distributive law between the *functor* $\exists$ and the monad $\boldsymbol{\exists}$ of type $\exists\boldsymbol{\exists} \to \boldsymbol{\exists}\exists$, called *cross-operator* in [19], meaning that the (η^{S}) and (μ^{S}) diagrams relative to the inner $\boldsymbol{\exists}$ commute. We provide the missing results:

▶ **Proposition 11.** *The unit* $\eta^{\boldsymbol{\exists}}$ *is nearly cartesian if and only if* C *is degenerate, i.e., the initial object* 0 *and the terminal object* 1 *are isomorphic.*

Proof sketch. If C is degenerate, then C is the category with a single object and a single arrow, and every natural transformation is nearly cartesian in such a category. Conversely, assume $\eta^{\boldsymbol{\exists}}$ is nearly cartesian. As $\eta^{\boldsymbol{\exists}}$ components are mono (see [16, Lemma 1 p.166]), this induces that $\eta^{\boldsymbol{\exists}}$ is cartesian, i.e., naturality squares are pullbacks. In particular, the left square below is a pullback. (Note that this square is the one that appears in the proof of

Proposition 8, because in Set we have $\{0\} \cong 1$ and $\{0,1\} \cong \Omega$.) Let $p : 1 \to \Omega^\Omega$ be the morphism that picks the maximal subobject $\Omega \subseteq \Omega$. The pasting law for pullbacks yields that $\Theta \cong \Phi$, where Θ and Φ are defined by the pullback squares on the right:

$$\begin{tikzcd} \Omega \arrow[r, "!_\Omega"] \arrow[d, "\eta^{\exists}_\Omega"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\eta^{\exists}_1"] \\ \Omega^\Omega \arrow[r, "\exists !_\Omega"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Theta \arrow[r] \arrow[d, "!_\Theta"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega \arrow[d, "\eta^{\exists}_\Omega"] \\ 1 \arrow[r, "p"'] & \Omega^\Omega \end{tikzcd} \qquad \begin{tikzcd} \Phi \arrow[r, "!_\Phi"] \arrow[d, "!_\Phi"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\eta^{\exists}_1"] \\ 1 \arrow[r, "\exists !_\Omega \circ p"'] & \Omega \end{tikzcd}$$

We can prove additionally that $\Theta \cong \Theta \times \Omega$ and $\Phi \cong 1$. Combining these results yields $\Omega \cong 1$, and this in turn implies that the topos is degenerate. ◀

▶ **Proposition 12.** *The multiplication $\mu^{\exists}$ is weakly cartesian.*

Proof. Mimicking the computation of Proposition 9 in the internal logic of C produces a valid proof, because the latter is a constructive intuitionistic proof, i.e., does not use either the axiom of choice or the law of excluded middle. ◀

▶ **Theorem 13.** *In any topos, there is a unique monotone weak distributive law of type $\exists\exists \to \exists\exists$, which is a distributive law exactly when the topos is degenerate.*

Proof. By Theorem 6, Propositions 10, 11, 12 and the fact that $F_{\exists} = \mathcal{G}$, the generalized Egli-Milner relation defines the unique monotone weak extension of $\exists$ to $\mathsf{Kl}(\exists)$, and is a monad extension iff the topos is degenerate. Applying Theorem 5 completes the proof. ◀

There is also a weak lifting of $\exists$ to the category $\mathsf{EM}(\exists)$ of internal complete join semi-lattices, implying in particular that the generalized determinization procedure described in Section 3 can be applied to $\exists\exists$-coalgebras in arbitrary toposes.

5 Compact Hausdorff Spaces

In this section, C is the category of compact Hausdorff spaces and continuous functions KHaus. As recalled in [10], KHaus is isomorphic to the Eilenberg-Moore category of the ultrafilter monad $\beta : \mathsf{Set} \to \mathsf{Set}$. This yields that KHaus is regular, complete, and that limits can be computed as in Set and given the initial topology afterwards. As KHaus is a pretopos (see, e.g., [17]), strong epis are just epis, that is, (continuous) surjections. Given an object X of KHaus, we denote its carrier set also by X and its topology by τ_X. Define VX to be the set of all closed subsets of X equipped with the *Vietoris topology*, i.e., the topology generated by the subbase

$$\Box U = \{A \in VX \mid A \subseteq U\} \qquad\qquad \Diamond U = \{A \in VX \mid A \cap U \neq \emptyset\} \tag{7}$$

where U ranges over τ_X. The mapping $X \mapsto VX$ extends into a monad $\mathbf{V}$ on KHaus called the *Vietoris monad* [10] in the same way as the powerset in Set. In concrete terms, V maps a continuous function $f : X \to Y$ to its direct image $Vf : VX \to VY$, $\eta^{\mathbf{V}} : 1 \to V$ takes singletons and $\mu^{\mathbf{V}} : VV \to V$ takes unions.

▶ Remark 14. It turns out that there is a monotone weak extension of β to $\mathsf{Kl}(\mathbf{P}) \cong \mathsf{Rel}$ and that the corresponding weak lifting is the Vietoris monad on $\mathsf{EM}(\beta) \cong \mathsf{KHaus}$. This is the main result of Garner's paper [10].

As KHaus is not a topos but only a pretopos, $\mathbf{V}$ cannot be obtained using the graph functor $\mathsf{KHaus} \to \mathsf{Rel}(\mathsf{KHaus})$ as a left adjoint. This is closely related to the fact that VX, being only part of a *weak* lifting, does not contain *all* subsets of X. Actually, $\mathsf{Rel}(\mathsf{KHaus})$ and $\mathsf{Kl}(\mathbf{V})$ correspond to the two essential ways of embedding KHaus into a relational category (see [4], where they are denoted by $\mathsf{KHaus}^{\mathsf{R}}$ and $\mathsf{KHaus}^{\mathsf{C}}$). Let X, Y be compact Hausdorff spaces and $R \subseteq X \times Y$ be a relation. Consider the following properties of R:

(i) $\forall A \in VX, R[A] \in VY$
(ii) $\forall B \in VY, R^{-1}[B] \in VX$
(iii) $\forall U \in \tau_Y, R^{-1}[U] \in \tau_X$

The relation R is *closed* if it satisfies properties (i) and (ii), or equivalently, if R is a closed subset of the product topology $\tau_{X\times Y}$. The relation R is *continuous* if it satisfies properties (i), (ii) and (iii). Note that these properties are preserved by the usual composition of relations and satisfied by identity relations. The following are straightforward results:

▶ **Proposition 15.** *The category of compact Hausdorff spaces and closed relations is isomorphic to* $\mathsf{Rel}(\mathsf{KHaus})$.

▶ **Proposition 16** (see [3, 4])**.** *The category of compact Hausdorff spaces and continuous relations is isomorphic to* $\mathsf{Kl}(\mathbf{V})$.

As a summary, we have the wide subcategory inclusions

$$\begin{tikzcd} \mathsf{KHaus} \arrow[r, "F_{\mathbf{V}}"'] \arrow[rr, bend left, "\mathcal{G}"] & \mathsf{Kl}(\mathbf{V}) \arrow[r, "\text{forget}", "\text{(iii)}"'] & \mathsf{Rel}(\mathsf{KHaus}) \end{tikzcd}$$

Any endofunctor on $\mathsf{Rel}(\mathsf{KHaus})$ that preserves continuous relations – i.e. preserves property (iii) – therefore restricts to an endofunctor on $\mathsf{Kl}(\mathbf{V})$. Similarly, given two such endofunctors on $\mathsf{Rel}(\mathsf{KHaus})$, any natural transformation whose components are continuous relations (e.g. every $\mathsf{Rel}(\alpha)$) automatically restricts to a natural transformation between their restrictions on $\mathsf{Kl}(\mathbf{V})$. Putting this together with Theorem 6 and the definition of monad extensions, we get

▶ **Proposition 17.** *Assume that T preserves continuous surjections, that T and $\mu^{\mathbf{T}}$ are nearly cartesian, and that* $\mathsf{Rel}(T)$ *preserves continuous relations. Then there is a monotone weak extension of* $\mathbf{T}$ *to* $\mathsf{Kl}(\mathbf{V})$*, and this is an extension if and only if $\eta^{\mathbf{T}}$ is nearly cartesian.*

Now we fix $\mathbf{T} = \mathbf{V}$ and proceed to verify the assumptions of Proposition 17.

▶ **Proposition 18.** *The Vietoris functor V preserves continuous surjections.*

Proof. For such a continuous surjective $f : X \to Y$, $Vf : VX \to VY$ is surjective because any $C \in VY$ satisfies $Vf(f^{-1}(C)) = C$, with $f^{-1}(C) \in VX$. ◀

▶ **Proposition 19.** *The Vietoris functor V is nearly cartesian.*

Proof. This is the same proof as for Proposition 7, with an additional check that if $A \in VA$ and $B \in VB$ then $C = \pi_1^{-1}(A) \cap \pi_2^{-1}(B)$ indeed is in VR, by continuity of π_1, π_2. ◀

▶ **Proposition 20.** *The unique relational extension* $\mathsf{Rel}(V)$ *preserves continuous relations.*

Proof. Assume that $R \subseteq X \times Y$ is a closed relation that satisfies condition (iii), denote its projections by $r_1 : R \to X$, $r_2 : R \to Y$ and show that $\mathsf{Rel}(V)R$ satisfies condition (iii). Sets of the form $\Box U_0 \cap \bigcap_{1 \le i \le n} \Diamond U_i$ with $n \in \omega$ and $U_i \in \tau_Y$ form a base of τ_{VY}, therefore the identity (8) below suffices to conclude because R satisfying condition (iii) makes the right-hand side an element of τ_{VY}.

$$(\mathsf{Rel}(V)R)^{-1}\left[\Box U_0 \cap \bigcap_{1 \le i \le n} \Diamond U_i\right] = \Box R^{-1}[U_0] \cap \bigcap_{1 \le i \le n} \Diamond R^{-1}[U_0 \cap U_i] \tag{8}$$

We now prove (8). A subset $C \in VX$ belongs to the left-hand side iff there exists $D \in VY$ such that

1. $\forall x \in C, \exists y \in D, (x, y) \in R$
2. $\forall y \in D, \exists x \in C, (x, y) \in R$
3. $D \subseteq U_0$
4. $\forall i \in \{1, ..., n\}, D \cap U_i \neq \emptyset$

A subset $C \in VX$ belongs to the right-hand side iff

5. $\forall x \in C, \exists y \in U_0, (x, y) \in R$
6. $\forall i \in \{1, ..., n\}, \exists (x_i, y_i) \in R, x_i \in C, y_i \in U_0 \cap U_i$

Let $C \in VX$ and $D \in VY$ such that $1. - 4.$ are satisfied. For any $x \in C$, using 1. and 3., we can find $y \in U_0$ such that $(x, y) \in R$, so that 5. holds. For any $i \in \{1, ..., n\}$, using 3. and 4. we find $y_i \in U_0 \cap U_i$, then using 2. we find $x_i \in C$ such that $(x_i, y_i) \in R$, so that 6. holds.

For the other direction we note that every compact Hausdorff space is regular and use the following property

▶ **Lemma 21** ([25, Theorem 14.3]). *A topological space Y is regular if and only if for every $U \in \tau_Y$ and every $y \in U$, there is a $W \in \tau_Y$ such that $y \in W$ and $\overline{W} \subseteq U$.*

Let $C \in VX$ such that $5. - 6.$ hold. For every $x \in C$ we fix $y_x \in U_0$ such that $(x, y_x) \in R$. For every $i \in \{1, ..., n\}$ we fix $(x_i, y_i) \in R$ such that $x_i \in C$ and $y_i \in U_0 \cap U_i$. Apply Lemma 21 to get $W_x \in \tau_Y$ such that $y_x \in W_x$ and $\overline{W_x} \subseteq U_0$ and $W_i \in \tau_Y$ such that $y_i \in W_i$ and $\overline{W_i} \subseteq U_0 \cap U_i$. For every $x \in C$, $(x, y_x) \in R$ and $y_x \in W_x$ so $x \in R^{-1}[W_x]$, hence $C \subseteq \bigcup_{x \in C} R^{-1}[W_x]$. As C is closed in a compact space, it is compact and we can extract a finite subcover $C \subseteq \bigcup_{1 \le k \le m} R^{-1}[W_{x_k}]$. Define the closed set

$$K = \bigcup_{1 \le i \le n} \overline{W_i} \cup \bigcup_{1 \le k \le m} \overline{W_{x_k}} \in VY \tag{9}$$

and consider $D = Vr_2((C \times K) \cap R) \in VY$. For any $x \in C$, there is $k \in \{1, ..., m\}$ and $y \in W_{x_k} \subseteq K$ such that $(x, y) \in R$, hence $y \in D$ and property 1. holds. Property 2. is immediate from the expression of D. As $\overline{W_i}, \overline{W_{x_k}} \subseteq U_0$, we have $D \subseteq K \subseteq U_0$ and property 3. holds. For any $i \in \{1, ..., n\}$, $(x_i, y_i) \in (C \times K) \cap R$ so $y_i \in D \cap U_i$ and property 4. holds. This achieves the proof. ◀

▶ **Proposition 22.** *The Vietoris unit $\eta^{\mathbf{V}}$ is not nearly cartesian.*

Proof. The counterexample of Proposition 8 still works by endowing the sets with the discrete topology. Note that the discrete topology on a finite set is always compact Hausdorff. ◀

A consequence is that there is no relational extension of $\mathbf{V}$ to $\mathsf{Rel}(\mathsf{KHaus})$, hence also no distributive law of type $\mathbf{VV} \to \mathbf{VV}$ can be obtained via Proposition 17. Further, there is no possible distributive law of this type because the counter-example for $\mathbf{PP} \to \mathbf{PP}$ presented in [14, Theorem 2.4] uses only finite sets: it is still valid in KHaus by endowing every set with the discrete topology again.

▶ **Proposition 23.** *The Vietoris multiplication μ^{V} is nearly cartesian.*

Proof. Consider $f : X \to Y$ and let P be the pullback of its μ^{V} naturality square. Surjectivity of the mediating map $h : VVX \to P$ amounts to proving that for every $C \in VX$ and $\mathcal{D} \in VVY$ such that $Vf(C) = {\mu^{\mathsf{V}}}_Y(\mathcal{D})$, there is a $\mathcal{C} \in VVX$ such that ${\mu^{\mathsf{V}}}_X(\mathcal{C}) = C$ and $VVf(\mathcal{C}) = \mathcal{D}$. However, our usual candidate $\mathcal{A} = \{C \cap f^{-1}(D) \mid D \in \mathcal{D}\}$ may not be a closed subset of VX. In its place we take

$$\mathcal{C} = (Vf)^{-1}(\mathcal{D}) \cap (\Diamond(C^c))^c \in VVX \tag{10}$$

Note that $\mathcal{C} = \{K \in VX \mid Vf(K) \in \mathcal{D} \text{ and } K \subseteq C\}$. Inclusions ${\mu^{\mathsf{V}}}_X(\mathcal{C}) \subseteq C$ and $VVf(\mathcal{C}) \subseteq \mathcal{D}$ are immediate. For the other ones, note that $\mathcal{A} \subseteq \mathcal{C}$, hence $C = {\mu^{\mathsf{P}}}_X(\mathcal{A}) \subseteq {\mu^{\mathsf{V}}}_X(\mathcal{C})$ and $\mathcal{D} = PPf(\mathcal{A}) \subseteq VVf(\mathcal{C})$. ◀

▶ **Theorem 24.** *There is a monotone weak distributive law of type* $\mathsf{VV} \to \mathsf{VV}$ *defined by*

$$\lambda_X(\mathcal{A}) = \left\{ B \in VX \mid B \subseteq \bigcup \mathcal{A} \text{ and } \forall A \in \mathcal{A}, A \cap B \neq \emptyset \right\} \tag{11}$$

Proof. Use Proposition 17 together with Propositions 18, 19, 20, 22, 23 to get that there is a monotone weak extension of V to $\mathsf{Kl}(\mathsf{V})$ defined by

$$\overline{V}R = \{(A, B) \in VX \times VY \mid \forall x \in A, \exists y \in B, (x, y) \in R \text{ and } \forall y \in B, \exists x \in A, (x, y) \in R\}$$

By Theorem 5, this corresponds to a weak distributive law with the wanted expression. ◀

Again, we also have a weak lifting of V to the category $\mathsf{EM}(\mathsf{V})$ – i.e., to continuous lattices, see [10, § 2.3] – whose existence notably allows generalized determinization of VV-coalgebras.

▶ Remark 25. The Vietoris monad restricts to the full subcategory of Stone spaces and continuous functions $\mathsf{Stone} \hookrightarrow \mathsf{KHaus}$, which regularly attracts the interest of the coalgebraic community ([15], [6]). Equation (11) clearly still defines a weak distributive law in Stone, which may be useful to understand better double Vietoris coalgebras as described in [5].

6 Conclusion

In this article, we have detailed how to obtain a triptych weak extension - weak distributive law - weak lifting using powerset-like monads. First we assembled results in the literature to show that the usual method for obtaining weak extensions from Set to Rel can be adapted to obtain weak extensions from any regular category C to its category of relations. We proved that weak self-distributivity of the Set powerset monad – known since the paper of Garner [10] – can actually be understood at the deeper level of toposes. Then we treated compact Hausdorff spaces, for which the Vietoris monad plays the role of a powerset. This case is particularly interesting because the category of closed relations and the Kleisli category of the Vietoris monad do not have exactly the same morphisms. In every case, we find that the unique monotone weak extension can be expressed with an Egli-Milner-shaped formula. Using the corresponding weak distributive law, we provide an application to automata theory: generalized determinization of alternating automata into non-deterministic automata. Here alternating automata are to be understood as double powerset coalgebras, living in any category previously considered.

To our knowledge, all previous examples of interesting weak distributive laws that are not distributive laws were exhibited in the category Set. Our work provides some first instances of such laws outside of Set. It would be an interesting research direction to find useful weak distributive laws that do not come from the extension result of Theorem 6, or that do not live into a set-like category. However, categories of relations being a rich framework, looking for more laws of the form $\mathbf{TP} \to \mathbf{PT}$ is also a promising direction. In particular, the category KHaus possesses a probability monad $\mathbf{R}$ called the Radon monad. Maybe is it possible to generalize the weak distributive law of type $\mathbf{DP} \to \mathbf{PD}$ presented in [11] (where $\mathbf{D}$ is the finitely supported distribution monad) to a continuous version $\mathbf{RV} \to \mathbf{VR}$ in KHaus. We leave this to future work.

Finally, we thank reviewers for bringing up the following remark. Although KHaus is not a topos, the Vietoris monad $\mathbf{V}$ may be thought of as classifying subobjects, in the sense that global elements of VX correspond to subobjects of X. It is an interesting question whether our weak distributive law can be generalized to (regular) categories admitting a monad classifying subobjects in this element-wise sense.

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