





Elementary Equivalence Versus Isomorphism in Semiring Semantics

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Abstract

We study the first-order axiomatisability of finite semiring interpretations or, equivalently, the question whether elementary equivalence and isomorphism coincide for valuations of atomic facts over a finite universe into a commutative semiring. Contrary to the classical case of Boolean semantics, where every finite structure is axiomatised up to isomorphism by a first-order sentence, the situation in semiring semantics is rather different, and depends on the underlying semiring. We prove that for a number of important semirings, including min-max semirings, and the semirings of positive Boolean expressions, there exist finite semiring interpretations that are elementarily equivalent but not isomorphic. The same is true for the polynomial semirings that are universal for the classes of absorptive, idempotent, and fully idempotent semirings, respectively. On the other side, we prove that for other, practically relevant, semirings such as the Viterby semiring \mathbb{V} , the tropical semiring \mathbb{T} , the natural semiring \mathbb{N} and the universal polynomial semiring $\mathbb{N}[X]$, all finite semiring interpretations are first-order axiomatisable, and thus elementary equivalence implies isomorphism.

2012 ACM Subject Classification Theory of computation \rightarrow Finite Model Theory

Keywords and phrases Semiring semantics, elementary equivalence, axiomatisability

Digital Object Identifier 10.4230/LIPIcs.ICALP.2021.133

Category Track B: Automata, Logic, Semantics, and Theory of Programming

Related Version *Previous Version*: <https://arxiv.org/abs/2102.05473>

1 Introduction

Semiring semantics is based on the idea to evaluate logical statements not just by *true* or *false*, but by values in some commutative semiring $(K, +, \cdot, 0, 1)$. In this context, the standard semantics appears as the special case when the Boolean semiring $\mathbb{B} = (\{\perp, \top\}, \vee, \wedge, \perp, \top)$ is used. Valuations in other semirings provide additional information, beyond the truth or falsity of a statement: the Viterbi-semiring $\mathbb{V} = ([0, 1]_{\mathbb{R}}, \max, \cdot, 0, 1)$ models *confidence scores*, the tropical semiring $\mathbb{T} = (\mathbb{R}_+^{\infty}, \min, +, \infty, 0)$ is used for *cost analysis*, and min-max-semirings (K, \max, \min, a, b) for a totally ordered set $(K, <)$ can model, for instance, different *access levels*. More generally, semirings of polynomials, such as $\mathbb{N}[X]$ or $\mathbb{B}[X]$, allow us to track the role of specific atomic facts for the evaluation of a logical statement, to describe evaluation strategies for a formula, and to determine which combinations of literals prove the truth of a formula.

Some of the motivation for the study of semiring semantics comes from the successful development of semiring provenance in database theory and related fields (see e.g. [5, 6, 9, 14, 15, 18, 19, 20]), and the fact that the typical applications of provenance analysis, such as confidence scores, cost analysis, proof counting, and the understanding of evaluation strategies are of importance in many other areas of logic as well. However, semiring provenance analysis for database queries had originally been largely confined to positive query languages, such as conjunctive queries, positive relational algebra, and Datalog, and the treatment



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48th International Colloquium on Automata, Languages, and Programming (ICALP 2021).

Editors: Nikhil Bansal, Emanuela Merelli, and James Worrell; Article No. 133; pp. 133:1–133:20

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



of negation poses non-trivial algebraic problems. Only recently, provenance analysis via semiring semantics has been extended to logics with negation, and in particular to full first-order logic [12, 13], by means of new algebraic constructions based on quotient semirings. Semiring semantics has also been studied for other logics, including modal logics, description logics, guarded logics, and fixed-point logic [1, 2, 3, 4], and this paper is part of a larger project devoted to a systematic study of semiring semantics for various logics. An important objective in this context is the understanding of the *model theory of semiring semantics*, and the development of model-theoretic methods for semiring interpretations.

It turns out that this is much more involved and diverse than for Boolean semantics. In the standard semantics, a model \mathfrak{A} assigns to each (instantiated) atomic formula a Boolean value, whereas K -interpretations π , for a suitable semiring K , generalise this by assigning to each literal a semiring value in K , where 0 is interpreted as *false* and all other semiring values as *nuances of true*. Interpreting disjunction by $+$ and conjunction by \cdot , we can extend π to provide semiring valuations $\pi[\![\varphi]\!] \in K$ for all first-order sentences φ , written in negation normal form. Semiring semantics thus gives a finer distinction of logical statements, and formulae that are equivalent in the Boolean sense (i.e. in the Boolean semiring) may have different valuations in other semirings. As a consequence, standard facts of classical (finite) model theory may lead to interesting and sometimes rather difficult questions in semiring semantics, and the answer may strongly depend on algebraic properties of the underlying semiring. Specific such questions that we study here concern the first-order axiomatisability of finite K -interpretations or, what amounts to the same, the relationship between isomorphism and elementary equivalence in this context.

It is a rather trivial fact of finite model theory that every finite structure \mathfrak{A} (with a finite vocabulary τ) can be axiomatised, up to isomorphism, by a first-order sentence $\chi_{\mathfrak{A}}$. In particular, two finite τ -structures \mathfrak{A} and \mathfrak{B} are isomorphic if, and only if, they are elementarily equivalent, in short $\mathfrak{A} \equiv \mathfrak{B}$, which means that they cannot be distinguished by any first-order sentence. Is this also the case for semiring interpretations? Notice that standard notions such as isomorphism and elementary equivalence generalise in a natural way from τ -structures to semiring interpretations, which raises, for any given semiring K , the following

► **Questions.**

1. *Are elementarily equivalent, finite K -interpretations always isomorphic?*
2. *Is every finite K -interpretation π_A first-order axiomatisable, in the sense that there is a set of axioms $\Phi_A \subseteq \text{FO}$ such that whenever $\pi_B[\![\varphi]\!] = \pi_A[\![\varphi]\!]$ for all $\varphi \in \Phi_A$, then $\pi_B \cong \pi_A$?*
3. *Does every finite K -interpretation admit an axiomatisation by a finite set of axioms?*
4. *Can every finite K -interpretation be axiomatised by a single first-order sentence?*

Clearly, the first two questions are equivalent, and a positive answer to the third question implies also positive ones to the first two. The converse is not necessarily true, because a first-order axiomatisation of a finite semiring interpretation might require an infinite collection of sentences, and, contrary to the Boolean case, it is a priori also not clear that an axiomatisation by a finite set of sentences implies an axiomatisation by a single sentence, because from the value of a conjunction we cannot necessarily infer the values of its components.

We shall prove that the answers to these questions strongly depend on the chosen semiring. There are in fact rather simple semirings, such as min-max semirings with at least three elements, for which one can construct examples of non-isomorphic K -interpretations which are, however, elementarily equivalent. The standard method for proving elementary equivalence in model theory, the Ehrenfeucht–Fraïssé method, seems not really available in

semiring semantics, an aspect that we shall discuss at the end of this paper. To establish elementary equivalence, we shall hence develop new methods based on classes of semiring homomorphisms and reduction arguments. Elementarily equivalent but non-isomorphic semiring interpretations also exist for powerful polynomial semirings, such as $\mathbb{S}[X]$ and $\mathbb{B}[X]$, which are universal for the classes of absorptive and idempotent semirings, respectively. On the other side, there are practically relevant semirings, such as the Viterby semiring \mathbb{V} , the tropical semiring \mathbb{T} , the natural semiring \mathbb{N} and the universal polynomial semiring $\mathbb{N}[X]$, for which any finite K -interpretation is first-order axiomatisable, thus elementary equivalence does indeed imply isomorphism. At least for \mathbb{V} and \mathbb{T} , finite axiomatisations are always possible, but not axiomatisations by a single sentence, so there exist semirings where the answers to questions (3) and (4) are different.

2 Semiring Interpretations

We briefly summarise semiring semantics for first-order logic, as introduced in [12].

► **Definition 1 (Semiring).** A commutative semiring $\mathbb{K} = (K, +, \cdot, 0, 1)$ is an algebraic structure with two binary operations such that $(K, +, 0)$ and $(K, \cdot, 1)$ are commutative monoids, multiplication distributes over addition and multiplication with zero annihilates elements. We may identify \mathbb{K} with its universe K if the operations are clear from the context.

In this paper, we only consider commutative semirings and simply call them “semirings” for convenience. Any class of semirings is implicitly restricted to commutative semirings only.

Let τ denote a finite relational vocabulary. We write $\text{Lit}_n(\tau)$ for the set of atoms $R\bar{z}$ and negated atoms $\neg R\bar{z}$ with $R \in \tau$ and where \bar{z} is any tuple of variables taken from $\{x_1, \dots, x_n\}$. For a universe A , we write $\text{Lit}_A(\tau)$ for the set of *instantiated* τ -literals $R\bar{a}$ and $\neg R\bar{a}$ with $\bar{a} \in A^{\text{arity}(R)}$.

► **Definition 2 (K -interpretation).** For a semiring K , a mapping $\pi: \text{Lit}_A(\tau) \rightarrow K$ is called a K -*interpretation* over the universe A with signature τ . We call π *model-defining* if exactly one of the values $\pi(L)$ and $\pi(\bar{L})$ is zero for all pairs of opposing literals $L, \bar{L} \in \text{Lit}_A(\tau)$. In that case, π *induces* the classical model \mathfrak{A}_π with $\mathfrak{A}_\pi \models L$ if, and only if, $\pi(L) \neq 0$.

► **Definition 3 (Isomorphism).** K -interpretations $\pi_A: \text{Lit}_A(\tau) \rightarrow K$ and $\pi_B: \text{Lit}_B(\tau) \rightarrow K$ are *isomorphic*, denoted as $\pi_A \cong \pi_B$, if there is a bijective mapping $\sigma: A \rightarrow B$ such that $\pi_A(L) = \pi_B(\sigma(L))$ for all $L \in \text{Lit}_A(\tau)$, where $\sigma(L) \in \text{Lit}_B(\tau)$ is defined by replacing each $a \in A$ occurring in L with $\sigma(a) \in B$. The mapping $\sigma: \pi_A \xrightarrow{\sim} \pi_B$ is called an *isomorphism*.

Given a K -interpretation $\pi: \text{Lit}_A(\tau) \rightarrow K$, a formula $\varphi(\bar{x}) \in \text{FO}(\tau)$ in negation normal form and an assignment $\bar{a} \subseteq A$, the semiring semantics $\pi[\varphi(\bar{a})]$ is straightforwardly defined by induction on $\text{FO}(\tau)$. Equalities are simply mapped to their truth values by $\pi[a = b] := 1$ if $a = b$ and 0 otherwise, and vice versa for inequalities. Similarly to the semantics of weighted logics introduced in [7], disjunctions and existential quantifiers are interpreted as sums, and conjunctions and universal quantifiers as products:

$$\begin{aligned} \pi[\psi(\bar{a}) \vee \vartheta(\bar{a})] &:= \pi[\psi(\bar{a})] + \pi[\vartheta(\bar{a})] & \pi[\psi(\bar{a}) \wedge \vartheta(\bar{a})] &:= \pi[\psi(\bar{a})] \cdot \pi[\vartheta(\bar{a})] \\ \pi[\exists x \vartheta(\bar{a}, x)] &:= \sum_{a \in A} \pi[\vartheta(\bar{a}, a)] & \pi[\forall x \vartheta(\bar{a}, x)] &:= \prod_{a \in A} \pi[\vartheta(\bar{a}, a)]. \end{aligned}$$

Our goal is to analyse classical model-theoretic concepts under semiring semantics.

► **Definition 4** (Elementary Equivalence). Two K -interpretations $\pi_A: \text{Lit}_A(\tau) \rightarrow K$ and $\pi_B: \text{Lit}_B(\tau) \rightarrow K$ are *elementarily equivalent*, denoted as $\pi_A \equiv \pi_B$, if $\pi_A \llbracket \psi \rrbracket = \pi_B \llbracket \psi \rrbracket$ holds for all sentences $\psi \in \text{FO}(\tau)$.

Clearly, the notions of isomorphism and elementary equivalence of K -interpretations are natural generalisations of the corresponding definitions for τ -structures. Further, it is obvious that, as in classical semantics, isomorphism implies elementary equivalence.

► **Lemma 5** (Isomorphism Lemma). Let $\pi_A: \text{Lit}_A(\tau) \rightarrow K$ and $\pi_B: \text{Lit}_B(\tau) \rightarrow K$ be two K -interpretations, $\bar{a} \in A^k$ and $\bar{b} \in B^k$ be k -tuples and $\sigma: \pi_A \xrightarrow{\sim} \pi_B$ an isomorphism with $\sigma(\bar{a}) = \bar{b}$. Then, $\pi_A \llbracket \varphi(\bar{a}) \rrbracket = \pi_B \llbracket \varphi(\bar{b}) \rrbracket$ holds for all $\varphi(\bar{x}) \in \text{FO}(\tau)$ with k free variables.

Coarser definitions may be conceivable in semirings K with more than two elements, such as replacing equality by a congruence relation $\sim \subseteq K \times K$. However, Definition 4 indirectly covers these variants, since any non-trivial congruence relation \sim on K induces a semiring homomorphism $h_\sim: K \rightarrow K/\sim$, which is compatible with FO-semantics as follows [12].

► **Lemma 6** (Fundamental Property). Let $\pi: \text{Lit}_A(\tau) \rightarrow K$ be a K -interpretation and $h: K \rightarrow L$ a semiring homomorphism. Then, $(h \circ \pi)$ is an L -interpretation such that $(h \circ \pi) \llbracket \varphi(\bar{a}) \rrbracket = h(\pi \llbracket \varphi(\bar{a}) \rrbracket)$ holds for all $\varphi(\bar{x}) \in \text{FO}(\tau)$ and $\bar{a} \subseteq A$.

3 Polynomial Semirings and the Universal Property

Semiring homomorphisms and the fundamental property open the possibility to reduce semiring semantics to the evaluation of polynomials. For a finite set X of abstract provenance tokens that are used to track atomic facts, consider the semiring $\mathbb{N}[X]$ of multivariate polynomials with indeterminates from X and coefficients from \mathbb{N} , whose generality is formalised by the following *universal property* [14].

► **Lemma 7** (Universal Property). For each commutative semiring K , every assignment $e: X \rightarrow K$ induces a unique homomorphism $h_e: \mathbb{N}[X] \rightarrow K$ with $h_e(x) = e(x)$ for $x \in X$.

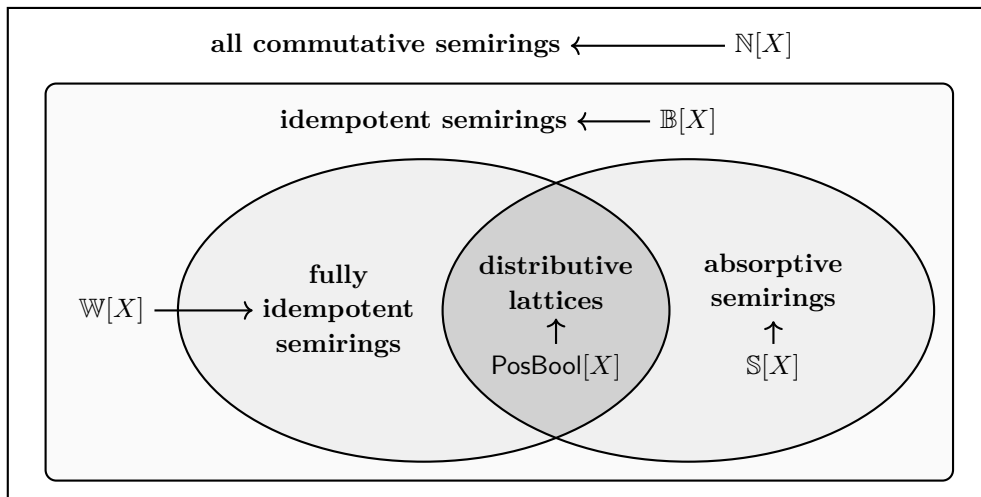
This property can be used to save computation resources, for example, if we would like to evaluate a sentence $\psi \in \text{FO}$ under several interpretations $(\pi_i)_{i \in I}$ sharing the same set of true literals. In such a scenario, we may build an $\mathbb{N}[X]$ -interpretation π that assigns a unique variable $x \in X$ to each positive literal. After computing the polynomial $p := \pi \llbracket \psi \rrbracket$ once, the semiring semantics $\pi_i \llbracket \psi \rrbracket$ for each $i \in I$ may be computed by plugging the positive literals' values into their corresponding variables and evaluating the polynomial p instead of starting from scratch.

The universality of $\mathbb{N}[X]$ also makes it relevant for model theory. Crucially, more restricted polynomial semirings can be used to capture smaller classes of semirings in the sense of the universal property. This is formalised in the following and summarised by Figure 1.

► **Definition 8** (Idempotence and Absorption). A semiring K is called *idempotent* if $a + a = a$ holds for all $a \in K$, that is, if addition is idempotent. It is *multiplicatively idempotent* if $a \cdot a = a$ for all $a \in K$. If both properties hold, we call K *fully idempotent*. Finally, a semiring K is *absorptive* if $a + ab = a$ holds for all $a, b \in K$.

■ By dropping coefficients from $\mathbb{N}[X]$, we get the semiring $\mathbb{B}[X]$ whose elements are finite sets of monomials. It has the universal property for the class of idempotent semirings.

- By collapsing exponents in $\mathbb{B}[X]$, we get the semiring $\mathbb{W}[X]$ of finite sums of monomials that are linear in each argument. It is sometimes called the Why-semiring. While $\mathbb{W}[X]$ is not fully idempotent itself due to $(x + y) \cdot (x + y) = x + xy + y$, it still holds the universal property for fully idempotent semirings.
- For absorptive semirings, we require absorptive polynomials as introduced in [5]. An absorptive polynomial is a sum of distinct monomials over a finite set of variables X , with absorption among monomials: “shorter” monomials absorb “longer” monomials. More formally, m_1 absorbs m_2 , denoted $m_1 \succeq m_2$, if it has smaller exponents, i.e. $m_2 = m \cdot m_1$ for some monomial m . For example, $xy^2 \succeq x^3y^2$ and $x \succeq xy$, but x^2y and xy^2 are incomparable. Applying absorption after each operation induces the semiring of absorptive polynomials $\mathbb{S}[X]$, which is universal for the class of all absorptive semirings.
- By collapsing exponents in $\mathbb{S}[X]$, we obtain yet another polynomial semiring $\text{PosBool}[X]$, which is universal for the fully idempotent and absorptive semirings. Those are precisely the semirings induced by distributive lattices, as shown in [17]. Incidentally, $\text{PosBool}[X]$ is the distributive lattice freely generated by the set X .



■ **Figure 1** Relationships between some classes of commutative semirings and their respective universal polynomial semirings, adapted from [17].

4 Separating Elementary Equivalence from Isomorphism

We shall now, for certain semirings K , provide examples of finite, non-isomorphic K -interpretations that are, however, elementarily equivalent. We thus provide negative answers to Question (1) from the introduction, and hence also to Questions (2) to (4). For instance, we claim that the following two K_4 -interpretations over the min-max-semiring with four elements, $K_4 = \{0, 1, 2, 3\}$, are elementarily equivalent, but not isomorphic.

$\pi_{PQ} :$	<table border="1" style="border-collapse: collapse; text-align: center;"> <tr> <td style="border: none;">A</td> <td style="border: none;">P</td> <td style="border: none;">Q</td> <td style="border: none;">$\neg P$</td> <td style="border: none;">$\neg Q$</td> </tr> <tr> <td style="border: none;">a</td> <td style="border: none;">1</td> <td style="border: none;">3</td> <td style="border: none;">0</td> <td style="border: none;">0</td> </tr> <tr> <td style="border: none;">b</td> <td style="border: none;">2</td> <td style="border: none;">1</td> <td style="border: none;">0</td> <td style="border: none;">0</td> </tr> <tr> <td style="border: none;">c</td> <td style="border: none;">3</td> <td style="border: none;">2</td> <td style="border: none;">0</td> <td style="border: none;">0</td> </tr> </table>	A	P	Q	$\neg P$	$\neg Q$	a	1	3	0	0	b	2	1	0	0	c	3	2	0	0
A	P	Q	$\neg P$	$\neg Q$																	
a	1	3	0	0																	
b	2	1	0	0																	
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$\pi_{QP} :$	<table border="1" style="border-collapse: collapse; text-align: center;"> <tr> <td style="border: none;">A</td> <td style="border: none;">P</td> <td style="border: none;">Q</td> <td style="border: none;">$\neg P$</td> <td style="border: none;">$\neg Q$</td> </tr> <tr> <td style="border: none;">a</td> <td style="border: none;">3</td> <td style="border: none;">1</td> <td style="border: none;">0</td> <td style="border: none;">0</td> </tr> <tr> <td style="border: none;">b</td> <td style="border: none;">1</td> <td style="border: none;">2</td> <td style="border: none;">0</td> <td style="border: none;">0</td> </tr> <tr> <td style="border: none;">c</td> <td style="border: none;">2</td> <td style="border: none;">3</td> <td style="border: none;">0</td> <td style="border: none;">0</td> </tr> </table>	A	P	Q	$\neg P$	$\neg Q$	a	3	1	0	0	b	1	2	0	0	c	2	3	0	0
A	P	Q	$\neg P$	$\neg Q$																	
a	3	1	0	0																	
b	1	2	0	0																	
c	2	3	0	0																	

Observe that π_{QP} is obtained from π_{PQ} by permuting the relations P and Q , or visually, by permuting the “columns”. Moreover, for both interpretations, the Q -column can be obtained by permuting the P -column. Informally, these properties ensure that the two interpretations are “sufficiently similar” so that no first-order sentence can distinguish them.

Clearly, π_{PQ} is not isomorphic to π_{QP} , as intended. However, the only tool we presented so far for proving elementary equivalence under semiring semantics is the Isomorphism Lemma itself, which is not directly applicable for obvious reasons. Hence, we shall develop another tool for proving elementary equivalence that enables the indirect use of the Isomorphism Lemma after switching to a different semiring via homomorphisms.

4.1 Separating Pairs of Homomorphisms

The central idea of the reduction technique is to “decompose” the semiring K via homomorphisms. Observe that if $\pi_A \not\equiv \pi_B$, then there is a witnessing sentence $\psi \in \text{FO}(\tau)$ with $\pi_A \llbracket \psi \rrbracket \neq \pi_B \llbracket \psi \rrbracket$, hence a pair of distinct elements $a, b \in K$ with $\pi_A \llbracket \psi \rrbracket = a \neq b := \pi_B \llbracket \psi \rrbracket$ exists. If we can find two homomorphisms $h_A, h_B: K \rightarrow K'$ with $h_A(a) \neq h_B(b)$, but we are sure that the corresponding K' -interpretations $(h_A \circ \pi_A)$ and $(h_B \circ \pi_B)$ are elementarily equivalent, then we can exclude (a, b) as a witness for $\pi_A \not\equiv \pi_B$. If we are able to provide enough pairs of homomorphisms so that each distinct pair (a, b) can be excluded, then $\pi_A \equiv \pi_B$ must hold. The following definition formalises the required properties.

► **Definition 9** (Separating Homomorphism Pairs). A set $S \subseteq \text{Hom}^2(K, K')$ of homomorphism pairs $h_A, h_B: K \rightarrow K'$ is called *separating* if for all $a, b \in K$ with $a \neq b$, there is a pair $(h_A, h_B) \in S$ such that $h_A(a) \neq h_B(b)$. S is called *diagonal* if $h_A = h_B$ for all pairs $(h_A, h_B) \in S$. In that case, we may write S as a subset of $\text{Hom}(K, K')$.

Note that a single *injective* homomorphism $h: K \rightarrow K'$ induces the diagonal separating set $S := \{(h, h)\}$ with just one element. Moreover, some semirings, such as $\text{PosBool}[X]$, can be completely decomposed into $K' := \mathbb{B}$ using a diagonal separating set of semiring homomorphisms as follows. Any subset $Y \subseteq X$ induces a unique homomorphism $h_Y: \text{PosBool}[X] \rightarrow \mathbb{B}$ by $h_Y(x) = \top$ for $x \in Y$ and $h_Y(x) = \perp$ for $x \in X \setminus Y$. Clearly, for any $p \in \text{PosBool}[X]$, we have that $h_Y(p) = \top$ if, and only if, p contains a monomial with only variables from Y .

► **Lemma 10.** *The set $S := \{h_Y \mid Y \subseteq X\} \subseteq \text{Hom}(\text{PosBool}[X], \mathbb{B})$ is a diagonal separating set of homomorphisms.*

Proof. Consider $p, q \in \text{PosBool}[X]$ such that $p \neq q$. Among the monomials that appear in one of the two polynomials p, q but not in the other, let m be one whose set Y of variables is minimal. By symmetry, we can assume that m appears in p but not in q . It follows that $h_Y(p) = \top$. We claim that $h_Y(q) = \perp$. Otherwise q must contain a monomial m' with only variables from Y . Since m' has less variables than m , m' must also be contained in p . But m' absorbs m , so m does not occur in p , a contradiction. ◀

On the other side, $\mathbb{N}[X]$, among other semirings, cannot be decomposed into \mathbb{B} by a diagonal separating set. For example, $h(x + xy) = h(x) \vee (h(x) \wedge h(y)) = h(x)$ for all homomorphisms $h: \mathbb{N}[X] \rightarrow \mathbb{B}$, but $x + xy \neq x$. The reason why a decomposition into \mathbb{B} would be useful for model theory is given by the following reduction technique.

► **Proposition 11** (Reduction Technique). *Let $\pi_A: \text{Lit}_A(\tau) \rightarrow K$ and $\pi_B: \text{Lit}_B(\tau) \rightarrow K$ be two K -interpretations, $\bar{a} \in A^k$ and $\bar{b} \in B^k$ be k -tuples and $S \subseteq \text{Hom}^2(K, K')$ a separating set of homomorphism pairs. Then, for any formula $\varphi(x_1, \dots, x_k) \in \text{FO}(\tau)$, we have that whenever $(h_A \circ \pi_A) \llbracket \varphi(\bar{a}) \rrbracket = (h_B \circ \pi_B) \llbracket \varphi(\bar{b}) \rrbracket$ for all $(h_A, h_B) \in S$, then also $\pi_A \llbracket \varphi(\bar{a}) \rrbracket = \pi_B \llbracket \varphi(\bar{b}) \rrbracket$.*

Proof. We show the contraposition. Suppose that $\pi_A[\varphi(\bar{a})] \neq \pi_B[\varphi(\bar{b})]$. Then, by definition of S , there exists a pair $(h_A, h_B) \in S$ such that $h_A(\pi_A[\varphi(\bar{a})]) \neq h_B(\pi_B[\varphi(\bar{b})])$. Applying the fundamental property yields $(h_A \circ \pi_A)[\varphi(\bar{a})] \neq (h_B \circ \pi_B)[\varphi(\bar{b})]$. \blacktriangleleft

► **Corollary 12.** For S as above, $(h_A \circ \pi_A) \equiv (h_B \circ \pi_B)$ for all $(h_A, h_B) \in S$ implies $\pi_A \equiv \pi_B$.

With the target semiring $K' := \mathbb{B}$, the corollary shows that proving equivalence in K may be reduced to proving equivalence in \mathbb{B} , which permits using results from standard semantics.

4.2 Applications to PosBool[X] and $\mathbb{W}[X]$

Consider the following two PosBool[X]-interpretations π_{xy}, π_{yx} with $X := \{x, y\}$ over the universe $A := \{a, b, c, d\}$ with four elements and a signature $\tau := \{P, Q\}$ with two unary relation symbols.

	A	P	Q	$\neg P$	$\neg Q$
$\pi_{xy} :$	a	0	y	x	0
	b	x	0	0	y
	c	y	x	0	0
	d	0	0	y	x

	A	P	Q	$\neg P$	$\neg Q$
$\pi_{yx} :$	a	y	0	0	x
	b	0	x	y	0
	c	x	y	0	0
	d	0	0	x	y

Thanks to Lemma 10, the four homomorphisms $S = \{h_\emptyset, h_{\{x\}}, h_{\{y\}}, h_{\{x,y\}}\}$ induce a separating set on PosBool[X] and with Corollary 12, it suffices to show that $(h \circ \pi_{xy}) \equiv (h \circ \pi_{yx})$ in \mathbb{B} for all $h \in S$ in order to prove $\pi_{xy} \equiv \pi_{yx}$ on PosBool[X]. Indeed, Figure 2 demonstrates that $(h \circ \pi_{xy}) \equiv (h \circ \pi_{yx})$ holds for all $h \in S$.

Thus, we conclude that $\pi_{xy} \equiv \pi_{yx}$ and due to $\pi_{xy} \not\equiv \pi_{yx}$, this shows that for finite PosBool[X]-interpretations, elementary equivalence does not necessarily imply isomorphism. Moreover, similar examples can be constructed for any distributive lattice semiring K thanks to the universal property of PosBool[X], by assigning $r, s \in K$ to the variables x, y .

	A	P	Q	$\neg P$	$\neg Q$
$\pi_{rs} :$	a	0	s	r	0
	b	r	0	0	s
	c	s	r	0	0
	d	0	0	s	r

	A	P	Q	$\neg P$	$\neg Q$
$\pi_{sr} :$	a	s	0	0	r
	b	0	r	s	0
	c	r	s	0	0
	d	0	0	r	s

Clearly, $\pi_{rs} \equiv \pi_{sr}$ holds as above, and the only requirement for $\pi_{rs} \not\equiv \pi_{sr}$ is that r and s must be distinct. This yields the following theorem.

► **Theorem 13.** For any distributive lattice semiring K with at least three elements, there is a pair of finite K -interpretations π_{rs}, π_{sr} over a universe with four elements and a signature with two unary relation symbols such that $\pi_{rs} \equiv \pi_{sr}$, but $\pi_{rs} \not\equiv \pi_{sr}$.

Note that the two K_4 -interpretations π_{PQ} and π_{QP} from the opening example of this section can be shown to be elementarily equivalent using a similar technique as above. In fact, the above theorem even shows that the opening example was not minimal and a counterexample with only three semiring elements in $K_3 = \{0, 1, 2\}$ exists.

We shall strengthen the result of Theorem 13 to the class of all fully idempotent semirings by simply regarding π_{xy} and π_{yx} as $\mathbb{W}[X]$ -interpretations instead of PosBool[X]-interpretations. However, the proof that $\pi_{xy} \equiv \pi_{yx}$ becomes more involved, since a diagonal separating set for $\mathbb{W}[X]$ into \mathbb{B} does not exist. Nevertheless, a suitable separating set can

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■ **Figure 2** Illustrations of $h \circ \pi_{xy}$ next to $h \circ \pi_{yx}$ for all $h \in S = \{h_{\emptyset}, h_{\{x\}}, h_{\{y\}}, h_{\{x,y\}}\}$.

be obtained by exploiting homomorphisms into $\mathbb{W}[X]$ itself. Consider any permutation $\sigma: X \rightarrow X$ of the variables. Surely, it induces an automorphism h_{σ} of $\mathbb{W}[X]$. In the previous example, if σ swaps the variables x and y , then applying h_{σ} to π_{xy} yields an interpretation that is isomorphic to π_{yx} , as illustrated below.

$(h_{\sigma} \circ \pi_{xy}) :$ <table border="1" style="border-collapse: collapse; text-align: center;"> <thead> <tr><th>A</th><th>P</th><th>Q</th><th>$\neg P$</th><th>$\neg Q$</th></tr> </thead> <tbody> <tr><td>a</td><td>0</td><td>x</td><td>y</td><td>0</td></tr> <tr><td>b</td><td>y</td><td>0</td><td>0</td><td>x</td></tr> <tr><td>c</td><td>x</td><td>y</td><td>0</td><td>0</td></tr> <tr><td>d</td><td>0</td><td>0</td><td>x</td><td>y</td></tr> </tbody> </table>	A	P	Q	$\neg P$	$\neg Q$	a	0	x	y	0	b	y	0	0	x	c	x	y	0	0	d	0	0	x	y	$\pi_{yx} :$ <table border="1" style="border-collapse: collapse; text-align: center;"> <thead> <tr><th>A</th><th>P</th><th>Q</th><th>$\neg P$</th><th>$\neg Q$</th></tr> </thead> <tbody> <tr><td>a</td><td>y</td><td>0</td><td>0</td><td>x</td></tr> <tr><td>b</td><td>0</td><td>x</td><td>y</td><td>0</td></tr> <tr><td>c</td><td>x</td><td>y</td><td>0</td><td>0</td></tr> <tr><td>d</td><td>0</td><td>0</td><td>x</td><td>y</td></tr> </tbody> </table>	A	P	Q	$\neg P$	$\neg Q$	a	y	0	0	x	b	0	x	y	0	c	x	y	0	0	d	0	0	x	y
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With this insight, we can construct a suitable separating set $S \subseteq \text{Hom}^2(\mathbb{W}[X], \mathbb{W}[X])$, starting with the pair $(h_{\sigma}, h_{\text{id}}) \in S$. This pair alone does not separate $\mathbb{W}[X]$, since we have $x \neq y$ in $\mathbb{W}[X]$, but $h_{\sigma}(x) = y = h_{\text{id}}(y)$. Hence, we add more homomorphisms by annihilating some variables, similarly to the construction for $\text{PosBool}[X]$ in Lemma 10. Fixing a permutation $\sigma: X \rightarrow X$, we want to construct a homomorphism h_{σ}^Y that annihilates all variables in $X \setminus Y$ and permutes the variables in Y . Observe that for each $x \in Y$ there is a minimal number $r(x) \geq 1$ such that $\sigma^{r(x)}(x) \in Y$. Formally, we define $\sigma^Y: X \rightarrow Y \cup \{0\}$ by setting $\sigma^Y(x) := \sigma^{r(x)}(x)$ for $x \in Y$, and $\sigma^Y(x) := 0$ for $x \in X \setminus Y$. Note that σ^Y induces a homomorphism $h_{\sigma}^Y: \mathbb{W}[X] \rightarrow \mathbb{W}[X]$.

► **Lemma 14.** $S := \{(h_{\sigma}^Y, h_{\text{id}}^Y) \mid Y \subseteq X\} \subseteq \text{Hom}^2(\mathbb{W}[X], \mathbb{W}[X])$ is a separating set of homomorphism pairs.

Proof. Suppose that $p \neq q$ for a pair $p, q \in \mathbb{W}[X]$. A monomial in $\mathbb{W}[X]$ can be identified with the set of its variables. Thus, without loss of generality, there is some $Y \subseteq X$ with $Y \in p$ and $Y \notin q$. Surely, $h_\sigma^Y(p)$ contains the monomial $h_\sigma^Y(Y) = Y$, but $h_{\text{id}}^Y(q)$ only contains monomials from q , hence it does not contain Y and $h_\sigma^Y(p) \neq h_{\text{id}}^Y(q)$. ◀

Before applying the reduction technique to obtain $\pi_{xy} \equiv \pi_{yx}$ from Corollary 12, it only remains to show that $(h_\sigma^Y \circ \pi_{xy}) \equiv (h_{\text{id}}^Y \circ \pi_{yx})$ for all $Y \subseteq X := \{x, y\}$. Since we have already illustrated $(h_\sigma^X \circ \pi_{xy}) \equiv (h_{\text{id}}^X \circ \pi_{yx})$, we only need to consider the cases where $Y \subsetneq X$. But then, at most one variable is contained in Y and the remaining variables are annihilated by h_σ^Y and h_{id}^Y , thus $(h_\sigma^Y \circ \pi_{xy}) \equiv (h_{\text{id}}^Y \circ \pi_{yx})$ clearly follows. The reduction technique then implies that $\pi_{xy} \equiv \pi_{yx}$ on $\mathbb{W}[X]$, which can naturally be lifted to all fully idempotent semirings thanks to the universal property.

► **Theorem 15.** *For any fully idempotent semiring K with at least three elements, there is a pair of finite K -interpretations π_{rs}, π_{sr} over a universe with four elements and a signature with two unary relation symbols such that $\pi_{rs} \equiv \pi_{sr}$, but $\pi_{rs} \not\equiv \pi_{sr}$.*

In conclusion, the proof of $\pi_{xy} \equiv \pi_{yx}$ on $\text{PosBool}[X]$ illustrates how elementary equivalence in semiring semantics can be reduced to elementary equivalence on \mathbb{B} by completely decomposing $\text{PosBool}[X]$ to \mathbb{B} with homomorphisms. Moreover, the proof of elementary equivalence on $\mathbb{W}[X]$ shows that it may even pay off to use separating sets of homomorphisms from $\mathbb{W}[X]$ to $\mathbb{W}[X]$ itself.

5 Characteristic Sentences

The results of the previous section raise the question whether it is possible to construct a similar example of non-isomorphic, but elementarily equivalent interpretations also for the most general semiring $\mathbb{N}[X]$, and lift it to all commutative semirings with at least three elements. In order to show that this is not possible, we draw inspiration from classical semantics, where for each finite τ -structure \mathfrak{A} with universe $A = \{a_1, \dots, a_n\}$ one can construct a *characteristic sentence* $\chi_{\mathfrak{A}}$ such that $\mathfrak{B} \models \chi_{\mathfrak{A}}$ if, and only if, $\mathfrak{A} \cong \mathfrak{B}$. The characteristic sentence is explicitly defined as

$$\chi_{\mathfrak{A}} := \exists x_1 \dots \exists x_n (\varphi(\bar{x}) \wedge \psi(\bar{x})) \quad \text{with}$$

$$\varphi(\bar{x}) := \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \wedge \forall y \bigvee_{i \leq n} y = x_i \quad \text{and} \quad \psi(\bar{x}) := \bigwedge \{L(\bar{x}) \in \text{Lit}_n(\tau) \mid \mathfrak{A} \models L(\bar{a})\}.$$

The subformula $\varphi(\bar{x})$ of this sentence asserts that the universe has precisely n elements assigned to the variables \bar{x} . Since $\varphi(\bar{x})$ uses only equalities and inequalities it can be used as-is for semiring semantics in any semiring.

► **Lemma 16.** *For every K -interpretation $\pi_B: \text{Lit}_B(\tau) \rightarrow K$ into an arbitrary semiring K and every tuple $\bar{b} = (b_1, \dots, b_n)$, we have that $\pi_B \llbracket \varphi(\bar{b}) \rrbracket = 1$ if $B = \{b_1, \dots, b_n\}$ and $b_i \neq b_j$ for $i \neq j$, and $\pi_B \llbracket \varphi(\bar{b}) \rrbracket = 0$, otherwise.*

Proof. Semiring interpretations evaluate equalities and inequalities to 0 and 1, so

$$\pi_B \llbracket \varphi(\bar{b}) \rrbracket = \prod_{i < j} \pi_B \llbracket b_i \neq b_j \rrbracket \cdot \prod_{b \in B} \left(\sum_{i \leq n} \pi_B \llbracket b = b_i \rrbracket \right)$$

evaluates to 1 if b_1, \dots, b_n is a distinct enumeration of all elements of B , and to 0 otherwise. ◀

On the other side, $\psi(\bar{a})$ is the conjunction of all true literals in \mathfrak{A} . Since \mathfrak{A} satisfies precisely one literal out of each pair of opposing literals L and \bar{L} , it is clear that $\psi(\bar{a})$ describes \mathfrak{A} up to isomorphism. However, this approach does not lift to arbitrary semiring interpretations π_A , since different literals in π_A may have different non-zero values, but conjunctions are interpreted as products, and it is in general impossible to trace the result back to the contributions of the literals.

5.1 The Viterbi Semiring

The Viterbi semiring $\mathbb{V} = ([0, 1]_{\mathbb{R}}, \max, \cdot, 0, 1)$ is used in confidence analysis, probabilistic parsing, and Hidden Markov Models (see [8, 10]). It is isomorphic to the tropical semiring $\mathbb{T} = (\mathbb{R}_{\neq}^{\infty}, \min, +, \infty, 0)$, used for instance for cost analysis and performance evaluation, via $x \mapsto e^{-x}$. Hence, all results that we establish for the Viterbi semiring also hold for the tropical semiring. We can illustrate the shortcomings of the characteristic sentences in their classical form by very simple \mathbb{V} -interpretations with one element.

$$\pi_{19} : \begin{array}{c|c|c|c|c} A & P & Q & \neg P & \neg Q \\ \hline a & 0.1 & 0.9 & 0 & 0 \end{array} \quad \pi_{91} : \begin{array}{c|c|c|c|c} A & P & Q & \neg P & \neg Q \\ \hline a & 0.9 & 0.1 & 0 & 0 \end{array}$$

They are clearly not isomorphic, but trying to construct χ_{19} from π_{19} as above would yield $\chi_{19} = \exists x(\varphi(x) \wedge \psi(x))$ with $\psi(x) = Px \wedge Qx$, hence $\pi_{19}[\chi_{19}] = 0.1 \cdot 0.9 = 0.9 \cdot 0.1 = \pi_{91}[\chi_{19}]$.

However, under semiring semantics, and especially on the Viterbi semiring \mathbb{V} , multiplication need not be idempotent; hence we can hope to distinguish two interpretations by simply repeating one of the literals. In the given example, we can set $\psi(x) := Px \wedge (Qx)^2$, which is short for $Px \wedge Qx \wedge Qx$, to obtain $\pi_{19}[\chi_{19}] = 0.1 \cdot 0.9^2 \neq 0.9 \cdot 0.1^2 = \pi_{91}[\chi_{19}]$. We now generalise this idea to arbitrary finite \mathbb{V} -interpretations. We shall associate with every finite \mathbb{V} -interpretation $\pi_A: \text{Lit}_A(\tau) \rightarrow \mathbb{V}$ and every $\varepsilon \in \mathbb{R}^+$ a characteristic sentence

$$\chi_{\pi_A, \varepsilon} := \exists x_1 \dots \exists x_n (\varphi(\bar{x}) \wedge \psi_{\varepsilon}(\bar{x})),$$

with $n := |A|$ and $\varphi(x)$ as before, but a more involved construction of $\psi_{\varepsilon}(\bar{x})$:

Let $\bar{a} = (a_1, \dots, a_n)$ be some fixed order on A and $L_1(\bar{a}), \dots, L_k(\bar{a})$ an arbitrary enumeration of the “true” literals in $\text{Lit}_A(\tau)$ with $\pi_A(L_i(\bar{a})) \neq 0$. Further, fix a sequence $f(1), \dots, f(k)$ of “exponents” in \mathbb{N} , where $f(1) = 1$ and $f(i+1)$ is chosen large enough so that

$$(*) \quad (1 - \varepsilon)^{f(i+1)} < \varepsilon^{f(1) + \dots + f(i)}.$$

Then, put $\psi_{\varepsilon}(\bar{x}) := \bigwedge_{i=1}^k L_i(\bar{x})^{f(i)}$, where “exponentiation” denotes repetition of a literal.

The idea is that $\chi_{\pi_A, \varepsilon}$ should characterise π_A up to isomorphism by repeating the literals in $\psi_{\varepsilon}(\bar{x})$ “sufficiently often” so that the contribution of each literal can be distinguished and changing the value for one literal surely alters the final value of $\psi_{\varepsilon}(\bar{x})$. Since the elements of \mathbb{V} are from $[0, 1]_{\mathbb{R}}$, the values of the literals can change by an arbitrarily small amount, hence the “exponents” $f(i)$ must depend on the “smallest possible change” ε . This intuition is formalised as follows.

► **Proposition 17.** *Let $\pi_A: \text{Lit}_A(\tau) \rightarrow \mathbb{V}$ and $\pi_B: \text{Lit}_B(\tau) \rightarrow \mathbb{V}$ be two finite, model-defining \mathbb{V} -interpretations, which induce the finite set of values*

$$V := \{\pi_A(L) \mid L \in \text{Lit}_A(\tau)\} \cup \{\pi_B(L) \mid L \in \text{Lit}_B(\tau)\}.$$

Then, for every $\varepsilon \in \mathbb{R}$ bounded by $0 < \varepsilon \leq \min\{|r - s| \mid r, s \in V, r \neq s\}$, we have that $\pi_A[\chi_{\pi_A, \varepsilon}] = \pi_B[\chi_{\pi_B, \varepsilon}]$ implies $\pi_A \cong \pi_B$.

Proof. Assume $\pi_A \llbracket \chi_{\pi_A, \varepsilon} \rrbracket = \pi_B \llbracket \chi_{\pi_A, \varepsilon} \rrbracket$. By construction, $\pi_A \llbracket \chi_{\pi_A, \varepsilon} \rrbracket > 0$, so $\pi_B \llbracket \chi_{\pi_A, \varepsilon} \rrbracket > 0$ as well. By Lemma 16, together with the fact that the existential quantifiers $\exists x_1 \dots \exists x_n$ in $\chi_{\pi_A, \varepsilon}$ are interpreted as max in the Viterbi semiring \mathbb{V} , this implies that $|A| = |B|$, and that we have enumerations $\bar{a} = (a_1, \dots, a_n)$ and $\bar{b} = (b_1, \dots, b_n)$ of the elements of A and B , such that

$$\pi_A \llbracket \chi_{\pi_A, \varepsilon} \rrbracket = \pi_A \llbracket \psi_\varepsilon(\bar{a}) \rrbracket = \pi_B \llbracket \psi_\varepsilon(\bar{b}) \rrbracket = \pi_B \llbracket \chi_{\pi_A, \varepsilon} \rrbracket.$$

Recall that $\pi_A \llbracket \psi_\varepsilon(\bar{a}) \rrbracket = \prod_{i=1}^k \pi_A(L_i(\bar{a}))^{f(i)} > 0$, hence $\pi_A(L_1(\bar{a})), \dots, \pi_A(L_k(\bar{a}))$ are all positive. Accordingly, $\pi_B \llbracket \psi_\varepsilon(\bar{b}) \rrbracket = \prod_{i=1}^k \pi_B(L_i(\bar{b}))^{f(i)} > 0$, so that $\pi_B(L_1(\bar{b})), \dots, \pi_B(L_k(\bar{b}))$ are positive as well. Given that π_A and π_B share the same signature and universe size, any permutations \bar{a}, \bar{b} of their elements yield the same number of positive literals, which is k by definition. We infer that all remaining literals in both interpretations are mapped to zero. Hence, for $i = 1, \dots, k$, let $r_i := \pi_A(L_i(\bar{a})) > 0$ and $s_i := \pi_B(L_i(\bar{b})) > 0$ be the values of the positive literals, then it only remains to show that $r_i = s_i$ for all $i \leq k$ in order to conclude that $\bar{a} \mapsto \bar{b}$ is indeed an isomorphism from π_A to π_B .

Towards a contradiction, assume that this is not the case and let j be the maximal index among $1, \dots, k$ with $r_j \neq s_j$. We can assume that $r_j < s_j$. Since the difference between the two values is at least ε and since $s_j \leq 1$, it follows that $r_j \leq s_j - \varepsilon \leq s_j - \varepsilon s_j = (1 - \varepsilon)s_j$. Further, we have $\varepsilon \leq s_i, r_i \leq 1$ for all i . It follows that

$$r_1^{f(1)} \dots r_j^{f(j)} \leq r_j^{f(j)} \leq (1 - \varepsilon)^{f(j)} s_j^{f(j)} \stackrel{*}{<} \varepsilon^{f(1) + \dots + f(j-1)} \cdot s_j^{f(j)} \leq s_1^{f(1)} \dots s_j^{f(j)}.$$

However, since $r_i = s_i$ for $i = j + 1, \dots, k$, this would imply that

$$\pi_A \llbracket \psi_\varepsilon(\bar{a}) \rrbracket = \prod_{i \leq k} r_i^{f(i)} \neq \prod_{i \leq k} s_i^{f(i)} = \pi_B \llbracket \psi_\varepsilon(\bar{b}) \rrbracket$$

and hence $\pi_A \llbracket \chi_{\pi_A, \varepsilon} \rrbracket \neq \pi_B \llbracket \chi_{\pi_A, \varepsilon} \rrbracket$. ◀

Notice that none of the sentences $\chi_{\pi_A, \varepsilon}$ characterises π_A alone, but the countable set $X_{\pi_A} := \{\chi_{\pi_A, \varepsilon} \mid \varepsilon \in \mathbb{Q}^+\}$ does so. No infinite \mathbb{V} -interpretation π_B agrees with π_A on any of the ε -characteristic sentences $\chi_{\pi_A, \varepsilon}$ due to $\varphi(\bar{x})$, whereas for each finite \mathbb{V} -interpretation π_B , one can calculate an $\varepsilon \in \mathbb{Q}$ to apply the proposition just proved.

► **Theorem 18.** *For finite \mathbb{V} -interpretations π_A and π_B , $\pi_A \equiv \pi_B$ implies $\pi_A \cong \pi_B$.*

As a consequence, there are indeed interesting semirings beyond the Boolean semiring \mathbb{B} , where elementary equivalence implies isomorphism on finite interpretations.

5.2 Finite Axiomatisability

The characteristic set X_{π_A} raises the question whether a finite set of sentences suffices to characterise a \mathbb{V} -interpretation π_A . We will answer this question positively using two observations. By Proposition 17, we observe that $\chi_{\pi_A, \varepsilon}$ characterises π_A up to isomorphism inside the class of \mathbb{V} -interpretations that only use values in $V = \{\pi_A(L) \mid L \in \text{Lit}_A(\tau)\}$, with $\varepsilon := \min\{|r - s| \mid r, s \in V, r \neq s\}$. Hence, π_A can be characterised by adding sentences to ensure that no values outside of V are used.

We will show that this is possible by building sentences that fix particular values $\pi_A(L)$.

► **Definition 19.** Let $\pi: \text{Lit}_A(\tau) \rightarrow \mathbb{V}$ be a finite \mathbb{V} -interpretation over a universe A with n elements and $\varphi(\bar{x}) \in \text{FO}(\tau)$ a formula with $k \in \mathbb{N}$ free variables. The sequence $(s_{\pi, \varphi}^i)_{1 \leq i \leq n^k}$ is defined as the non-increasingly sorted sequence of the values $\pi \llbracket \varphi(\bar{a}) \rrbracket$ for $\bar{a} \in A^k$.

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In particular, $s_{\pi,\varphi}^1$ is the largest possible value $\pi[\llbracket\varphi(\bar{a})\rrbracket]$; further, $s_{\pi,\varphi}^2 \leq s_{\pi,\varphi}^1$ is either the second largest one, or equal to $s_{\pi,\varphi}^1$ if the maximal value is shared by two distinct tuples $\bar{a}, \bar{b} \in A^k$, and so on. We construct a series of sentences that fix the values $(s_{\pi,\varphi}^i)_{1 \leq i \leq n^k}$.

► **Lemma 20** (Sorting Lemma). *For $\varphi(\bar{x}) \in \text{FO}(\tau)$ with k free variables and $1 \leq i \leq n^k$, let*

$$\psi_\varphi^i := \exists \bar{x}_1 \dots \exists \bar{x}_i \left(\bigwedge_{1 \leq j < \ell \leq i} \bar{x}_j \neq \bar{x}_\ell \wedge \bigwedge_{j=1}^i \varphi(\bar{x}_j) \right)$$

where $\bar{x}_1, \dots, \bar{x}_i$ are k -tuples of variables. Then, for any \mathbb{V} -interpretation $\pi: \text{Lit}_A(\tau) \rightarrow \mathbb{V}$ over a universe with n elements, we have that

$$\pi[\llbracket\psi_\varphi^i\rrbracket] = \prod_{1 \leq j \leq i} s_{\pi,\varphi}^j \quad \text{for } 1 \leq i \leq n^k.$$

Proof. Recall that existential quantifiers are interpreted as max on \mathbb{V} . Due to monotonicity of multiplication, the maximum $\pi[\llbracket\psi_\varphi^i\rrbracket]$ is achieved by picking the i pairwise distinct tuples $\bar{a}_1, \dots, \bar{a}_i$ that yield the largest values $\pi[\llbracket\varphi(\bar{a}_j)\rrbracket] = s_{\pi,\varphi}^j$ and inserting them for $\bar{x}_1, \dots, \bar{x}_i$. Clearly, this yields $\pi[\llbracket\psi_\varphi^i\rrbracket] = \prod_{1 \leq j \leq i} s_{\pi,\varphi}^j$. ◀

By observing that \mathbb{V} is cancellative, i.e. $ab = ac$ implies $b = c$ for all $a, b, c \in \mathbb{V}$ with $a \neq 0$, we may disentangle the products $\prod_{1 \leq j \leq i} s_{\pi,\varphi}^j$ to draw the following conclusion.

► **Corollary 21.** *Let $\varphi(\bar{x})$ be as above, and consider two \mathbb{V} -interpretations $\pi: \text{Lit}_A(\tau) \rightarrow \mathbb{V}$ and $\pi': \text{Lit}_B(\tau) \rightarrow \mathbb{V}$ with $|A| = |B| = n$. If π and π' agree on $\Psi := \{\psi_\varphi^i \mid 1 \leq i \leq n^k\}$, then $s_{\pi,\varphi}^i = s_{\pi',\varphi}^i$ for all $1 \leq i \leq n^k$. In other words, the values $\pi[\llbracket\varphi(\bar{a})\rrbracket]$ for $\bar{a} \in A^k$ and $\pi'[\llbracket\varphi(\bar{b})\rrbracket]$ for $\bar{b} \in B^k$ are the same, up to permutation.*

If $R \in \tau$ is a k -ary relation, pick $\varphi(\bar{x}) := R\bar{x}$ and construct $\Psi_R := \{\psi_{R\bar{x}}^i \mid 1 \leq i \leq n^k\}$ according to the Sorting Lemma. Similarly, construct $\Psi_{\neg R}$ from $\varphi(\bar{x}) := \neg R\bar{x}$. Then, define

$$\Psi_\tau := \bigcup_{R \in \tau} \Psi_R \cup \bigcup_{R \in \tau} \Psi_{\neg R}.$$

Clearly, any two \mathbb{V} -interpretations over τ that agree on Ψ_τ use the same set of values from \mathbb{V} . Putting this together with the characteristic sentences $\chi_{\pi,\varepsilon}$ from Proposition 17 provides a finite axiomatisation of any \mathbb{V} -interpretation.

► **Theorem 22.** *Let $\pi: \text{Lit}_A(\tau) \rightarrow \mathbb{V}$ be a finite \mathbb{V} -interpretation. Then, $\Psi_\tau \cup \{\chi_{\pi,\varepsilon}\}$ is a finite axiomatisation of π up to isomorphism, where $\varepsilon := \min\{|r - s| \mid r, s \in \pi(\text{Lit}_A(\tau)), r \neq s\}$.*

Proof. Let $\pi': \text{Lit}_B(\tau) \rightarrow \mathbb{V}$ agree with π on all sentences in $\Psi_\tau \cup \{\chi_{\pi,\varepsilon}\}$. Due to the construction of $\chi_{\pi,\varepsilon}$, π' is finite and $|A| = |B|$. Since π' agrees with π on Ψ_τ , we have $\{\pi(L) \mid L \in \text{Lit}_A(\tau)\} = \{\pi'(L) \mid L \in \text{Lit}_B(\tau)\}$. Thus, we can invoke Proposition 17 by observing that $V = \pi(\text{Lit}_A(\tau))$ and conclude that $\pi[\llbracket\chi_{\pi,\varepsilon}\rrbracket] = \pi'[\llbracket\chi_{\pi,\varepsilon}\rrbracket]$ implies $\pi \cong \pi'$. ◀

Under classical semantics, any finite axiom system $\Phi \subseteq \text{FO}(\tau)$ can be collapsed to a single axiom $\psi := \bigwedge \Phi$, but this is not the case in semiring semantics. To illustrate this, we shall show that there are \mathbb{V} -interpretations that cannot be axiomatised up to isomorphism by a single sentence.

► **Proposition 23.** *There exist \mathbb{V} -interpretations $\pi: \text{Lit}_A(\tau) \rightarrow \mathbb{V}$ such that, for every sentence $\psi \in \text{FO}(\tau)$, there exists an interpretation $\pi': \text{Lit}_A(\tau) \rightarrow \mathbb{V}$ such that $\pi \not\cong \pi'$, but $\pi[\llbracket\psi\rrbracket] = \pi'[\llbracket\psi\rrbracket]$.*

Proof. Take an interpretation with just two atoms Pa and Qa and with values $\pi(Pa) = p$ and $\pi(Qa) = q$ such that $0 < p, q < 1$ are multiplicatively independent real numbers, i.e. $k = \ell = 0$ is the only solution to $p^k q^\ell = 1$ with $k, \ell \in \mathbb{Z}$. Let $\pi_{\mathbb{B}}$ be the corresponding $\mathbb{B}[x, y]$ -interpretation, with $\pi_{\mathbb{B}}(Pa) = x$ and $\pi_{\mathbb{B}}(Qa) = y$. A sentence $\psi \in \text{FO}$ is evaluated under $\pi_{\mathbb{B}}$ to a polynomial $\pi_{\mathbb{B}}[\psi] \in \mathbb{B}[x, y]$, and by the universal property for idempotent semirings, the homomorphism $h: \mathbb{B}[x, y] \rightarrow \mathbb{V}$ induced by $h(x) = p$ and $h(y) = q$ maps $\pi_{\mathbb{B}}[\psi]$ to $\pi[\psi]$. Writing $\pi_{\mathbb{B}}[\psi]$ as a sum of monomials $m = x^i y^j$, we conclude that $\pi[\psi] = p^i q^j$ is the maximal value $m(p, q)$ for the monomials m occurring in $\pi_{\mathbb{B}}[\psi]$. Since p, q are multiplicatively independent, no other monomial can take the same value, i.e. $m'(p, q) < m(p, q)$ for all other monomials m' in $\pi_{\mathbb{B}}[\psi]$. We now can certainly find a value $r \neq p$ that is sufficiently close to p , and a value s such that $r^i s^j = p^i q^j$, i.e. $m(r, s) = m(p, q)$, but $m'(r, s) < m(r, s)$ for all other monomials m' in $\pi_{\mathbb{B}}[\psi]$. For the \mathbb{V} -interpretation π' with $\pi'(Pa) = r$ and $\pi'(Qa) = s$ this implies that $\pi'[\psi] = r^i s^j = p^i q^j = \pi[\psi]$, but clearly, $\pi' \not\equiv \pi$. \blacktriangleleft

This result can be strengthened in many directions. It holds, in fact, for almost all \mathbb{V} -interpretations, as long as they do not map all literals to either 0 or 1. Further, we shall exploit the isomorphism of \mathbb{V} and \mathbb{T} in order to prove explicit lower bounds on the number of axioms that are needed to characterise an interpretation, depending on the number of literals mapped to multiplicatively independent values.

5.3 Lower Bound for Axiomatisations of \mathbb{T} - and \mathbb{V} -interpretations

Recall that $\mathbb{V} = ([0, 1]_{\mathbb{R}}, \max, \cdot, 0, 1)$ is isomorphic to $\mathbb{T} = (R_+^{\infty}, \min, +, \infty, 0)$ via isomorphisms $\sigma_{\mathbb{V} \rightarrow \mathbb{T}}(a) = -\log_b(a)$ for any fixed base $b \in \mathbb{R}_{>1}$, and the corresponding inverse isomorphisms $\sigma_{\mathbb{T} \rightarrow \mathbb{V}}(a) = b^{-a}$ for any fixed $b \in \mathbb{R}_{>1}$. We formulate our result in terms of \mathbb{T} .

▶ Theorem 24. *Let $\pi: \text{Lit}_A(\tau) \rightarrow \mathbb{T}$ be any finite, model-defining \mathbb{T} -interpretation with $|A| = n$ and $|\text{Lit}_A(\tau)| = 2\ell$, such that its finite values $\pi(\text{Lit}_A(\tau)) \setminus \{\infty\}$ are linearly independent over \mathbb{Q} . Then, for any set of sentences $\Psi \subseteq \text{FO}(\tau)$ with $|\Psi| < \ell$, there is an interpretation $\pi': \text{Lit}_A(\tau) \rightarrow \mathbb{T}$ such that $\pi[\psi] = \pi'[\psi]$ for all $\psi \in \Psi$, but $\pi \not\equiv \pi'$.*

Proof. Since π is model-defining, there are ℓ literals L in $\text{Lit}_A(\tau)$ with $\pi(L) \neq \infty$, which we call the positive literals. Choose $X := \{x_1, \dots, x_\ell\}$ and construct the $\mathbb{B}[X]$ -interpretation $\pi_{\mathbb{B}}: \text{Lit}_A(\tau) \rightarrow \mathbb{B}[X]$ by assigning a unique variable to each of the positive literals. Clearly, there is a homomorphism $h: \mathbb{B}[X] \rightarrow \mathbb{T}$ with $h \circ \pi_{\mathbb{B}} = \pi$ induced by mapping each variable to the original value $\pi(L)$ of the corresponding literal.

Enumerate $\Psi = \{\psi_1, \dots, \psi_j\}$ arbitrarily with $j < \ell$ and construct the polynomials $p_i := \pi_{\mathbb{B}}[\psi_i] \in \mathbb{B}[X]$ for $1 \leq i \leq j$. By the fundamental property, $\pi[\psi_i] = h(p_i)$ holds for $1 \leq i \leq j$.

We assume without loss of generality that $p_i \neq 0$ for all $1 \leq i \leq j$, and we will construct π' with the same positive literals as π . Thus, p_i contains a monomial m_i so that $h(m_i)$ is minimal among $\{h(m) : m \in p_i\}$. This monomial is unique thanks to the linear independence of the values of π , which guarantees that $h(m) \neq h(m')$ for $m \neq m'$. Suppose for a contradiction that $h(m) = h(m')$. We may write

$$h(m) = h\left(\prod_{i=1}^{\ell} x_i^{m(x_i)}\right) = \sum_{i=1}^{\ell} m(x_i) \cdot h(x_i),$$

which implies that $h(m)$ is a linear combination of the values of π . In particular, $h(m) = h(m')$ implies that $h(m - m') = 0$, hence $m(x_i) - m'(x_i) = 0$ for all $1 \leq i \leq j$ due to linear independence.

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We conclude that there is a sufficiently small $\varepsilon \in \mathbb{R}_{>0}$ such that changing the numbers $h(x_i)$ by less than ε does not affect the monomial order. In other words, view the values $\bar{v} := (h(x_1), \dots, h(x_\ell)) \in \mathbb{R}_{\geq 0}^\ell$ as a vector and notice that any $\bar{w} \in \mathbb{R}_{\geq 0}^\ell$ with $|\bar{v} - \bar{w}| < \varepsilon$ preserves the monomial order, so that if we construct $h': \mathbb{B}[X] \rightarrow \mathbb{T}$ induced by $h'(x_i) = w_i$ for $1 \leq i \leq \ell$, we have $h(m) < h(m')$ if, and only if, $h'(m) < h'(m')$.

To complete the proof, it remains to ensure that $h'(p_i) = h(p_i)$ stays the same for all $1 \leq i \leq j$. By the above considerations, it suffices to ensure that $h'(m_i) = h(m_i)$ for the corresponding maximal monomials m_1, \dots, m_j . Each of these monomials induces one condition $h(m_i) - h'(m_i) = 0$, which translates to a linear equation

$$h(m_i) - h'(m_i) = \sum_{i=1}^{\ell} m_i(x_i)(h(x_i) - h'(x_i)) = \sum_{i=1}^{\ell} m_i(x_i) \cdot (v_i - w_i) = 0$$

on $(\bar{v} - \bar{w})$.

Since there are only $j < \ell$ equations and ℓ variables, the solution space is at least one-dimensional, meaning that we can pick $\bar{w} \neq \bar{v}$ adequately with $|\bar{v} - \bar{w}| < \varepsilon$ to satisfy all equations and obtain $h'(p_i) = h(p_i)$ for all $1 \leq i \leq j$. Note that due to linear independence, none of the entries from \bar{v} was zero, hence it is possible to ensure that \bar{w} only has positive entries. We thus can pick $\pi' := h' \circ \pi_{\mathbb{B}}$ with the desired properties. ◀

This result translates to \mathbb{V} thanks to isomorphism. Linear independence of values from \mathbb{T} as \mathbb{Q} -vectors translates to multiplicative independence of the corresponding values from \mathbb{V} .

5.4 The Semirings \mathbb{N} and $\mathbb{N}[X]$

We will now provide a similar analysis of axiomatisability for the most general semiring $\mathbb{N}[X]$ by taking a detour via \mathbb{N} . For the construction of the characteristic sentences for \mathbb{N} , we shall need the following combinatorial lemma.

► **Lemma 25.** *For any two natural numbers k, c with $c > 1$, there exists a exponent e such that, for any two non-decreasing sequences $r_1 \leq r_2 \leq \dots \leq r_k$ and $s_1 \leq s_2 \leq \dots \leq s_k$ of k natural numbers, with $r_k, s_k < c$, the equation $r_1^e + \dots + r_k^e = s_1^e + \dots + s_k^e$ implies that the two sequences are the same, i.e. $r_i = s_i$ for all $i \leq k$.*

Proof. Choose e large enough so that $(c/(c-1))^e > k$. Towards a contradiction, assume that there are two *distinct* sequences $r_1 \leq r_2 \leq \dots \leq r_k < c$ and $s_1 \leq s_2 \leq \dots \leq s_k < c$ such that $r_1^e + \dots + r_k^e = s_1^e + \dots + s_k^e$. Let j be the maximal index with $r_j \neq s_j$. Thanks to additive cancellation, we can remove the summands with index $i > j$ to obtain that $r_1^e + \dots + r_j^e = s_1^e + \dots + s_j^e$. By symmetry we can assume that $r_j < s_j$. Since $s_j < c$, it follows that $s_j > (c/(c-1))r_j$ and hence $s_j^e > k \cdot r_j^e$. But this implies that

$$r_1^e + \dots + r_j^e \leq j \cdot r_j^e \leq k \cdot r_j^e < s_j^e \leq s_1^e + \dots + s_j^e,$$

contradicting the equation above. ◀

► **Lemma 26.** *Let $(r_1, \dots, r_k), (s_1, \dots, s_k) \in \mathbb{N}^k$ be strictly bounded by c , that is $r_i, s_i < c$ for all $i \leq k$. Then, there is an exponent e depending only on c and k such that*

$$\sum_{i=1}^k r_i^e = \sum_{i=1}^k s_i^e$$

implies that there is a permutation $\sigma \in S_k$ such that $r_i = s_{\sigma(i)}$ for all $1 \leq i \leq k$.

Proof. Sort both sequences non-decreasingly, that is, permute them with $\rho, \tau \in S_k$ so that $r_{\rho(1)} \leq \dots \leq r_{\rho(k)}$ and $s_{\tau(1)} \leq \dots \leq s_{\tau(k)}$. By the previous Lemma, there is a suitable e such that $r_{\rho(i)} = s_{\tau(i)}$ for all $i \leq k$. Then, $\sigma := \tau \circ \rho^{-1}$ is the desired permutation. ◀

We are now ready to construct characteristic sentences for finite \mathbb{N} -interpretations $\pi_A: \text{Lit}_A(\tau) \rightarrow \mathbb{N}$. For $n = |A|$, let $L_1(\bar{x}), \dots, L_k(\bar{x})$ be an enumeration of all literals in $\text{Lit}_n(\tau)$. For any constant $q \in \mathbb{N}$ we define the q -characteristic sentence $\chi_{\pi_A, q}$ as

$$\chi_{\pi_A, q} := \exists x_1 \dots \exists x_n (\varphi(\bar{x}) \wedge \psi_q(\bar{x}))^e, \quad \text{with } \psi_q(\bar{x}) := \bigvee_{i=1}^k q^{i-1} \cdot L_i(\bar{x}),$$

with $\varphi(\bar{x})$ as given before and e is an exponent that depends on q, n and τ , according to Lemma 26. The notation $q^{i-1} \cdot L_i(\bar{x})$ denotes a disjunctive repetition of the literal $L_i(\bar{x})$ for q^{i-1} times.

The idea of this construction is similar to the one for the Viterbi semiring. While ε -characteristic sentences work for \mathbb{V} -interpretations where the differences of two distinct values are at least ε , q -characteristic sentences work for \mathbb{N} -interpretations with values less than q . With this in mind, we can explain the construction of $\psi_q(\bar{x})$ as follows. If all values in π_A are less than q , we can picture the value

$$\pi_A[\psi_q(\bar{a})] = \sum_{i=1}^k q^{i-1} \pi_A(L_i(\bar{a})),$$

to be in a number system with radix q , hence the values $\pi_A(L_i(\bar{a}))$ can be seen as digits. Thus, it is immediately clear that for any \mathbb{N} -interpretation π_B with universe enumerated by \bar{b} and values less than q , $\pi_A[\psi_q(\bar{a})] = \pi_B[\psi_q(\bar{b})]$ implies that $\bar{a} \xrightarrow{\sim} \bar{b}$ is an isomorphism between π_A and π_B , since the corresponding “digits” $\pi_A(L_i(\bar{a}))$ and $\pi_B(L_i(\bar{b}))$ for each $1 \leq i \leq k$ have to be the same.

The only remaining problem is the fact that existential quantifiers $\exists x_1 \dots \exists x_n$ from $\chi_{\pi_A, q}$ are interpreted as a sum in \mathbb{N} . Thus, the value $\pi_A[\chi_{\pi_A, q}]$ is not induced by a single variable assignment \bar{a} . The exponent e is used to separate the contributions of different variable assignments to the sum on the basis of Lemma 26.

▶ **Theorem 27.** *Let π_A and π_B are finite \mathbb{N} -interpretations with values less than q . Then $\pi_A[\chi_{\pi_A, q}] = \pi_B[\chi_{\pi_A, q}]$ implies that $\pi_A \cong \pi_B$.*

Proof. Clearly, $\varphi(\bar{x})$ takes care of the number of elements, hence we can assume \bar{a} and \bar{b} enumerate the universes of π_A and π_B . Now, we have

$$\pi_A[\chi_{\pi_A, q}] = \sum_{\sigma \in S_n} \pi_A[\psi_q(\sigma(\bar{a}))]^e = \sum_{\sigma \in S_n} \pi_B[\psi_q(\sigma(\bar{b}))]^e = \pi_B[\chi_{\pi_A, q}].$$

Recall that $\psi_q(\bar{x})$ is constructed as a number with k “digits”, where the digits are the values of the literals $\pi_A(L_i(\bar{a}))$ and $\pi_B(L_i(\bar{b}))$, which are bounded by q . Hence, $\pi_A[\psi_q(\sigma(\bar{a}))]$ and $\pi_B[\psi_q(\sigma(\bar{b}))]$ are less than $c := q^k$, which only depends on q, n and τ . By Lemma 26, there is a sufficiently large e so that $\sum_{\sigma \in S_n} \pi_A[\psi_q(\sigma(\bar{a}))]^e = \sum_{\sigma \in S_n} \pi_B[\psi_q(\sigma(\bar{b}))]^e$ implies that both sums share the same summands. In particular, there are permutations $\sigma_A, \sigma_B \in S_n$ such that $\pi_A[\psi_c(\sigma_A(\bar{a}))] = \pi_B[\psi_c(\sigma_B(\bar{b}))]$.

Thanks to the construction of $\psi_c(\bar{x})$, this yields $\pi_A(L_i(\sigma_A(\bar{a}))) = \pi_B(L_i(\sigma_B(\bar{b})))$ for all literals L_i of $\text{Lit}_n(\tau)$. Thus, $\sigma_B \circ (\bar{a} \mapsto \bar{b}) \circ \sigma_A^{-1}$ is an isomorphism from π_A to π_B . ◀

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We can further use the q -characteristic sentences also for $\mathbb{N}[X]$ -interpretations instead of \mathbb{N} -interpretations. Let $X_k = \{x_1, \dots, x_k\}$, and let $\mathbb{N}[X_k](C, n)$ denote the set of polynomials $p \in \mathbb{N}[X_k]$ with coefficients less than C and exponents less than n . If we choose a suitable variable assignment $X_k \rightarrow \mathbb{N}$, we can obtain a homomorphism that assigns unique values to all polynomials in $\mathbb{N}[X_k](C, n)$.

► **Lemma 28.** *The variable assignment $x_i \mapsto C^{n^{i-1}}$ for $1 \leq i \leq k$ defines a homomorphism $h: \mathbb{N}[X_k] \rightarrow \mathbb{N}$ which induces a bijection from $\mathbb{N}[X_k](C, n)$ to $\{0, \dots, c - 1\} \subseteq \mathbb{N}$ where $c := C^{n^k}$.*

Proof. We proceed by induction on the number of variables $k \in \mathbb{N}$. The base case $k = 0$ is trivial, since $\mathbb{N}[\emptyset] \cong \mathbb{N}$ and the empty assignment induces the corresponding isomorphism. For $k > 0$, notice that $\mathbb{N}[X_k] \cong \mathbb{N}[X_{k-1}][x_k]$. Hence, each $p \in \mathbb{N}[X_k](C, n)$ may be written as

$$p = \sum_{i=0}^{n-1} q_i x_k^i, \quad \text{where } q_i \in \mathbb{N}[X_{k-1}](C, n).$$

Thus, applying the induced homomorphism h yields

$$h(p) = \sum_{i=0}^{n-1} h(q_i) h(x_k)^i = \sum_{i=0}^{n-1} h'(q_i) (C^{n^{k-1}})^i,$$

where $h': \mathbb{N}[X_{k-1}] \rightarrow \mathbb{N}$ is induced by $x_i \mapsto C^{n^{i-1}}$ for $1 \leq i < k$. By induction hypothesis the restriction $h'|_{\mathbb{N}[X_{k-1}](C, n)}$ is a bijection from $\mathbb{N}[X_{k-1}](C, n)$ to $\{0, \dots, C^{n^{k-1}} - 1\}$. Clearly, $h(p)$ may be seen as a number with n digits $h'(q_i) \in \{0, \dots, C^{n^{k-1}} - 1\}$ for $0 \leq i < n$ in the number system with radix $C^{n^{k-1}}$. Thus, any number in $\{0, \dots, C^{n^k} - 1\}$ can be uniquely represented as $h(p)$ for $p \in \mathbb{N}[X_k](C, n)$, which completes the proof. ◀

► **Corollary 29.** *For finite $\mathbb{N}[X]$ -interpretations π_A and π_B whose values are contained in $\mathbb{N}[X](C, n)$, $\pi_A \llbracket \chi_{\pi_A, c} \rrbracket = \pi_B \llbracket \chi_{\pi_A, c} \rrbracket$ implies $\pi_A \cong \pi_B$ with $c := C^{n^{|X|}}$.*

Proof. Transform π_A and π_B to \mathbb{N} -interpretations $\pi'_A := h \circ \pi_A$ and $\pi'_B := h \circ \pi_B$ by applying the homomorphism from above. The fundamental property yields $\pi'_A \llbracket \chi_{\pi_A, c} \rrbracket = \pi'_B \llbracket \chi_{\pi_A, c} \rrbracket$. Since $h|_{\mathbb{N}[X](C, n)}$ is a bijection from $\mathbb{N}[X](C, n)$ to $\{0, \dots, c\}$, the values of π'_A and π'_B are less than c , hence we can invoke Theorem 27 to conclude $\pi'_A \cong \pi'_B$. Now, the injectivity of h on $\mathbb{N}[X](C, n)$ yields $\pi_A \cong \pi_B$. ◀

Similarly to the implications of Proposition 17 on the Viterbi semiring \mathbb{V} , we conclude that finite $\mathbb{N}[X]$ -interpretations π_A are characterised by the set $X_{\pi_A} := \{\chi_{\pi_A, c} \mid c \in \mathbb{N}, c > 1\}$ of characteristic sentences. The obvious consequence is that no counterexamples exist on $\mathbb{N}[X]$.

► **Theorem 30.** *For finite $\mathbb{N}[X]$ -interpretations π_A and π_B , $\pi_A \equiv \pi_B$ implies $\pi_A \cong \pi_B$.*

These results highlight the importance of cancellation for the construction of characteristic sentences. \mathbb{V} and $\mathbb{N}[X]$ allow multiplicative cancellation, and $\mathbb{N}[X]$ even allows additive cancellation. We shall use this observation in the search for counterexamples on semirings that do *not* admit cancellation.

6 Cancellation

There are indeed counterexamples for a large class of semirings that break cancellation, including the polynomial semirings $\mathbb{B}[X]$ and $\mathbb{S}[X]$.

► **Definition 31.** Let K be an idempotent semiring. A witness that K breaks cancellation is a triple $a, b, c \in K \setminus \{0\}$ such that

- (1) $a + b = a + c = a$ and
- (2) $ab = ac$, but $b \neq c$.¹

For any such triple, we define the following two non-isomorphic K -interpretations.

$$\pi_b : \begin{array}{c|c|c} A & R & \neg R \\ \hline d & a & 0 \\ e & b & 0 \end{array} \quad \pi_c : \begin{array}{c|c|c} A & R & \neg R \\ \hline d & a & 0 \\ e & c & 0 \end{array}$$

► **Lemma 32.** The K -interpretations π_b and π_c are elementarily equivalent.

Proof. Consider the $\mathbb{B}[X]$ interpretation

$$\pi : \begin{array}{c|c|c} A & R & \neg R \\ \hline d & x & 0 \\ e & y & 0 \end{array}$$

Let $h_b, h_c: \mathbb{B}[X] \rightarrow K$ be the unique homomorphisms induced by $x \mapsto a, y \mapsto b$ and $x \mapsto a, y \mapsto c$ respectively. Obviously, $\pi_b = h_b \circ \pi$ and $\pi_c = h_c \circ \pi$, hence, for each sentence $\psi \in \text{FO}(\{R\})$, the fundamental property yields $\pi_b \llbracket \psi \rrbracket = h_b(\pi \llbracket \psi \rrbracket)$ and $\pi_c \llbracket \psi \rrbracket = h_c(\pi \llbracket \psi \rrbracket)$. In fact, if we set $p := \pi \llbracket \psi \rrbracket$, we have $\pi_b \llbracket \psi \rrbracket = h_b(p)$ and $\pi_c \llbracket \psi \rrbracket = h_c(p)$, hence both interpretations evaluate the same polynomial p under their own homomorphism.

It remains to show that $h_b(p) = h_c(p)$. The automorphism \bar{h} of $\mathbb{B}[X]$ induced by swapping the variables x and y yields the $\mathbb{B}[X]$ -interpretation

$$\bar{h} \circ \pi = \bar{\pi} : \begin{array}{c|c|c} A & R & \neg R \\ \hline d & y & 0 \\ e & x & 0 \end{array}$$

Clearly, $\pi \cong \bar{\pi}$ by swapping d and e , hence $p = \pi \llbracket \psi \rrbracket = \bar{\pi} \llbracket \psi \rrbracket = \bar{h}(\pi \llbracket \psi \rrbracket) = \bar{h}(p)$. In other words, p is invariant under swapping variables, so for each pair i, j we have that $x^i y^j \in p$ if, and only if, $x^j y^i \in p$. In particular, $x^i \in p \Leftrightarrow y^i \in p$ (*).

Since p is finite and all exponents are less than some $d \in \mathbb{N}$, we may write p as

$$p = \sum_{i,j < d, x^i y^j \in p} x^i y^j = \sum_{i < d, x^i \in p} x^i + \sum_{j < d, y^j \in p} y^j + \sum_{0 < i,j < d, x^i y^j \in p} x^i y^j.$$

Set $I := \{i < d \mid x^i \in p\} \stackrel{*}{=} \{j < d \mid y^j \in p\}$ and $M := \{x^i y^j \mid 0 < i, j < d, x^i y^j \in p\}$. Then,

$$p = \sum_{i \in I} (x^i + y^i) + \sum_{m \in M} m.$$

For each $m \in M$, $h_b(m) = a^i b^j = a^i c^j = h_c(m)$ due to condition (2) and $i > 0$. More precisely, if $i > 0$, we can invoke (2) inductively to transform ab^j into ac^j due to commutativity of multiplication. We now invoke condition (1) for each $i \in I$. For $z \in \{b, c\}$ and using idempotence of K (i), this yields

¹ Strictly speaking, condition (2) suffices for breaking cancellation. Condition (1) imposes a further restriction on the witness, which is needed for the proof of Lemma 32.

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$$\begin{aligned} a^i + z^i &\stackrel{(1)}{=} (a + z)^i + z^i = \sum_{j=0}^i a^{i-j} z^j + z^i = \sum_{j=0}^{i-1} a^{i-j} z^j + z^i + z^i \\ &\stackrel{(i)}{=} \sum_{j=0}^{i-1} a^{i-j} z^j + z^i = a^i. \end{aligned}$$

Hence, $h_b(x^i + y^i) = a^i + b^i = a^i = a^i + c^i = h_c(x^i + y^i)$ for each $i \in I$ holds as well. Together, we have

$$h_b(p) = \sum_{i \in I} h_b(x^i + y^i) + \sum_{m \in M} h_b(m) = \sum_{i \in I} h_c(x^i + y^i) + \sum_{m \in M} h_c(m) = h_c(p),$$

which completes the proof, since ψ was arbitrary and $\pi_b[\psi] = h_b(p) = h_c(p) = \pi_c[\psi]$. ◀

Since $\pi_b \not\equiv \pi_c$ clearly holds for π_b and π_c from Definition 31, we can apply Lemma 32 to construct counterexamples in many idempotent semirings, such as $\mathbb{B}[X]$ itself and $\mathbb{S}[X]$. The only requirement for this approach is the existence of an appropriate witness (a, b, c) .

► **Theorem 33.** *For $X \supseteq \{x, y\}$, there exists a pair of elementarily equivalent, but non-isomorphic K -interpretations in the shape of π_b and π_c for both $K = \mathbb{B}[X]$ and $K = \mathbb{S}[X]$.*

Proof. For $\mathbb{B}[X]$, choose $a := x + y + x^2 + xy + y^2$, $b := x^2 + y^2$ and $c := x^2 + xy + y^2$ to obtain the following pair of $\mathbb{B}[X]$ -interpretations.

$$\pi_b : \begin{array}{c|cc} A & R & \neg R \\ \hline d & x + y + x^2 + xy + y^2 & 0 \\ e & x^2 + y^2 & 0 \end{array} \quad \pi_c : \begin{array}{c|cc} A & R & \neg R \\ \hline d & x + y + x^2 + xy + y^2 & 0 \\ e & x^2 + xy + y^2 & 0 \end{array}$$

To prove the desired properties, we only have to check conditions (1) and (2) from Definition 31 and then invoke Lemma 32. Condition (1) is obvious, and for (2), it suffices to expand the products ab and ac to calculate that

$$ab = ac = x^3 + xy^2 + x^2y + y^3 + x^4 + x^3y + x^2y^2 + xy^3 + y^4.$$

In $\mathbb{S}[X]$, we may use the “absorptive versions” $a := x + y$, $b := x^2 + y^2$ and $c := x^2 + xy + y^2$ and apply the same argument. ◀

Finally, we will apply Lemma 32 to the Łukasiewicz semiring $\mathbb{L} = ([0, 1]_{\mathbb{R}}, \max, \star, 0, 1)$, which is used in many-valued logic. A witness that \mathbb{L} breaks cancellation is easy to obtain due to the multiplication \star being defined as $a \star b := \max\{0, a + b - 1\}$, which “cuts off” at 0.

► **Theorem 34.** *There exists a pair of elementarily equivalent, but non-isomorphic \mathbb{L} -interpretations.*

Proof. \mathbb{L} is idempotent and the triple $a := 1/2, b := 1/3, c := 1/4 \in \mathbb{L}$ satisfies the conditions (1) and (2) from Definition 31. Together with Lemma 32, we conclude that the \mathbb{L} -interpretations π_b and π_c , as constructed in the definition from the triple (a, b, c) , are elementarily equivalent, but non-isomorphic. ◀

7 Conclusion and Outlook

Our analysis of first-order axiomatisations and elementary equivalence of finite semiring interpretations has revealed some remarkable differences between semiring semantics and classical Boolean semantics. Depending on the underlying semiring, there may exist finite semiring interpretations that are elementarily equivalent without being isomorphic. Indeed, this phenomenon happens already in very simple cases such as for min-max semirings with three elements. On the other side, there are relevant semirings, used for instance in provenance analysis in databases such as the tropical semiring or the Viterbi semiring, where every finite interpretation is first-order axiomatisable, and in fact even by a finite set of axioms. However, and this is again an interesting difference to Boolean semantics, a finite axiomatisation does not imply an axiomatisation by a single axiom.

Also for the semirings of polynomials, fundamental for a general provenance analysis that reveals which combinations of atomic facts are responsible for the truth of a logical statement, the picture is not unique. While the most general semiring $\mathbb{N}[X]$, freely generated by X , admits axiomatisations of all finite interpretations, so that elementary equivalence implies isomorphism, this is not the case for the semirings $\text{PosBool}[X]$, $\mathbb{S}[X]$, $\mathbb{B}[X]$ and $\mathbb{W}[X]$ which are universal for important subclasses of semirings.

In the study of elementary equivalence for semiring semantics, it turns out that there is no *straightforward* adaptation of Ehrenfeucht–Fraïssé games, or their generalisations such as Hellas bijective pebble game [16], to semiring interpretations. Whatever the specific protocol of possible moves in such games may be, they always result in a *localisation*, in the sense that some tuples in the two structures are picked that are indistinguishable on the atomic level. As shown by the very simple example of the interpretations π_{PQ} and π_{QP} at the beginning of Sect. 4, this is not possible in semiring semantics. Although the two interpretations are elementarily equivalent, no element of the first “looks the same” as any element of the second, so any kind of localisation would result in a winning play of the Spoiler. It is an intriguing open question how elementary equivalence of semiring interpretations can be captured by a different notion of comparison games or back-and-forth systems à la Fraïssé. This not being available (yet), we have established elementary equivalence by different methods, based on homomorphisms, which we believe to be of independent interest.

There are many other model-theoretic issues that deserve to be studied in semiring semantics. While we have limited ourselves here to *finite* semiring interpretations, the study of semiring semantics over infinite universes is of course very interesting as well. It requires certain restrictions on the underlying semirings, concerning existence and appropriate algebraic properties of infinite sums and products, but there are useful semirings satisfying such properties. A particularly interesting question is what kind of compactness and preservation results are possible in such contexts.

We finally remark that an altogether different approach to semiring interpretations would consider them as two-sorted structures, one sort being a finite or infinite structure (or just a set), the second one consisting of the semiring, with functions from the first to the second sort giving the semiring interpretation of the literals. This is very similar to the approach of metafinite model theory [11], and to get a reasonable logical theory it is important that the elements of the second sort, here the semiring, are treated differently than the elements of the first sort. In particular, quantifiers should range only over the first sort, and operations on the second sort are just the algebraic semiring operations and their aggregates, together with equalities between terms. Such an approach is certainly useful for a number of questions and permits the study of semiring interpretations via classical Boolean semantics. In particular,

once semiring values are directly accessible in the logic, the construction of characteristic sentences, axiomatising a finite structure up to isomorphism, can easily be translated into such a setting. However, such an internalisation of the semiring values, from the meta-level of truth values into the structures under consideration, does not really capture the essence of semiring semantics.

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