Smooth Approximations and Relational Width Collapses

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Abstract

We prove that relational structures admitting specific polymorphisms (namely, canonical pseudo-WNU operations of all arities $n \geq 3$) have low relational width. This implies a collapse of the bounded width hierarchy for numerous classes of infinite-domain CSPs studied in the literature. Moreover, we obtain a characterization of bounded width for first-order reducts of unary structures and a characterization of MMSNP sentences that are equivalent to a Datalog program, answering a question posed by Bienvenu *et al.*. In particular, the bounded width hierarchy collapses in those cases as well.

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1 Introduction

Local consistency checking is an algorithmic technique that is central in computer science. Intuitively speaking, it consists in propagating local information through a structure so as to infer global information (consider, e.g., computing the transitive closure of a relation as deriving global information from local one). Local consistency checking has a prominent role in the area of constraint satisfaction, where one is given a set of variables V and constraints and one has to find a satisfying assignment $h\colon V \to D$ for the constraints. In this setting, the local consistency algorithm can be used to decrease the size of the search space efficiently or even to correctly solve some constraint satisfaction problems in polynomial time (for example, 2-SAT or Horn-SAT). However, the use of local consistency methods is not limited to constraint satisfaction. Indeed, local consistency checking is also used for such problems as the graph isomorphism problem, where it is is known as the Weisfeiler-Leman algorithm. Again, the technique can be used to derive implied constraints that an isomorphism between two graphs has to satisfy so as to narrow down the search space, but local consistency is in fact powerful enough to solve the graph isomorphism problem over any non-trivial minor-closed class of graphs [36]. Notably, the best algorithm for graph isomorphism to date also uses local consistency as a subroutine [3]. Finally, local consistency can be used to solve games involved in formal verification such as parity games and mean-payoff games [16].

One of the reasons for the ubiquity of local consistency is that its underlying principles can be described in many different languages, such as the language of category theory [1], in the language of finite model theory (by Spoiler-Duplicator games [38] or by homomorphism duality [2]), and logical definability (in Datalog, or infinitary logics with bounded number of variables). For constraint satisfaction problems over a finite template, the power of local consistency checking can additionally be characterised algebraically. More precisely, there are conditions on the set of *polymorphisms* of a template $\mathbb{A}$ such that local consistency correctly solves its constraint satisfaction problem CSP($\mathbb{A}$) if, and only if, the polymorphisms of $\mathbb{A}$ satisfy these conditions. Moreover, whenever local consistency correctly solves CSP($\mathbb{A}$), where $\mathbb{A}$ is finite, then in fact only a very restricted form of local consistency checking is needed [4]. This fact is known as the *collapse of the bounded width hierarchy*, and it has strong consequences both for complexity and logic. On the one hand, the collapse gives efficient algorithms that are able to solve *all* the CSPs that are solvable by local consistency methods, and in fact this gives a polynomial-time algorithm solving instances of the *uniform CSP*. On the other hand, this collapse induces collapses in all the areas mentioned at the beginning of this paragraph.

Many natural problems from computer science can only be phrased as CSPs where the template is infinite. This is the case for linear programming, some reasoning problems in artificial intelligence such as ontology-mediated data access, or even problems as simple to formulate as the digraph acyclicity problem. In order to understand the power of local consistency in more generality it is thus necessary to consider its use for infinite-domain CSPs. Infinite-domain CSPs with an ω-categorical template form a very general class of problems for which the algebraic approach from the finite case can be extended, and numerous results in the recent years have shown the power of this approach. An algebraic characterisation of local consistency checking for infinite-domain CSPs is, however, missing. In fact, the negative results of [19], refined in [35], show that no purely algebraic description of local consistency is possible for CSPs with ω-categorical templates; this is even the case for temporal CSPs [20]. These negative results are to be compared with the recent result by Mottet and Pinsker [45] that did provide an algebraic description of local consistency for several subclasses of ω-categorical templates.

In the finite, the algebraic characterisation of local consistency relies on a set of algebraic tools whose development eventually led to the solutions of the Feder-Vardi dichotomy conjecture. Bulatov's proof of the Feder-Vardi conjecture [30] builds on his theory of edge-colored algebras, that were also used in his characterisation of bounded width [29]; Zhuk's proof [50, 51] relies on the concept of absorption, which was developed by Barto and Kozik in their effort to prove the bounded width conjecture [5, 7]. Comparable algebraic tools, or a general theory, are at the moment missing in the theory of infinite-domain CSPs, even with an ω-categorical template. The most general results obtained so far use *canonical operations*, which behave like operations on finite sets, and for which it is sometimes possible to mimic the universal-algebraic approach to finite-domain CSPs. Canonical operations alone do not seem to be sufficient in full generality and a characterisation of their applicability is also missing, but on the positive side their applicability covers a vast majority of the results that were proved in the area. The application of canonical operations to approach the question of local consistency for infinite-domain CSPs has only been started recently [18, 45, 49].

1.1 Results

In the present paper, we focus on applying the theory of canonical functions to study the power of local consistency checking for constraint satisfaction problems over ω-categorical templates. Our objective is two-fold: on the one hand, we wish to obtain generic sufficient conditions that imply that local consistency solves a given CSP, and on the other hand we wish to understand the amount of locality needed for local consistency to solve the CSP, as measured by the so-called *relational width*. The definitions of all concepts mentioned in this section can be found in the preliminaries.

In order to solve the first objective, we build on recent work by Mottet and Pinsker [45] and expand the use of their *smooth approximations* to fully suit *equational (non-)affineness*, which is roughly the algebraic situation imposed by local consistency solvability. The main technical contribution is a new *loop lemma* that exploits deep algebraic tools from the finite [6] and, assuming the use of canonical functions is unfruitful, allows to obtain the existence of polymorphisms of every arity $n \geq 2$ and satisfying certain strong symmetry conditions. Using this loop lemma, we are able to obtain a characterisation of bounded width for particular classes of templates, namely for first-order reducts of unary structures and for certain structures related to the logic MMSNP. In particular, we obtain a decidable necessary and sufficient condition for an MMSNP sentence to be equivalent to a Datalog program, solving an open problem from [11, 34].

▶ **Theorem 1.** *The Datalog-rewritability problem for MMSNP is decidable, and is 2NExpTime-complete.*

In order to solve the second objective, we prove that sufficiently locally consistent instances of a given CSP can be turned into locally consistent instances of a finite-domain CSP. If the finite-domain CSP has bounded width then it has relational width $(2, 3)$ by [4], which allows us to obtain a collapse of bounded width for structures whose clone of canonical polymorphisms satisfy suitable identities, thus obtaining a similar collapse as in the finite case. In particular, it turns out that the relational width of a structure then only depends on certain simple parameters of the structure whose automorphism group is considered in the notion of canonicity.

▶ **Theorem 2.** *Let $k, \ell \geq 1$, and let $\mathbb{A}$ be a first-order reduct of a k-homogeneous ℓ-bounded ω-categorical structure $\mathbb{B}$.*

- *If the clone of* $\mathrm{Aut}(\mathbb{B})$*-canonical polymorphisms of* $\mathbb{A}$ *contains pseudo-WNUs modulo* $\overline{\mathrm{Aut}(\mathbb{B})}$ *of all arities $n \geq 3$, then $\mathbb{A}$ has relational width $(2k, \max(3k, \ell))$.*
- *If the clone of* $\mathrm{Aut}(\mathbb{B})$*-canonical polymorphisms of* $\mathbb{A}$ *contains pseudo-totally symmetric operations modulo* $\overline{\mathrm{Aut}(\mathbb{B})}$ *of all arities, then $\mathbb{A}$ has relational width $(k, \max(k+1, \ell))$.*

Note that every finite structure $\mathbb{A}$ with domain $\{a_1, \ldots, a_n\}$ is a first-order reduct of the structure $(\{a_1, \ldots, a_n\}; \{a_1\}, \ldots, \{a_n\})$, which is easily seen to be 1-homogeneous and 2-bounded. Thus the width obtained in Theorem 2 coincides with the width given by Barto's collapse result from [4].

As a corollary of Theorem 2, we obtain a collapse of the bounded width hierarchy for first-order reducts of the unary structures mentioned above, as well as of numerous other structures studied in the literature [22, 18, 17, 39].

▶ **Corollary 3.** *Let $\mathbb{A}$ be a structure that has bounded width. If $\mathbb{A}$ is a first-order reduct of:*

- *the universal homogeneous graph $\mathbb{G}$ or tournament $\mathbb{T}$, or of a unary structure, then $\mathbb{A}$ has relational width at most $(4, 6)$;*
- *the universal homogeneous K_n-free graph $\mathbb{H}_n$, where $n \geq 3$, then at most $(2, n)$;*
- $(\mathbb{N}; =)$*, the countably infinite equivalence relation with infinitely many equivalence classes $\mathbb{C}_\omega^\omega$, or the universal homogeneous partial order $\mathbb{P}$, then at most $(2, 3)$.*

Proof. A first-order reduct of $\mathbb{G}$ or $\mathbb{T}$ has bounded width if and only if the algebraic condition in the first item of Theorem 2 is satisfied [45]. Since both $\mathbb{G}$ and $\mathbb{T}$ are 2-homogeneous and 3-bounded our claim follows. First-order reducts of $\mathbb{H}_n$, $(\mathbb{N}; =)$ or $\mathbb{C}_\omega^\omega$ have bounded width if and only if the condition in the second item of Theorem 2 is satisfied, by [17], [14] and [26], respectively. Since $\mathbb{H}_n$ is 2-homogeneous and n-bounded, and since both $(\mathbb{N}; =)$ and $\mathbb{C}_\omega^\omega$ are 2-homogeneous and 3-bounded, the claimed bound follows.

By appeal to Theorems 2 and 23 in the present paper our claim holds for first-order reducts of unary structures.

Finally, a first-order reduct of $\mathbb{P}$ with bounded width is either homomorphically equivalent to a first-order reduct of $(\mathbb{Q}; <)$ or it satisfies the algebraic condition in the second item of Theorem 2 [39]. In the latter case we are done by Theorem 2, in the former we appeal to the syntactical characterization of first-order reducts of $(\mathbb{Q}; <)$. Indeed, such a structure has bounded width iff it is definable by a conjunction of so-called Ord-Horn clauses [20]. It then follows by [28] that a first-order reduct of $(\mathbb{Q}; <)$ with bounded width has relational width $(2, 3)$. The result for $\mathbb{P}$ follows. ◀

The following example shows that for some of the structures under consideration, the bounds on relational width provided by Corollary 3 are tight.

▶ **Example 4.** To show the tightness of the first bound in the case of the universal homogeneous graph, we exhibit a first-order reduct $\mathbb{A}$ such that for all $i \leq j$ with $1 \leq j < 4$ or $1 \leq i < 6$ there exists a non-trivial, (i, j)-minimal instance of $\mathrm{CSP}(\mathbb{A})$ that has no solution. Let $\mathbb{B} := (A; E)$ be the universal homogeneous graph with edge relation E, and let $N := (A^2 \setminus E) \cap \neq$. Consider the first-order reduct $\mathbb{A} := (A; R_=, R_{\neq})$ of $\mathbb{B}$, where $R_= := \{(a, b, c, d) \in A^4 \mid E(a, b) \wedge E(c, d) \text{ or } N(a, b) \wedge N(c, d)\}$ and $R_{\neq} := \{(a, b, c, d) \in A^4 \mid E(a, b) \wedge N(c, d) \text{ or } N(a, b) \wedge E(c, d)\}$.

It can be seen that $\mathbb{A}$ has bounded width, so that Theorem 2 implies that $\mathbb{A}$ has relational width $(4, 6)$. It is easy to see that the instance

$$\Phi = R_=(v_1, v_2, v_3, v_4) \wedge R_{\neq}(v_1, v_2, v_3, v_4)$$

is non-trivial, (i,j)-minimal for all $i \leq j$ with $1 \leq i \leq 3$, and has no solution. Moreover, the $(4,5)$-minimal instance equivalent to the instance

$$\Psi = R_{=}(v_1,v_2,v_3,v_4) \wedge R_{\neq}(v_3,v_4,v_5,v_6) \wedge R_{\neq}(v_1,v_2,v_5,v_6)$$

is non-trivial and has no solution. It follows that the exact relational width of $\mathbb{A}$ is $(4,6)$.

The bounds on relational width provided by the second and third item of Corollary 3 are easily seen to be tight as well. Indeed, let $n \geq 3$, let $\mathbb{B} := (A; E)$ be the universal homogeneous K_n-free graph, let $N := (A^2 \backslash E) \cap \neq$ and let $\mathbb{A} := (A; E, N)$. $\mathbb{A}$ is preserved by canonical pseudo-totally symmetric operations modulo $\overline{\mathrm{Aut}(\mathbb{B})}$ of all arities and has therefore relational width $(2,n)$ by Theorem 2. But the non-trivial, $(2,n-1)$-minimal instance

$$\Phi = \bigwedge_{1 \leq i \neq j \leq n} E(v_i, v_j)$$

has no solution; moreover, the instance $\Psi = E(v_1,v_2) \wedge N(v_1,v_2)$ is non-trivial, $(1,j)$-minimal for every $j \geq 1$ and has no solution either.

For the other structures from Corollary 3, the tightness of the bound can be shown similarly.

1.2 Related results

Local consistency for ω-categorical structures was studied for the first time in [13] where basic notions were introduced and some basic results provided. First-order reducts of certain k-homogeneous ℓ-bounded structures with bounded width were characterized in [45, 20].

A structure $\mathbb{A}$ has *bounded strict width* [33] if not only CSP($\mathbb{A}$) is solvable by local consistency, but moreover every partial solution of a locally consistent instance can be extended to a total solution. The articles [49] and [48] give the upper bound $(2,\ell)$ on the relational width for some classes of 2-homogeneous, ℓ-bounded structures under the stronger assumption of bounded strict width; it also follows from [49] that first-order reducts of $\mathbb{H}_n$ with bounded width have relational width at most $(2,n)$.

1.3 Organisation of the present article

In Section 2 we provide the basic notions and definitions. The reduction to the finite using canonical functions which leads to the collapse of the bounded width hierarchy is given in Section 3. We then extend the algebraic theory of smooth approximations in Section 4 before applying it to first-order reducts of unary structures and MMSNP in Section 5. For lack of space, some proofs are omitted and can be found in the full version of the paper.

2 Preliminaries

2.1 Structures and model-theoretic notions

For sets B, I, the *orbit* of a tuple $b \in B^I$ under the action of a permutation group $\mathcal{G}$ on B is the set $\{\alpha(b) \mid \alpha \in \mathcal{G}\}$. A countable structure $\mathbb{B}$ is *ω-categorical* if its automorphism group $\mathrm{Aut}(\mathbb{B})$ is *oligomorphic*, i.e., for all $n \geq 1$, the number of orbits of the action of $\mathrm{Aut}(\mathbb{B})$ on n-tuples is finite. For $\ell \geq 1$, we say that $\mathbb{B}$ is *ℓ-bounded* if for every finite $\mathbb{X}$, if all substructures $\mathbb{Y}$ of $\mathbb{X}$ of size at most ℓ embed in $\mathbb{B}$, then $\mathbb{X}$ embeds in $\mathbb{B}$. For $k \geq 1$, we say that $\mathbb{B}$ is *k-homogeneous* if for all tuples a, b of arbitrary finite length, if all k-subtuples of a

and b are in the same orbit under $\mathrm{Aut}(\mathbb{B})$, then a and b are in the same orbit under $\mathrm{Aut}(\mathbb{B})$. We say that $\mathbb{B}$ is *unary* if all relations in its signature are unary. A *first-order reduct* of a structure $\mathbb{B}$ is a structure on the same domain whose relations have a first-order definition in $\mathbb{B}$.

2.2 Polymorphisms, clones and identities

A *polymorphism* of a relational structure $\mathbb{A}$ is a homomorphism from some finite power of $\mathbb{A}$ to $\mathbb{A}$. The set of all polymorphisms of a structure $\mathbb{A}$ is denoted by $\mathrm{Pol}(\mathbb{A})$; it is a *function clone*, i.e., a set of finitary operations on a fixed set which contains all projections and which is closed under arbitrary compositions.

If $\mathcal{C}$ is a function clone, then we denote the domain of its functions by C; we say that $\mathcal{C}$ *acts on C*. The clone $\mathcal{C}$ also naturally acts (componentwise) on C^l for any $l \geq 1$, on any invariant subset S of C (by restriction), and on the classes of any invariant equivalence relation $\sim$ on an invariant subset S of C (by its action on representatives of the classes). We write $\mathcal{C} \curvearrowright C^l$, $\mathcal{C} \curvearrowright S$ and $\mathcal{C} \curvearrowright S/{\sim}$ for these actions. Any action $\mathcal{C} \curvearrowright S/{\sim}$ is called a *subfactor* of $\mathcal{C}$, and we also call the pair $(S, \sim)$ a subfactor. A subfactor $(S, \sim)$ is *minimal* if $\sim$ has at least two classes and no proper subset of S intersecting at least two $\sim$-classes is invariant under $\mathcal{C}$. For a clone $\mathcal{C}$ acting on a set X and $Y \subseteq X$ we write $\langle Y \rangle_{\mathcal{C}}$ for the smallest $\mathcal{C}$-invariant subset of X containing Y.

For $n \geq 1$, a k-ary operation f defined on the domain C of a permutation group $\mathcal{G}$ is *n-canonical* with respect to $\mathcal{G}$ if for all $a_1, \ldots, a_k \in C^n$ and all $\alpha_1, \ldots, \alpha_k \in \mathcal{G}$ there exists $\beta \in \mathcal{G}$ such that $f(a_1, \ldots, a_k) = \beta \circ f(\alpha_1(a_1), \ldots, \alpha_k(a_k))$. In particular, f induces an operation on the set $C^n/\,\mathcal{G}$ of $\mathcal{G}$-orbits of n-tuples. If all functions of a function clone $\mathcal{C}$ are n-canonical with respect to $\mathcal{G}$, then $\mathcal{C}$ acts on $C^n/\,\mathcal{G}$ and we write $\mathcal{C}^n \,/\, \mathcal{G}$ for this action; if $\mathcal{G}$ is oligomorphic then $\mathcal{C}^n \,/\, \mathcal{G}$ is a function clone on a finite set. A function is *canonical* with respect to a permutation group $\mathcal{G}$ if it is n-canonical with respect to $\mathcal{G}$ for all $n \geq 1$. We say that it is *diagonally canonical* if it satisfies the definition of canonicity in case $\alpha_1 = \cdots = \alpha_k$.

We write $\mathcal{G}_{\mathcal{C}}$ to denote the largest permutation group contained in a function clone $\mathcal{C}$, and say that $\mathcal{C}$ is oligomorphic if $\mathcal{G}_{\mathcal{C}}$ is oligomorphic. For $n \geq 1$, the *n-canonical (canonical) part* of $\mathcal{C}$ is the clone of those functions of $\mathcal{C}$ which are n-canonical (canonical) with respect to $\mathcal{G}_{\mathcal{C}}$. We write $\mathcal{C}_n^{\mathrm{can}}$ and $\mathcal{C}^{\mathrm{can}}$ for these sets which form themselves function clones.

For a set of functions $\mathcal{F}$ over the same fixed set C we write $\overline{\mathcal{F}}$ for the set of those functions g such that for all finite subsets F of C, there exists a function in $\mathcal{F}$ which agrees with g on F. We say that f *locally interpolates* g modulo $\mathcal{G}$, where f, g are k-ary functions and $\mathcal{G}$ is a permutation group all of which act on the same domain, if $g \in \overline{\{\beta \circ f(\alpha_1, \ldots, \alpha_k) \mid \beta, \alpha_1, \ldots, \alpha_k \in \mathcal{G}\}}$. Similarly, we say that f *diagonally interpolates* g modulo $\mathcal{G}$ if f locally interpolates g with $\alpha_1 = \cdots = \alpha_k$. If $\mathcal{G}$ is the automorphism group of a *Ramsey structure* in the sense of [12], then every function on its domain locally (diagonally) interpolates a canonical (diagonally canonical) function modulo $\mathcal{G}$ [25, 21]. We say that a clone $\mathcal{D}$ locally interpolates a clone $\mathcal{C}$ modulo a permutation group $\mathcal{G}$ if for every $g \in \mathcal{D}$ there exists $f \in \mathcal{C}$ such that g locally interpolates f modulo $\mathcal{G}$. A clone $\mathcal{C}$ is a *model-complete core* if its unary functions are equal to $\overline{\mathcal{G}_{\mathcal{C}}}$. A structure $\mathbb{A}$ is called a model- complete core if its polymorphism clone is.

A function f is *idempotent* if $f(x, \ldots, x) = x$ for all values x of its domain; a function clone is idempotent if all of its functions are. A k-ary operation w is called a *weak near-unanimity (WNU)* operation if it satisfies the set of identities containing an equation for each pair of terms in $\{w(x, \ldots, x, y), w(x, y, \ldots, x), \ldots, w(y, x, \ldots, x)\}$. It is called *totally symmetric* if $w(x_1, \ldots, x_k) = w(y_1, \ldots, y_k)$ whenever $\{x_1, \ldots, x_k\} = \{y_1, \ldots, y_k\}$ (where $1 \leq |\{x_1, \ldots, x_k\}| \leq k$). Each set of identities also has a *pseudo*-variant obtained by composing each term appearing in the identities with a distinct unary function symbol. For example, a ternary pseudo-WNU operation f satisfies the identities: $e_1 \circ f(y,x,x) = e_2 \circ f(x,y,x)$, $e_3 \circ f(y,x,x) = e_4 \circ f(x,x,y)$ and $e_5 \circ f(x,y,x) = e_6 \circ f(x,x,y)$. If $\mathcal{C}$ is a function clone and $\mathcal{U} \subseteq \mathcal{C}$ is a set of unary functions, then $\mathcal{C}$ *satisfies a set of pseudo-identities modulo* $\mathcal{U}$ if it satisfies the identities in such a way that the unary function symbols are assigned values in $\mathcal{U}$.

An arity-preserving map $\xi\colon \mathcal{C} \to \mathcal{D}$ between function clones is called a *clone homomorphism* if it preserves projections, i.e., maps every projection in $\mathcal{C}$ to the corresponding projection in $\mathcal{D}$, and compositions, i.e., it satisfies $\xi(f \circ (g_1,\dots,g_n)) = \xi(f) \circ (\xi(g_1),\dots,\xi(g_n))$ for all $n,m \geq 1$ and all n-ary $f \in \mathcal{C}$ and m-ary $g_1,\dots,g_n \in \mathcal{C}$. An arity-preserving map ξ is a *minion homomorphism* if it preserves compositions with projections, i.e., compositions where $g_1,\dots,g_n$ are projections. We say that a function clone $\mathcal{C}$ is *equationally trivial* if it has a clone homomorphism to the clone $\mathcal{P}$ of projections over the two-element domain, and *equationally non-trivial* otherwise. We also say that $\mathcal{C}$ is *equationally affine* if it has a clone homomorphism to an *affine clone*, i.e., a clone of affine maps over a finite module. It is known that a finite idempotent clone is either equationally affine or it contains WNU operations of all arities $n \geq 3$ ([43], this stronger version is attributed to E. Kiss in [40, Theorem 2.8]). Similarly, if $\mathbb{A}$ is an ω-categorical model-complete core, then $\mathrm{Pol}(\mathbb{A})^{\mathrm{can}}$ is either equationally affine, or it contains pseudo-WNU operations modulo $\overline{\mathrm{Aut}(\mathbb{A})}$ of all arities $n \geq 3$ (see [24, 45] for the lift of the corresponding result from the finite).

If $\mathcal{C}$, $\mathcal{D}$ are function clones and $\mathcal{D}$ has a finite domain, then a clone (or minion) homomorphism $\xi\colon \mathcal{C} \to \mathcal{D}$ is *uniformly continuous* if for all $n \geq 1$ there exists a finite subset F of C^n such that $\xi(f) = \xi(g)$ for all n-ary $f,g \in \mathcal{C}$ which agree on F.

A first-order formula is called a *primitive-positive* (pp-)formula if it is built exclusively from atomic formulae, existential quantifiers, and conjunction. A relation is pp-definable in a structure $\mathbb{B}$ if it is first-order definable by a pp-formula; in that case, it is invariant under $\mathrm{Pol}(\mathbb{B})$. Any ω-categorical model-complete core pp-defines all orbits of n-tuples with respect to its own automorphism group, for all $n \geq 1$.

2.3 CSP, Relational Width, Minimality

A *CSP instance* over a set A is a pair $\mathcal{I} = (\mathcal{V},\mathcal{C})$, where $\mathcal{V}$ is a finite set of variables, and $\mathcal{C}$ is a set of *constraints*; each constraint $C \in \mathcal{C}$ is a subset of A^U for some non-empty $U \subseteq \mathcal{V}$ (U is the *scope* of C). We say that $\mathcal{I}$ is an *instance of* $\mathrm{CSP}(\mathbb{A})$ if for every $C \in \mathcal{C}$ with scope U, there exists an enumeration $u_1,\dots,u_k$ of the elements of U and a k-ary relation R of $\mathbb{A}$ such that for all $f\colon U \to A$ we have $f \in C \Leftrightarrow (f(u_1),\dots,f(u_k)) \in R$. Given a constraint $C \subseteq A^U$ and $K \subseteq U$, the *projection of C onto K* is defined by $C|_K := \{f|_K \colon f \in C\}$.

▶ **Definition 5.** *Let $1 \leq k \leq \ell$. We say that an instance $\mathcal{I}$ over $\mathcal{V}$ of* $\mathrm{CSP}(\mathbb{A})$ *is* (k,ℓ)-minimal *if both of the following hold:*

- *every non-empty subset of at most ℓ variables in $\mathcal{V}$ is the scope of some constraint in $\mathcal{I}$;*
- *for every at most k-element subset of variables $K \subseteq \mathcal{V}$ and any two constraints $C_1, C_2 \in \mathcal{I}$ whose scopes contain K, the projections of C_1 and C_2 onto K coincide.*

We say that an instance $\mathcal{I}$ of the CSP is *non-trivial* if it does not contain any empty constraint. Otherwise, $\mathcal{I}$ is *trivial*.

Let $1 \leq k \leq \ell$. Clearly not every instance $\mathcal{I}$ over variables $\mathcal{V}$ of $\mathrm{CSP}(\mathbb{A})$ is (k,ℓ)-minimal. However, every instance $\mathcal{I}$ is *equivalent* to a (k,ℓ)-minimal instance $\mathcal{I}'$ in the sense that $\mathcal{I}$ and $\mathcal{I}'$ have the same set of solutions. In particular we have that if $\mathcal{I}'$ is trivial, then $\mathcal{I}$ has no solutions. Moreover, if $\mathbb{A}$ is ω-categorical, then $\mathcal{I}'$ can be computed in time polynomial in the size of $\mathcal{I}$. Indeed, it is enough to introduce a new constraint A^L for every set $L \subseteq \mathcal{V}$ with at most ℓ elements to satisfy the first condition. Then the algorithm removes tuples (in fact, orbits of tuples with respect to $\mathrm{Aut}(\mathbb{A})$) from the constraints in the instance as long as the second condition is not satisfied. Since $\mathbb{A}$ is ω-categorical and thus every relation in $\mathcal{I}$ is a union of a finite number of orbits of tuples with respect to $\mathrm{Aut}(\mathbb{A})$, the algorithm terminates.

▶ **Definition 6.** *Let $1 \leq k \leq \ell$. A relational structure $\mathbb{A}$ has* relational width (k, ℓ) *if every non-trivial (k, ℓ)-minimal instance of $\mathbb{A}$ has a solution. $\mathbb{A}$ has* bounded width *if it has relational width (k, ℓ) for some natural numbers $k \leq \ell$.*

▶ **Theorem 7** ([7])**.** *Let $\mathcal{I}$ be a non-trivial $(2, 3)$-minimal CSP instance over a finite set. Suppose that the constraints of $\mathcal{I}$ are preserved by WNUs of all arities $m \geq 3$. Then $\mathcal{I}$ has a solution.*

▶ **Theorem 8** ([32, 33])**.** *Let $\mathcal{I}$ be a non-trivial $(1, 1)$-minimal CSP instance over a finite set. Suppose that the constraints of $\mathcal{I}$ are preserved by totally symmetric polymorphisms of all arities. Then $\mathcal{I}$ has a solution.*

2.4 Smooth Approximations

We are going to apply the fundamental theorem of smooth approximations [45] to lift an action of a function clone to a larger clone.

▶ **Definition 9.** (Smooth approximations) *Let A be a set, $n \geq 1$, and let $\sim$ be an equivalence relation on a subset S of A^n. We say that an equivalence relation η on some set S' with $S \subseteq S'$* approximates *$\sim$ if the restriction of η to S is a (possibly non-proper) refinement of $\sim$. We call η an* approximation *of $\sim$.*

For a permutation group $\mathcal{G}$ acting on A and leaving η as well as the $\sim$-classes invariant, we say that the approximation η is smooth *if each equivalence class C of $\sim$ intersects some equivalence class C' of η such that $C \cap C'$ contains a $\mathcal{G}$-orbit.*

▶ **Theorem 10** (The fundamental theorem of smooth approximations [45])**.** *Let $\mathcal{C} \subseteq \mathcal{D}$ be function clones on a set A, and let $\mathcal{G}$ be a permutation group on A such that $\mathcal{D}$ locally interpolates $\mathcal{C}$ modulo $\mathcal{G}$. Let $\sim$ be a $\mathcal{C}$-invariant equivalence relation on $S \subseteq A$ with $\mathcal{G}$-invariant classes and finite index, and η be a $\mathcal{D}$-invariant smooth approximation of $\sim$ with respect to $\mathcal{G}$. Then there exists a uniformly continuous minion homomorphism from $\mathcal{D}$ to $\mathcal{C} \curvearrowright S/{\sim}$.*

3 Collapses in the Relational Width Hierarchy

▶ **Definition 11.** *Let $\mathcal{I} = (\mathcal{V}, \mathcal{C})$ be a CSP instance over A. Let $\mathcal{G}$ be a permutation group on A, let $k \geq 1$, and let $\mathcal{O}$ be the set of orbits of k-tuples under $\mathcal{G}$. Let $\mathcal{I}_{\mathcal{G},k}$ be the following instance over $\mathcal{O}$:*

- *The variable set of $\mathcal{I}_{\mathcal{G},k}$ is the set $\binom{\mathcal{V}}{k}$ of k-element subsets of $\mathcal{V}$. Thus, every variable K of $\mathcal{I}_{\mathcal{G},k}$ is meant to take a value in $\mathcal{O}$, and we consider that the values for K are* K-orbits, *i.e., orbits of maps $f \colon K \to A$ under the natural action of $\mathcal{G}$.*
- *For every constraint $C \subseteq A^U$ in $\mathcal{I}$, $\mathcal{I}_{\mathcal{G},k}$ contains the constraint $C_{\mathcal{G},k} \subseteq \mathcal{O}^{\binom{U}{k}}$ defined by*

$$C_{\mathcal{G},k} = \left\{ g \colon \binom{U}{k} \to \mathcal{O} \mid \exists f \in C \;\; \forall K \in \binom{U}{k} \;\; (f|_K \in g(K)) \right\}.$$

Note that the notation $f|_K \in g(K)$ makes sense precisely because $g(K)$ is a K-orbit. Observe that if $\mathcal{I}$ is non-trivial, then so is $\mathcal{I}_{\mathcal{G},k}$.

▶ **Lemma 12.** *Let $1 \leq a \leq b$. If $\mathcal{I}$ is (ak, bk)-minimal, then $\mathcal{I}_{\mathcal{G},k}$ is (a,b)-minimal.*

Note that for every solution h of $\mathcal{I}$, the map $\chi_h \colon \binom{\mathcal{V}}{k} \to \mathcal{O}$ defined by $K \mapsto \{\alpha h|_K \mid \alpha \in \mathcal{G}\}$ defines a solution to $\mathcal{I}_{\mathcal{G},k}$. The next lemma proves that every solution to $\mathcal{I}_{\mathcal{G},k}$ is of the form χ_h for some solution h of $\mathcal{I}$, provided that $\mathcal{I}$ is (k,ℓ)-minimal and that $\mathcal{G} = \mathrm{Aut}(\mathbb{B})$ for some ℓ-bounded k-homogeneous structure $\mathbb{B}$.

▶ **Lemma 13.** *Let $1 \leq k < \ell$. Let $\mathbb{B}$ be ℓ-bounded and k-homogeneous, let $\mathbb{A}$ be a first-order reduct of $\mathbb{B}$, and let $\mathcal{I}$ be a (k,ℓ)-minimal instance of $\mathrm{CSP}(\mathbb{A})$. Then every solution to $\mathcal{I}_{\mathrm{Aut}(\mathbb{B}),k}$ lifts to a solution of $\mathcal{I}$.*

Proof. Let $h \colon \binom{\mathcal{V}}{k} \to \mathcal{O}$ be a solution to $\mathcal{I}_{\mathrm{Aut}(\mathbb{B}),k}$. Recall that for any $K \in \binom{\mathcal{V}}{k}$, we view $h(K)$ as a K-orbit, and one can therefore restrict $h(K)$ to any $L \subseteq K$ by setting $h(K)|_L := \{f|_L \mid f \in h(K)\}$. Note that since $\mathcal{I}$ is (k,k)-minimal, we have $h(K)|_{K\cap K'} = h(K')|_{K\cap K'}$ for all $K, K' \in \binom{\mathcal{V}}{k}$.

We now define an equivalence relation $\sim$ on $\mathcal{V}$. Suppose first that $k = 1$. Then every orbit of $\mathbb{B}$ must be a singleton (for any orbit with two elements a, b, the pairs (a,a) and (a,b) are not in the same orbit but their entries are, so that $\mathbb{B}$ is not 1-homogeneous). In that case, we identify $\mathcal{O}$ with the domain B itself, and set $x \sim y$ if and only if $h(\{x\}) = h(\{y\})$; that is, $\sim$ is essentially the kernel of h.

Suppose next that $k \geq 2$, and set $x \sim y$ iff there is $K \in \binom{\mathcal{V}}{k}$ containing x, y such that $h(K)|_{\{x,y\}}$ consists of constant maps. It can be seen that one could equivalently ask that this holds for *all* K containing x, y by 2-minimality, and that this is indeed an equivalence relation by $(2,3)$-minimality of $\mathcal{I}$. Moreover, h descends to $\binom{\mathcal{V}/\sim}{k}$: if $K' = \{[v_1]_\sim, \ldots, [v_k]_\sim\}$ is a k-element set, define $\tilde{h}(K') := h(\{v_1, \ldots, v_k\})$. The definition of $\tilde{h}$ does not depend on the choice of representatives, by the very definition of $\sim$.

Define a finite structure $\mathbb{C}$ with domain $\mathcal{V}/\sim$ in the signature of $\mathbb{B}$ as follows. Let $K = \{[v_1]_\sim, \ldots, [v_k]_\sim\}$. The orbit $\tilde{h}(K)$ describes an atomic type on the elements of K; one defines $\mathbb{C}$ such that its substructure induced by K has the same atomic type. This is a well-defined construction by the previous paragraphs.

Finally, note that all substructures of $\mathbb{C}$ of size at most ℓ embed into $\mathbb{B}$. Indeed, let $\mathbb{L}$ be an ℓ-element substructure of $\mathbb{C}$, and let $L' \subseteq V$ be an ℓ-element set containing one representative for each element of $\mathbb{L}$. By (k,ℓ)-minimality of $\mathcal{I}$, there exists $C \subseteq A^{L'}$ in $\mathcal{I}$, and a corresponding $C_{\mathrm{Aut}(\mathbb{B}),k} \subseteq \mathcal{I}_{\mathrm{Aut}(\mathbb{B}),k}$. Thus, $h|_{\binom{L'}{k}} \in C_{\mathrm{Aut}(\mathbb{B}),k}$, so that there exists $g \in C$ such that for all $K \in \binom{L'}{k}$, $g|_K \in h(K)$. Thus g corresponds to an embedding of every k-element substructure of $\mathbb{L}$ into $\mathbb{B}$, and since $\mathbb{B}$ is k-homogeneous, g is an embedding of $\mathbb{L}$ into $\mathbb{B}$. Finally, since $\mathbb{B}$ is ℓ-bounded, it follows that there exists an embedding e of $\mathbb{C}$ into $\mathbb{B}$.

It remains to check that the composition of e with the canonical projection $\mathcal{V} \to \mathcal{V}/\sim$ is a solution to $\mathcal{I}$, which is trivial since the relations of $\mathbb{A}$ are definable in $\mathbb{B}$. ◀

Every operation f that is canonical with respect to a group $\mathcal{G}$ induces an operation on the set orbits of k-tuples under $\mathcal{G}$, by definition. We denote this operation by $f_{\mathcal{G},k}$.

▶ **Lemma 14.** *Let f be a polymorphism of $\mathbb{A}$ that is canonical with respect to $\mathcal{G}$. Every constraint $C_{\mathcal{G},k}$ in $\mathcal{I}_{\mathcal{G},k}$ is preserved under $f_{\mathcal{G},k}$.*

Finally, this allows us to prove Theorem 2 from the introduction.

Proof of Theorem 2. Suppose that the assumption of the first item of Theorem 2 is satisfied. Let $\mathcal{I}$ be a non-trivial $(2k, \max(3k,\ell))$-minimal instance of $\mathrm{CSP}(\mathbb{A})$, and let $\mathcal{I}_{\mathrm{Aut}(\mathbb{B}),k}$ be the associated instance of Definition 11. By Lemma 12, $\mathcal{I}_{\mathrm{Aut}(\mathbb{B}),k}$ is a $(2,3)$-minimal instance, and it is non-trivial by definition. The constraints of $\mathcal{I}_{\mathrm{Aut}(\mathbb{B}),k}$ are preserved by WNUs of all arities $m \geq 3$ (Lemma 14). By Theorem 7, $\mathcal{I}_{\mathrm{Aut}(\mathbb{B}),k}$ admits a solution and since $\mathcal{I}$ is $(k, \max(3k, \ell))$-minimal, this solution lifts to a solution of $\mathcal{I}$ by Lemma 13. Thus, $\mathbb{A}$ has width $(2k, \max(3k, \ell))$.

Suppose now that the assumption in the second item is satisfied. By the same reasoning but using Theorem 8 instead of Theorem 7, given a $(k, \max(k+1, \ell))$-minimal instance $\mathcal{I}$, the associated instance $\mathcal{I}_{\mathrm{Aut}(\mathbb{B}),k}$ is $(1,1)$-minimal and therefore has a solution. Since $\mathcal{I}$ is $(k, \max(k+1, \ell))$-minimal, this solution lifts to a solution of $\mathcal{I}$. ◀

4 A New Loop Lemma for Smooth Approximations

We refine the algebraic theory of smooth approximations from [45]. Building on deep algebraic results from [6] on finite idempotent algebras that are equationally non-trivial, we lift some of the theory from binary symmetric relations to cyclic relations of arbitrary arity.

4.1 The loop lemma

▶ **Definition 15.** *The* linkedness congruence *of a binary relation $R \subseteq A \times B$ is the equivalence relation λ_R on $\mathrm{proj}_{(2)}(R)$ defined by $(b, b') \in \lambda_R$ iff there are $k \geq 0$ and $a_0, \ldots, a_{k-1} \in A$ and $b = b_0, \ldots, b_k = b' \in B$ such that $(a_i, b_i) \in R$ and $(a_i, b_{i+1}) \in R$ for all $i \in \{0, \ldots, k-1\}$. We say that R is* linked *if it is non-empty and λ_R relates any two elements of $\mathrm{proj}_{(2)}(R)$.*

If A is a set and $m \geq 2$, then we call a relation $R \subseteq A^m$ cyclic *if it is invariant under cyclic permutations of the components of its tuples. The* support *of R is its projection on any argument. We apply the same terminology as above to any cyclic R, viewing R as a binary relation between $\mathrm{proj}_{(1,\ldots,m-1)}(R)$ and $\mathrm{proj}_{(m)}(R)$.*

If R is invariant under an oligomorphic group action on $A \times B$, then there is an upper bound on the length k to witness $(b, b') \in \lambda_R$, and therefore λ_R is pp-definable from R; in particular, it is invariant under any function clone acting on $A \times B$ and preserving R.

▶ **Definition 16.** *Let $\mathcal{G}$ be a permutation group on a set A. A* pseudo-loop with respect to $\mathcal{G}$ *is a tuple of elements of A all of whose components belong to the same $\mathcal{G}$-orbit [46, 9, 10]. If $\mathcal{G}$ contains only the identity function, then a pseudo-loop is called a* loop.

▶ **Theorem 17** (Consequence of the proof of Theorem 4.2 in [6])**.** *Let $\mathcal{C}$ be an idempotent function clone on a finite domain that is equationally non-trivial. Then any $\mathcal{C}$-invariant cyclic linked relation on its domain contains a loop.*

The following is a generalization of [45, Theorem 10] from binary symmetric relations to arbitrary cyclic relations.

▶ **Proposition 18.** *Let $n \geq 1$, and let $\mathcal{D}$ be an oligomorphic function clone on a set A which is a model-complete core. Let $\mathcal{C} \subseteq \mathcal{D}_n^{\mathrm{can}}$ be such that $\mathcal{C}^n/\mathcal{G}_{\mathcal{D}}$ is equationally non-trivial. Let $(S, \sim)$ be a minimal subfactor of the action $\mathcal{C}^n$ with $\mathcal{G}_{\mathcal{D}}$-invariant $\sim$-classes. Then for every $\mathcal{D}$-invariant cyclic relation R with support $\langle S \rangle_{\mathcal{D}}$ one of the following holds:*

1. *The linkedness congruence of R is a $\mathcal{D}$-invariant approximation of $\sim$.*
2. *R contains a pseudo-loop with respect to $\mathcal{G}_{\mathcal{D}}$.*

Proof. Let R be given, and denote its arity by m. Assuming that (1) does not hold, we prove (2).

Denote by $\mathcal{O}$ the set of orbits of n-tuples under the action of $\mathcal{G}_{\mathcal{D}}$ thereon. Let R' be the relation obtained by considering R as a relation on $\mathcal{O}$, i.e.,

$$R' := \{(O_1, \dots, O_m) \in \mathcal{O}^m \mid R \cap (O_1 \times \dots \times O_m) \neq \emptyset\}.$$

Thus, R' is an m-ary cyclic relation with support $S' \subseteq \mathcal{O}$, and R' contains a loop if and only if R satisfies (2).

By assumption, the action $\mathcal{C}^n/\mathcal{G}_{\mathcal{D}}$ is equationally non-trivial; moreover, it is idempotent since $\mathcal{D}$ is a model-complete core. Note also that R', and in particular S', are preserved by this action. It is therefore sufficient to show that R' is linked and apply Theorem 17.

Recall that we consider R also as a binary relation between $\mathrm{proj}_{m-1}(R)$ and $\langle S\rangle_{\mathcal{D}}$; similary, we consider R' as a binary relation between $\mathrm{proj}_{m-1}(R')$ and S'. By the oligomorphicity of $\mathcal{D}$, the linkedness congruence λ_R of R is invariant under $\mathcal{D}$.

By our assumption that (1) does not hold, there exist $c, d \in S$ which are not $\sim$-equivalent and such that $\lambda_R(c, d)$ holds; otherwise, λ_R would be an approximation of $\sim$. This implies that the orbits O_c, O_d of c, d are related via $\lambda_{R'}$. By the minimality of $(S, \sim)$, we have that $\langle S\rangle_{\mathcal{D}} = \langle\{c, d\}\rangle_{\mathcal{D}}$. Since $\mathcal{D}$ is a model-complete core, it preserves the $\mathcal{G}_{\mathcal{D}}$-orbits, and it follows that any tuple in $\langle S\rangle_{\mathcal{D}} = \langle\{c, d\}\rangle_{\mathcal{D}}$ is λ_R-related to a tuple in the orbit of c. Hence, $\lambda_{R'} = (S')^2$, and thus R' is linked. Theorem 17 therefore implies that R' contains a loop, and hence R contains a pseudo-loop with respect to $\mathcal{G}_{\mathcal{D}}$, which is what we had to show. ◀

The following is a generalization of Lemma 12 in [45] from binary relations and functions to relations and functions of higher arity.

▶ **Lemma 19.** *Let $n \geq 1$, and let $\mathcal{D}$ be an oligomorphic polymorphism clone on a set A that is a model-complete core. Let $\sim$ be an equivalence relation on a set $S \subseteq A^n$ with $\mathcal{G}_{\mathcal{D}}$-invariant classes. Let $m \geq 1$, and let P be an m-ary relation on $\langle S\rangle_{\mathcal{D}}$. Suppose that every m-ary $\mathcal{D}$-invariant cyclic relation R on $\langle S\rangle_{\mathcal{D}}$ which contains a tuple in P with components in at least two $\sim$-classes contains a pseudo-loop with respect to $\mathcal{G}_{\mathcal{D}}$.*

Then there exists an m-ary $f \in \mathcal{D}$ such that for all $a_1, \dots, a_m \in A^n$ we have that if the tuple $(f(a_1, \dots, a_m), f(a_2, \dots, a_m, a_1), \dots, f(a_m, a_1, \dots, a_{m-1}))$ is in P, then it intersects at most one $\sim$-class.

5 Applications: Collapses of the bounded width hierachies for some classes of infinite structures

We now apply the algebraic results of Section 4 and the theory of smooth approximations to obtain a characterisation of bounded width for CSPs of first-order reducts of unary structures ($k = 2, \ell = 2$) and for CSPs in MMSNP (where k and ℓ are arbitrarily large). Moreover, the results of Section 3 then imply a collapse of the bounded width hierarchy for such CSPs.

5.1 Unary Structures

▶ **Lemma 20** (Proposition 6.5 in [18]). *Let $\mathbb{A}$ be a first-order expansion of a stabilized partition $(\mathbb{N}; V_1, \dots, V_r)$. For every $f \in \mathrm{Pol}(\mathbb{A})$ there exists $g \in \mathrm{Pol}(\mathbb{A})^{\mathrm{can}}$ which is locally interpolated by f modulo $\mathrm{Aut}(\mathbb{A})$.*

▶ **Proposition 21** (Proposition 6.6 in [18]). *Let $\mathbb{A}$ be a first-order expansion of a stabilized partition $(\mathbb{N}; V_1, \dots, V_r)$, and assume it is a model-complete core. Suppose that $\mathrm{Pol}(\mathbb{A})$ contains a binary operation whose restriction to V_i is injective for all $1 \leq i \leq r$. Then the following are equivalent:*

- $\mathrm{Pol}(\mathbb{A})^{\mathrm{can}}$ *is equationally affine;*
- $\mathrm{Pol}(\mathbb{A})^{\mathrm{can}} \curvearrowright \mathbb{N}/\mathrm{Aut}(\mathbb{A})$ *is equationally affine.*

▶ **Lemma 22** (Subset of the proof of Proposition 6.6 [18]). *Let $\mathbb{A}$ be a first-order expansion of a stabilized partition $(\mathbb{N}; V_1, \ldots, V_r)$, and assume it is a model-complete core. If* $\mathrm{Pol}(\mathbb{A})$ *has no continuous clone homomorphism to $\mathcal{P}$, then it contains operations of all arities whose restrictions to V_i are injective for all $1 \leq i \leq r$.*

▶ **Theorem 23.** *Let $\mathbb{A}$ be a first-order reduct of a unary structure, and assume that $\mathbb{A}$ is a model-complete core. Then one of the following holds:*

- $\mathrm{Pol}(\mathbb{A})^{\mathrm{can}}$ *is not equationally affine, or equivalently, it contains pseudo-WNUs modulo* $\mathrm{Aut}(\mathbb{A})$ *of all arities $n \geq 3$;*
- $\mathrm{Pol}(\mathbb{A})$ *has a uniformly continuous minion homomorphism to an affine clone.*

In the first case, $\mathbb{A}$ has relational width $(4, 6)$ by Theorem 2, and in the second case it does not have bounded width by results from [23, 41]. Theorem 23 gives a characterization of bounded width for *all* first-order reducts of unary structures, since this class is closed under taking model-complete cores by Lemma 6.7 in [18].

The two items of Theorem 23 are invariant under expansions of $\mathbb{A}$ by a finite number of constants. Thus, by Proposition 6.8 in [18], one can assume that $\mathbb{A}$ is a first-order expansion of $(\mathbb{N}; V_1, \ldots, V_r)$ where $V_1, \ldots, V_r$ form a partition of $\mathbb{N}$ in which every set is either a singleton or infinite. Such partitions were called *stabilized partitions* in [18], and we shall also call the structure $(\mathbb{N}; V_1, \ldots, V_r)$ a stabilized partition.

Proof of Theorem 23. Let $\mathbb{A}$ as in Theorem 23 be given; by the remark preceding this proof, we may without loss of generality assume that $\mathbb{A}$ is a first-order expansion of a stabilized partition $(\mathbb{N}; V_1, \ldots, V_r)$. Assume henceforth that $\mathrm{Pol}(\mathbb{A})^{\mathrm{can}}$ is equationally affine; we show that $\mathrm{Pol}(\mathbb{A})$ has a uniformly continuous minion homomorphism to an affine clone.

If $\mathrm{Pol}(\mathbb{A})$ has a continuous clone homomorphism to $\mathcal{P}$, then we are done. Assume therefore the contrary; then by Lemma 22, $\mathrm{Pol}(\mathbb{A})$ contains for all $k \geq 2$ a k-ary operation whose restriction to V_i is injective for all $1 \leq i \leq r$. In particular, Proposition 21 applies, and thus $\mathrm{Pol}(\mathbb{A})^{\mathrm{can}} \curvearrowright \mathbb{N}/\mathrm{Aut}(\mathbb{A})$ is equationally affine. Let $(S, \sim)$ be a minimal subfactor of $\mathrm{Pol}(\mathbb{A})^{\mathrm{can}}$ such that $\mathrm{Pol}(\mathbb{A})^{\mathrm{can}}$ acts on the $\sim$-classes as an affine clone; the fact that this exists is well-known (see, e.g., Proposition 3.1 in [47]).

Let R be any $\mathrm{Pol}(\mathbb{A})$-invariant cyclic relation with support $\langle S \rangle_{\mathrm{Pol}(\mathbb{A})}$, containing a tuple with components in pairwise distinct $\mathrm{Aut}(\mathbb{A})$-orbits and which intersects at least two $\sim$-classes. By Proposition 18, R either gives rise to a $\mathrm{Pol}(\mathbb{A})$-invariant approximation of $\sim$, or it contains a pseudo-loop with respect to $\mathrm{Aut}(\mathbb{A})$. In the first case, the presence of the tuple required above implies smoothness of the approximation: if $t \in R$ is such a tuple, $c \in S$ appears in t, and $d \in S$ belongs to the same $\mathrm{Aut}(\mathbb{A})$-orbit as c, then there exists an element of $\mathrm{Aut}(\mathbb{A})$ which sends c to d and fixes all other elements of t. Hence, c and d are linked in R, and the entire $\mathrm{Aut}(\mathbb{A})$-orbit of c is contained in a class of the linkedness relation of R. Thus, $\mathrm{Pol}(\mathbb{A})$ admits a uniformly continuous minion homomorphism to an affine clone by Theorem 10.

Hence we may assume that for any R as above the second case holds. We are now going to show that this leads to a contradiction, finishing the proof of Theorem 23. By Lemma 19 applied with any $m \geq 2$ and P the set of m-tuples with entries in pairwise distinct $\mathrm{Aut}(\mathbb{A})$-orbits within $\langle S \rangle_{\mathrm{Pol}(\mathbb{A})}$, we obtain an m-ary function $f \in \mathrm{Pol}(\mathbb{A})$ with the property that the tuple $(f(a_0, \ldots, a_{m-1}), \ldots, f(a_1, \ldots, a_{m-1}, a_0))$ intersects at most one $\sim$-class whenever it has entries in pairwise distinct $\mathrm{Aut}(\mathbb{A})$-orbits, for all $a_0, \ldots, a_{m-1} \in S$. Let $(\mathbb{A}, <)$ be the expansion of $\mathbb{A}$ by a linear order that is convex with respect to the partition $V_1, \ldots, V_r$ and dense and without endpoints on every infinite set of the partition. The structure $(\mathbb{A}, <)$ can be seen to be a *Ramsey structure*, since $\mathrm{Aut}(\mathbb{A}, <)$ is isomorphic as a permutation group to the action of the product $\prod_{i=1}^{r} \mathrm{Aut}(V_i; <)$, and each of the groups of the product is either trivial or the automorphism group of a Ramsey structure [37]. By diagonal interpolation we may assume that f is diagonally canonical with respect to $\mathrm{Aut}(\mathbb{A}, <)$. Let $a, a' \in A^m$ be so that a_i, a'_i belong to the same orbit with respect to $\mathrm{Aut}(\mathbb{A})$ for all $1 \leq i \leq m$. Then there exists $\alpha \in \mathrm{Aut}(\mathbb{A}, <)$ such that $\alpha(a) = \alpha(a')$, and hence $f(a)$ and $f(a')$ lie in the same $\mathrm{Aut}(\mathbb{A})$-orbit by diagonal canonicity; hence f is 1-canonical with respect to $\mathrm{Aut}(\mathbb{A})$. Applying Lemma 20, we obtain a canonical function $g \in \mathrm{Pol}(\mathbb{A})^{\mathrm{can}}$ which acts like f on $\mathbb{N}/\mathrm{Aut}(\mathbb{A})$. The property of f stated above then implies for g that $g(a_0, \ldots, a_{m-1}) \sim g(a_1, \ldots, a_{m-1}, a_0)$ for all $a_0, \ldots, a_{m-1} \in S$ such that the values $g(a_0, \ldots, a_{m-1}), \ldots, g(a_{m-1}, a_0, \ldots, a_{m-2})$ lie in pairwise distinct $\mathrm{Aut}(\mathbb{A})$-orbits.

By the choice of $(S, \sim)$ we have that $\mathrm{Pol}(\mathbb{A})^{\mathrm{can}}$ acts on $S/{\sim}$ by affine functions over a finite module. We use the symbols $+, \cdot$ for the addition and multiplication in the corresponding ring, and also $+$ for the addition in the module and $\cdot$ for multiplication of elements of the module with elements of the ring. We denote by 1 the multiplicative identity of the ring, by -1 its additive inverse, and identify their powers in the additive group with the non-zero integers. The domain of the module is $S/{\sim}$, and we denote the identity element of its additive group by $[a_0]_\sim$. Pick an arbitrary element $[a_1]_\sim \neq [a_0]_\sim$ from $S/{\sim}$, and let $m \geq 2$ be its order in the additive group of the module, i.e., the minimal positive number such that $m \cdot [a_1]_\sim = [a_0]_\sim$. Let $g \in \mathrm{Pol}(\mathbb{A})^{\mathrm{can}}$ be the m-ary operation obtained in the preceding paragraph. If the values $g(a_0, \ldots, a_{m-1}), \ldots, g(a_{m-1}, a_0, \ldots, a_{m-2})$ lie in pairwise distinct $\mathrm{Aut}(\mathbb{A})$-orbits, then computing indices modulo m we have that $g([a_0]_\sim, \ldots, [a_{m-1}]_\sim), \ldots, g([a_{m-1}]_\sim, \ldots, [a_{m+m-1}]_\sim)$ are all equal. If on the other hand they do not, then $g([a_k]_\sim, \ldots, [a_{k+m-1}]_\sim) = g([a_{k+j}]_\sim, \ldots, [a_{k+j+m-1}]_\sim)$ for some $0 \leq k < m$ and $1 \leq j < m$. Hence, in either case we may assume the latter equation holds. By assumption, g acts on $S/{\sim}$ as an affine map, i.e., as a map of the form $(x_0, \ldots, x_{m-1}) \mapsto \sum_{i=0}^{m-1} c_i \cdot x_i$, where $c_0, \ldots, c_{m-1}$ are elements of the ring which sum up to 1. We compute (with indices to be read modulo m)

$$\begin{aligned}
[a_0]_\sim &= g([a_{k+j}]_\sim, \ldots, [a_{k+j+m-1}]_\sim) + (-1) \cdot g([a_k]_\sim, \ldots, [a_{k+m-1}]_\sim) \\
&= \sum_{i=0}^{m-1} c_i \cdot [a_{k+j+i}]_\sim + (-1) \cdot \sum_{i=0}^{m-1} c_i \cdot [a_{k+i}]_\sim \\
&= \sum_{i=0}^{m-1} c_i \cdot (k+i+j) \cdot [a_1]_\sim + (-1) \cdot \sum_{i=0}^{m-1} c_i \cdot (k+i) \cdot [a_1]_\sim \\
&= \left(\sum_{i=0}^{m-1} c_i \right) \cdot j \cdot [a_1]_\sim = j \cdot [a_1]_\sim.
\end{aligned}$$

But $j \cdot [a_1]_\sim \neq [a_0]_\sim$ since the order of $[a_1]_\sim$ equals $m > j$, a contradiction. ◀

5.2 MMSNP

MMSNP is a fragment of existential second order logic that was discovered by Feder and Vardi in their seminal paper [33]. We prefer not to define the syntax of MMSNP, and rather define it using a correspondence between MMSNP sentences and certain coloring problems. We refer to [15] for a precise definition of all the terms employed here.

Let τ be a relational signature, let σ be a unary signature whose relations are called the *colors*, and let $\mathcal{F}$ be a finite set of finite connected $(\tau \cup \sigma)$-structures whose vertices have exactly one color. We call $\mathcal{F}$ a *colored obstruction set* in the following. The problem $\mathrm{FPP}(\mathcal{F})$ takes as input a τ-structure $\mathbb{G}$ and asks whether there exists a $(\tau \cup \sigma)$-expansion $\mathbb{G}^*$ of $\mathbb{G}$ whose vertices are all colored with exactly one color and such that for every $\mathbb{F} \in \mathcal{F}$, there exists no homomorphism from $\mathbb{F}$ to $\mathbb{G}^*$. The connection between MMSNP and FPP is shown in [42, Corollary 3.7]: every MMSNP sentence Φ is equivalent to a union $\mathrm{FPP}(\mathcal{F}_1) \cup \cdots \cup \mathrm{FPP}(\mathcal{F}_p)$, in the sense that a τ-structure $\mathbb{G}$ satisfies Φ iff it is a yes-instance for one of the problems $\mathrm{FPP}(\mathcal{F}_i)$. We say Φ is *connected* if it is equivalent to a single $\mathrm{FPP}(\mathcal{F})$.

Every set $\mathcal{F}$ as above has a *strong normal form* $\mathcal{G}$ such that $\mathrm{FPP}(\mathcal{F}) = \mathrm{FPP}(\mathcal{G})$. We say $\mathcal{F}$ is *precolored* if for every symbol $M \in \sigma$, there is an associated unary symbol $P_M \in \tau$, and moreover if $\mathcal{F}$ contains for every $M \neq M'$ a 1-element structure whose vertex belongs to P_M and M'. Every $\mathcal{F}$ has a *standard precoloration*, obtained by enlarging τ with the necessary symbols and enlarging $\mathcal{F}$ with the associated obstructions.

It was shown in [15, Definition 4.3] that for every set $\mathcal{F}$ in strong normal form, there exists an ω-categorical τ-structure $\mathbb{A}_{\mathcal{F}}$ such that for any finite τ-structure $\mathbb{B}$, $\mathbb{B}$ is a yes-instance of $\mathrm{FPP}(\mathcal{F})$ iff there exists an injective homomorphism from $\mathbb{B}$ to $\mathbb{A}_{\mathcal{F}}$, and such that:

- If $\mathcal{F}$ is precolored, then the orbits of the elements of $\mathbb{A}_{\mathcal{F}}$ under $\mathrm{Aut}(\mathbb{A}_{\mathcal{F}})$ correspond to the colors of $\mathcal{F}$ and to the corresponding predicates in τ. In particular, the action of $\mathrm{Pol}(\mathbb{A}_{\mathcal{F}})_1^{\mathrm{can}}$ on $\mathrm{Aut}(\mathbb{A}_{\mathcal{F}})$-orbits of elements is idempotent [15, Proposition 7.1].
- Every $f \in \mathrm{Pol}(\mathbb{A}_{\mathcal{F}})$ locally interpolates an operation $g \in \mathrm{Pol}(\mathbb{A}_{\mathcal{F}})_1^{\mathrm{can}}$, and there exists a linear order $<$ on $\mathbb{A}_{\mathcal{F}}$ such that every f diagonally interpolates an operation f' that is diagonally canonical with respect to $\mathrm{Aut}(\mathbb{A}_{\mathcal{F}}, <)$.

We finally solve the Datalog-rewritability problem for MMSNP and prove that a connected sentence Φ is equivalent to a Datalog-program iff the action of $\mathrm{Pol}(\mathbb{A}_{\mathcal{F}})_1^{\mathrm{can}}$ on $\mathrm{Aut}(\mathbb{A}_{\mathcal{F}})$-orbits of elements is not equationally affine, where $\mathcal{F}$ is any strong normal form for Φ.

The following proposition is proved in [15] in the case where $m = 2$. Only small alterations to the proof are needed to prove the more general version, so we omit it.

▶ **Proposition 24.** *Let $\mathcal{F}$ be a precolored obstruction set and in normal form. Let $\mathbb{B}$ have a homomorphism to $\mathbb{A}_{\mathcal{F}}$ and let $m \geq 1$. There exists an embedding e of $\{1, \dots, m\} \times \mathbb{B}$, the disjoint union of m copies of $\mathbb{B}$, into $\mathbb{A}_{\mathcal{F}}$ such that $(e(i_1, a_1), \dots, e(i_m, a_m))$ and $(e(j_1, b_1), \dots, e(j_m, b_m))$ are in the same orbit under* $\mathrm{Aut}(\mathbb{A}_{\mathcal{F}}, <)$ *provided that:*

- *a_k and b_k are in the same color for all $k \in \{1, \dots, m\}$*
- *a_k and a_ℓ are in distinct colors for all $k \neq \ell$,*
- *$\{i_1, \dots, i_m\} = \{j_1, \dots, j_m\} = \{1, \dots, m\}$.*

The following proposition shows that for the question of Datalog-rewritability, one can reduce to the precolored case without loss of generality. The same proposition was shown in [15] for the P/NP-complete dichotomy, with $\mathcal{P}$ replacing affine clones in the statement.

▶ **Proposition 25.** *Let $\mathcal{F}$ be a colored obstruction set in strong normal form and let $\mathcal{G}$ be its standard precoloration. There is a uniformly continuous minion homomorphism from* $\mathrm{Pol}(\mathbb{A}_{\mathcal{G}})$ *to an affine clone if and only if there is a uniformly continuous minion homomorphism from* $\mathrm{Pol}(\mathbb{A}_{\mathcal{F}})$ *to an affine clone.*

Proof. It is shown in [15] that $\mathrm{Pol}(\mathbb{A}_{\mathcal{G}})$ has a uniformly continuous minion homomorphism to $\mathrm{Pol}(\mathbb{A}_{\mathcal{F}})$ and that $\mathrm{Pol}(\mathbb{A}_{\mathcal{F}}, \neq)$ has a uniformly continuous minion homomorphism to $\mathrm{Pol}(\mathbb{A}_{\mathcal{G}})$. Thus, it suffices to show that if $\mathrm{Pol}(\mathbb{A}_{\mathcal{F}}, \neq)$ has a uniformly continuous minion homomorphism to an affine clone, then so does $\mathrm{Pol}(\mathbb{A}_{\mathcal{F}})$.

Let $p \geq 2$ be prime and let R_0 and R_1 be the relations defined by $\{(x, y, z) \in \mathbb{Z}_p \mid x + y + z = i \bmod p\}$ for $i \in \{0, 1\}$. For an arbitrary ω-categorical structure $\mathbb{B}$, it is known that the existence of a uniformly continuous minion homomorphism $\mathrm{Pol}(\mathbb{B})$ to an affine clone is equivalent to the existence of a p such that the relational structure $(\mathbb{Z}_p; R_0, R_1)$ has a *pp-construction* in $\mathbb{B}$.

Suppose that $(\mathbb{Z}_p; R_0, R_1)$ has a pp-construction in $(\mathbb{A}_{\mathcal{F}}, \neq)$. Thus, there is $n \geq 1$ and pp-formulas $\phi_0(\mathbf{x}, \mathbf{y}, \mathbf{z}), \phi_1(\mathbf{x}, \mathbf{y}, \mathbf{z})$ defining relations S_0, S_1 such that $(A^n; S_0, S_1)$ and $(\mathbb{Z}_p; R_0, R_1)$ are homomorphically equivalent; we take n to be minimal with the property that such pp-formulas exist. Since R_0 and R_1 are totally symmetric relations (i.e., the order of the entries in a tuple does not affect its membership into any of R_0 or R_1), we can assume that S_0 and S_1 are, too, and that the formulas pp-defining them are syntactically invariant under permutation of the block of variables $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

We first claim that ϕ_i does not contain any equality atom or any inequality atom $x_j \neq y_j$ for $j \in \{1, \ldots, n\}$ (so that by symmetry, also $y_j \neq z_j$ and $x_j \neq z_j$ do not appear). Let $h\colon (\mathbb{Z}_p; R_0, R_1) \to (A^n; S_0, S_1)$ be a homomorphism. Since $(0, 0, 0) \in R_0$, we have that $(h(0), h(0), h(0))$ satisfies ϕ_0, and therefore the listed inequality atoms cannot appear. The same holds for ϕ_1, by considering $(h(0), h(0), h(1))$ and its permutations.

In order to rule out equalities, we proceed as in [15]. Suppose that ϕ_0 contains $x_i = x_j$ for $i \neq j$. Then the entries i and j of $h(q)$ are equal, for any $q \in \{0, \ldots, p-1\}$, since every q belongs to the support of R_0. Thus, one can also add $x_i = x_j$ to ϕ_1, since the structure defined by the modified formula still admits a homomorphism from $(\mathbb{Z}_p; R_0, R_1)$. By existentially quantifying x_j, y_j, z_j in ϕ_0 and ϕ_1, one obtains a pp-construction of some $(A^{n-1}; S_0', S_1')$ that is still homomorphically equivalent to $(\mathbb{Z}_p; R_0, R_1)$, a contradiction to the minimality of n. If ϕ_0 contains $x_i = y_j$ for $j \neq i$, then it also contains $y_j = z_i$ and $z_i = x_j$ since we enforced that ϕ_0 is syntactically symmetric. By transitivity, we obtain that $x_i = x_j$ is implied by ϕ_0 and we are back in the first case. Suppose now that ϕ_0 contains $x_i = y_i$. Then the ith entry of $h(0)$ and $h(q)$ are equal, for all $q \in \{0, \ldots, p-1\}$, since for all q there exists r such that $(0, q, r) \in R_0$. Thus we can again reduce n by fixing the ith coordinate.

Let ψ_i be the formula obtained from ϕ_i by removing the possible inequality literals, and let T_i be defined by ψ_i in $\mathbb{A}_{\mathcal{F}}$. We claim that $(A^n; T_0, T_1)$ and $(A^n; S_0, S_1)$ are homomorphically equivalent, which concludes the proof. Since ϕ_i implies ψ_i, we have that $(A^n; S_0, S_1)$ is a (non-induced) substructure of $(A^n; T_0, T_1)$, and therefore it homomorphically maps to $(A^n; T_0, T_1)$ by the identity map. For the other direction, we prove the result by compactness and show that every finite substructure $\mathbb{B}$ of $(A^n; T_0, T_1)$ has a homomorphism to $(A^n; S_0, S_1)$. Let $\mathbf{b}^1, \ldots, \mathbf{b}^m$ be the elements of $\mathbb{B}$. Let $\mathbb{C}$ be the τ-structure over precisely $n \cdot m$ elements $\{c^i_j \mid i, j\}$ corresponding to the entries of $\mathbf{b}^i_j$, whose relations are pulled back from $\mathbb{A}_{\mathcal{F}}$ under the map $\pi\colon c^i_j \mapsto b^i_j$. Note that no structure from $\mathcal{F}$ has a homomorphism to $\mathbb{C}$ (otherwise, we would obtain a homomorphism to $\mathbb{A}_{\mathcal{F}}$ by composition with π), and thus $\mathbb{C}$ admits an injective homomorphism g to $\mathbb{A}_{\mathcal{F}}$. We claim that if $(\mathbf{b}^i, \mathbf{b}^j, \mathbf{b}^k) \in T_0$ then $(g(\mathbf{c}^i), g(\mathbf{c}^j), g(\mathbf{c}^k)) \in S_0$. Indeed, suppose that $(\mathbf{b}^i, \mathbf{b}^j, \mathbf{b}^k)$ satisfies ψ_0. Then by construction $(g(\mathbf{c}^i), g(\mathbf{c}^j), g(\mathbf{c}^k))$ satisfies ψ_0. Moreover, by injectivity of g, we have $g(c^i_r) \neq g(c^j_s)$ as long as $i \neq j$ or $r \neq s$. Consider any inequality atom in ϕ_0. By our first claim, it is not of the form $x_r \neq y_r$, and therefore it is satisfied by $(g(\mathbf{c}^i), g(\mathbf{c}^j), g(\mathbf{c}^k))$. Thus, $(g(\mathbf{c}^i), g(\mathbf{c}^j), g(\mathbf{c}^k))$ satisfies ϕ_0. The same reasoning for ϕ_1 shows that g induces a homomorphism $\mathbb{B} \to (A^n; S_0, S_1)$ by mapping $\mathbf{b}^i$ to $g(\mathbf{c}^i)$. ◀

▶ **Lemma 26.** *Let $(S, \sim)$ be a subfactor of $\mathrm{Pol}(\mathbb{A}_{\mathcal{F}})^{\mathrm{can}}_1$ with $\mathrm{Aut}(\mathbb{A}_{\mathcal{F}})$-invariant $\sim$-classes. Let $m \geq 2$, and let $f \in \mathrm{Pol}(\mathbb{A}_{\mathcal{F}})$ be as in Lemma 19: for all $a_1, \ldots, a_m \in A_{\mathcal{F}}$ we have that if the entries of the tuple $(f(a_1, \ldots, a_m), f(a_2, \ldots, a_m, a_1), \ldots, f(a_m, a_1, \ldots, a_{m-1}))$ all belong to different colors, then it intersects at most one $\sim$-class. Let $O_0, \ldots, O_{m-1} \in S$ be pairwise distinct orbits under $\mathrm{Aut}(\mathbb{A}_{\mathcal{F}})$. There exists $g \in \mathrm{Pol}(\mathbb{A}_{\mathcal{F}})^{\mathrm{can}}_1$ that is locally interpolated by f and that satisfies*

$$g(O_k, \ldots, O_{k+m-1}) \sim g(O_{j+k}, \ldots, O_{j+k+m-1}) \tag{$\star$}$$

for some $0 \leq k < m$ and $1 \leq j < m$.

Proof. Recall that the expansion of $\mathbb{A}_\mathcal{F}$ by a generic linear order is a Ramsey structure [15]. Thus, f diagonally interpolates a function $g \in \mathrm{Pol}(\mathbb{A}_\mathcal{F})$ with the same properties and which is diagonally canonical with respect to $\mathrm{Aut}(\mathbb{A}_\mathcal{F}, <)$, and without loss of generality we can therefore assume that f is itself diagonally canonical.

Let $\mathbb{B} := \{0, \dots, m-1\} \times \mathbb{A}_\mathcal{F}$ be the disjoint union of m copies of $\mathbb{A}_\mathcal{F}$ and let e be an embedding of $\{0, \dots, m-1\} \times \mathbb{B}$ into $\mathbb{A}_\Phi$ with the properties stated in Proposition 24. Let $e_i(x) := e(i, x)$, which is a self-embedding of $\mathbb{A}_\mathcal{F}$. Consider $f'(x_0, \dots, x_{m-1}) := f(e_0 x_0, \dots, e_{m-1} x_{m-1})$, and note that f' is 1-canonical when restricted to m-tuples where all entries are in pairwise distinct orbits. Let g be obtained by canonising f' with respect to $\mathrm{Aut}(\mathbb{A}_\mathcal{F}, <)$. In particular $g \in \mathrm{Pol}(\mathbb{A}_\mathcal{F})_1^{\mathrm{can}}$ and $g(O_k, \dots, O_{k+m-1})$ and $f'(O_k, \dots, O_{k+m-1})$ are in S and $\sim$-equivalent for all k.

As in the proof of Theorem 23, there are suitable $0 \leq k < m$ and $1 \leq j < m$ such that

$$f(e_k O_k, \dots, e_{k+m-1} O_{k+m-1}) \sim f(e_{k+j} O_{k+j}, \dots, e_{k+j+m-1} O_{j+k+m-1})$$

holds, where indices are computed modulo m. Then

$$\begin{aligned} g(O_k, \dots, O_{k+m-1}) &\sim f(e_0 O_k, \dots, e_{m-1} O_{k+m-1}) \\ &\sim f(e_k O_k, \dots, e_{k+m-1} O_{k+m-1}) && (\star) \\ &\sim f(e_{k+j} O_{k+j}, \dots, e_{k+j+m-1} O_{k+j+m-1}) \\ &\sim f(e_0 O_{k+j}, \dots, e_{m-1} O_{k+j+m-1}) && (\star) \\ &\sim g(O_{k+j}, \dots, O_{k+j+m-1}), \end{aligned}$$

where the equivalences marked $(\star)$ hold by the fact that f is diagonally canonical with respect to $\mathrm{Aut}(\mathbb{A}_\mathcal{F}, <)$ and by Proposition 24. ◀

The following theorem gives a characterization of Datalog-rewritability in terms of precolored normal forms. The proof is similar to that of Theorem 23.

▶ **Theorem 27.** *Let Φ be a connected MMSNP τ-sentence, let $\mathcal{F}$ be an equivalent colored obstruction set and suppose that $\mathcal{F}$ is precolored and in strong normal form. The following are equivalent:*

1. *$\neg\Phi$ is equivalent to a Datalog program;*
2. *$\mathrm{Pol}(\mathbb{A}_\mathcal{F})$ does not have a uniformly continuous minion homomorphism to an affine clone;*
3. *The action of $\mathrm{Pol}(\mathbb{A}_\mathcal{F})_1^{\mathrm{can}}$ on $\mathrm{Aut}(\mathbb{A}_\mathcal{F})$-orbits of elements is not equationally affine;*
4. *$\mathbb{A}_\mathcal{F}$ has relational width $(k, \max(k+1, \ell))$, where k and ℓ are such that $\mathbb{A}_\mathcal{F}$ is k-homogeneous ℓ-bounded.*

Proof. (1) implies (2) by general principles [41, 8].

(2) implies (3). We do the proof by contraposition. The proof is essentially the same as in the case of reducts of unary structures (Theorem 23). Suppose that $\mathrm{Pol}(\mathbb{A}_\mathcal{F})_1^{\mathrm{can}} \curvearrowright \mathbb{A}_\mathcal{F}/\mathrm{Aut}(\mathbb{A}_\mathcal{F})$ is equationally affine and let $(S, \sim)$ be a minimal module for this action.

Let $m \geq 2$ and let R be an m-ary cyclic relation invariant under $\mathrm{Pol}(\mathbb{A}_\mathcal{F})$ and containing a tuple $(a_1, \dots, a_m)$ whose entries are pairwise distinct. By Proposition 18, either the linkedness congruence of R defines an approximation of $\sim$, or R contains a pseudoloop modulo $\mathrm{Aut}(\mathbb{A}_\mathcal{F})$. In the first case, the approximation is smooth and we obtain a uniformly continuous minion homomorphism from $\mathrm{Pol}(\mathbb{A}_\mathcal{F})$ to a clone of affine maps. Any such clone admits a uniformly continuous minion homomorphism to $\mathbb{Z}_p$ for some p, and by composition this gives us a uniformly continuous minion homomorphism $\mathrm{Pol}(\mathbb{A}_\mathcal{F}) \to \mathbb{Z}_p$.

So let us assume that for all $m \geq 2$, every such relation R contains a pseudoloop. By applying Lemma 19, we obtain a polymorphism f such that for all $a_1, \ldots, a_m$, if $f(a_1, \ldots, a_m), \ldots, f(a_m, a_0, \ldots, a_{m-1})$ are pairwise distinct, then they intersect at most one $\sim$-class. As in the proof of Theorem 23, pick an arbitrary $a_1 \in S$ such that $[a_1]_\sim$ is not the zero element of the module $S/{\sim}$. Let $m \geq 2$ be its order, and let O_i be the orbit of $i \cdot [a_1]_\sim$, for $i \in \{0, 1, \ldots, m-1\}$. By Lemma 26, we obtain $g \in \mathrm{Pol}(\mathbb{A}_{\mathcal{F}})_1^{\mathrm{can}}$ such that

$$g(O_k, \ldots, O_{k+m-1}) \sim g(O_{j+k}, \ldots, O_{j+k+m-1})$$

for some $k \in \{0, \ldots, m-1\}$ and $j \in \{1, \ldots, m-1\}$. The same computation as in Theorem 23 then gives a contradiction and concludes the proof.

(3) implies (4). First, note that $\mathbb{A}_{\mathcal{F}}$ is infinite, and therefore $k \geq 2$. Let $\mathcal{I}$ be a non-trivial $(k, \max(k+1, \ell))$-minimal instance of $\mathbb{A}_{\mathcal{F}}$. Let $\mathcal{G}$ be $\mathrm{Aut}(\mathbb{A}_{\mathcal{F}})$. Consider the instance $\mathcal{I}_{\mathcal{G},1}$ as in Definition 11. Thus, the variables of $\mathcal{I}_{\mathcal{G},1}$ are the same as the variables of $\mathcal{I}$ (up to the natural bijection between $\mathcal{V}$ and $\binom{\mathcal{V}}{1}$) and the values for the variables are taken from the set of colors of $\mathcal{F}$. By Lemma 12, $\mathcal{I}_{\mathcal{G},1}$ is $(2,3)$-minimal, and from (3) and Lemma 14 we obtain that it has a solution h. Note that we cannot use Lemma 13 to obtain a solution to $\mathcal{I}$, since we only considered $\mathcal{I}_{\mathcal{G},1}$. Let $\mathbb{B}$ be the τ-structure described by $\mathcal{I}$ (i.e., $\mathbb{B}$ is the *canonical database* of $\mathcal{I}$). Let $\mathbb{B}^*$ be the $(\tau \cup \sigma)$-expansion of $\mathbb{B}$ obtained by coloring the vertices of $\mathbb{B}$ according to h. Since $\mathcal{I}$ is (k, ℓ)-minimal, it can be seen that $\mathbb{B}^*$ does not contain any homomorphic copy of $\mathbb{F} \in \mathcal{F}$, so that $\mathbb{B}$ admits a homomorphism to $\mathbb{A}_{\mathcal{F}}$, i.e., $\mathcal{I}$ has a solution in $\mathbb{A}_{\mathcal{F}}$.

(4) implies (1). Trivial. ◀

Combining Proposition 25, Theorem 27, and known facts about MMSNP and normal forms [15], this allows us to obtain Theorem 1 from the introduction.

▶ **Theorem 1.** *The Datalog-rewritability problem for MMSNP is decidable, and is 2NExpTime-complete.*

Proof. Let Φ be an MMSNP sentence, which is equivalent to a disjunction $\Phi_1 \vee \cdots \vee \Phi_p$ of connected MMSNP sentences [15, Proposition 3.2]. Moreover, if p is minimal then $\neg\Phi$ is equivalent to a Datalog program iff every $\neg\Phi_i$ is equivalent to a Datalog program (see, e.g., Proposition 3.3 in [15], for a proof of a similar fact).

By Theorem 4.3 in [15], one can compute for every Φ_i a coloured obstruction set $\mathcal{F}_i$ that is in strong normal form. Let $\mathcal{G}_i$ be the standard precoloration of $\mathcal{F}_i$. By Proposition 25, one has a uniformly continuous minion homomorphism from $\mathrm{Pol}(\mathbb{A}_{\mathcal{G}_i})$ to an affine clone iff one has one from $\mathrm{Pol}(\mathbb{A}_{\mathcal{F}_i})$ to an affine clone. Then, by Theorem 27, we get that deciding Datalog-rewritability for $\mathcal{G}_i$ is equivalent to deciding whether $\mathrm{Pol}(\mathbb{A}_{\mathcal{G}_i})_1^{\mathrm{can}} \curvearrowright \mathbb{A}_{\mathcal{G}_i}/\mathrm{Aut}(\mathbb{A}_{\mathcal{G}_i})$ is equationally non-affine, which is known to be decidable in polynomial time since $\mathrm{Pol}(\mathbb{A}_{\mathcal{G}_i})_1^{\mathrm{can}} \curvearrowright \mathbb{A}_{\mathcal{G}_i}/\mathrm{Aut}(\mathbb{A}_{\mathcal{G}_i})$ is idempotent.

The computation of a strong normal form is costly and can be performed in 2-ExpSpace. In order to obtain a 2NExpTime-algorithm, we rather compute a normal form $\mathcal{F}_i$ for Φ_i (by Lemma 3.1 in [15]), which can be done in doubly exponential-time. The consequence of not working with a strong normal form is that the clone $\mathrm{Pol}(\mathbb{A}_{\mathcal{F}_i})_1^{\mathrm{can}} \curvearrowright \mathbb{A}_{\mathcal{F}_i}/\mathrm{Aut}(\mathbb{A}_{\mathcal{F}_i})$ is not a core; its core is the action considered for the strong normal form. Deciding whether such a clone admits a minion homomorphism to an affine clone is in NP [31, Corollary 6.8]. We obtain overall a 2NExpTime algorithm. The complexity lower bound is Theorem 18 in [27]. ◀

References

1 Samson Abramsky, Anuj Dawar, and Pengming Wang. The pebbling comonad in finite model theory. In *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017*, pages 1–12. IEEE Computer Society, 2017. doi:10.1109/LICS.2017.8005129.

2 Albert Atserias, Andrei A. Bulatov, and Víctor Dalmau. On the power of k -consistency. In Lars Arge, Christian Cachin, Tomasz Jurdzinski, and Andrzej Tarlecki, editors, *Automata, Languages and Programming, 34th International Colloquium, ICALP 2007, Wroclaw, Poland, July 9-13, 2007, Proceedings*, volume 4596 of *Lecture Notes in Computer Science*, pages 279–290. Springer, 2007. doi:10.1007/978-3-540-73420-8_26.

3 László Babai. Graph isomorphism in quasipolynomial time [extended abstract]. In Daniel Wichs and Yishay Mansour, editors, *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2016, Cambridge, MA, USA, June 18-21, 2016*, pages 684–697. ACM, 2016. doi:10.1145/2897518.2897542.

4 Libor Barto. The collapse of the bounded width hierarchy. *J. Log. Comput.*, 26(3):923–943, 2016. doi:10.1093/logcom/exu070.

5 Libor Barto and Marcin Kozik. Congruence distributivity implies bounded width. *SIAM J. Comput.*, 39(4):1531–1542, 2009. doi:10.1137/080743238.

6 Libor Barto and Marcin Kozik. Absorbing subalgebras, cyclic terms, and the constraint satisfaction problem. *Log. Methods Comput. Sci.*, 8(1), 2012. doi:10.2168/LMCS-8(1:7)2012.

7 Libor Barto and Marcin Kozik. Constraint satisfaction problems solvable by local consistency methods. *J. ACM*, 61(1):3:1–3:19, 2014. doi:10.1145/2556646.

8 Libor Barto, Jakub Opršal, and Michael Pinsker. The wonderland of reflections. *Israel Journal of Mathematics*, 223(1):363–398, 2018.

9 Libor Barto and Michael Pinsker. The algebraic dichotomy conjecture for infinite domain constraint satisfaction problems. In *Proceedings of the 31th Annual IEEE Symposium on Logic in Computer Science – LICS'16*, pages 615–622, 2016. Preprint arXiv:1602.04353.

10 Libor Barto and Michael Pinsker. Topology is irrelevant. *SIAM Journal on Computing*, 49(2):365–393, 2020.

11 Meghyn Bienvenu, Balder ten Cate, Carsten Lutz, and Frank Wolter. Ontology-based data access: A study through disjunctive datalog, csp, and MMSNP. *ACM Trans. Database Syst.*, 39(4):33:1–33:44, 2014. doi:10.1145/2661643.

12 Manuel Bodirsky. Ramsey classes: Examples and constructions. In *Surveys in Combinatorics. London Mathematical Society Lecture Note Series 424*. Cambridge University Press, 2015. Invited survey article for the British Combinatorial Conference; arXiv:1502.05146.

13 Manuel Bodirsky and Víctor Dalmau. Datalog and constraint satisfaction with infinite templates. *J. Comput. Syst. Sci.*, 79(1):79–100, 2013. doi:10.1016/j.jcss.2012.05.012.

14 Manuel Bodirsky and Jan Kára. The complexity of equality constraint languages. *Theory of Computing Systems*, 3(2):136–158, 2008. A conference version appeared in the proceedings of Computer Science Russia (CSR'06).

15 Manuel Bodirsky, Florent R. Madelaine, and Antoine Mottet. A universal-algebraic proof of the complexity dichotomy for monotone monadic SNP. In Anuj Dawar and Erich Grädel, editors, *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018*, pages 105–114. ACM, 2018. doi:10.1145/3209108.3209156.

16 Manuel Bodirsky, Barnaby Martin, and Antoine Mottet. Discrete temporal constraint satisfaction problems. *J. ACM*, 65(2):9:1–9:41, 2018. doi:10.1145/3154832.

17 Manuel Bodirsky, Barnaby Martin, Michael Pinsker, and András Pongrácz. Constraint satisfaction problems for reducts of homogeneous graphs. *SIAM J. Comput.*, 48(4):1224–1264, 2019. A conference version appeared in the Proceedings of the 43rd International Colloquium on Automata, Languages, and Programming, ICALP 2016, pages 119:1-119:14. doi:10.1137/16M1082974.

18 Manuel Bodirsky and Antoine Mottet. A dichotomy for first-order reducts of unary structures. *Log. Methods Comput. Sci.*, 14(2), 2018. doi:10.23638/LMCS-14(2:13)2018.

19 Manuel Bodirsky, Antoine Mottet, Miroslav Olšák, Jakub Opršal, Michael Pinsker, and Ross Willard. Topology is relevant (in a dichotomy conjecture for infinite-domain constraint satisfaction problems). In *34th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2019, Vancouver, BC, Canada, June 24-27, 2019*, pages 1–12. IEEE, 2019. doi:10.1109/LICS.2019.8785883.

20 Manuel Bodirsky, Wied Pakusa, and Jakub Rydval. Temporal constraint satisfaction problems in fixed-point logic. In Holger Hermanns, Lijun Zhang, Naoki Kobayashi, and Dale Miller, editors, *LICS '20: 35th Annual ACM/IEEE Symposium on Logic in Computer Science, Saarbrücken, Germany, July 8-11, 2020*, pages 237–251. ACM, 2020. doi:10.1145/3373718.3394750.

21 Manuel Bodirsky and Michael Pinsker. Canonical Functions: a Proof via Topological Dynamics. To appear. Preprint arXiv:1610.09660.

22 Manuel Bodirsky and Michael Pinsker. Schaefer's theorem for graphs. *J. ACM*, 62(3):19:1–19:52, 2015. doi:10.1145/2764899.

23 Manuel Bodirsky and Michael Pinsker. Topological Birkhoff. *Transactions of the American Mathematical Society*, 367:2527–2549, 2015.

24 Manuel Bodirsky, Michael Pinsker, and András Pongrácz. Projective clone homomorphisms. *Journal of Symbolic Logic*, 2019. doi:10.1017/jsl.2019.23.

25 Manuel Bodirsky, Michael Pinsker, and Todor Tsankov. Decidability of definability. *Journal of Symbolic Logic*, 78(4):1036–1054, 2013. A conference version appeared in the Proceedings of the Twenty-Sixth Annual IEEE Symposium on Logic in Computer Science (LICS) 2011, pages 321–328.

26 Manuel Bodirsky and Michał Wrona. Equivalence constraint satisfaction problems. In *Computer Science Logic (CSL'12) - 26th International Workshop/21st Annual Conference of the EACSL, CSL 2012, September 3-6, 2012, Fontainebleau, France*, pages 122–136, 2012. doi:10.4230/LIPIcs.CSL.2012.122.

27 Pierre Bourhis and Carsten Lutz. Containment in monadic disjunctive datalog, mmsnp, and expressive description logics. In Chitta Baral, James P. Delgrande, and Frank Wolter, editors, *Principles of Knowledge Representation and Reasoning: Proceedings of the Fifteenth International Conference, KR 2016, Cape Town, South Africa, April 25-29, 2016*, pages 207–216. AAAI Press, 2016. URL: http://www.aaai.org/ocs/index.php/KR/KR16/paper/view/12847.

28 Hans-Jürgen Bückert and Bernhard Nebel. Reasoning about temporal relations: a max-imal tractable subclass of allen's interval algebra. *J. ACM*, 42(1):43–66, 1995.

29 Andrei A. Bulatov. Bounded relational width, 2009. URL: https://www2.cs.sfu.ca/~abulatov/papers/relwidth.pdf.

30 Andrei A. Bulatov. A dichotomy theorem for nonuniform csps. In Chris Umans, editor, *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, 2017*, pages 319–330. IEEE Computer Society, 2017. doi:10.1109/FOCS.2017.37.

31 Hubie Chen and Benoît Larose. Asking the metaquestions in constraint tractability. *ACM Trans. Comput. Theory*, 9(3):11:1–11:27, 2017. doi:10.1145/3134757.

32 Víctor Dalmau and Justin Pearson. Closure functions and width 1 problems. In *Proceedings of the International Conference on Principles and Practice of Constraint Programming (CP)*, pages 159–173, 1999.

33 Tomás Feder and Moshe Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory. *SIAM J. Comput.*, 28(1):57–104, 1998. doi:10.1137/S0097539794266766.

34 Cristina Feier, Antti Kuusisto, and Carsten Lutz. Rewritability in monadic disjunctive datalog, mmsnp, and expressive description logics. *Log. Methods Comput. Sci.*, 15(2), 2019. doi:10.23638/LMCS-15(2:15)2019.

35 Pierre Gillibert, Julius Jonušas, Michael Kompatscher, Antoine Mottet, and Michael Pinsker. Hrushovski's encoding and ω-categorical CSP monsters. In Artur Czumaj, Anuj Dawar, and Emanuela Merelli, editors, *47th International Colloquium on Automata, Languages, and Programming, ICALP 2020, July 8-11, 2020, Saarbrücken, Germany (Virtual Conference)*, volume 168 of *LIPIcs*, pages 131:1–131:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.ICALP.2020.131.

36 Martin Grohe. Fixed-point definability and polynomial time on graphs with excluded minors. *J. ACM*, 59(5):27:1–27:64, 2012. doi:10.1145/2371656.2371662.

37 Alexander Kechris, Vladimir Pestov, and Stevo Todorčević. Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups. *Geometric and Functional Analysis*, 15(1):106–189, 2005.

38 Phokion G. Kolaitis and Moshe Y. Vardi. Conjunctive-query containment and constraint satisfaction. *J. Comput. Syst. Sci.*, 61(2):302–332, 2000. doi:10.1006/jcss.2000.1713.

39 Michael Kompatscher and Trung Van Pham. A complexity dichotomy for poset constraint satisfaction. *FLAP*, 5(8):1663–1696, 2018. URL: https://www.collegepublications.co.uk/downloads/ifcolog00028.pdf.

40 Marcin Kozik, Andrei Krokhin, Matt Valeriote, and Ross Willard. Characterizations of several Maltsev conditions. *Algebra universalis*, 73(3-4):205–224, 2015. doi:10.1007/s00012-015-0327-2.

41 Benoit Larose and László Zádori. Bounded width problems and algebras. *Algebra Universalis*, 56(3-4):439–466, 2007.

42 Florent R. Madelaine and Iain A. Stewart. Constraint satisfaction, logic and forbidden patterns. *SIAM J. Comput.*, 37(1):132–163, 2007. doi:10.1137/050634840.

43 Miklós Maróti and Ralph McKenzie. Existence theorems for weakly symmetric operations. *Algebra Universalis*, 59(3), 2008.

44 Antoine Mottet, Tomáš Nagy, Michael Pinsker, and Michał Wrona. Smooth approximations and relational width collapses, 2021. arXiv:2102.07531.

45 Antoine Mottet and Michael Pinsker. Smooth approximations and CSPs over finitely bounded homogeneous structures. *CoRR*, abs/2011.03978, 2020. arXiv:2011.03978.

46 Michael Pinsker, Pierre Gillibert, and Julius Jonušas. Pseudo-loop conditions. *Bulletin of the London Mathematical Society*, 51(5):917–936, 2019.

47 Matthew A. Valeriote. A subalgebra intersection property for congruence distributive varieties. *Canadian Journal of Mathematics*, 61(2):451–464, 2009. doi:10.4153/CJM-2009-023-2.

48 Michał Wrona. On the relational width of first-order expansions of finitely bounded homogeneous binary cores with bounded strict width. In Holger Hermanns, Lijun Zhang, Naoki Kobayashi, and Dale Miller, editors, *LICS '20: 35th Annual ACM/IEEE Symposium on Logic in Computer Science, Saarbrücken, Germany, July 8-11, 2020*, pages 958–971. ACM, 2020. doi:10.1145/3373718.3394781.

49 Michał Wrona. Relational width of first-order expansions of homogeneous graphs with bounded strict width. In Christophe Paul and Markus Bläser, editors, *37th International Symposium on Theoretical Aspects of Computer Science, STACS 2020, March 10-13, 2020, Montpellier, France*, volume 154 of *LIPIcs*, pages 39:1–39:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.STACS.2020.39.

50 Dmitriy Zhuk. A proof of CSP dichotomy conjecture. In Chris Umans, editor, *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, 2017*, pages 331–342. IEEE Computer Society, 2017. doi:10.1109/FOCS.2017.38.

51 Dmitriy Zhuk. A proof of the CSP dichotomy conjecture. *J. ACM*, 67(5):30:1–30:78, 2020. doi:10.1145/3402029.