Depth-First Search in Directed Planar Graphs, Revisited

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Abstract

We present an algorithm for constructing a depth-first search tree in planar digraphs; the algorithm can be implemented in the complexity class $\mathsf{AC}^1(\mathsf{UL} \cap \text{co-}\mathsf{UL})$, which is contained in AC^2. Prior to this (for more than a quarter-century), the fastest uniform deterministic parallel algorithm for this problem was $O(\log^{10} n)$ (corresponding to the complexity class $\mathsf{AC}^{10} \subseteq \mathsf{NC}^{11}$).

We also consider the problem of computing depth-first search trees in other classes of graphs, and obtain additional new upper bounds.

2012 ACM Subject Classification Theory of computation → Complexity classes; Theory of computation → Parallel algorithms

Keywords and phrases Depth-First Search, Planar Digraphs, Parallel Algorithms, Space-Bounded Complexity Classes

Digital Object Identifier 10.4230/LIPIcs.MFCS.2021.7

Related Version *Full Version*: https://eccc.weizmann.ac.il/report/2020/074/

Funding *Eric Allender*: Supported in part by NSF Grant CCF-1909216 and CCF-1909683.
Archit Chauhan: Partially supported by a grant from Infosys foundation and TCS PhD fellowship.
Samir Datta: Partially supported by a grant from Infosys foundation and SERB-MATRICS grant MTR/2017/000480.

1 Introduction

Depth-first search trees (DFS trees) constitute one of the most useful items in the algorithm designer's toolkit, and for this reason they are a standard part of the undergraduate algorithmic curriculum around the world. When attention shifted to parallel algorithms in the 1980's, the question arose of whether NC algorithms for DFS trees exist. An early negative result was that the problem of constructing the *lexicographically least* DFS tree in a given digraph is complete for P [20]. But soon thereafter significant advances were made in developing parallel algorithms for DFS trees, culminating in the RNC^7 algorithm of Aggarwal, Anderson, and Kao [1]. This remains the fastest parallel algorithm for the problem of constructing DFS trees in general graphs, in the probabilistic setting, or in the setting of nonuniform circuit complexity. It remains unknown if this problem lies in (deterministic) NC (and we do not solve that problem here).

More is known for various restricted classes of graphs. For directed acyclic graphs (DAGs), the lexicographically-least DFS tree from a given vertex can be computed in AC^1 [10]. (See also [11, 7, 13, 19, 16, 15].) For undirected planar graphs, an AC^1 algorithm for DFS trees was presented by Hagerup [14]. For more general planar directed graphs Kao and Klein presented an AC^{10} algorithm. Kao subsequently presented an AC^5 algorithm for DFS in *strongly connected* planar digraphs. In stating the complexity results for this prior work in terms of complexity classes (such as $\mathsf{AC}^1, \mathsf{AC}^{10}$, etc.), we are ignoring an aspect that was of particular interest to the authors of this earlier work: minimizing the number of processors. This is because our focus is on classifying the complexity of constructing DFS trees in terms of complexity classes. Thus, if we reduce the complexity of a problem from AC^{10} to AC^2, then we view this as a significant advance, even if the AC^2 algorithm uses many more processors (so long as the number of processors remains bounded by a polynomial). Indeed, our algorithms rely on the logspace algorithm for undirected reachability [21], which does not directly translate into a processor-efficient algorithm. We suspect that our approach can be modified to yield a more processor-efficient AC^3 algorithm, but we leave that for others to investigate.

1.1 Our Contributions

First, we observe that, given a DAG G, computation of a DFS tree in G logspace reduces to the problem of reachability in G. Thus, for general DAGs, computation of a DFS tree lies in NL, and for planar DAGs, the problem lies in $\mathsf{UL} \cap \mathsf{co\text{-}UL}$ [8, 23]. For classes of graphs where the reachability problem lies in L, so does the computation of DFS trees. One such class of graphs is planar DAGs with a single source (see [2], where this class of graphs is called SMPDs, for **S**ingle-source, **M**ultiple-sink, **P**lanar **D**AGs).

For undirected planar graphs, it was shown in [4] that the approach of Hagerup's AC^1 DFS algorithm [14] can be adapted in order to show that construction of a DFS tree in a planar *undirected* graph logspace-reduces to computing the distance between two nodes in a planar digraph. Since this latter problem lies in $\mathsf{UL} \cap \mathsf{co\text{-}UL}$ [24], so does the problem of DFS for planar *undirected* graphs.

Our main contribution in the current paper is to show that a more sophisticated application of the ideas in [14] leads to an $\mathsf{AC}^1(\mathsf{UL} \cap \mathsf{co\text{-}UL})$ algorithm for construction of DFS trees in planar *directed* graphs. (That is, we show DFS trees can be constructed by unbounded fan-in log-depth circuits that have oracle gates for a set in $\mathsf{UL} \cap \mathsf{co\text{-}UL}$.[1]) Since $\mathsf{UL} \subseteq \mathsf{NL} \subseteq \mathsf{SAC}^1 \subseteq \mathsf{AC}^1$, the $\mathsf{AC}^1(\mathsf{UL} \cap \mathsf{co\text{-}UL})$ algorithm can be implemented in AC^2. Thus this is a significant improvement over the best previous parallel algorithm for this problem: the AC^{10} algorithm of [18], which has stood for 28 years.

2 Preliminaries

We assume that the reader is familiar with depth-first search trees (DFS trees).

We further assume that the reader is familiar with the standard complexity classes L, NL and P (see e.g. the text [6]). We will also make frequent reference to the logspace-uniform circuit complexity classes NC^k and AC^k. NC^k is the class of problems for which there is a logspace-uniform family of circuits $\{C_n\}$ consisting of AND, OR, and NOT gates, where the AND and OR gates have fan-in two and each circuit C_n has depth $O(\log^k n)$. (The logspace-uniformity condition implies that each C_n has only $n^{O(1)}$ gates.) AC^k is defined similarly, although the AND and OR gates are allowed unbounded fan-in. An equivalent characterization of AC^k is in terms of concurrent-read concurrent-write PRAMs with running time $O(\log^k n)$, using $n^{O(1)}$ processors. For more background on these circuit complexity classes, see, e.g., the text [26].

[1] An earlier version of this work claimed a stronger upper bound, but there was an error in one of the lemmas in that version [3].

A nondeterministic Turing machine is said to be *unambiguous* if, on every input x, there is at most one accepting computation path. If we consider logspace-bounded nondeterministic Turing machines, then unambiguous machines yield the class UL. A set A is in $\mathsf{co\text{-}UL}$ if and only if its complement lies in UL.

The construction of DFS trees is most naturally viewed as a *function* that takes a graph G and a vertex v as input, and produces as output an encoding of a DFS tree in G rooted at v. But the complexity classes mentioned above are all defined as sets of *languages*, instead of as sets of *functions*. Since our goal is to place DFS tree construction into the appropriate complexity classes, it is necessary to discuss how the complexity of functions fits into the framework of complexity classes.

When $\mathcal{C}$ is one of $\{\mathsf{L}, \mathsf{P}\}$, it is fairly obvious what is meant by "f is computable in $\mathcal{C}$"; the classes of logspace-computable functions and polynomial-time-computable functions should be familiar to the reader. However, the reader might be less clear as to what is meant by "f is computable in NL". As it turns out, essentially all of the reasonable possibilities are equivalent. Let us denote by FNL the class of functions that are computable in NL; it is shown in [17] each of the three following conditions is equivalent to "$f \in \mathsf{FNL}$".

1. f is computed by a logspace machine with an oracle from NL.
2. f is computed by a logspace-uniform NC^1 circuit family with oracle gates for a language in NL.
3. $f(x)$ has length bounded by a polynomial in $|x|$, and the set $\{(x, i, b) :$ the i^{th} bit of $f(x)$ is $b\}$ is in NL.

Rather than use the unfamiliar notation "FNL", we will abuse notation slightly and refer to certain functions as being "computable in NL".

The proof of the equivalence above relies on the fact that NL is closed under complement. Thus it is far less clear what it should mean to say that a function is "computable in UL" since it remains an open question if UL is closed under complement (although it is widely conjectured that $\mathsf{UL} = \mathsf{NL}$) [22, 5]). However the proof from [17] carries over immediately to the class $\mathsf{UL} \cap \mathsf{co\text{-}UL}$. That is, the following conditions are equivalent:

1. f is computed by a logspace machine with an oracle from $\mathsf{UL} \cap \mathsf{co\text{-}UL}$.
2. f is computed by a logspace-uniform NC^1 circuit family with oracle gates for a language in $\mathsf{UL} \cap \mathsf{co\text{-}UL}$.
3. $f(x)$ has length bounded by a polynomial in $|x|$, and the set $\{(x, i, b) :$ the i^{th} bit of $f(x)$ is $b\}$ is in $\mathsf{UL} \cap \mathsf{co\text{-}UL}$.

Thus, if any of those conditions hold, we will say that "f is computable in $\mathsf{UL} \cap \mathsf{co\text{-}UL}$".

The important fact that the composition of two logspace-computable functions is also logspace-computable (see, e.g., [6]) carries over with an identical proof to the functions computable in L^C for any oracle C. Thus the class of functions computable in $\mathsf{UL} \cap \mathsf{co\text{-}UL}$ is also closed under composition. We make implicit use of this fact frequently when presenting our algorithms. For example, we may say that a colored labeling of a graph G is computable in $\mathsf{UL} \cap \mathsf{co\text{-}UL}$, and that, given such a colored labeling, a decomposition of the graph into layers is also computable in logspace, and furthermore, that – given such a decomposition of G into layers – an additional coloring of the smaller graphs is computable in $\mathsf{UL} \cap \mathsf{co\text{-}UL}$, etc. The reader need not worry that a logspace-bounded machine does not have adequate space to store these intermediate representations; the fact that the final result is also computable in $\mathsf{UL} \cap \mathsf{co\text{-}UL}$ follows from closure under composition. In effect, the bits of these intermediate representations are re-computed each time we need to refer to them.

Finally, we will consider AC^k circuits augmented with oracle gates for an oracle in $\mathsf{UL} \cap \mathsf{co\text{-}UL}$, which we denote by $\mathsf{AC}^k(\mathsf{UL} \cap \mathsf{co\text{-}UL})$.

3 DFS in DAGs logspace reduces to Reachability

In this section, we observe that constructing the lexicographically-least DFS tree in a DAG G can be done in logspace given an oracle for reachability in G. But first, let us define what we mean by the lexicographically-first DFS tree in G:

▶ **Definition 1.** *Let G be a DAG, with the neighbours of the vertices given in some order in the input. (For example, with adjacency lists, we can consider the ordering in which the neighbors are presented in the list). Then the lexicographic first DFS traversal of G is the traversal with the (very natural) condition that the children of every vertex are explored in the order given in the input. For details, see the full version [3].*

The lemma we need is the following:

▶ **Lemma 2.** *Construction of lexicographic least DFSs tree in DAGs logspace reduces to reachability. In particular, DFS in general DAGs, planar DAGs, and planar DAGs with single source(SMPDs) lie in classes* NL*,* UL ∩ co-UL*, and* L *respectively.*

The correctness of this lemma is shown by the proof of Theorem 11 of [10] for general DAGs. The extension for the other two classes is a consequence of planar reachability in UL ∩ co-UL [24] and of SMPD reachability in L [2]. We defer the details to the full version [3].

4 Overview of the Algorithm

The main algorithmic insight that led us to the current algorithm was a generalization of the layering algorithm that Hagerup developed for *undirected* graphs [14]. We show that this approach can be modified to yield a useful decomposition of *directed* graphs, where the layers of the graph have a restricted structure that can be exploited. More specifically, the strongly-connected components of each layer are what we call *meshes*, which enable us easily to construct paths (which will end up being paths in the DFS trees we construct) whose removal partitions the graph into significantly smaller strongly connected components.

The high-level structure of the algorithm is thus:

1. Construct a planar embedding of G.
2. Partition the planar graph G into layers (each of which is surrounded by a directed cycle).
3. Identify one such cycle C that has properties that will allow us to partition the graph into smaller weakly connected components.
4. Depending on which properties C satisfies, create a path p from the exterior face either to a vertex on C or to one of the meshes that reside in the layer just inside C. Removal of p partitions G into weakly connected components, where each strongly-connected component therein is smaller than G by a constant factor.
5. Let the vertices on this path p be $v_1, v_2, \ldots, v_k$. The DFS tree will start with the path p, and append DFS trees for subgraphs $G_1, G_2, \ldots, G_k$ to this path, where G_i consists of all of the vertices that are reachable from v_i that are not reachable from v_j for any $j > i$. (This is obviously a tree, and it will follow that it is a DFS tree.) Further, decompose each G_i into a DAG of strongly-connected components. Build a DFS of that DAG, and then work on building DFS trees of the remaining (smaller) strongly-connected components.
6. Each of the steps above can be accomplished in UL ∩ co-UL, which means that there is an AC^0 circuit with oracle gates from UL ∩ co-UL that takes G as input and produces the list of much smaller graphs $G_1, \ldots, G_k$, as well as the path p that forms the spine of the DFS tree. We now recursively apply this procedure (in parallel) to each of these smaller graphs. The construction is complete after $O(\log n)$ phases, yielding the desired AC^1(UL ∩ co-UL) circuit family.

In the exposition below, we first layer the graph in terms of clockwise cycles (which we will henceforth call red cycles), and obtain a decomposition of the original graph into smaller pieces. We then apply a nested layering in terms of counterclockwise cycles (which we will henceforth call blue cycles); ultimately we decompose the graph into units that are structured as a DAG, which we can then process using the tools from the earlier sections of the paper. The more detailed presentation follows.

4.1 Degree Reduction and Expansion

▶ **Definition 3** (of $\mathbf{Exp}^{\circlearrowright}(G)$ and $\mathbf{Exp}^{\circlearrowleft}(G)$)**.** *Let G be a planar digraph. The "expanded" digraph $\mathbf{Exp}^{\circlearrowright}(G)$ (respectively, $\mathbf{Exp}^{\circlearrowleft}(G)$) is formed by replacing each vertex v of total degree $d(v) > 3$ by a clockwise (respectively, counterclockwise) cycle C_v on $d(v)$ vertices such that the endpoint of the i-th edge incident on v is now incident on the the i-th vertex of the cycle.*

$\mathbf{Exp}^{\circlearrowright}(G)$ and $\mathbf{Exp}^{\circlearrowleft}(G)$ each have maximum degree bounded by 3; i.e., they are *subcubic*. Next we define the clockwise (and counterclockwise) dual for such a graph and also a notion of distance.

Recall that for an undirected plane graph H, the dual (multigraph) H^* is formed by placing, for every edge $e \in E(H)$, a dual edge e^* between the face(s) on either side of e (see Section 4.6 from [12] for more details). Faces f of H and the vertices f^* of H^* correspond to each other as do vertices v of H and faces v^* of H^*.

▶ **Definition 4** (of Duals $G^{\circlearrowright}$ and $G^{\circlearrowleft}$)**.** *Let G be a plane digraph, then the clockwise dual $G^{\circlearrowright}$ (respectively, counterclockwise dual $G^{\circlearrowleft}$) is a weighted bidirected version of the undirected dual of the underlying undirected graph of G in a way so that the orientation formed by rotating the corresponding directed edge of G in a clockwise (respectively, counterclockwise) way gets a weight of 1 and the other orientation gets weight 0. We inherit the definition of dual vertices and faces from the underlying undirected dual.*

▶ **Definition 5.** *For a plane subcubic digraph G, let f_0 be the external face. Define the type $\mathbf{type}^{\circlearrowright}(f)$ (respectively, $\mathbf{type}^{\circlearrowleft}(f)$) of a face to be the singleton set consisting of the distance at which f lies from f_0 in $G^{\circlearrowright}$: $\{d^{\circlearrowright}(f_0, f)\}$ (respectively, $\{d^{\circlearrowleft}(f_0, f)\}$). Generalise this to edges e by defining $\mathbf{type}^{\circlearrowright}(e)$ (respectively $\mathbf{type}^{\circlearrowleft}(e)$) as the set consisting of the union of the $\mathbf{type}^{\circlearrowright}$ (respectively, $\mathbf{type}^{\circlearrowleft}$) of the two faces adjacent to e, and to vertices v by defining as the $\mathbf{type}^{\circlearrowright}(v)$ (respectively $\mathbf{type}^{\circlearrowleft}(v)$) union of the $\mathbf{type}^{\circlearrowright}$ (respectively, $\mathbf{type}^{\circlearrowleft}$) of the faces incident on the vertex v.*

The following is a direct consequence of subcubicity and the triangle inequality:

▶ **Lemma 6.** *In every subcubic graph G, the cardinality $|\mathbf{type}^{\circlearrowright}(x)|, |\mathbf{type}^{\circlearrowleft}(x)|$ where x is a face, edge or a vertex is at least one and at most 2 and in the latter case consists of consecutive non-negative integers.*

Further, if $v \in V(G)$ is such that $|\mathbf{type}^{\circlearrowright}(v)| = 2$, then there exist unique $u, w \in V(G)$, such that $(u, v), (v, w) \in E(G)$ and $|\mathbf{type}^{\circlearrowright}(u, v)| = |\mathbf{type}^{\circlearrowright}(v, w)| = 2$.

For proof, see Appendix A.1.

▶ **Definition 7.** *For a plane subcubic graph G as above, we refer to vertices and edges with a type of cardinality two in $G^{\circlearrowright}$ (respectively, in $G^{\circlearrowleft}$) as red (respectively, blue) while the ones with a cardinality of one as white. The resulting colored graphs are called $\mathbf{red}(G)$ and $\mathbf{blue}(G)$ respectively.*

We will see later how to apply both the duals in G to get red and blue layerings of a given input graph.

Also note that a red (respectively blue) edge must have red (respectively blue) end points, as they are adjacent to the same faces as the edge between is.

We enumerate some properties of $\mathbf{red}(G)$, $\mathbf{blue}(G)$ (G is subcubic):

▶ **Lemma 8.**

1. *Red vertices and edges in* $\mathbf{red}(G)$ *form disjoint clockwise cycles.*
2. *No clockwise cycle in* $\mathbf{red}(G)$ *consists of only white edges (and hence white vertices).*

Similar properties hold for $\mathbf{blue}(G)$*.*

Proof.

1. Firstly, note that a red edge must have red end point vertices, as they are adjacent to the same faces that the edge between them is adjacent to. It is immediate from Lemma 6 that if v is a red vertex, it has exactly one red incoming edge and one red outgoing edge, proving that they form disjoint cycles. Now consider a red cycle C. The type of each edge of C must be the same, since if there are two consecutive edges in C of different types, it would make the common vertex adjacent to at least three vertices of different types contradicting lemma 6. This means that the distance in $G^{\circlearrowright}$ of each face bordering the "outside" of C from the external face is one less than the distance of each face bordering the "inside" of C. But in any *counterclockwise* cycle, the distance in $G^{\circlearrowright}$ from the external face to both sides of C are the same (by the way distances are defined in $G^{\circlearrowright}$). Thus C is clockwise.
2. Suppose C is a clockwise cycle. Consider the shortest path in $G^{\circlearrowright}$ from the external face to a face enclosed by C. From the Jordan curve theorem (Theorem 4.1.1 [12]), it must cross the cycle C. The edge dual to the crossing must be red. ◀

The definitions above, which apply only to subcubic plane graphs, can now be extended to a general plane graph G, by considering the subcubic graphs $\mathbf{Exp}^{\circlearrowright}(G)$ (and $\mathbf{Exp}^{\circlearrowleft}(G)$). But first, we must make a simple observation about $\mathbf{red}(\mathbf{Exp}^{\circlearrowright}(G))$ (respectively about $\mathbf{blue}(\mathbf{Exp}^{\circlearrowleft}(G))$).

▶ **Lemma 9.** *Let $v \in V(G)$ be a vertex of degree more than 3. Let C_v be the corresponding expanded cycle in $\mathbf{Exp}^{\circlearrowright}(G)$. Suppose at least one edge of C_v is white in $\mathbf{red}(\mathbf{Exp}^{\circlearrowright}(G))$ then there is a unique red cycle C that shares edges with C_v.*

Proof. First we note that C_v does not contain anything inside it since it is an expanded cycle. By lemma 8 we know that C_v has at least one red edge. Suppose it shares one or more edges with a red cycle R_1. Since both cycles are clockwise and C_v has nothing inside, the cycle R_1 must enclose C_v. Now suppose there is another red cycle R_2 that shares one or more edges with C_v. Then R_2 must also enclose C_v. But two cycles cannot enclose a cycle whilst sharing edges with it without touching each other, which contradicts the above lemma that all red cycles in a subcubic graph are vertex disjoint. ◀

The last two lemmas allow us to consistently contract the red cycles in $\mathbf{red}(\mathbf{Exp}^{\circlearrowright}(G))$:

▶ **Definition 10.** *The colored graph $\mathbf{Col}^{\circlearrowright}(G)$ (respectively, $\mathbf{Col}^{\circlearrowleft}(G)$) is obtained by labeling a degree more than 3 vertex $v \in V(G)$ as red iff the cycle C_v in $\mathbf{red}(\mathbf{Exp}^{\circlearrowright}(G))$ has at least one red edge and at least one white edge. Else the color of v is white. All the low degree vertices and edges of G inherit their colors from $\mathbf{red}(\mathbf{Exp}^{\circlearrowright}(G))$. The coloring of $\mathbf{Col}^{\circlearrowleft}(G)$ is similar.*

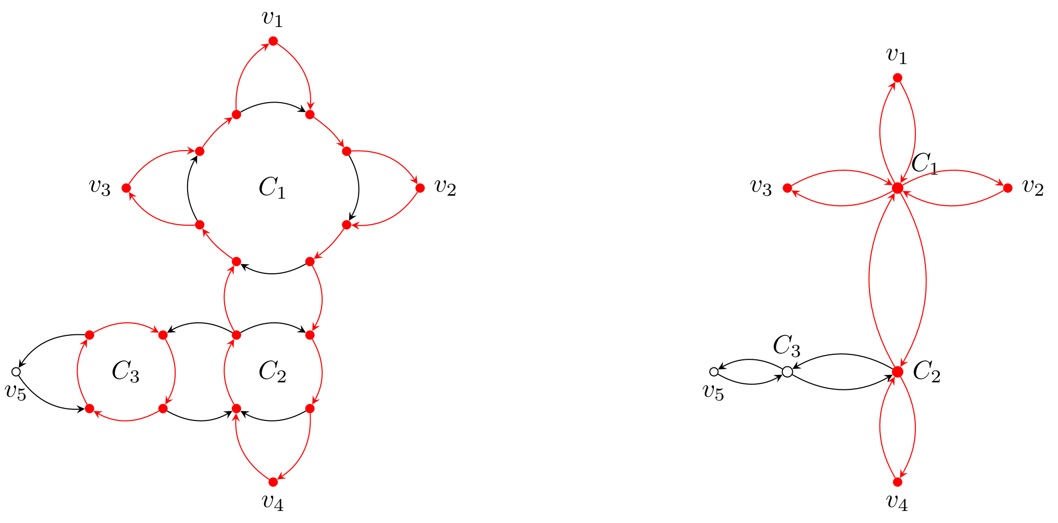

Figure 1 An example of contracting expanded cycles. The figure on right shows the graph after contracting the expanded cycles C_1, C_2, C_3 according to definition 10.

We can now characterize the colorings in the graph $\mathbf{Col}^{\circlearrowright}(G)$:

▶ **Lemma 11.** *The following hold:*
1. *A red cycle in* $\mathbf{Col}^{\circlearrowright}(G)$ *is vertex disjoint from every red cycle contained in its interior.*
2. *Every* 2*-connected component of the red subgraph of* $\mathbf{Col}^{\circlearrowright}(G)$ *is a simple clockwise cycle.*

We defer the proof to Section A.1.

Although the above lemmas have been proved for the clockwise dual, they also hold for counterclockwise dual with red replaced by blue.

4.2 Layering the colored graphs

▶ **Definition 12.** *Let* $x \in V(\mathbf{Col}^{\circlearrowright}(G)) \cup E(\mathbf{Col}^{\circlearrowright}(G))$. *Let* $\ell^{\circlearrowright}(x)$ *be one more than the minimum integer that occurs in* $\mathbf{type}^{\circlearrowright}(x')$, *for each* $x' \in V(\mathbf{Exp}^{\circlearrowright}(G)) \cup E(\mathbf{Exp}^{\circlearrowright}(G))$ *that is contracted to* x. *Further let* $\mathcal{L}^k(\mathbf{Col}^{\circlearrowright}(G)) = \{x \in V(\mathbf{Col}^{\circlearrowright}(G)) \cup E(\mathbf{Col}^{\circlearrowright}(G)) : \ell^{\circlearrowright}(x) = k\}$. *Similarly define,* $\ell^{\circlearrowleft}(x), \mathcal{L}^k(\mathbf{Col}^{\circlearrowleft}(G))$. *We call* $\mathcal{L}^k(\mathbf{Col}^{\circlearrowright}(G))$ *the* k^{th} *layer of the graph.*

See Fig 11 for an example. It is easy to see the following from Lemma 11:

▶ **Proposition 13.** *For every* $x \in V(\mathbf{Col}^{\circlearrowright}(G)) \cup E(\mathbf{Col}^{\circlearrowright}(G))$ *the quantity* $\ell^{\circlearrowright}(x)$ *is one more than the number of red cycles that strictly enclose* x *in* $\mathbf{Col}^{\circlearrowright}(G)$. *All the vertices and edges of a red cycle of* $\mathbf{Col}^{\circlearrowright}(G)$ *lie in the same layer* $\mathcal{L}^{k+1}(\mathbf{Col}^{\circlearrowright}(G))$ *for the enclosure depth* k *of the cycle.*

We had already noted above that the red subgraph of G had simple clockwise cycles as its biconnected components. We note a few more lemmas about the structure of a layer of G:

▶ **Lemma 14.** *We have:*
1. *A red cycle in a layer* $\mathcal{L}^{k+1}(\mathbf{Col}^{\circlearrowright}(G))$ *does not contain any vertex/edge of the same layer inside it.*
2. *Any clockwise cycle in a layer consists of only red vertices and edges.*

Dually, a blue cycle in a layer does not contain any vertex or edge of the same layer inside it.

▶ Remark 15. Notice that the conclusion in the second part of the lemma fails to hold if we allow cycles spanning more than one layer.

Proof. The first part is a direct consequence of proposition 13. For the second part we mimic the proof of the second part of Lemma 8. Consider a clockwise cycle $C \subseteq \mathcal{L}^{k+1}(\mathbf{Col}^{\circlearrowright}(G))$ that contains a white edge e. Every face adjacent to C from the outside must have $\mathbf{type}^{\circlearrowright} = k$ because C is contained in layer $k+1$. Then the $\mathbf{type}^{\circlearrowright}$ of the faces on either side of e is the same and therefore must be k. Let f be a face enclosed by C that has $\mathbf{type}^{\circlearrowright}(f) = k$. Thus it must be adjacent to a face of $\mathbf{type}^{\circlearrowright} = k-1$. But this contradicts that every face inside and adjacent to C must have $\mathbf{type}^{\circlearrowright}$ at least k. ◀

The lemmas above show that the strongly connected components of the red subgraph of a layer consist of red cycles touching each other without nesting, in a tree like structure. This prompts the following definition:

▶ **Definition 16.** *For a red cycle $R \subseteq \mathcal{L}^{k}(\mathbf{Col}^{\circlearrowright}(G))$ we denote by G_R, the graph induced by vertices of $\mathcal{L}^{k+1}(\mathbf{Col}^{\circlearrowright}(G))$ enclosed by R.*

Now we combine Definitions 10 and 12:

▶ **Definition 17.** *Each vertex or edge $x \in V(G) \cup E(G)$ gets a red layer number $k+1$ if it belongs to $\mathcal{L}^{k+1}(\mathbf{Col}^{\circlearrowright}(G))$ and a blue layer number $l+1$, if it belongs to $\mathcal{L}^{l+1}(\mathbf{Col}^{\circlearrowleft}(G_R))$ where $R \subseteq \mathcal{L}^{k}(\mathbf{Col}^{\circlearrowright}(G))$ is the red cycle immediately enclosing x.*

Moreover this defines the colored graph $\mathbf{Col}(G)$ by giving x the color red if it is red in $\mathbf{Col}^{\circlearrowright}(G)$ and/or blue in $\mathbf{Col}^{\circlearrowleft}(G_R)$ (notice it could be both red and blue) and lastly white if it is white in both the graphs. In this case, we say that x belongs to sublayer $\mathcal{L}^{k+1,l+1}(\mathbf{Col}(G))$.

By Proposition 13, we can also say that a sublayer $\mathcal{L}^{k+1,l+1}(\mathbf{Col}(G))$ thus consists of edges/vertices that are strictly enclosed inside k red cycles and inside l blue cycles that are contained *inside* the first enclosing red cycle.

We'll see some observations and lemmas regarding the structure of a sublayer now.

Since every edge/vertex in $\mathcal{L}^{k+1,l+1}(\mathbf{Col}(G))$ has the same red AND blue layer number, it is clear that there can be no nesting of colored cycles. Also we have:

▶ **Lemma 18.** *Every clockwise cycle in a sublayer $\mathcal{L}^{k+1,l+1}(\mathbf{Col}(G))$ consists of all red edges and vertices and any every counterclockwise cycle in the sublayer consists of all blue vertices and edges. (Some edges/vertices of the cycle can be both red as well as blue)*

Proof. This is a direct consequence of Lemma 14 applied to the sublayer $\mathcal{L}^{k+1,l+1}(\mathbf{Col}(G))$, which is a (counterclockwise) layer in graph G_R for some red cycle R. ◀

Thus we can refer to clockwise cycles and counterclockwise cycles as red and blue cycles respectively.

▶ **Definition 19.** *For a red or blue colored cycle C of layer $\mathcal{L}^{k,l}(\mathbf{Col}(G))$, we denote by G_C the graph induced by vertices of $\mathcal{L}^{k',l'}(\mathbf{Col}(G))$ enclosed by C, where $\{k', l'\}$ is $\{k+1, 1\}$ or $\{k, l+1\}$ according to whether C is a red or a blue cycle respectively.*

Note that:

▶ **Proposition 20.** *Two cycles of the same color in $\mathcal{L}^{k+1,l+1}(G)$ cannot share an edge.*

This is since neither is enclosed by the other as they belong to the same layer, and as they also have the same orientation. Cycles of different colors can share edges but we note:

▶ **Lemma 21.** *Two cycles of a sublayer $\mathcal{L}^{k+1,l+1}(\mathbf{Col}(G))$ can only share one contiguous segment of edges.*

Proof. Let a red cycle R and a blue cycle B in a sublayer share two vertices u, v but let the paths $R(u, v), B(u, v)$ in the two cycles be disjoint. Notice that the graph $(R \setminus R(u, v)) \cup B(u, v)$ is also a clockwise cycle that encloses the edges of $R(u, v)$ contradicting the first part of Lemma 14. ◀

We consider the strongly connected components of a sublayer and note the following lemmas regarding them:

▶ **Lemma 22.** *The trivial strongly connected components of a sublayer (those that consist of a single vertex) are white vertices. The non-trivial strongly connected components of a sublayer have the following properties:*
1. *Every vertex/edge in them is blue or red (possibly both).*
2. *Every face, except possibly the outer face, is a directed cycle.*
3. *Every face other than the outer face has at least one edge adjacent to the outer face.*

We defer the proof to Section A.1. The strongly connected components of a sublayer hence consist of intersecting red and blue facial cycles, with every face having at least one boundary edge adjacent to the outer face of the component.

▶ **Definition 23.** *We call the strongly connected components of a sublayer $\mathcal{L}(k, l)$* **meshes**.

5 Mesh Properties

▶ **Definition 24.** *Given a subgraph H of G embedded in the plane, we define the* closure *of H, denoted by $\widetilde{H}$, to be the induced graph on the vertices of H together with the vertices of G that lie in the interior of faces of H (except for the outer face of H).*

For convenience, we call a face of a graph that is not the outer face an *internal face*.

From Lemmas 18 and 22, we have a bijection: every face of a mesh, except possibly its outer face, is a directed cycle, and every directed cycle in a mesh is the boundary of a face of the mesh.

▶ **Definition 25.** *Let $0 < \alpha < 1$. An α* separator *of a digraph H that is a subgraph of a digraph G is a set of vertices of H whose removal from H separates $\widetilde{H}$ into subgraphs, where no strongly connected component has size greater than $\alpha|G|$. A* path separator *is a sequence of vertices $\langle v_1, \ldots, v_n \rangle$ that is a separator and also is a directed path.*

▶ **Definition 26.** *Let G be a graph and let M be a mesh in a sublayer G. For an internal face f of M, we define $wt(f)$ to be $|V(\widetilde{f})|$. Let $wt(H)$ where H is a subgraph of M be defined as $|V(\widetilde{H})|$.*

▶ **Definition 27.** *For a mesh M, we call a vertex that is adjacent to the outer face of M an* external vertex*, and a vertex that is not adjacent to the outer face an* internal vertex*. Also, we call vertices of degree more than two* junction vertices*.*

If $p = \langle v_1, v_2, \ldots, v_k \rangle$ is a directed path such that $v_2, \ldots, v_{k-1}$ are all vertices of degree two, but v_1, v_k have degree more than two, then we call p a **segment***. We call v_k the* out junction neighbour *of v_1 and v_1 the* in junction neighbour *of v_k.*

We call a **segment** *with all edges adjacent to the outer face an* external **segment***, and a* **segment** *with no edge adjacent to the outer face an* internal **segment***. If the end points of an* internal **segment** *are both internal vertices also, we call the segment an* **i-i-segment***.*

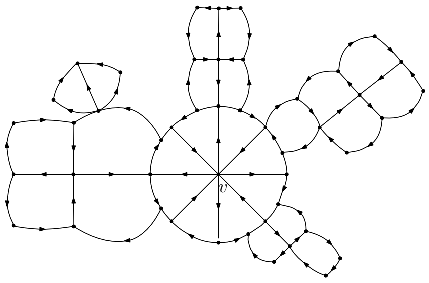

Figure 2 An example of a mesh.

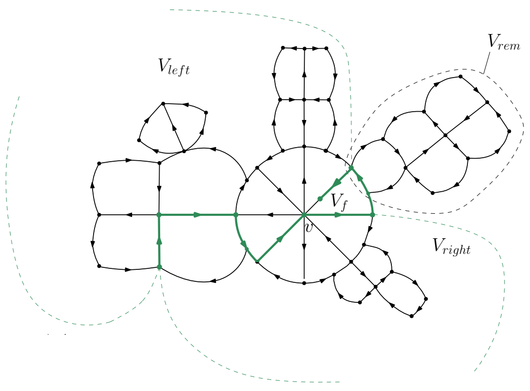

Figure 3 An example of a path separator. The vertex v is a central node, and the green path is a separator.

The rest of this section is devoted to a proof of the following, which asserts that we can construct a path separator in a mesh, assuming that no internal face of the mesh is too large.

▶ **Lemma 28.** *Suppose $wt(f) < wt(G)/12$ holds for every internal face f of a mesh M that is a subgraph of G. Then from any external vertex r of M, we can find (in $\mathsf{UL} \cap \mathsf{co\text{-}UL}$) an $\frac{11}{12}$ path separator of M, starting at r.*

The high level idea is that using a clique sum decomposition of $2, 3$-cliques (see figure 9) we find a "central" vertex v in the mesh M, such that we can find a path from the external vertex r to v, and then extend the path around one of the faces adjacent to v to get the path separator (all faces are directed cycles by lemma 22). Because every face touches the outer face and weight of every face is small by the hypothesis of the lemma, we can always find a face adjacent to v to encircle such that removing the path leaves no large (weakly) connected component. The vertices of M with degree two (in-degree 1 and out-degree 1 because M is strongly connected) are not important since they can be seen as just "subdivision" vertices. Now we will look at the structure of a mesh around an internal junction vertex, and the way the rest of the mesh is attached to that structure. Also, we state here that we will abuse the notion of 3-connected components by ignoring the non-junction vertices for convenience.

▶ **Lemma 29.** *If v is an internal junction vertex of a mesh and $e_1, \ldots, e_k$ are the edges adjacent to v in the cyclic order of embedding, then the edges alternate in directions i.e. if e_1 is outgoing from v, then e_2 is incoming to v and e_3 is outgoing and so on. Consequently, v has even degree (at least 4).*

▶ **Definition 30.** *Let v be an internal junction vertex of degree $2d$ in a* **mesh** *M, and let its junction neighbours be $(u_1, w_1, u_2, w_2, \ldots, u_d, w_d)$ in clockwise order starting from edge $\langle u_1, v\rangle$(the w_i's are out neighbours, and u_i's the in neighbours, since junction neighbours alternate).*

Every adjacent pair of edges incident to v borders a face that is not the outer face. Let $f_{u,v,w}$ denote the face bordered by v and the junction neighbours u and w of v which are adjacent in cyclic order around v. The boundary of $f_{u,v,w}$ can be written as three disjoint parts (except for endpoints), **segment** *(u, v) +* **segment** *(v, w) + $petal_{w,u}$, where the third part denotes a simple path from w to u along the face boundary. We will use the notation $petal_{w,u}$ to denote the corresponding boundary for any face $f_{u,v,w}$ adjacent to v. We define* **flower**(v) *as $\bigcup\{$vertices on boundary of faces adjacent to $v\}$(See figure 4).*

We note the following property of petals whose proof is deferred to AppendixA.2.

▶ **Proposition 31.** *For all adjacent junction neighbour pairs w_i, u_j of internal vertex v, $petal_{w_i,u_j}$ are disjoint, except possibly the end points.*

For an internal junction vertex v, the union of the petals around $\mathbf{flower}(v)$ thus form an undirected cycle around v, with at least four alternations in directions. Now we define bridges of the cycle, which roughly, are components of M we get after removing $\mathbf{flower}(v)$, leaving the points of attachment intact. We use the formal definition of bridges from [25]:

▶ **Definition 32.** *For a subgraph H of M, a* vertex of attachment *of H is a vertex of H that is incident with some edge of M not belonging to H. Let J be an undirected cycle of M. We define a* **bridge** *of J in M as a subgraph B of M with the following properties:*
1. *each vertex of attachment of B is a vertex of J.*
2. *B is not a subgraph of J.*
3. *no proper subgraph of B has both the above properties.*

We denote by **2-bridge**, *bridges with exactly two vertices of attachment to the specified cycle, and by* **3-bridge**, *bridges with three or more vertices of attachment.*

Note that for the cycle formed by petals of $\mathbf{flower}(v)$, the vertex v along with paths leading to/ coming from $\mathbf{flower}(v)$ also form a bridge, but we call that a trivial bridge and do not take it into consideration.

▶ **Lemma 33.**
1. *The vertices of attachment of a* **2-bridge** *of* $\mathbf{flower}(v)$ *must both lie on one* **petal** *of* $\mathbf{flower}(v)$.
2. *The vertices of attachment of a* **3-bridge** *of* $\mathbf{flower}(P)$ *can lie on one or, at most two adjacent petals. Moreover, in the latter case the junction neighbour of v common to both petals must be a vertex of attachment of the* **3-bridge**.
3. *For an internal vertex v, and an external vertex r of M, let $p = \langle r, \ldots, u_1, v\rangle$ be a simple path from r to v, where u_1 is an in junction neighbour of v. Let the other junction neighbours of v be named as in Definition 30 in cyclic order from u_1. For $j \in \{i, i+1\}$, consider an extended path of p, $p_{w_i,u_j} = \langle r, \ldots, u_1, v, w_i\rangle + petal_{w_i,u_j} + \langle u_j, \ldots, v\rangle$, excluding the last edge incident to v in the sequence. That is, p_{w_i,u_j} goes from r to v, then to an out junction neighbour w_i, and then wraps around f_{u_j,v,w_i} by taking $petal_{w_i,u_j}$ and then the segment back towards v from u_j. If there is a bridge of* $\mathbf{flower}(v)$ *of which u_1 is a point of attachment and also includes the edge of p incoming to u_1, we denote it by B_{in}. The set $V(\widetilde{M}) \setminus V(p_{w_i,u_j})$ can be partitioned into four disconnected parts, say V_{left} and V_{right}, V_f, V_{rem} such that:*

$$\begin{aligned} V_{left} =& (\{\widetilde{f}_{u_1,v,w_1} \cup \widetilde{f}_{u_2,v,w_1} \cup \widetilde{f}_{u_2,v,w_2} \ldots \cup \widetilde{f}_{u_i,v,w_{i-1}}\} \cup \{\widetilde{f}_{u_i,v,w_i} \textit{if } j = i+1\} \\ & \cup \{\textit{vertices in closure of } \textbf{bridges} \textit{ attached to the petals of these faces, excluding } B_{in}\} \\ & \cup \{\textit{the “left” part of } B_{in}.\textit{(See figure 8)} \}) \setminus V(p_{w_i,u_j}) \end{aligned}$$

$$\begin{aligned} V_{right} =& (\{\widetilde{f}_{u_i,v,w_{i+1}} \cup \widetilde{f}_{u_{i+2},v,w_{i+1}} \ldots \cup \widetilde{f}_{u_d,v,w_d}\} \cup \{\widetilde{f}_{u_{i+1},v,w_i} \textit{if } j = i\} \\ & \cup \{\textit{vertices in closure of } \textbf{bridges} \textit{ attached to petals petals these faces, excluding } B_{in}\} \\ & \cup \{\textit{the “right” part of } B_{in}.\textit{(See figure 8)} \} \setminus V(p_{w_i,u_j}) \end{aligned}$$

$$V_f = \widetilde{f}_{u_j,v,w_i} \setminus V(p_{w_i,u_j})$$

$$\begin{aligned} V_{rem} = & (\bigcup \{\textit{vertices in closure of all } \textbf{bridges} \textit{ that have vertices} \\ & \textit{of attachment only in } petal_{w_i,u_j}\}) \setminus V(p_{w_i,u_j}). \end{aligned}$$

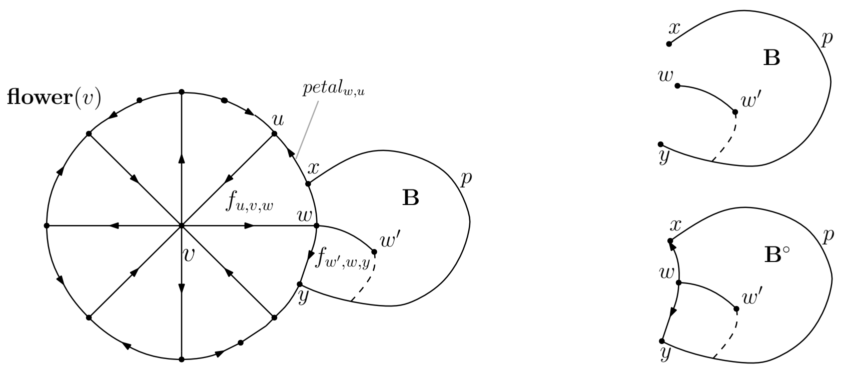

Figure 4 A vertex v and **flower**(v). B is a **bridge** with two points of attachment x, y on two different petals of **flower**(v). On the right are drawn the **bridge** B itself, and its closed version $\mathbf{B}^\circ$. The only way the boundary of $f_{w',w,y}$ can have an external edge is if it touches B, making w a point of attachment of B also.

such that there is no undirected path between any vertex of one of these four sets to any vertex of another. The path p_{w_i,u_i} is therefore a path separator that gives these components.

We introduce another notation for an extension of a bridge:

▶ **Definition 34.** *For a* **bridge** *B of* **flower**(v)*, we define* $\mathbf{B}^\circ$ *as B along with* **segments** *of* **flower**(v) *that lie between consecutive vertices of attachment of B. We call this the* **closed bridge** *of B.*

Now we will give definitions/lemmas regarding the "internal structure" of meshes, that will be useful to define the "center" of a mesh.

▶ **Definition 35.** *For a mesh M, we call its* **internal-skeleton***, denoted by $I(M)$, the induced subgraph on the vertices of* **i-i-segments** *of M.(See figure 6)*

▶ **Lemma 36.**
1. *For a* **mesh** *M, the graph $I(M)$ is a forest.*
2. *If H is a 3-connected induced subgraph of M(ignoring subdivision vertices), then $I(H)$ is a tree.*

We state a well-known proposition about a vertex separator in a tree T with weighted nodes, without the proof.

▶ **Proposition 37.** *Suppose T is a tree with each node having a weight assigned to it. Let $wt(T')$ denote sum of weights of each node in a subgraph T' of T. Then there exists a node v_c or a pair of adjacent nodes v_{c_1}, v_{c_2}, such that after removing it (or them in case of a pair), no connected component in the remaining forest has weight more than $\frac{1}{2}wt(T)$.*

We will next give a procedure to define a "center" of a mesh.

▶ **Definition 38.** *For a mesh M, let T_M denote the tree obtained by the 1, 2-clique sum decomposition of M. The nodes of T_M are of two types, clique nodes (cut vertices or separating pairs), and piece nodes, which are either 3-connected parts or cycles. Every piece node is adjacent to a clique node and vice-versa. (See [9, Section 3.1] for background about this decomposition.)*

Consider the $\frac{1}{2}$ separator node of T_M as described in Proposition 37. If it is a separating pair, a cut vertex, or a face cycle, we call that subgraph the **center** *of M.*

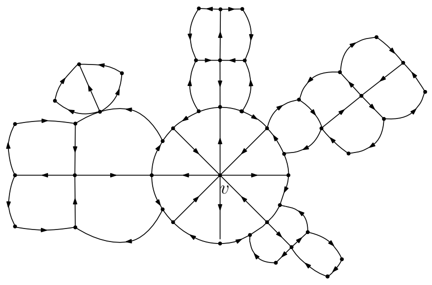

Figure 5 An example of a mesh.

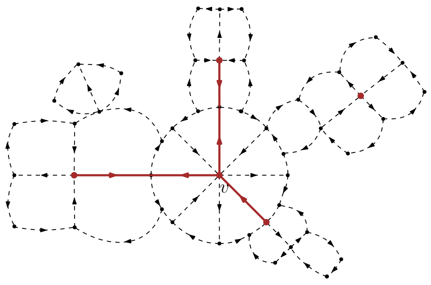

Figure 6 The internal skeleton of the mesh. One of its components is a single node.

If it is a 3-connected node P, look at its internal skeleton $I(P)$. We construct a new graph $I'(P)$ which is isomorphic to $I(P)$ but has edges directed differently. let u, v be two adjacent internal junction vertices of M. To give direction to a **segment** *(u, v) in $I'(P)$, we consider the unique* **bridge** *B of* **flower**(u) *that contains v as a point of attachment; we denote the* **closed bridge** *of B by $\mathbf{B}^{\circ}_{u}(v)$. $\mathbf{B}^{\circ}_{v}(u)$ is defined analogously. We orient (u, v) in the direction of the heavier of $\mathbf{B}^{\circ}_{u}(v)$ and $\mathbf{B}^{\circ}_{v}(u)$ (breaking ties arbitrarily), where the weights of $\mathbf{B}^{\circ}_{u}(v), \mathbf{B}^{\circ}_{v}(u)$ are $|\widetilde{\mathbf{B}^{\circ}_{u}(v)}|$ and $|\widetilde{\mathbf{B}^{\circ}_{v}(u)}|$, respectively.*

The **center** *of M is defined to be* **flower**(v) *in this case, where v is the sink node of $I'(P)$.*

We show why $I'(P)$ cannot have more than one sink.

▶ **Lemma 39.** *The tree $I'(P)$ defined above will have exactly one sink vertex.*

▶ **Lemma 40.** *If the* **center** *of M is* **flower**(v)*, and w is an out neighbor of v, then $wt(\mathbf{B}^{\circ}_{v}(w)) \leq \frac{1}{2}(wt(\widetilde{M} - wt(V_{rem}(u, w)))$, where u is either of the two in neighbors of v that are adjacent to w around* **flower**(v).

Proof. Since the **center** is **flower**(v), we have that $wt(\mathbf{B}^{\circ}_{v}(w)) \leq wt(\mathbf{B}^{\circ}_{w}(v))$. But $V_{rem}(u, w)$ has empty intersection with each of $\mathbf{B}^{\circ}_{v}(w)$ and $\mathbf{B}^{\circ}_{w}(v)$. Thus $wt(\mathbf{B}^{\circ}_{v}(w)) + wt(\mathbf{B}^{\circ}_{w}(v)) \leq wt(\widetilde{M}) - wt(V_{rem}(u, w))$. The lemma follows. ◀

▶ **Lemma 41.**

1. *If the* **center** *of M is not of the form* **flower**(v) *where v is an internal node of a 3-connected component, then removing it from $\widetilde{M}$ disconnects $\widetilde{M}$ into weakly connected components, each with weight less than $\frac{1}{2}wt(\widetilde{M})$.*
2. *If the* **center** *of M is* **flower**(v) *for an internal node v of a 3-connected component P, then on removing* **flower**(v) *from $\widetilde{M}$, no weakly connected component has weight more than $\frac{1}{2}wt(\widetilde{M})$.*

Proof.

1. This follows from the vertex separator lemma for trees with weighted vertices.
2. This follows from the v being the sink node of $I'(P)$. ◀

▶ **Lemma 42.** *For every possible path p_{w_i,u_j} around v as defined in Lemma 33, V_{rem} consists of a disjoint union of weakly-connected components, each of which has weight $\leq \frac{1}{2}(wt(M))$.*

For a path p_{w_i,u_j} (where $j \in \{i, i+1\}$) we sometimes use the notation $V_{rem}(w_i, u_j)$ to specify the petal where the bridges of V_{rem} are attached.

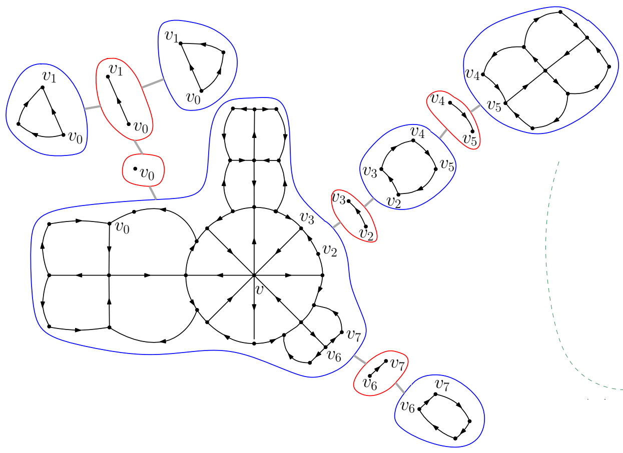

Figure 7 The tree decomposition of the mesh using 1,2-clique sums. The nodes encircled red are clique separator nodes.

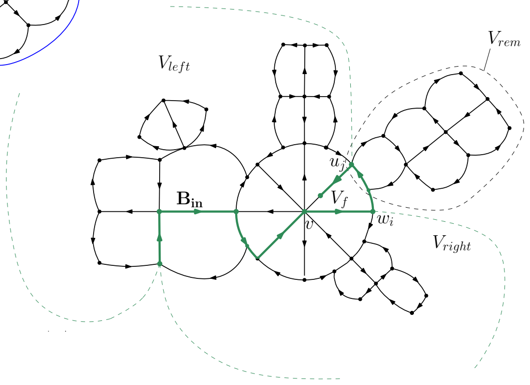

Figure 8 An example of a path separator. The vertex v is a central node, and the green path is a separator.

5.1 Mesh Separator Algorithm

Now we give the algorithm to find an α separator in a mesh $M(G)$, assuming the hypothesis of Lemma 28.

1. Find the decomposition tree, T_M of M with 2-cliques and 1-cliques as the separating sets.
2. Find the **center** of the mesh M. It will either be a cut vertex, a separating pair, a cycle, or **flower**(v) for some internal vertex v.
3. If it is a cut vertex, we just find a path from the root r to it. If it is a separating pair (u, v), both the vertices must lie on a same face, which is a directed cycle. In both this case, and also the case in which the **center** is a cycle, find a path from the root to any vertex of the face that touches it the first time, and then extend the path by encircling the cycle.
4. If it is **flower**(v) for some internal vertex v, find a path $p = \langle r, \dots, u_1, v\rangle$ to v. Let the junction neighbours of v in clockwise order starting from (u_1, v), be $w_1, u_2, w_2, \dots, w_d$, with the w's being out junction neighbours and the u's being in junction neighbours. Starting clockwise from segment $\langle u, v\rangle$, find the first index i and $j \in \{i, i+1\}$ s.t. after removing the extended path p_{w_i,u_j}, (defined in Lemma 33) the remaining strongly connected components are smaller than $\frac{11}{12}wt(G)$.

The algorithm above can clearly be implemented in logspace with an oracle for planar reachability, and thus it can be implemented in $\mathsf{UL} \cap \mathsf{co\text{-}UL}$.

It remains to show that the "first i" mentioned in the final step actually exists. For the proof see Appendix A.3:

▶ **Lemma 43.** *If the* **center** *of M is* **flower**(v) *for some internal vertex v, then there will always exist an adjacent face f_{u_i,v,w_i} s.t. the path p_{w_i,u_i} is a $\frac{11}{12}$-separator.*

6 Path separator in a planar digraph

Having seen how to construct a path separator in a **mesh**, we now show how to use that to construct an $\frac{11}{12}$ separator in any planar digraph.

1. Given a graph G, first embed the graph so that the root r lies on the outer face. Through the root, draw a virtual directed cycle C_0 that encloses the entire graph, and orient it, say clockwise. Find the layering described in Section 4 and output it on a transducer. Cycle C_0 will be colored red and will be in the sublayer $(0, 0)$.

2. In the laminar family of red/blue cycles, find the cycle C s.t. $wt(C)$ is more than $|G|/12$, but no colored cycle C' in the interior of C has the same property. Such a cycle will clearly exist (it could be the virtual cycle C_0). Let the sublayer of C be (k, l).
3. Find a path p from the root r to any vertex r_C of the cycle C such that no other vertex of C is in the path. As seen above in Lemma 22, the graph in the interior of C and belonging to the immediately next sublayer ($(k+1, l)$ if C is clockwise and $(k, l+1)$ if C is counter-clockwise) is a DAG of meshes. There are two cases possible:
   a. The graph $\widetilde{C}$ has no strongly connected components of weight larger than $|G|/12$. In this case we simply extend the path p from r_C by encircling the cycle C till the last vertex and stop.
   b. The graph $\widetilde{C}$ has a strongly connected component of weight more than $|G|/12$. In this case, we extend p from r_C by encircling C till the last vertex u on C that can reach any such component M_C. Then extend the path from u to any vertex of M_C and apply the mesh separator lemma (Lemma 28) to obtain the desired separator. (Observe that M_C satisfies the hypothesis of Lemma 28.)

▶ **Lemma 44.** *The path p obtained by the above procedure is an $\frac{11}{12}$ separator.*

Proof. We look at the two cases:
1. In this case it is clear that the interior and exterior of cycle C are disconnected by p. The exterior of C has size $\leq \frac{11}{12}|G|$ (by definition of C), and in its interior every strongly-connected component has weight at most $|G|/12$. Thus this satisfies the definition of an $\frac{11}{12}$ separator.
2. We took the last edge in C from r_C that can reach the mesh M_C, and extended the path to M_C. Thus after removing p, one weakly-connected component consists of the exterior of G, along with (possibly) some vertices in the interior of C that cannot reach any "large" mesh in the interior. Since M_C has weight greater than $\frac{1}{12}|G|$, no strongly-connected component embedded outside of M_C can have weight more than $\frac{11}{12}|G|$. Also, after removing path p, Lemma 28 guarantees that no other strongly-connected component will have weight more than $\frac{11}{12}|G|$. Thus this is an $\frac{11}{12}$ separator.

   Hence overall we can guarantee an $\frac{11}{12}$ path separator in G. ◀

7 Building a DFS tree using path separators

We give a recursive divide and conquer algorithm for DFS:
1. Given a planar drawing of G and a root vertex on the outer face r, find an $\frac{11}{12}$ path separator $p = \langle r, v_1, v_2..v_k \rangle$. Path p is included in the DFS tree.
2. Let $R(v)$ denote the set of vertices of G reachable from v. Now for every vertex v_i in p compute in parallel: $R'(v_i) = R(v) \backslash (\bigcup_{j=i+1}^{k} R(v_j))$ Our DFS will correspond to first traveling along p to v_k, doing DFS on $R(v_k)$, and then while backtracking on p, do DFS on $R'(v_i)$ for i from $k-1$ downto 1. Given G, the encodings of p and $R'(v_i)$ can all be computed in $\mathsf{AC}^0(\mathsf{UL} \cap \mathsf{co\text{-}UL})$.
3. For any v_i, $R'(v_i)$ can be written as a DAG of SCCs (strongly connected components), where each SCC is smaller than $\frac{11}{12}|G|$. In $\mathsf{AC}^0(\mathsf{UL} \cap \mathsf{co\text{-}UL})$ we can compute this DAG and we can compute an encoding of the tuple (i, M, v) where M is a SCC in $R'(i)$ and v is a vertex in M. Recursively, in parallel, we compute a DFS tree of M for each tuple (i, M, v), using v is the root. Now we need to show how to sew together (some of) these trees, to form a DFS tree for G with root r.

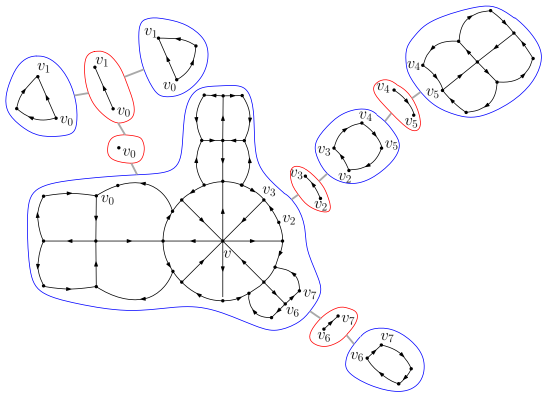

Figure 9 The tree decomposition of the mesh using 1,2-clique sums. The nodes encircled red are clique separator nodes.

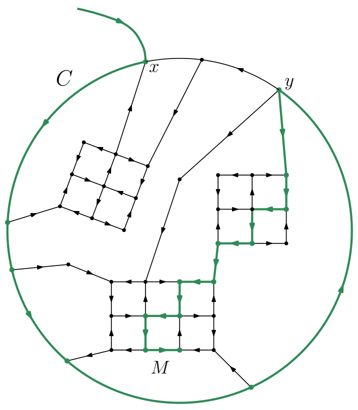

Figure 10 The cycle C is a cycle satisfying the property stated in step 2 of the algorithm. The mesh M in the next sublayer is heavy, so we find a path from the last vertex on C that can reach M (in this case y), and then apply the algorithm of previous section on M.

4. Given a triple (i, M, v), let $x_0, x_1, \ldots, x_r$ be the order in which the vertices of M appear in a DFS traversal where the root $x_0 = v$. Our DFS will correspond to first following the edges from x_0 that lead to other SCCs in $R(v_i)$. (No vertex reachable in this way can reach any x_j, or else that vertex would also be in M.) And then we will move on to x_1 and repeat the process, etc. Thus let $R''_{i,M,v}(x_j) = (R'(x_j)\backslash M)\backslash(\bigcup_{k<j} R'(x_k))$.
   Our DFS tree is composed by computing a DFS tree T of the DAG of meshes (considering each mesh to be a vertex) using the algorithm of Section 3. A logspace machine can do a DFS traversal of T, starting with the node containing v_i as the root, and using (as auxiliary information) the DFS tree that was computed for (i, M, v_i). If this traversal contains an edge (M, M') (where M and M' are SCCs in $R'(v_i)$), then there is exactly one j such that there is an edge from x_j in the DFS tree for (i, M, v_i) to a vertex (call it $v_{M'}$) in $M' \cap R''_{i,M,v}(x_j)$. [Namely, x_j is the first vertex in this tree that has an edge to M'.] The edge from x_j to $v_{M'}$ will be in our DFS tree, as will the DFS tree that was computed for $(i, M', v_{M'})$. We then continue the traversal of T, and process each node of the DAG in the same way. All of this can be accomplished in $\mathsf{AC}^0(\mathsf{UL} \cap \mathsf{co\text{-}UL})$.
5. The final DFS tree for R_i consists of all of the edges that appear in the trees for tuples (i, M, v) that were utilized in the traversal of T. The tree for G consists of p together with the trees for each R_i.

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A Appendix: Omitted details

A.1 Proofs from Section 4

We first need a simple lemma:

▶ **Lemma 45.** *Suppose (f_1, f_2) is a dual edge with weight 1 (and (f_2, f_1) is of weight 0) then,* $d^{\circlearrowright}(f_0, f_1) \leq d^{\circlearrowright}(f_0, f_2) \leq d^{\circlearrowright}(f_0, f_1) + 1$.

Proof. From the triangle inequality $d^{\circlearrowright}(f_0, f_1) \leq d^{\circlearrowright}(f_0, f_2) + d^{\circlearrowright}(f_2, f_1) = d^{\circlearrowright}(f_0, f_2)$. Similarly, $d^{\circlearrowright}(f_0, f_2) \leq d^{\circlearrowright}(f_0, f_1) + d^{\circlearrowright}(f_1, f_2) \leq d^{\circlearrowright}(f_0, f_1) + 1$. ◀

Proof of Lemma 6. Since each vertex $v \in V(G)$ of a subcubic graph is incident on at most 3 faces the only case in which $|\mathbf{type}^{\circlearrowright}(v)| > 2$ corresponds to three distinct faces f_1, f_2, f_3 being incident on a vertex. But here the undirected dual edges form a triangle such that in the directed dual the edges with weight 1 are oriented either as a cycle or acyclically. In the former case by three applications of the first half of Lemma 45 we get that $d^{\circlearrowright}(f_0, f_1) \leq d^{\circlearrowright}(f_0, f_2) \leq d^{\circlearrowright}(f_0, f_3) \leq d^{\circlearrowright}(f_0, f_1)$, hence all 3 distances are the same. Therefore $|\mathbf{type}^{\circlearrowright}(v)| = 1$.

In the latter case, suppose the edges of weight 1 are $(f_1, f_2), (f_2, f_3), (f_1, f_3)$, then by Lemma 45 we get: $d^{\circlearrowright}(f_0, f_1) \leq d^{\circlearrowright}(f_0, f_2), d^{\circlearrowright}(f_0, f_3) \leq d^{\circlearrowright}(f_0, f_1) + 1$. Thus, both $d^{\circlearrowright}(f_0, f_2), d^{\circlearrowright}(f_0, f_3)$ are sandwiched between two consecutive values $d^{\circlearrowright}(f_0, f_1), d^{\circlearrowright}(f_0, f_1)+1$. Hence $d^{\circlearrowright}(f_0, f_1)$, $d^{\circlearrowright}(f_0, f_2)$, $d^{\circlearrowright}(f_0, f_3)$ must take at most two distinct values, and thus $|\mathbf{type}^{\circlearrowright}(v)| \leq 2$. Moreover either $\mathbf{type}^{\circlearrowright}(f_1) \neq \mathbf{type}^{\circlearrowright}(f_2) = \mathbf{type}^{\circlearrowright}(f_3)$ or $\mathbf{type}^{\circlearrowright}(f_1) = \mathbf{type}^{\circlearrowright}(f_2) \neq \mathbf{type}^{\circlearrowright}(f_3)$. Let e_1, e_2, e_3 be such that, ${e_1}^{\circlearrowright} = (f_2, f_3), {e_2}^{\circlearrowright} = (f_1, f_3), {e_3}^{\circlearrowright} = (f_1, f_2)$. Then the two cases correspond to $|\mathbf{type}^{\circlearrowright}(e_1)| = |\mathbf{type}^{\circlearrowright}(e_2)| = 2, |\mathbf{type}^{\circlearrowright}(e_3)| = 1$ and to $|\mathbf{type}^{\circlearrowright}(e_1)| = 1, |\mathbf{type}^{\circlearrowright}(e_2)| = |\mathbf{type}^{\circlearrowright}(e_3)| = 2$ respectively. Noticing that e_1, e_3 are both incoming or both outgoing edges of v completes the proof for the clockwise case. The proof for the counterclockwise case is formally identical. ◀

Proof of Lemma 11. For $v \in V(G)$, let $C_v \subseteq \mathbf{Exp}^{\circlearrowright}(G)$ be the expanded cycle. If it has a red vertex it is immediately enclosed by a unique red cycle R in $\mathbf{Exp}^{\circlearrowright}(G)$ by Lemma 9. Assuming C_v is not all red, it consists of alternating red subpaths and white subpaths. On contracting C_v we get a collection of clockwise red cycles outside sharing a common cut-vertex v. Notice that the new collection of red cycles consists of edges that R did not share with C_v. Also notice that (as a thought experiment) if we contracted the C_v's that share a vertex with R, one at a time we would get an edge-disjoint set of red cycles with distinct cut vertices. Therefore, in $\mathbf{Col}^{\circlearrowright}(G)$, the red subgraph consists of a collection of connected components, each of which is a remnant of exactly one red cycle in $\mathbf{Exp}^{\circlearrowright}(G)$; these connected components consist of red cycles that touch externally at cut vertices. Hence both parts of the lemma follow. ◀

Proof of Lemma 22.

1. In a non-trivial strongly connected graph every vertex and edge lies on a cycle and therefore by Lemma 18 must be colored red or blue (or both).
2. Suppose there is a face f the boundary of which is not a directed cycle. Look at a directed dual (say clockwise) of the strongly connected component (just the component independently). This dual must be a DAG since the primal is strongly connected. The vertex f^* in the dual corresponding to face f of the strongly connected component has in degree at least one and out degree at least one since it has boundary edges of both orientations, hence the edges adjacent to f^* do not form a directed cut of the dual.
   Let o^* denote the dual vertex corresponding to the outer face of the SCC. In order to prove the claim, it is sufficient to show the existence of a directed cut C^* that separates f^* and o^*, since it would imply by cut cycle duality that there is a directed cycle C in the primal SCC that encloses the face f w.r.t the outer face and since the boundary of f is not a directed cycle, C must strictly enclose at least one edge of the boundary of f contradicting Lemma 14. To see the cut, consider a topological sort ordering of the dual (it is a DAG). Let the number of a dual vertex v^* in the ordering be denoted by $n(v^*)$. W.l.o.g, let $n(f^*) < n(o^*)$. Consider the partition of the dual vertices:

   $$A = \{v^* \mid n(v^*) \leq n(f^*)\},\ B = \{v^* \mid n(v^*) > n(f^*)\}$$

   By definition of topological sort, all edges across this partition must be directed from A to B, hence it is a directed cut, and therefore it must also contain a subset which is a minimal directed cut. But clearly the minimal cut is not the set of edges adjacent to f^* since it has both out and in degree at least one, hence proving the claim. Hence every face in the SCC of a sublayer must be a directed (hence colored) cycle (by Lemma 18).
3. Let H be an SCC of the sublayer. We observed from the proof above that no vertex in the dual of H, except possibly the vertex corresponding to the outer face of H, can have both in degree and out degree more than one. (i.e. every dual vertex, except the outer face is a source or a sink). Therefore if any dual vertex f^* has a directed path to o^* or vice versa, then the path must be an edge and we are done. Suppose there is no directed path from f^* to o^* and w.l.o.g. let f^* be a source. Consider the trivial directed cut C_1:

   $$A = \{f^*\},\ B = V(H) \setminus A$$

   This is a cut since there are no edges from B to A, and this cut clearly corresponds to the directed cycle which is the boundary of face f in H.
   Now consider the cut C_2:

   $$A' = \{v^* \mid v^* \text{is reachable from } f^*\},\ B' = V(H) \setminus A'$$

   Clearly this is a f^*-o^* cut with no edge from a vertex in A' to a vertex in B' and $o^* \in B'$. But this f^*-o^* cut is different from C_1 since f^* is a source vertex and hence A' has at least one more vertex than just f^*. Hence this corresponds to a directed cycle in H that strictly encloses at least some edge of f, and we again get a contradiction of Lemma 14. ◀

A.2 Details for Section 5

Proof. (of Lemma 29 Let e_i, e_{i+1} be two edges adjacent to v, that are also adjacent in the cyclic order of the drawing. Since they are adjacent in the drawing, they must enclose between them, a region, and hence a face, which is not the outer face. But the boundary of

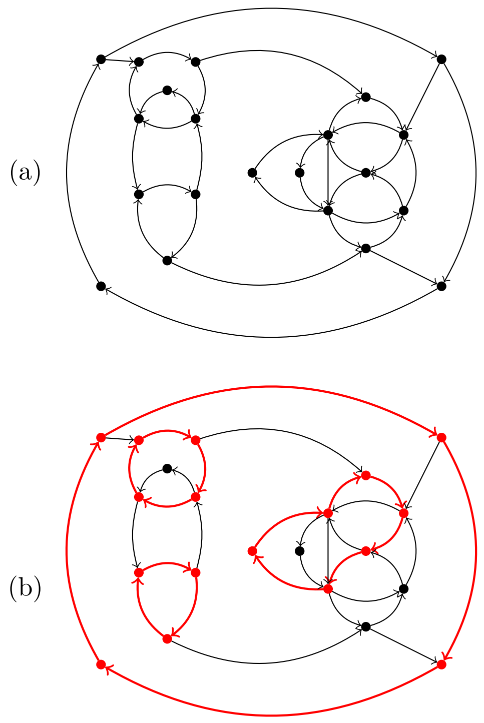

Figure 11 Figure (a) is a graph G. Figure (b) is the graph in (a) after labelling red edges using clockwise dual. We omit the cycle expansion and contraction procedure here.

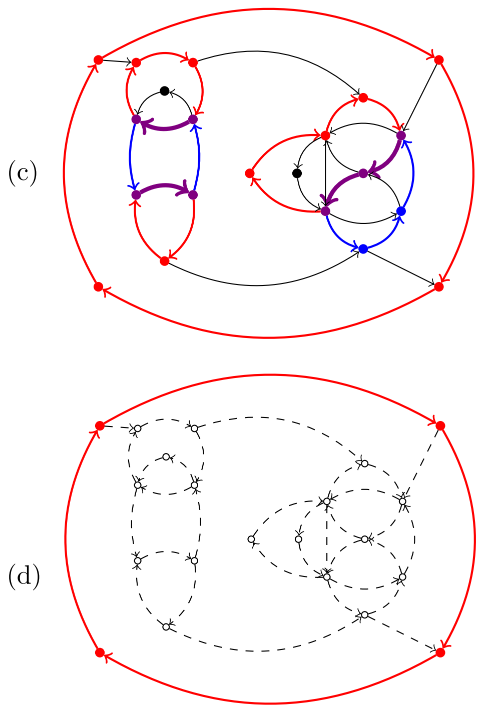

Figure 12 Figure (c) shows G after applying blue labellings to each red layer we obtained in the previous figure. The vertices and edges colored purple are those that are red as well as blue. Figure (d) represents the sublayer $(1, 1)$. The dashed edges and empty vertices are not part of the layer.

every non-outer face in a mesh is a directed cycle, hence v, e_i, e_{i+1} lie on a directed cycle, with both edges adjacent to v. Hence one of them must be an out edge from v, and the other incident towards v. ◀

Proof of Proposition 31. Petals of two faces must be internally disjoint because the corresponding faces share the vertex v and two faces cannot have a non-contiguous intersection, by Lemma 21. ◀

Proof of Lemma 33.

1. Let x, y be the two vertices of attachment of the **2-bridge** B on **flower**(v). Since bridges are connected graphs without the edges of the corresponding cycle(by 3^{rd} property of definition 32), there must be an undirected path, p in the **bridge** connecting x, y, without using any edge of **flower**(v). If x and y were *not* on the same petal, then this path along with the other petals in **flower**(v), must clearly enclose a junction neighbour of v, say w (see Figure 4). Thus w is not adjacent to the outer face. Now since w is an internal junction vertex, and two of its adjacent faces are also adjacent to v, look at another face f adjacent to w and not adjacent to v. (Internal junction vertices have at least four adjacent faces.) The boundary of this face cannot touch B since that would make it a part of B and consequently w a vertex of attachment of B to **flower**(v). Therefore the boundary of f is enclosed within the paths p and the part of **flower**(v) that is also enclosed by p. Therefore f has no external edge, contradicting Lemma 22.

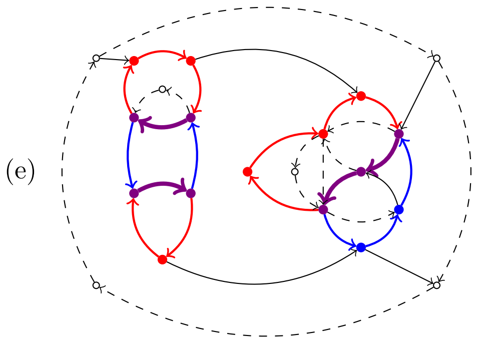

Figure 13 Figure (e) figure represents the sublayer $(2, 1)$.

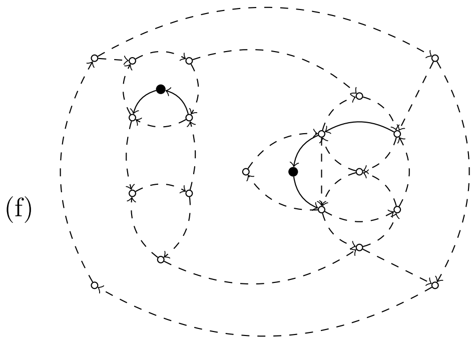

Figure 14 Figure (f) represents the sublayer $(3, 1)$.

2. Let $x_1, x_2, \ldots, x_k$ be the vertices of attachment of the bridge B on **flower**(v), in the cyclic order of boundary of **flower**(v). Clearly if the vertices of attachment lie on more than two petals of v, then at least one petal will be completely enclosed by B, which is not possible since every petal must have at least one external edge. Lets say they lie on two adjacent petals, and the junction neighbour common to both of them is w. By the same argument as above, w must have an edge other than those of adjacent petals of v, that connect it to B. Therefore w must be a vertex of attachment of B to **flower**(v).
3. First we note that $petal_{w_i,u_j}$ will have an external vertex in it since the boundary of every face has at least one external vertex (Lemma 22), and segments (u_j, v) and (v, w_i) are internal. Let z be an external vertex on $petal_{w_i,u_j}$. The path p starts at external vertex r, comes to u_1, v, w_i, and reaches external vertex z on its way back to v. It will clearly divide $\widetilde{M}$ into at least two parts by Jordan Curve theorem. Since p_{w_i,u_j} is just a wrap around the face f_{u_j,v,w_i} after z, is clear that since $w_1, u_2, \ldots, w_{i-1}$ and everything connected to them after removing p lie in one region, which we call V_{left}, and $w_{i+1}, u_{i+2}, \ldots, w_d$ and everything connected to them after removing p lie in another, and vertices of $\widetilde{f}_{u,v,w}$ lie in another disconnected region since p wraps around $f_{u,v,w}$. ◀

Proof of Lemma 36.

1. Suppose there were an undirected cycle in M of all internal segments, then this cycle must enclose a face whose boundaries are also all internal segments. This contradicts Lemma 22 as it states that every face must have at least one external edge, and hence segment. Hence there can be no cycle (directed or undirected) consisting of all internal segments, and consequently, no cycle (directed or undirected) of all internal vertices.
2. Let H be a 3-connected induced subgraph of M. By definition, $I(H)$ is obtained from M by removing all external edges and external non-junction vertices. Suppose $I(H)$ is not a tree, and hence consists of two or more disconnected trees. Let T_1 and T_2 be any two trees in $I(H)$. Let x be a vertex in T_1 and y be a vertex in T_2. Since H is 3-connected, there must be at least three disjoint paths(undirected) between x and y. Clearly in a planar graph, if there are three disjoint paths between two vertices, one of the paths must be strictly enclosed in the closed region formed by other two. Therefore there must a path between x and y that is strictly enclosed inside the boundary of H, and hence does not contain any edge or vertex adjacent to the outer face of H. Hence x and y cannot become disconnected after removing external edges and external non-junction vertices leading to a contradiction that $I(H)$ is disconnected. Therefore $I(H)$ must be a tree. ◀

Proof of Lemma 39. Suppose $I'(P)$ has two junction vertices x and y that are sinks. They cannot be adjacent, so consider the unique undirected path in $I'(P)$ between x and y. There must be a source z on the path. Let neighbours of z be x', y', lying on the path from x to z and from z to y respectively.

Let $\mathbf{B}^{\circ}_{z}(x')$ and $\mathbf{B}^{\circ}_{z}(y')$ denote the **bridges** of **flower**(z) with points of attachments x' and y' respectively. Then by the orientations of the edges we have: $|\mathbf{B}^{\circ}_{z}(x')| \geq |\mathbf{B}^{\circ}_{x'}(z)|$ which gives$|\mathbf{B}^{\circ}_{z}(x')| > |\mathbf{B}^{\circ}_{z}(y')|$since $\mathbf{B}^{\circ}_{z}(y')$ is clearly a proper subgraph of $\mathbf{B}^{\circ}_{x'}(z)$ and $|\mathbf{B}^{\circ}_{z}(y')| \geq |\mathbf{B}^{\circ}_{y'}(z)|$ which gives$|\mathbf{B}^{\circ}_{z}(y')| > |\mathbf{B}^{\circ}_{z}(x')|$ which is clearly a contradiction. ◀

Proof of Lemma 42. A (weakly connected) component of V_{rem} is a bridge, attached to $petal_{w_i,u_i}$ or to $petal_{w_i,u_{i+1}}$ via its vertices of attachment. In the clique sum decomposition, these vertices of attachment will always contain a 1 or 2 separating clique, since if a bridge is attached to a petal via three or more nodes, the first and the last vertices of attachment form a separating pair that separates the bridge from **flower**(v). Hence it is a branch remaining in T_M after removing the $3-connected$ piece node that is central in T_M. Since every branch after removal of the central piece of T_M has weight $\leq \frac{1}{2}(wt(M))$, every (weakly) connected component of V_{rem} has weight $\leq \frac{1}{2}(wt(M))$. ◀

A.3 Mesh Separator Algorithm

Proof of Lemma 43. We have the following two cases

1. For some i and $j \in \{i, i+1\}$, p_{w_i,u_j}, $wt(V_{rem}(w_i, u_j)) \geq \frac{1}{2}wt(M)$.
   Then by Lemma 42, p_{w_i,u_j} separates $V_{rem}(w_i, u_j)$ from the rest of the graph, and also every weakly connected component in $V_{rem}(w_i, u_j)$ has weight $\leq \frac{1}{2}wt(M)$. Hence every weakly connected component in M after removing p_{w_i,u_j} has weight $\leq \frac{1}{2}wt(M)$.
2. For every $p_{w_i,u_j}, wt(V_{rem}(w_i, u_j)) \leq \frac{1}{2}wt(M)$.
   We know that for any index i and $j \in \{i, i+1\}$, if $f = f_{u_j,v,w_i}$, then $wt(V_f) \leq wt(G)/12$ by the hypothesis of Lemma 28. Starting clockwise from p_{u_1,w_1}, at first V_{left} is small, and on shifting from p_{w_i,u_i} to $p_{w_i,u_{i+1}}$ or from $p_{w_i,u_{i+1}}$ to $p_{w_{i+1},u_{i+1}}$, the increase in V_{left} is bounded above by $wt(V_f) + wt(V_{rem}(w_i, u_j))$ + wt$(\mathbf{B}^{\circ}_{v}(w_i))$. Recall that
   a. $wt(V_f) \leq wt(G)/12$ (by the hypothesis of Lemma 28).
   b. $wt(V_{rem}(w_i, u_j)) \leq \frac{1}{2}wt(M)$ (by hypothesis for this case).
   c. $wt(\mathbf{B}^{\circ}_{v}(w_i)) \leq \frac{1}{2}(wt(M) - wt(V_{rem}(w_i, u_j)))$ (by Lemma 40).

   Thus the addition to V_{left} in each iteration is $\leq \frac{1}{12}wt(G) + wt(V_{rem}(w_i, u_j)) + \frac{1}{2}(wt(M)) - \frac{1}{2}(wt(V_{rem}(w_i, u_j))))$, which is equal to $\frac{1}{12}wt(G) + \frac{1}{2}wt(V_{rem}(w_i, u_i)) + \frac{1}{2}(wt(M)) \leq \frac{1}{12}wtG + \frac{3}{4}wt(M)$. Thus we can stop the first time $wt(V_{left})$ is greater than $wt(G)/12$. So, we have $wt(V_{left}) \leq \frac{2}{12}wt(G) + \frac{3}{4}wt(M) \leq \frac{11}{12}wt(G)$, and $wt(V_{right}) \leq \frac{11}{12}wt(M)$, and $wt(V_f) \leq \frac{1}{12}wt(M)$, and $wt(v_{rem}) \leq \frac{1}{2}wt(M)$. Thus we have an upper bound of $\frac{11}{12}wt(G)$ on all the disconnected components. Hence p_{x_i,w_i} is a $\frac{11}{12}$ path separator. ◀