Graph Characterization of the Universal Theory of Relations

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Abstract

The equational theory of relations can be characterized using graphs and homomorphisms. This result, found independently by Freyd and Scedrov and by Andréka and Bredikhin, shows that the equational theory of relations is decidable. In this paper, we extend this characterization to the whole universal first-order theory of relations. Using our characterization, we show that the positive universal fragment is also decidable.

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1 Introduction

Binary relations are a versatile mathematical object, used to model graphs, programs, databases, etc. It is then a natural task to understand the laws governing them. Since the seminal work of Tarski [14], this task has occupied researchers for several decades [15, 12, 3, 9, 11, 10, 2, 4, 13].

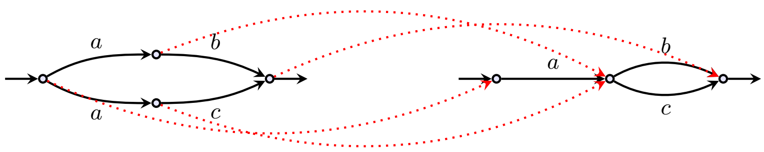
Relations usually come with a certain number of standard operations: union $\cup$, intersection $\cap$, composition $\cdot$, converse $^\circ$ etc. We are interested in containment between terms built with these operations with respect to their relational interpretations. When a containment between two terms t and u holds, we say that $t \geq u$ is a *valid inequation for relations* and write $\mathcal{R}el \models t \geq u$. For instance, an emblematic valid inequation is the following one:

$$(a \cdot b) \cap (a \cdot c) \geq a \cdot (b \cap c)$$

This law is valid because no matter how we interpret the letters a, b and c as relations, the relation denoted by the term $a \cdot (b \cap c)$ will be contained in the relation denoted by the term $(a \cdot b) \cap (a \cdot c)$. A very simple way to check that this inequation is valid relies on the following characterization ([1, Thm. 1], [7, p. 208]):

$$\mathcal{R}el \models t \geq u \qquad \Leftrightarrow \qquad \mathcal{G}(t) \triangleright \mathcal{G}(u) \qquad (\star)$$

In this theorem, $\mathcal{G}(t)$ and $\mathcal{G}(u)$ are finite graphs associated to the terms t and u respectively, and $\triangleright$ denotes the existence of a *graph homomorphism*. For example, the validity of the law above is witnessed by this homomorphism (in red) from the graph of $(a \cdot b) \cap (a \cdot c)$ to the graph of $a \cdot (b \cap c)$:



Using this characterization, we can show that relational validity of inequations is decidable.

As maybe noticed by the reader, inequations are implicitly universally quantified. They actually form a fragment of the more general *universal first-order* formulas. The latter comprises *universal positive formulas* which are basically disjunctions of inequations, and *Horn formulas* which are implications between inequations.

Universal first-order formulas have received a lot of attention in the model theory community. They enjoy for example the Łoś–Tarski theorem [8, Thm.5.4.4], which states that the set of universal first-order formulas is exactly the set of first-order formulas preserved under taking substructures.

In this paper, we give a graph characterization for those universal first-order formulas which are valid for relations, generalizing the characterization ($\star$). To this end, we proceed in three steps. First, we provide a characterization of relational validity for positive universal formulas. Based on this, we show that relational validity is decidable for this fragment. As a second step, we characterize relational validity for Horn formulas. Finally we combine the techniques used for both fragments to characterize validity for all universal first-order formulas. Before presenting our results, we start by recalling some background in Section 2.

2 Preliminaries

2.1 Universal theory of relations

We let $a, b \dots$ range over the letters of an alphabet A. *Terms* are generated by this syntax:

$$t, u ::= t \cdot u \mid t \cap u \mid t^\circ \mid 1 \mid \top \mid a \qquad a \in A$$

We denote the set of terms by $\mathcal{T}$. We often write tu for $t \cdot u$, and assign priorities to symbols so that $ab \cap c$, $a \cap b^\circ$ and ab° parse respectively as $(a \cdot b) \cap c$, $a \cap (b^\circ)$ and $a \cdot (b^\circ)$.

First-order formulas are generated by the following syntax:

$$\varphi, \psi := t \geq u \mid \neg(t \geq u) \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists a.\varphi \mid \forall a.\varphi \qquad t, u \in \mathcal{T}, a \in A.$$

Formulas of the form $t \geq u$ are called *inequations*. We extend the operation of negation $\neg$ to all formulas in the standard way, for instance $\neg(\varphi \wedge \psi) = \neg\varphi \vee \neg\psi$. Implication $\varphi \Rightarrow \psi$ is a shortcut for $\neg\varphi \vee \psi$. Free and bound variables are defined as usual, and we call *sentence* a formula without free variables.

A *universal formula* is a formula from the syntax above which does not use existential quantification. A *generalized Horn formula* is a formula of the following form, where $\forall\vec{a}$ denotes a sequence of universal quantifications:

$$\forall\vec{a}.\ \bigwedge_{j\in J}(v_j \geq w_j) \Rightarrow \bigvee_{i\in I}(t_i \geq u_i)$$

We generally write it as follows, where $\mathcal{H}$ is the set of inequations $\{v_j \geq w_j, j \in J\}$:

$$\forall\vec{a}.\ \mathcal{H} \Rightarrow \bigvee_{i\in I}(t_i \geq u_i)$$

We call $\mathcal{H}$ its *hypothesis* and $\bigvee_{i\in I}(t_i \geq u_i)$ its *conclusion*. A *Horn formula* is a generalized Horn formula whose conclusion contains a single disjunct. We write it like this:

$$\forall\vec{a}.\ \mathcal{H} \Rightarrow t \geq u$$

A *Positive universal formula* is a generalized Horn formula whose set of hypothesis is empty. It looks like this:

$$\forall \vec{a}.\ \bigvee_{i\in I}(t_i \geq u_i)$$

A *universal inequation* is a positive universal formula with a single disjunct. We will sometimes call it simply inequation. It looks like this:

$$\forall \vec{a}.\ t \geq u$$

In the rest of the paper, we will be interested only on universal sentences, this is why we will omit the universal quantification in front of our formulas.

Note that every universal formula can be written as the conjunction of generalized Horn formulas. In the rest of the paper, we will mainly focus on the latter.

Let us define *relational validity* for generalized Horn sentences. An *interpretation* σ is a function $\sigma : A \to \mathcal{P}(B \times B)$ mapping letters into relations over a base set B. We can extend σ to all terms $\sigma : \mathcal{T} \to \mathcal{P}(B \times B)$, by interpreting the operations $\cdot, \cap, ^\circ, 1$ and $\top$ on relations as follows:

$$\begin{array}{lcll}
R \cdot S & = & \{(x,y) \mid \exists z.(x,z) \in R \text{ and } (z,y) \in S\} & \text{(Composition)}\\
R \cap S & = & \{(x,y) \mid (x,y) \in R \text{ and } (x,y) \in S\} & \text{(Intersection)}\\
R^\circ & = & \{(x,y) \mid (y,x) \in R\} & \text{(Converse)}\\
1 & = & \{(x,x) \mid x \in B\} & \text{(Identity)}\\
\top & = & \{(x,y) \mid x,y \in B\} & \text{(Full relation)}
\end{array}$$

Let σ be an interpretation as above. An inequation $t \geq u$ is *true under* σ, noted $\sigma \models t \geq u$, if $\sigma(t) \supseteq \sigma(u)$. A set of inequations $\mathcal{H}$ *are true under* σ, noted $\sigma \models \mathcal{H}$, if this is the case for every inequation in $\mathcal{H}$. A generalized Horn sentence

$$\varphi := (\mathcal{H} \Rightarrow \bigvee_{i\in I}(t_i \geq u_i))$$

is *true under* σ, noted $\sigma \models \varphi$ if either $\sigma \not\models \mathcal{H}$ or there exists $i \in I$ such that $\sigma \models t_i \geq u_i$. We say that φ is *valid* for relations, noted $\mathcal{R}el \models \varphi$, if φ is true under all interpretations, using all possible base sets B.

Here are respectively a universal inequation (1), a positive universal sentence (2), and a Horn sentence (3), that are all valid for relations:

$$a(ba \cap 1)b \geq ab \cap 1 \tag{1}$$

$$\big(\top c \top \cap ab \cap ad \geq a(b \cap d)\big) \ \vee\ \big(d \geq ac\big) \tag{2}$$

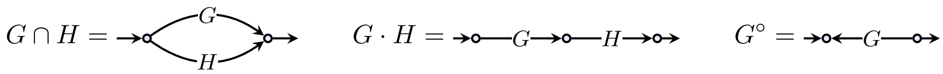
$$ef^\circ \geq \top \ \Rightarrow\ (ae \cap cf)(e^\circ b \cap f^\circ d) \geq ab \cap cd \tag{3}$$

We will see in the upcoming sections how to check their validity.

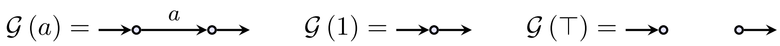
2.2 Graph characterization of the inequational theory of relations

Let A be an alphabet. A *2-pointed labeled graph* is a structure (V, E, ι, o) where V is a set of vertices, $E \subseteq V \times A \times V$ is a set of edges and ι and o are two distinguished vertices called the *input* and *output*. We simply call them *graphs* in the sequel; we depict them as expected, with unlabeled ingoing and outgoing arrows to denote the input and the output, respectively. We denote by $\mathcal{G}r$ the set of finite graphs. If G is a graph and x, y two of its vertices, we denote by (x, G, y) the graph obtained from G by forgetting the original input and output of G, and considering x and y as the new input and output respectively.

We define the following operations of graphs.

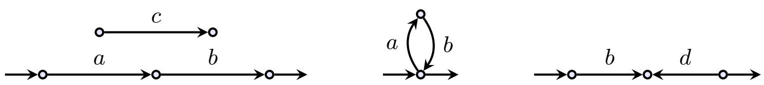


We associate to every term $t \in \mathcal{T}$ a graph $\mathcal{G}(t)$ called the *graph of* t, by letting



and by interpreting the operations $\cdot, \cap$ and $^\circ$ on graphs as above.

▶ **Example 1.** The graphs $\mathcal{G}(\top c \top \cap ab)$, $\mathcal{G}(ab \cap 1)$ and $\mathcal{G}(bd^\circ)$ are respectively the following:



Graph homomorphisms play a central role in the paper, they are defined as follows:

▶ **Definition 2** (Graph homomorphism). *Given two graphs $G = \langle V, E, \iota, o\rangle$ and $G' = \langle V', E', \iota', o'\rangle$, a* (graph) homomorphism *$h : G \to H$ is a mapping from $V \to V'$ that preserves labeled edges, ie. if $(x, a, y) \in E$ then $(h(x), a, h(y)) \in E'$, and preserves input and output, ie. $h(\iota) = \iota'$ and $h(o) = o'$.*
The image of G by h, *denoted $h(G)$, is the graph $\langle h(V), E'', \iota', o'\rangle$ where*

$$E'' = \{(h(x), a, h(y)) \mid (x, a, y) \in E\}.$$

We write $G \triangleright H$ if there exists a graph homomorphism from G to H, and $G \hookrightarrow H$ if there exists an injective *graph homomorphism from G to H. In the later case, we usually consider G as an actual subgraph of H.*

Our starting point was this characterization of the inequational theory of relations:

▶ **Theorem 3** ([1, Thm. 1], [7, p. 208]). *For all terms u, v,*

$$\mathcal{R}el \models u \geq v \quad \textit{iff} \quad \mathcal{G}(u) \triangleright \mathcal{G}(v)$$

2.3 Graphs and interpretations

We state below the main lemma (Lemma 6) that was used to prove Theorem 3, which will be useful for us too. But first, let us explicit a link between graphs and interpretations.

▶ **Definition 4** (Graphs and interpretations). *Let $\sigma : A \to \mathcal{P}(B \times B)$ be an interpretation. The* graph associated to σ, $\mathrm{G}(\sigma)$, *is the graph whose set of vertices is B and*

$$(x, a, y) \text{ is an edge of } \mathrm{G}(\sigma) \quad \textit{iff} \quad (x, y) \in \sigma(a).$$

Conversely if $G = (V, E)$ is a graph, the interpretation associated to G, $\mathcal{I}(G)$, *is the function*

$$\begin{array}{rcl} A & \to & \mathcal{P}(V \times V) \\ a & \mapsto & \{(x, y) \mid (x, a, y) \in E\} \end{array}$$

In the above definition, graphs are considered without distinguished input and output.

▶ Remark 5. The functions $\mathcal{G}$ and $\mathcal{I}$ are inverses of each other: $\mathcal{I} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{I}$ are the identity function on interpretations and graphs respectively.

Recall that (x, G, y) is the graph G where x and y are chosen to be the input and output.

▶ **Lemma 6** ([1], Lemma 3). *Let t be a term, $\sigma : A \to \mathcal{P}(B \times B)$ be an interpretation and $x, y \in B$. We have that:*

$$\sigma(t) \ni (x, y) \quad \textit{iff} \quad \mathcal{G}(t) \triangleright (x, \mathrm{G}(\sigma), y)$$

3 Characterizing the positive universal theory of relations

Given two graphs G and H, we define $G \oplus H$ as the disjoint union of G and H, whose input and output are those of G. Note that $\oplus$ is associative, but not commutative. However, note that the following holds:

$$G \oplus H \oplus K = G \oplus K \oplus H$$

$$G \triangleright H \oplus H \oplus K \quad \Leftrightarrow \quad G \triangleright H \oplus K$$

Now we can state our first characterization theorem:

▶ **Theorem 7.** *For all terms t_i, u_i where $i \in [1, n]$, the following holds*

$$\mathcal{R}el \models \bigvee_{i \in [1,n]} (t_i \geq u_i) \qquad \textit{iff} \qquad \bigvee_{i \in [1,n]} (\mathcal{G}(t_i) \triangleright \mathcal{G}(u_i) \oplus G)$$

where $\quad G = \mathcal{G}(u_1) \oplus \cdots \oplus \mathcal{G}(u_n)$.

Using the remark above, the case of two disjuncts can be formulated as follows

$$\mathcal{R}el \models (t_0 \geq u_0) \ \vee \ (t_1 \geq u_1) \qquad \textit{iff} \qquad \mathcal{G}(t_0) \triangleright \mathcal{G}(u_0) \oplus \mathcal{G}(u_1) \ \ \textit{or} \ \ \mathcal{G}(t_1) \triangleright \mathcal{G}(u_1) \oplus \mathcal{G}(u_0)$$

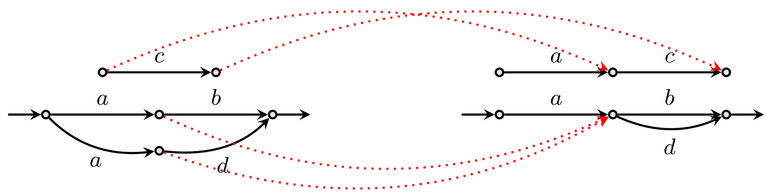
Before proving Theorem 7, let us see an example of its application.

▶ **Example 8.** The validity of the following positive universal sentence

$$\big(\top c \top \cap ab \cap ad \geq a(b \cap d)\big) \quad \vee \quad \big(d \geq ac\big) \qquad\qquad (2)$$

is witnessed by this homomorphism depicted below:

$$\mathcal{G}(\top c \top \cap ab \cap ad) \triangleright \mathcal{G}(a(b \cap d)) \oplus \mathcal{G}(ac)$$



▶ Remark 9. Surprisingly, this characterization tells us that only one left-hand-side (lhs) of the disjuncts of a positive universal sentence plays a role in its validity. For instance, in the sentence (2) above, we can replace the lhs of the second inequation, d, by any term without affecting the validity.

Proof. We show here the case of binary disjunctions to lighten notations. The general case works exactly in the same way.

($\Rightarrow$) Suppose that $\mathcal{R}el \models (t_0 \geq u_0) \ \vee \ (t_1 \geq u_1)$, let us show that either

$$\mathcal{G}(t_0) \triangleright \mathcal{G}(u_0) \oplus \mathcal{G}(u_1) \quad \text{or} \quad \mathcal{G}(t_1) \triangleright \mathcal{G}(u_1) \oplus \mathcal{G}(u_0)$$

Let G be the graph (without specified input and output) which is the disjoint union of $\mathcal{G}(u_0)$ and $\mathcal{G}(u_1)$, and let σ be the interpretation associated to G. We denote by G_0 the graph $\mathcal{G}(u_0) \oplus \mathcal{G}(u_1)$ and by G_1 the graph $\mathcal{G}(u_1) \oplus \mathcal{G}(u_0)$. To conclude the proof of this direction, we show that, for $i = 0, 1$:

$$\sigma(t_i) \supseteq \sigma(u_i) \quad \Rightarrow \quad \mathcal{G}(t_i) \triangleright G_i$$

Suppose that $\sigma(t_0) \supseteq \sigma(u_0)$, the other case is treated symmetrically. Let ι and o be respectively the vertices corresponding to the input and output of $\mathcal{G}(u_0)$ in G. We have that $\mathcal{G}(u_0) \rhd (\iota, G, o)$, then by Lemma 6, $\sigma(u_0) \ni (\iota, o)$. Thus, $\sigma(t_0) \ni (\iota, o)$ and again by Lemma 6, $\mathcal{G}(t_0) \rhd (\iota, G, o)$. But (ι, G, o) is G_0 and this remark concludes the proof.

($\Leftarrow$) Suppose that $G(t_0) \rhd \mathcal{G}(u_0) \oplus \mathcal{G}(u_1)$ and let us show that:

$$\mathcal{R}el \models (t_0 \geq u_0) \ \vee \ (t_1 \geq u_1)$$

The other case is treated symmetrically. Let $\sigma : A \to \mathcal{P}(B \times B)$ be an interpretation, and let G be its graph. We distinguish two cases. We have either:

$$\forall x, y \in B, \qquad \mathcal{G}(u_1) \not\rhd (x, G, y)$$

In this case, by Lemma 6, there is no pair (x, y) such that $(x, y) \in \sigma(u_1)$, hence $\sigma(t_1) \supseteq \sigma(u_1)$ is vacuously true.

Suppose now that there is x_1 and y_1 in B such that $\mathcal{G}(u_1) \rhd (x_1, G, y_1)$, let h_1 be such homomorphism. Notice the following:

$$\forall x, y \in B, \qquad \mathcal{G}(u_0) \rhd (x, G, y) \quad \Rightarrow \quad \mathcal{G}(u_0) \oplus \mathcal{G}(u_1) \rhd (x, G, y) \qquad (\dagger)$$

Indeed, if h_0 is a homomorphism from $\mathcal{G}(u_0)$ to (x, G, y), then we can combine it with h_1 to get a homomorphism from $\mathcal{G}(u_0) \oplus \mathcal{G}(u_1)$ to (x, G, y).

Let us show that $\sigma(t_0) \supseteq \sigma(u_0)$. If $\sigma(u_0) \ni (x, y)$, then by Lemma 6, we have that $\mathcal{G}(u_0) \rhd (x, G, y)$. Using the remark ($\dagger$), we get that $\mathcal{G}(u_0) \oplus \mathcal{G}(u_1) \rhd (x, G, y)$. By our hypothesis, we know that $\mathcal{G}(t_0) \rhd \mathcal{G}(u_0) \oplus \mathcal{G}(u_1)$, thus $\mathcal{G}(t_0) \rhd (x, G, y)$. We conclude that $\sigma(t_0) \ni (x, y)$, and this ends the proof of our first characterization theorem. ◀

Testing the existence of a homomorphism between finite graphs is decidable. Hence, we get as a corollary of Theorem 7 that:

▶ **Theorem 10.** *The positive universal theory of relations is decidable.*

4 Characterizing the Horn theory of relations

To give a characterization of the Horn theory of relations, we need to generalize the homomophism relation between graphs to take into account some set of hypothesis.

A *context* is a graph with a distinguished edge labeled by a special letter $\bullet$, called its *hole*. If G is a graph and C a context, then $C[G]$ is the graph obtained by "plugging G in the hole" of C, that is, $C[G]$ is the graph obtained as the disjoint union of G and C, where we identify the input (resp. output) of G with the input (resp. output) of the edge labeled by $\bullet$ in C, and we remove the edge of C labeled $\bullet$.

▶ **Definition 11** (The relation $\rhd_{\mathcal{H}}$). *Let $\mathcal{H}$ be a set of inequations. We define the relation $>_{\mathcal{H}}$ on graphs as follows. We set $G >_{\mathcal{H}} H$ if and only if there is a context C and an inequation $(t \geq u) \in \mathcal{H}$ such that*

$$G = C[\mathcal{G}(t)] \quad \textit{and} \quad H = C[\mathcal{G}(u)]$$

We define $\rhd_{\mathcal{H}}$ as the transitive closure of $\rhd \cup >_{\mathcal{H}}$.

In the definition above, the graphs G, H and C are not necessarily the graphs of some terms.

We can state now the main theorem of this section:

▶ **Theorem 12.** *For all terms t, u and set of inequations $\mathcal{H}$, we have:*

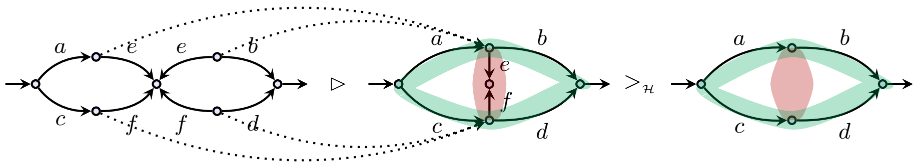
$$\mathcal{R}el \models (\mathcal{H} \Rightarrow t \geq u) \qquad \textit{iff} \qquad \mathcal{G}(t) \rhd_{\mathcal{H}} \mathcal{G}(u)$$

Hence, in order to show that a Horn sentence $(\mathcal{H} \Rightarrow t \geq u)$ is valid, we need to find a sequence of graphs $G_0, \ldots, G_n$ such that $G_0 = \mathcal{G}(t)$, $G_n = \mathcal{G}(u)$ and for every $i \in [0, n-1]$ the graphs G_i and G_{i+1} are either related by homomorphism or by the relation $>_{\mathcal{H}}$. We say that this sequence *witnesses* the validity of this Horn sentence.

▶ **Example 13.** The validity of the following Horn sentence:

$$ef^\circ \geq \top \;\Rightarrow\; (ae \cap cf)(e^\circ b \cap f^\circ d) \geq ab \cap cd \qquad (3)$$

is witnessed by the following sequence:



We start by applying a homomorphism represented by the dotted lines, then we factorize the obtained graph into a context (in green) and an inner graph (in red) which is the graph of ef°, the lhs of the hypothesis $ef^\circ \geq \top$. We replace it by the graph of the rhs $\top$, which is the empty graph. Doing so, we get the graph of $ab \cap cd$

Notice that the intermediary graph is *not* the graph of a term.

▶ Remark 14. One may wonder whether Theorem 12 leads to a decidability result for the Horn theory of relations. Actually, the latter is undecidable, as it subsumes the word problem for monoids [6, Thm.4.5].

The next two subsections are dedicated to the proof of Theorem 12.

4.1 From $\rhd_{\mathcal{H}}$ to validity

In this section we prove the right-to-left implication of Theorem 12. But first, let us show the following lemma, which says that $\rhd_{\mathcal{H}}$ collapses to $\rhd$ if the target graph is the graph of an interpretation making $\mathcal{H}$ true.

▶ **Lemma 15.** *Let $\mathcal{H}$ be a set of inequations and σ an interpretation. If the inequations $\mathcal{H}$ are true under σ, then for every graph G:*

$$G \rhd_{\mathcal{H}} (x, \mathrm{G}(\sigma), y) \qquad \textit{iff} \qquad G \rhd (x, \mathrm{G}(\sigma), y)$$

Proof. The right-to-left direction is trivial. We prove the other direction by induction on the length of a sequence witnessing that $G \rhd_{\mathcal{H}} (x, \mathrm{G}(\sigma), y)$. The most interesting base case is when, for some graph H:

$$G >_{\mathcal{H}} H \rhd (x, \mathrm{G}(\sigma), y) \qquad \text{(BC)}$$

The other two base cases are: $G \rhd (x, \mathrm{G}(\sigma), y)$, which is trivial, and $G >_{\mathcal{H}} (x, \mathrm{G}(\sigma), y)$, which can be seen as a particular case of the interesting base case, by taking H to be $(x, \mathrm{G}(\sigma), y)$. The inductive step is easy, as the composition of two homomorphisms is a homomorphism. Now, let us prove the interesting base case. Suppose that there is a graph H satisfying (BC), and let us find a homomorphism from G to $(x, \mathrm{G}(\sigma), y)$.

Since $G >_{\mathcal{H}} H$, there is an inequation $(t \geq u) \in \mathcal{H}$ and a context C such that $G = C[\mathcal{G}(t)]$ and $H = C[\mathcal{G}(u)]$. We have also that $H \rhd (x, \mathrm{G}(\sigma), y)$, so let h be a homomorphism:

$$h : C[\mathcal{G}(u)] \to (x, \mathrm{G}(\sigma), y)$$

Let x' and y' be respectively the image of the input and the output of $\mathcal{G}(u)$ by h. By considering the restriction of h to $\mathcal{G}(u)$, we have that $\mathcal{G}(u) \rhd (x', \mathrm{G}(\sigma), y')$. Hence, by Lemma 6, we have that $(x', y') \in \sigma(u)$. As $\mathcal{H}$ is true under σ, we have also that $(x', y') \in \sigma(t)$, and again by Lemma 6, $\mathcal{G}(t) \rhd (x', \mathrm{G}(\sigma), y')$. Let us denote by k a homomorphism:

$$k : \mathcal{G}(t) \to (x', \mathrm{G}(\sigma), y')$$

With these ingredients, we construct a homomorphism f from $G = C[\mathcal{G}(t)]$ to $(x, \mathrm{G}(\sigma), y)$ as follows: the restriction of f to C is h and the restriction of f to $\mathcal{G}(t)$ is k. It is easy to check that f is indeed an homomorphism, and this ends the proof. ◀

We can now prove the right-to-left direction of Theorem 12.

Proof of Theorem 12 (⇐). Suppose that $\mathcal{G}(t) \rhd_{\mathcal{H}} \mathcal{G}(u)$. Let σ be an interpretation satisfying $\mathcal{H}$ and suppose that $(x, y) \in \sigma(u)$.

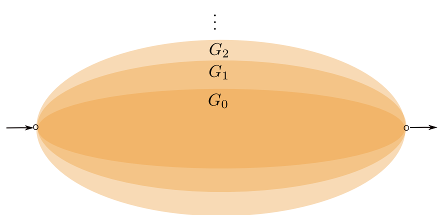
$$\begin{array}{rcll} \sigma(u) \ni (x,y) & \Rightarrow & \mathcal{G}(u) \rhd (x, \mathrm{G}(\sigma), y) & \text{Lem. 6} \\ & \Rightarrow & \mathcal{G}(t) \rhd_{\mathcal{H}} (x, \mathrm{G}(\sigma), y) & \text{By hypothesis} \\ & \Rightarrow & \mathcal{G}(t) \rhd (x, \mathrm{G}(\sigma), y) & \text{Lem. 15} \\ & \Rightarrow & \sigma(t) \ni (x,y) & \text{Lem. 6} \end{array}$$ ◀

4.2 From validity to $\rhd_{\mathcal{H}}$

The main ingredient to prove the left-to-right direction of Theorem 12 is to construct, given a set of hypothesis $\mathcal{H}$, an interpretation making them true. For that we start from an arbitrary graph and "saturate" it by the hypothesis $\mathcal{H}$, then we iterate this construction ω-times and take the *limit graph*. The desired interpretation will be the interpretation associated to this graph. In the sequel, we define the notions of graph limit and saturation, then we proceed to the proof of our theorem.

4.2.1 Limit of a sequence of graphs

When we consider an increasing sequence of graphs $(G_i)_{i\in\omega}$, that is, $G_i \hookrightarrow G_{i+1}$ for every $i \in \omega$, the notion of limit is clear: it is just the union of the graphs G_i, its input and output being respectively the common input and output of the graphs G_i; we denote it by $\lim_{i\in\omega} G_i$. We denote by $\theta_i : G_i \to \lim_{i\in\omega} G_i$ the natural injection of G_i into the limit graph, we call it the *limit injection for G_i*. Here is an illustration of this construction:



In the following, we extend this notion of limit to the case where the graphs G_i and G_{i+1} are related by an arbitrary homomorphism, not necessarily an injective one. Let us start with an observation.

Let $G_0 \xrightarrow{h_0} G_1 \xrightarrow{h_1} G_2 \dots$ be a sequence of *finite* graphs related by homomorphism. Let $(H_i)_{i\in\omega}$ be the successive images of G_0 by these homomorphisms, that is:

$$H_0 = G_0, \qquad \text{and} \qquad H_{i+1} = h_i(H_i) \quad \text{for } i \geq 0.$$

At some point, the image of G_0 will stabilize, in other words there is an index s such that, for all $i > s$ the function $k_i : H_i \to H_{i+1}$, the restriction of h_i to H_i is a bijection. We call *stabilization index of* G_0 the least index s satisfying this property, we denote it by s_0. We call *stable image of* G_0 the graph H_{s_0} and we denote it by $S(G_0)$.

We define in the same way the *stabilization index of* G_i, and denote it s_i: it is the least index starting from which the homomorphisms h_j for $j > s_i$ do not merge nodes coming from G_i. We define similarly the stable image of G_i and denote it by $S(G_i)$.

Note that if $i \leq j$ then $s_i \leq s_j$ and $S(G_i) \hookrightarrow S(G_j)$. By considering the sequence of the stable images of the graphs G_i, we can now define the limit of this sequence:

▶ **Definition 16** (Limit of a sequence of graphs)**.** *Let $(G_i)_{i\in\omega}$ be a sequence of finite graphs such that there is a homomorphism $h_i : G_i \to G_{i+1}$ for every $i \in \omega$. As the sequence of stable images $(S(G_i))_{i\in\omega}$ is increasing, its limit $\lim\limits_{i\in\omega} S(G_i)$ is well defined. For every $i \in \omega$, let θ_i be the limit injection $\theta_i : S(G_i) \to \lim\limits_{i\in\omega} S(G_i)$.*

We define the limit of the sequence $(G_i)_{i\in\omega}$ as follows:

$$\lim_{i\in\omega} G_i = \lim_{i\in\omega} S(G_i)$$

For every $i < j \in \omega$, we denote by $h_{[i,j]}$ the homomorphism $h_{[i,j]} : G_i \to G_j$ obtained as the composition $h_{j-1} \circ \dots \circ h_i$. We denote by $\pi_i : G_i \to \lim\limits_{i\in\omega} G_i$ the homomorphism $\theta_{s_i} \circ h_{[i,s_i]}$. We call π_i the limit homomorphism *for G_i.*

▶ **Example 17.** Consider the sequence of terms $(t_i)_{i\in\omega}$ defined by:

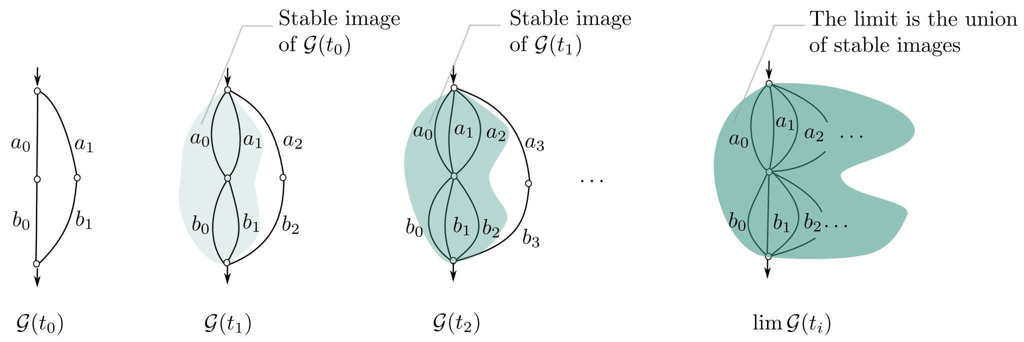
$$t_i = (\bigcap_{k=0}^{i} a_k \cdot \bigcap_{k=0}^{i} b_k) \cap (a_{i+1} \cdot b_{i+1}).$$

There is a (unique) homomorphism $h_i : \mathcal{G}(t_i) \to \mathcal{G}(t_{i+1})$. The limit of the sequence of graphs $(\mathcal{G}(t_i))_{i\in\omega}$ related by the homomorphisms $(h_i)_{i\in\omega}$, converges to the graph of this "term"[1]:

$$\bigcap_{k=0}^{\infty} a_k \cdot \bigcap_{k=0}^{\infty} b_k$$

Here is an illustration of this example:

[1] This is not really a term since it contains infinite intersections, but it is clear how to define the graphs of such generalized terms.



Note that the limit does not depend only on the sequence of graphs, but also on the homomorphisms relating them. Consider for instance the following sequence of terms.

$$u_i = (\bigcap_{k=0}^{i} a_k \cdot \bigcap_{k=0}^{i} b_k) \cap \bigcap_{k=0}^{i+1} (a_k \cdot b_k).$$

If we consider the injections $\iota_i : \mathcal{G}(u_i) \to \mathcal{G}(u_{i+1})$, then the sequence $(\mathcal{G}(u_i))_{i\in\omega}$ related by the homomorphisms $(\iota_i)_{i\in\omega}$ converges to the graph of this "term":

$$(\bigcap_{k=0}^{\infty} a_k \cdot \bigcap_{k=0}^{\infty} b_k) \cap \bigcap_{k=0}^{\infty} (a_k \cdot b_k)$$

But if we consider the homomorphisms $k_i : \mathcal{G}(u_i) \to \mathcal{G}(u_{i+1})$ which merges all the inner nodes[2] of $\mathcal{G}(u_i)$, we obtain as limit the graph of this "term":

$$\bigcap_{k=0}^{\infty} a_k \cdot \bigcap_{k=0}^{\infty} b_k$$

▶ Remark 18. This notion of limit is a well known concept of category theory. Since the category of graphs and graph homomorphisms is cocomplete, every infinite sequence of homomorphisms has a colimit, unique up to isomorphism. We made the choice to give an explicit definition for the readers which are, as the author, not familiar with category theory.

Here are some properties satisfied by the limit of a sequence of graphs.

▶ **Proposition 19.** *Let $(G_i)_{i\in\omega}$ be a sequence of graphs, and $h_i : G_i \to G_{i+1}$. Let G_ω be their graph limit, π_i be the limit homomorphism for G_i and H be a* finite *graph.*

1. *For every $i \in \omega$, if $H \triangleright (x, G_i, y)$ then $H \triangleright (\pi_i(x), G_\omega, \pi_i(y))$.*
2. *Conversely, if $H \triangleright (x, G_\omega, y)$ then $H \triangleright (x', G_i, y')$ for some i, x', y' satisfying $\pi_i(x') = x$ and $\pi_i(y') = y$.*
3. *In particular, we have that: $H \triangleright G_\omega \Leftrightarrow \exists i \in \omega,\ H \triangleright G_i$.*

Proof. Property (1) is trivial. Indeed, if $h : H \to (x, G_i, y)$ is a homomorphism then $h_i \circ h : H \to (\pi_i(x), G_\omega, \pi_i(y))$ is also a homomorphism.

Suppose that $H \triangleright (x, G_\omega, y)$. Since H is finite, there is $i \in \omega$ such that $H \triangleright (x', S(G_i), y')$ where $\pi_i(x') = x$ and $\pi_i(y') = y$. But $S(G_i)$ is a subgraph of some G_j, where $j \in \omega$. Hence $H \triangleright (x', G_j, y')$. ◀

[2] That is, nodes different from the input and the output.

4.2.2 Saturation by hypothesis

Let $G, H \in \mathcal{G}r$ and let x, y be two vertices of G. We denote by $G[H/xy]$ the graph obtained from G by merging the input of H with x and its output with y. The input and output of $G[H/xy]$ are those of G.

▶ Remark 20. Note that if the input and output of H are equal, then the operation $G[H/xy]$ merges the nodes x, y. Note also that $G \rhd G[H/xy]$, but this homomorphism is not necessarily injective because of the possible merge of x and y.

▶ **Definition 21** (Saturation). *Let $\mathcal{H}$ be a finite set of inequations, $G \in \mathcal{G}r$ and V its set of vertices. Let $T \subseteq V \times V \times \mathcal{G}r$ be the set of triplets satisfying:*

$$(x, y, H) \in T \qquad \textit{iff} \qquad \exists (t \geq u) \in \mathcal{H}, \quad \mathcal{G}(u) \rhd (x, G, y) \quad \textit{and} \quad H = \mathcal{G}(t)$$

Let $(x_i, y_i, H_i)_{i \leq n}$ be an enumeration of T. The saturation of G by $\mathcal{H}$ *is the graph denoted $Sat_{\mathcal{H}}(G)$ and defined as:*

$$Sat_{\mathcal{H}}(G) = G[H_0/x_0 y_0] \dots [H_n/x_n y_n]$$

In words, a triplet (x, y, H) is in T means that in the graph G, we "identified" the graph $\mathcal{G}(u)$, the rhs of a hypothesis in $\mathcal{H}$, between the nodes x and y. The graph H is $\mathcal{G}(t)$, the graph of the lhs of this hypothesis. To make G "agree" with hypothesis $\mathcal{H}$, we need to plug H between x and y. When we do that for all the triplets in T, we obtain the saturation of G by $\mathcal{H}$. Now, let us make some properties of saturation explicit.

▶ **Proposition 22.** *Let G be a graph and $\mathcal{H}$ a set of inequations.*

1. *For every inequation $(t \geq u) \in \mathcal{H}$, we have:*

$$\mathcal{G}(u) \rhd (x, G, y) \;\Rightarrow\; \mathcal{G}(t) \rhd (x, Sat_{\mathcal{H}}(G), y)$$

2. $G \rhd Sat_{\mathcal{H}}(G)$.
3. $Sat_{\mathcal{H}}(G) \rhd_{\mathcal{H}} G$.

Proof. To prove (1), suppose that $(t \geq u) \in \mathcal{H}$ and $\mathcal{G}(u) \rhd (x, G, y)$. This means that the triplet $(x, y, \mathcal{G}(t))$ is in the set T of Definition 21. Hence $Sat_{\mathcal{H}}(G)$ is of the form $K[\mathcal{G}(t)/xy]$. It is clear then that $\mathcal{G}(t) \rhd (x, Sat_{\mathcal{H}}(G), y)$.

Property (2) is a consequence of Remark 20 above. For property (3), we will show that if $(x, y, H) \in T$, where T is as in definition 21, then $G[H/xy] \rhd_{\mathcal{H}} G$. The result will be an iteration of this argument for all elements of T. By definition of T, there is a hypothesis $(t \geq u)$ such that $\mathcal{G}(u) \rhd (x, G, y)$ and $H = \mathcal{G}(t)$. Let us denote by k a homomorphism from $\mathcal{G}(u)$ to (x, G, y). Let C be the context obtained from G by adding an edge labeled $\bullet$ between x and y. We have that:

$$G[H/xy] = C[\mathcal{G}(t)] >_{\mathcal{H}} C[\mathcal{G}(u)] \rhd G$$

Indeed, the equality and inequation $>_{\mathcal{H}}$ are trivially true. To justify the $\rhd$ inequation, we define a homomorphism from $C[\mathcal{G}(u)]$ to G as follows: its restriction to $\mathcal{G}(u)$ its is the homomorphism k, and its restriction to C is the identity. ◀

Now, we can define the ω-saturation of a graph by a set of hypothesis.

▶ **Definition 23** (ω-saturation)**.** *If G is a graph and $\mathcal{H}$ a set of inequations, we define the sequence $(Sat^i_{\mathcal{H}}(G))_{i\in\omega}$ as the successive iterations of G by saturation by the hypothesis $\mathcal{H}$:*

$$Sat^0_{\mathcal{H}}(G) = G, \qquad Sat^{i+1}_{\mathcal{H}}(G) = Sat(Sat^i_{\mathcal{H}}(G)) \quad (i \in \omega).$$

By Proposition 22 (2), we have that $Sat^i_{\mathcal{H}}(G) \triangleright Sat^{i+1}_{\mathcal{H}}(G)$. The limit is then well defined by Definition 16. We define the ω-saturation of G by $\mathcal{H}$ *as the graph:*

$$Sat^{\omega}_{\mathcal{H}}(G) = \lim_{i\in\omega} Sat^i_{\mathcal{H}}(G).$$

The ω-saturation satisfies the following property. It says that given a set of inequations $\mathcal{H}$, if we start from an arbitrary graph G (so in general, the inequations of $\mathcal{H}$ are not true under the interpretation associated to G) and we ω-saturate it by $\mathcal{H}$, then the inequations from $\mathcal{H}$ are true under the interpretation associated to the obtained graph.

▶ **Proposition 24.** *Let $\mathcal{H}$ be a set of inequations, G be a graph, and σ the interpretation associated to $Sat^{\omega}_{\mathcal{H}}(G)$. The inequations from $\mathcal{H}$ are true under σ.*

Proof. We denote by G^ω the graph $Sat^{\omega}_{\mathcal{H}}(G)$, by G^i the graph $Sat^i_{\mathcal{H}}(G)$ for every $i \in \omega$ and by σ the interpretation associated to G^ω. By Remark 5, the graph associated to σ is G^ω. Let $\pi_i : G^i \to G^\omega$ be the limit homomorphism for G_i.

Let $(t \geq u) \in \mathcal{H}$, let us show that $\sigma(t) \supseteq \sigma(u)$. Suppose that $\sigma(u) \ni (x, y)$.

$\sigma(u) \ni (x, y)$	$\Longrightarrow$	$\mathcal{G}(u) \triangleright (x, G^\omega, y)$	Lem. 6
	$\overset{\exists i,x',y'}{\Longrightarrow}$	$\mathcal{G}(u) \triangleright (x', G^i, y')$, $x = \pi_i(x')$ and $y = \pi_i(y')$	Prop. 19 (2)
	$\Longrightarrow$	$\mathcal{G}(t) \triangleright (x', G^{i+1}, y')$	Prop. 22 (1)
	$\Longrightarrow$	$\mathcal{G}(t) \triangleright (x, G^\omega, y)$	Prop. 19 (1)
	$\Longrightarrow$	$\sigma(t) \ni (x, y)$	Lem. 6

And this concludes the proof. ◀

We can go back to the proof of Theorem 12.

Proof of Theorem 12 ($\Leftarrow$). Suppose that $\mathcal{R}el \models \mathcal{H} \Rightarrow t \geq u$ and let us show that $\mathcal{G}(t) \triangleright_{\mathcal{H}} \mathcal{G}(u)$. We denote by $\mathcal{G}(u)^\omega$ the graph $Sat^{\omega}_{\mathcal{H}}(\mathcal{G}(u))$, by $\mathcal{G}(u)^i$ the graph $Sat^i_{\mathcal{H}}(\mathcal{G}(u))$ for every $i \in \omega$, and by σ be the interpretation associated to $\mathcal{G}(u)^\omega$. By Proposition 24, the inequations $\mathcal{H}$ are true under σ. Note that the graph associated to σ is $\mathcal{G}(u)^\omega$ and that the input and output of $\mathcal{G}(u)^\omega$ are those of $\mathcal{G}(u)$, let us denote them by ι and o respectively.

By Poposition 19 (1), we have $\mathcal{G}(u) \triangleright \mathcal{G}(u)^\omega$. It follows that:

$\mathcal{G}(u) \triangleright \mathcal{G}(u)^\omega$	$\Rightarrow$	$\sigma(u) \ni (\iota, o)$	Lem. 6
	$\Rightarrow$	$\sigma(t) \ni (\iota, o)$	By hypothesis
	$\Rightarrow$	$\mathcal{G}(t) \triangleright \mathcal{G}(u)^\omega$	Lem. 6
	$\Rightarrow$	$\mathcal{G}(t) \triangleright \mathcal{G}(u)^i$ for some $i \in \omega$	Prop. 19 (3)
	$\Rightarrow$	$\mathcal{G}(t) \triangleright_{\mathcal{H}} \mathcal{G}(u)$	Prop. 22 (3)

This ends the proof of Theorem 12. ◀

4.3 Characterizing the universal theory of relations

We characterize now the validity of the generalized Horn sentences. The proof is a mix of the techniques used to prove Theorems 7 and 12.

▶ **Theorem 25.** *For all terms t_i, u_i where $i \in [1, n]$, and set of inequations $\mathcal{H}$, the following holds:*

$$\mathcal{R}el \models \mathcal{H} \Rightarrow \bigvee_{i\in[1,n]} (t_i \geq u_i) \qquad \textit{iff} \qquad \bigvee_{i\in[1,n]} (\mathcal{G}(t_i) \rhd_{\mathcal{H}} \mathcal{G}(u_i) \oplus G)$$

where $G = \mathcal{G}(u_1) \oplus \cdots \oplus \mathcal{G}(u_n)$.

Proof. As for Theorem 7, we prove this result in the case of binary dijunctions, that is:

$$\mathcal{R}el \models \mathcal{H} \Rightarrow (t_0 \geq u_0) \vee (t_1 \geq u_1) \quad \text{iff} \quad G(t_0) \rhd_{\mathcal{H}} \mathcal{G}(u_0) \oplus G(u_1) \text{ or } G(t_1) \rhd_{\mathcal{H}} \mathcal{G}(u_1) \oplus G(u_0)$$

($\Rightarrow$) Suppose that $\mathcal{R}el \models \mathcal{H} \Rightarrow (t_0 \geq u_0) \ \vee \ (t_1 \geq u_1)$, let us show that either

$$G(t_0) \rhd_{\mathcal{H}} \mathcal{G}(u_0) \oplus G(u_1) \quad \text{or} \quad G(t_1) \rhd_{\mathcal{H}} \mathcal{G}(u_1) \oplus G(u_0)$$

We set $G = \mathcal{G}(u_0) \oplus \mathcal{G}(u_1)$, and let G^ω denote the graph $Sat^\omega_{\mathcal{H}}(G)$, G^i denote the graph $Sat^i_{\mathcal{H}}(G)$ for every $i \in \omega$, and let σ be the interpretation associated to G^ω. By Proposition 24, the inequations $\mathcal{H}$ are true under σ. Hence, we have either $\sigma(t_0) \supseteq \sigma(u_0)$ or $\sigma(t_1) \supseteq \sigma(u_1)$. Let us study the former case, the latter being symmetric.

Suppose that $\sigma(t_0) \supseteq \sigma(u_0)$. Let ι and o be respectively the input and output of G^ω. Notice that $\mathcal{G}(u_0) \rhd (\iota, G^\omega, o)$, it follows that:

$$\begin{array}{rcll}
\mathcal{G}(u_0) \rhd (\iota, G^\omega, o) & \Rightarrow & \sigma(u_0) \ni (\iota, o) & \text{Lem. 6} \\
& \Rightarrow & \sigma(t_0) \ni (\iota, o) & \text{By hypothesis} \\
& \Rightarrow & \mathcal{G}(t_0) \rhd G^\omega & \text{Lem. 6} \\
& \Rightarrow & \mathcal{G}(t_0) \rhd G^i \ \text{ for some } i \in \omega & \text{Prop. 19 (3)} \\
& \Rightarrow & \mathcal{G}(t_0) \rhd_{\mathcal{H}} G & \text{Prop. 22 (3)}
\end{array}$$

This concludes the proof of this direction.

($\Leftarrow$) Suppose that $G(t_0) \rhd_{\mathcal{H}} \mathcal{G}(u_0) \oplus G(u_1)$ and let us show that:

$$\mathcal{R}el \models \mathcal{H} \Rightarrow (t_0 \geq u_0 \ \vee \ t_1 \geq u_1)$$

Note that the other case is symmetric. Let $\sigma : A \to \mathcal{P}(B \times B)$ be an interpretation under which $\mathcal{H}$ is true, and let G be its graph. We distinguish two cases. We have either:

$$\forall x, y \in B, \qquad \mathcal{G}(u_1) \not\rhd (x, G, y)$$

In this case, by Lemma 6, there is no pair (x, y) such that $(x, y) \in \sigma(u_1)$, hence $\sigma(t_1) \supseteq \sigma(u_1)$ is vacuously true.

Suppose now that there is x_1 and y_1 in B such that $\mathcal{G}(u_1) \rhd (x_1, G, y_1)$. Notice the following:

$$\forall x, y \in B, \qquad \mathcal{G}(u_0) \rhd (x, G, y) \quad \Rightarrow \quad \mathcal{G}(u_0) \oplus \mathcal{G}(u_1) \rhd (x, G, y)$$

By using Lemma 6 and this remark, we get that if $\sigma(u_0) \ni (x, y)$ then $\mathcal{G}(t_0) \rhd_{\mathcal{H}} (x, G, y)$. By Lemma 15, and since the inequations $\mathcal{H}$ are true under σ, we have that $\mathcal{G}(t_0) \rhd (x, G, y)$. Hence, by Lemma 6, we get $\sigma(t_0) \ni (x, y)$ which concludes the proof. ◀

As every universal sentence can be written as the conjunction of some generalized Horn sentences, Theorem 25 gives us a characterization of the validity of all universal sentences.

5 Conclusion

We end this paper by some concluding remarks and open problems.

By characterizing the universal theory of relations, we characterized also their existential theory. Now, can we characterize the full first-order theory of relations using graphs and homomorphisms?

Another direction of work is to extend the syntax of terms. For instance, we could add the operations of union and Kleene star. In this case, terms are interpreted, not by a single graph as we did here, but by a set of graphs as in [4, Def. 4]. Graph homomorphism is generalized to the relation $\blacktriangleright$ between sets of graphs as follows:

$$C \blacktriangleright D \qquad \Leftrightarrow \qquad \forall H \in D,\ \exists G \in C,\ G \rhd H$$

With these interpretations, Theorem 7 can be easily adapted when union is added to the syntax. However, it is not clear how to adapt it in the presence of the Kleene star. Theorem 12 seems hard to adapt both for the union and the Kleene star extensions.

Even if Theorems 12 and 25 do not give decidability for the corresponding theories, we can wonder whether it can be obtained under some restrictions on the hypothesis $\mathcal{H}$. For instance, is it the case when the hypothesis $\mathcal{H}$ form a Noetherian rewriting system?

We can easily adapt this work to the realm of conjunctive queries. Indeed, terms can be replaced by conjunctive queries and inequations between terms by equivalence between conjunctive queries $Q_1 \equiv Q_2$. For example, by adapting Theorem 7 we get the decidability of the following problem:

Input: Conjunctive queries Q_1, Q_2, Q_3 and Q_4.
Output: Do we have $(Q_1 \equiv Q_2) \vee (Q_3 \equiv Q_4)$?

which generalizes the result of Chandra and Merlin [5].

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