A Note on the Join of Varieties of Monoids with LI

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Abstract

In this note, we give a characterisation in terms of identities of the join of **V** with the variety of finite locally trivial semigroups **LI** for several well-known varieties of finite monoids **V** by using classical algebraic-automata-theoretic techniques. To achieve this, we use the new notion of essentially-**V** stamps defined by Grosshans, McKenzie and Segoufin and show that it actually coincides with the join of **V** and **LI** precisely when some natural condition on the variety of languages corresponding to **V** is verified.

This work is a kind of rediscovery of the work of J. C. Costa around 20 years ago from a rather different angle, since Costa's work relies on the use of advanced developments in profinite topology, whereas what is presented here essentially uses an algebraic, language-based approach.

2012 ACM Subject Classification Theory of computation → Formal languages and automata theory; Theory of computation → Algebraic language theory

Keywords and phrases Varieties of monoids, join, LI

Digital Object Identifier 10.4230/LIPIcs.MFCS.2021.51

Acknowledgements I want to thank Thomas Place, who suggested the link between essentially-**V** stamps and $\mathbf{V} \vee \mathbf{LI}$, but also Luc Segoufin who started the discussion with Thomas Place and encouraged me to write the present article. My thanks go as well to the anonymous referees for their helpful comments and suggestions. Finally, I want to mention that the introductions of Jean-Éric Pin's future book on algebraic automata theory and of Marc Zeitoun's works cited in the references have been helpful inspirations for my own introduction.

1 Introduction

One of the most fundamental problems in finite automata theory is the one of *characterisation*: given some subclass of the class of regular languages, find out whether there is a way to characterise those languages using some class of finite objects. This problem is often linked to and motivated by the problem of *decidability*: given some subclass of the class of regular languages, find out whether there exists an algorithm testing the membership of any regular language in that subclass. The obvious approach to try to find a characterisation of a class of regular languages would be to look for properties shared by all the minimal finite automata of those languages. If we find such characterising properties, we can then ask whether they can be checked by an algorithm to answer the problem of decidability for this class of languages. However, one of the most fruitful approaches of those two problems has been the *algebraic approach*, in which we basically replace automata with morphisms into monoids: a language L over an alphabet Σ is then said to be recognised by a morphism φ into a monoid M if and only if L is the inverse image by φ of a subset of M. Under this notion of recognition, each language has a minimal morphism recognising it, the *syntatic morphism* into the *syntactic monoid* of that language, that are minimal under some notion of division. The fundamental result on which this algebraic approach relies is that a language is regular if and only if its syntactic monoid is finite. One can thus try to find a characterisation of some class of regular languages by looking at the algebraic properties of the syntactic monoids of these languages.

And many such characterisations that are decidable were indeed successfully obtained since Schützenberger's seminal work in 1965 [18]. His famous result, that really started the field of *algebraic automata theory*, states that the star-free regular languages are exactly

46th International Symposium on Mathematical Foundations of Computer Science (MFCS 2021).
Editors: Filippo Bonchi and Simon J. Puglisi; Article No. 51; pp. 51:1–51:16
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

those whose syntactic monoids are finite and aperiodic. Another important early result in that vein is the one of Simon [19] characterising the piecewise testable languages as exactly those having a finite $\mathfrak{J}$-trivial syntactic monoid. Eilenberg [12] was the first to prove that such algebraic characterisations actually come as specific instances of a general bijective correspondence between varieties of finite monoids and varieties of languages – classes of, respectively, finite monoids and regular languages closed under natural operations. Thus, a class of regular languages can indeed be characterised by the syntactic monoids of these languages, as soon as it verifies some nice closure properties. Eilenberg's result was later completed by Reiterman's theorem [17], that uses a notion of identities defined using profinite topology and states that a class of finite monoids is a variety of finite monoids if and only if it is defined by a set of profinite identities. Therefore, one can always characterise the variety of finite monoids associated to a variety of languages by a set of profinite identities and, additionally, this characterisation often leads to decidability, especially when this set is finite. A great deal of research works have been conducted to characterise varieties of finite monoids or semigroups by profinite identities (see the book of Almeida [3] for an overview; see also the book chapter by Pin [15] for more emphasis on the "language" part).

A kind of varieties of finite monoids or semigroups that has attracted many research efforts aiming for characterisations through identities are the varieties defined as the join of two other varieties. Given two varieties of finite monoids $\mathbf{V}$ and $\mathbf{W}$, the *join of* $\mathbf{V}$ *and* $\mathbf{W}$, denoted by $\mathbf{V} \vee \mathbf{W}$, is the least variety of finite monoids containing both $\mathbf{V}$ and $\mathbf{W}$. One of the main motivations to try to understand $\mathbf{V} \vee \mathbf{W}$ is that the variety of languages corresponding to it by the Eilenberg correspondence, $\mathcal{L}(\mathbf{V} \vee \mathbf{W})$, is the one obtained by considering direct products of automata recognising languages from both $\mathcal{L}(\mathbf{V})$ and $\mathcal{L}(\mathbf{W})$, the varieties of languages corresponding to, respectively, $\mathbf{V}$ and $\mathbf{W}$. This is a fundamental operation on automata, and while it is straightforward that $\mathcal{L}(\mathbf{V} \vee \mathbf{W})$ is simply the least variety of languages containing both $\mathcal{L}(\mathbf{V})$ and $\mathcal{L}(\mathbf{W})$, this does not at all furnish a decidable characterisation of $\mathcal{L}(\mathbf{V} \vee \mathbf{W})$, let alone a set of identities defining $\mathbf{V} \vee \mathbf{W}$. Generally speaking, the problem of finding a set of identities defining $\mathbf{V} \vee \mathbf{W}$ is difficult (see [3, 23]): in fact, there exist two varieties of finite semigroups that have a decidable membership problem but whose join has an undecidable membership problem [1]. However, sets of identities have been found for many specific joins: have a look at [2, 4, 6, 22, 21, 7, 9, 10] for some examples.

In this paper, we give a general method to find a set of identities defining the join of an arbitrary variety of finite monoids $\mathbf{V}$ and the *variety of finite locally trivial semigroups* $\mathbf{LI}$, as soon as one has a set of identities defining $\mathbf{V}$ and $\mathbf{V}$ verifies some criterion. Joins of that sort have been studied quite a lot in the literature we mentioned in the previous paragraph (e.g. in [6, 21, 9, 10]), but while these works usually rely heavily on profinite topology with some in-depth understanding of the structure of the elements of the so-called free pro-$\mathbf{V}$ monoids and free pro-$\mathbf{LI}$ semigroups, we present a method that reduces the use of profinite topology to the minimum and that relies mainly on algebraic and language-theoretic techniques. The variety $\mathbf{LI}$ is well-known to correspond to the class of languages for which membership only depends on bounded-length prefixes and suffixes of words. In [13], McKenzie, Segoufin and the author introduced the notion of *essentially*-$\mathbf{V}$ stamps (surjective morphisms $\varphi\colon \Sigma^* \to M$ for Σ an alphabet and M a finite monoid) to characterise the built-in ability that programs over monoids in $\mathbf{V}$ have to treat separately some constant-length beginning and ending of a word. Informally said, a stamp is essentially-$\mathbf{V}$ when it behaves like a stamp into a monoid of $\mathbf{V}$ as soon as a sufficiently long beginning and ending of the input word has been fixed. Our method builds on two results, that we prove in this article.

1. The first result is a characterisation in terms of identities of the class $\mathbf{EV}$ of essentially-$\mathbf{V}$ stamps given a set of identities E defining $\mathbf{V}$: a stamp is in $\mathbf{EV}$ if and only if it satisfies all identities $x^\omega yuzt^\omega = x^\omega yvzt^\omega$ for $u = v$ an identity in E and where x, y, z, t do appear neither in u nor in v.
2. The second result says that $\mathbf{EV}$ and $\mathbf{V} \vee \mathbf{LI}$ do coincide if and only if $\mathbf{V}$ verifies some criterion, that can be formulated in terms of quotient-expressibility in $\mathcal{L}(\mathbf{V})$: any language $L \in \mathcal{L}(\mathbf{V})$ must, for an arbitrary choice of x, y, be such that the quotient $u^{-1}Lv^{-1}$ for u and v long enough can be expressed as the quotient $(xu)^{-1}K(vy)^{-1}$ for a $K \in \mathcal{L}(\mathbf{V})$.

Using these results, we can find a set of identities defining $\mathbf{V} \vee \mathbf{LI}$ as soon as a set of identities defining $\mathbf{V}$ is known by proving that $\mathbf{V}$ verifies the criterion in point 2. Note that actually, for technical reasons, we work with the so-called *ne*-variety of stamps corresponding to $\mathbf{V} \vee \mathbf{LI}$ rather than directly with the variety of finite semigroups $\mathbf{V} \vee \mathbf{LI}$, but this is not a problem since a variety of finite semigroups can always be seen as an *ne*-variety of stamps and vice versa. We apply this method to reprove characterisations of the join of $\mathbf{LI}$ with each of the well-known varieties of finite monoids $\mathbf{R}$, $\mathbf{L}$, $\mathbf{J}$ and any variety of finite groups.

The author noticed after proving those results that his work actually forms a kind of rediscovery of the work of J. C. Costa in [9]. He defines an operator U associating to each set of identities E the exact same new set $U(E)$ of identities as in point 1. Costa then defines a property of cancellation for varieties of finite semigroups such that for any $\mathbf{V}$ verifying it, $U(E)$ defines $\mathbf{V} \vee \mathbf{LI}$ for E defining $\mathbf{V}$. He finally uses this result to derive characterisations of $\mathbf{V} \vee \mathbf{LI}$ for all the cases we are treating in our paper and many more.

What is, then, the contribution of our article? In a nutshell, it does mainly use algebraic and language-theoretic techniques while Costa's work relies heavily on profinite topology. In our setting, once the stage is set, all proofs are quite straightforward without real difficulties and rely on classical language-theoretic characterisations of the varieties under consideration. This is to contrast with Costa's work, that for instance draws upon the difficult analysis of the elements of free pro-$\mathbf{R}$ monoids by Almeida and Weil [5] to characterise $\mathbf{R} \vee \mathbf{LI}$.

Organisation of the article. Section 2 is dedicated to the necessary preliminaries. In Section 3, we recall the definition of essentially-$\mathbf{V}$ stamps and prove the characterisation by identities of point 1 above. Section 4 is then dedicated to the necessary and sufficient criterion for $\mathbf{EV}$ and $\mathbf{V} \vee \mathbf{LI}$ to coincide presented in point 2 and finally those results are applied to specific cases in Section 5. We finish with a short conclusion.

2 Preliminaries

We briefly introduce the mathematical material used in this paper. For the basics and the classical results of automata theory, we refer the reader to the two classical references of the domain by Eilenberg [11, 12] and Pin [14]. For definitions and results specific to varieties of stamps and associated profinite identities, see the articles by Straubing [20] and by Pin and Straubing [16]. We also assume some basic knowledge of topology.

General notations. Let $i \in \mathbb{N}$ be a natural number. We shall denote by $[i]$ the set of all natural numbers $n \in \mathbb{N}$ verifying $1 \leq n \leq i$.

Words and languages. Let Σ be a finite alphabet. We denote by Σ^* the set of all finite words over Σ. We also denote by Σ^+ the set of all finite non empty words over Σ, the empty word being denoted by ε. Our alphabets and words will always be finite, without further mention of this fact. Given a word $w \in \Sigma^*$, we denote its length by $|w|$ and the set of letters it contains by $\mathrm{alph}(w)$. Given $n \in \mathbb{N}$, we denote by $\Sigma^{\geq n}$, Σ^n and $\Sigma^{<n}$ the set of words over Σ of length, respectively, at least n, exactly n and less than n.

A *language over* Σ is a subset of Σ^*. A language is *regular* if it is recognised by a deterministic finite automaton. The *quotient of a language L over Σ relative to the words u and v over* Σ is the language, denoted by $u^{-1}Lv^{-1}$, of the words w such that $uwv \in L$.

Monoids, semigroups and varieties. A *semigroup* is a non-empty set equipped with an associative law that we will write multiplicatively. A *monoid* is a semigroup with an identity. An example of a semigroup is Σ^+, the free semigroup over Σ. Similarly Σ^* is the free monoid over Σ. A *morphism φ from a semigroup S to a semigroup* T is a function from S to T such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in S$. A morphism of monoids additionally requires that the identity is preserved. A semigroup T is a *subsemigroup* of a semigroup S if T is a subset of S and is equipped with the restricted law of S. Additionally the notion of submonoids requires the presence of the identity. A semigroup T *divides* a semigroup S if T is the image by a semigroup morphism of a subsemigroup of S. Division of monoids is defined in the same way. The *Cartesian (or direct) product* of two semigroups is simply the semigroup given by the Cartesian product of the two underlying sets equipped with the Cartesian product of their laws. An element s of a semigroup is *idempotent* if $ss = s$.

A *variety of finite monoids* is a non-empty class of finite monoids closed under Cartesian product and monoid division. A *variety of finite semigroups* is defined similarly. When dealing with varieties, we consider only finite monoids and semigroups, so we will drop the adjective finite when talking about varieties in the rest of this article.

Varieties of stamps. Let $f\colon \Sigma^* \to \Gamma^*$ be a morphism from the free monoid over an alphabet Σ to the free monoid over an alphabet Γ, that we might call an *all*-morphism. We say that f is an ne-*morphism* (non-erasing morphism) whenever $f(\Sigma) \subseteq \Gamma^+$.

We call *stamp* a surjective morphism $\varphi\colon \Sigma^* \to M$ for Σ an alphabet and M a finite monoid. We say that a stamp $\varphi\colon \Sigma^* \to M$ all-*divides* (respectively ne-*divides*) a stamp $\psi\colon \Gamma^* \to N$ whenever there exists an *all*-morphism (respectively *ne*-morphism) $f\colon \Sigma^* \to \Gamma^*$ and a surjective morphism $\alpha\colon \mathfrak{Im}(\psi \circ f) \to M$ such that $\varphi = \alpha \circ \psi \circ f$. The *direct product* of two stamps $\varphi\colon \Sigma^* \to M$ and $\psi\colon \Sigma^* \to N$ is the stamp $\varphi \times \psi\colon \Sigma^* \to K$ such that K is the submonoid of $M \times N$ generated by $\{(\varphi(a), \psi(a)) \mid a \in \Sigma\}$ and $\varphi \times \psi(a) = (\varphi(a), \psi(a))$ for all $a \in \Sigma$.

An all-*variety of stamps* (respectively ne-*variety of stamps*) is a non-empty class of stamps closed under direct product and *all*-division (respectively *ne*-division).

We will often use the following characteristic index of stamps, defined in [8]. Consider a stamp $\varphi\colon \Sigma^* \to M$. As M is finite there is a $k \in \mathbb{N}_{>0}$ such that $\varphi(\Sigma^{2k}) = \varphi(\Sigma^k)$: this implies that $\varphi(\Sigma^k)$ is a semigroup. The least such k is called the *stability index* of φ.

Varieties of languages. A language L over an alphabet Σ is *recognised by a monoid M* if there is a morphism $\varphi\colon \Sigma^* \to M$ and $F \subseteq M$ such that $L = \varphi^{-1}(F)$. We also say that φ *recognises* L. It is well known that a language is regular if and only if it is recognised by a finite monoid. The *syntactic congruence* of L, denoted by $\sim_L$, is the equivalence relation on Σ^* defined by $u \sim_L v$ for $u, v \in \Sigma^*$ whenever for all $x, y \in \Sigma^*$, $xuy \in L$ if and only if $xvy \in L$. The quotient $\Sigma^*/{\sim_L}$ is a monoid, called *the syntactic monoid of* L, that recognises L via the *syntactic morphism η_L of* L sending any word u to its equivalence class $[u]_{\sim_L}$ for $\sim_L$. A stamp $\varphi\colon \Sigma^* \to M$ recognises L if and only if there exists a surjective morphism $\varphi\colon M \to \Sigma^*/{\sim_L}$ verifying $\eta_L = \alpha \circ \varphi$.

A *class of languages* $\mathcal{C}$ is a correspondence that associates a set $\mathcal{C}(\Sigma)$ to each alphabet Σ. A *(*all*-)variety of languages* (respectively an ne-*variety of languages*) $\mathcal{V}$ is a non-empty class of regular languages closed under Boolean operations, quotients and inverses of *all*-morphisms (respectively *ne*-morphisms). A classical result of Eilenberg [12, Chapter VII, Section 3] says that there is a bijective correspondence between varieties of monoids and varieties of languages: to each variety of monoids $\mathbf{V}$ we can bijectively associate $\mathcal{L}(\mathbf{V})$ the variety of languages whose syntactic monoids belong to $\mathbf{V}$. This was generalised by Straubing [20] to varieties of stamps: to each *all*-variety (respectively *ne*-variety) of stamps $\mathbf{V}$ we can bijectively associate $\mathcal{L}(\mathbf{V})$ the *all*-variety (respectively *ne*-variety) of languages whose syntactic morphisms belong to $\mathbf{V}$. Given two *all*-varieties (respectively *ne*-varieties) of stamps $\mathbf{V_1}$ and $\mathbf{V_2}$, we have $\mathbf{V_1} \subseteq \mathbf{V_2} \Leftrightarrow \mathcal{L}(\mathbf{V_1}) \subseteq \mathcal{L}(\mathbf{V_2})$.

For $\mathbf{V}$ a variety of monoids, we define $\langle \mathbf{V} \rangle_{all}$ the *all*-variety of all stamps $\varphi\colon \Sigma^* \to M$ such that $M \in \mathbf{V}$. Of course, in that case $\mathcal{L}(\mathbf{V}) = \mathcal{L}(\langle \mathbf{V} \rangle_{all})$. Similarly, for $\mathbf{V}$ a variety of semigroups, we define $\langle \mathbf{V} \rangle_{ne}$ the *ne*-variety of all stamps $\varphi\colon \Sigma^* \to M$ such that $\varphi(\Sigma^+) \in \mathbf{V}$. In that case, we consider $\mathcal{L}(\mathbf{V})$ to be the *ne*-variety of languages corresponding to $\langle \mathbf{V} \rangle_{ne}$. The operations $\langle \cdot \rangle_{all}$ and $\langle \cdot \rangle_{ne}$ form bijective correspondences between varieties of monoids and *all*-varieties of stamps and between varieties of semigroups and *ne*-varieties of stamps, respectively (see [20]).

Identities. Let Σ be an alphabet. Given $u, v \in \Sigma^*$, we set

$$r(u, v) = \min\{|M| \mid \exists \varphi\colon \Sigma^* \to M \text{ stamp s.t. } \varphi(u) \neq \varphi(v)\}$$

and $d(u, v) = 2^{-r(u,v)}$, using the conventions that $\min \emptyset = +\infty$ and $2^{-\infty} = 0$. Then d is a metric on Σ^*. The completion of the metric space (Σ^*, d), denoted by $(\widehat{\Sigma^*}, \widehat{d})$, is a metric monoid called the *free profinite monoid on* Σ^*. Its elements are all the formal limits $\lim_{n\to\infty} x_n$ of Cauchy sequences $(x_n)_{n\geq 0}$ in (Σ^*, d) and the metric d on Σ^* extends to a metric $\widehat{d}$ on $\widehat{\Sigma^*}$ defined by $\widehat{d}(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = \lim_{n\to\infty} d(x_n, y_n)$ for Cauchy sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ in (Σ^*, d). Note that, when it is clear from the context, we usually do not make the metric explicit when talking about a metric space. One important example of elements of $\widehat{\Sigma^*}$ is given by the elements $x^\omega = \lim_{n\to\infty} x^{n!}$ for all $x \in \Sigma^*$.

Every finite monoid M is considered to be a complete metric space equipped with the discrete metric d defined by $d(m, n) = \begin{cases} 0 & \text{if } m = n \\ 1 & \text{otherwise} \end{cases}$ for all $m, n \in M$. Every stamp $\varphi\colon \Sigma^* \to M$ extends uniquely to a uniformly continuous morphism $\widehat{\varphi}\colon \widehat{\Sigma^*} \to M$ with $\widehat{\varphi}(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \varphi(x_n)$ for every Cauchy sequence $(x_n)_{n\geq 0}$ in Σ^*. Similarly, every *all*-morphism $f\colon \Sigma^* \to \Gamma^*$ extends uniquely to a uniformly continuous morphism $\widehat{f}\colon \widehat{\Sigma^*} \to \widehat{\Gamma^*}$ with $\widehat{f}(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$ for every Cauchy sequence $(x_n)_{n\geq 0}$ in Σ^*.

For $u, v \in \widehat{A^*}$ with A an alphabet, we say that a stamp $\varphi\colon \Sigma^* \to M$ all-*satisfies* (respectively ne-*satisfies*) the identity $u = v$ if for every *all*-morphism (respectively *ne*-morphism) $f\colon A^* \to \Sigma^*$, it holds that $\widehat{\varphi} \circ \widehat{f}(u) = \widehat{\varphi} \circ \widehat{f}(v)$. Given a set of identities E, we denote by $[\![E]\!]_{all}$ (respectively $[\![E]\!]_{ne}$) the class of stamps *all*-satisfying (respectively *ne*-satisfying) all the identities of E. When $[\![E]\!]_{all}$ (respectively $[\![E]\!]_{ne}$) is equal to an *all*-variety (respectively *ne*-variety) of stamps $\mathbf{V}$, we say that E all-*defines* (respectively ne-*defines*) $\mathbf{V}$.

▶ **Theorem 1** ([16, Theorem 2.1]). *A class of stamps is an* all*-variety (respectively* ne*-variety) of stamps if and only if it can be* all*-defined (respectively* ne*-defined) by a set of identities.*

To give some examples, the classical varieties of monoids $\mathbf{J}$, $\mathbf{R}$ and $\mathbf{L}$ can be characterised by identities in the following way:

$$\begin{aligned}
\langle \mathbf{R}\rangle_{all} &= [\![(ab)^\omega a = (ab)^\omega]\!]_{all} = [\![(ab)^\omega a = (ab)^\omega]\!]_{ne}\\
\langle \mathbf{L}\rangle_{all} &= [\![b(ab)^\omega = (ab)^\omega]\!]_{all} = [\![b(ab)^\omega = (ab)^\omega]\!]_{ne}\\
\langle \mathbf{J}\rangle_{all} &= [\![(ab)^\omega a = (ab)^\omega, b(ab)^\omega = (ab)^\omega]\!]_{all} = [\![(ab)^\omega a = (ab)^\omega, b(ab)^\omega = (ab)^\omega]\!]_{ne} .
\end{aligned}$$

Finite locally trivial semigroups and the join operation. The variety $\mathbf{LI}$ of finite locally trivial semigroups is well-known to verify $\langle \mathbf{LI}\rangle_{ne} = [\![x^\omega y x^\omega = x^\omega]\!]_{ne}$ and to be such that for any alphabet Σ, the set $\mathcal{L}(\mathbf{LI})(\Sigma)$ consists of all Boolean combinations of languages of the form $u\Sigma^*$ or $\Sigma^* u$ for $u \in \Sigma^*$, or equivalently of all languages of the form $U\Sigma^* V \cup W$ with $U, V, W \subseteq \Sigma^*$ finite (see [14, p. 38]).

Given a variety of monoids $\mathbf{V}$, the join of $\mathbf{V}$ and $\mathbf{LI}$, denoted by $\mathbf{V} \vee \mathbf{LI}$, is the inclusion-wise least variety of semigroups containing both $\mathbf{V}$ and $\mathbf{LI}$. In fact, a finite semigroup S belongs to $\mathbf{V} \vee \mathbf{LI}$ if and only if there exist $M \in \mathbf{V}$ and $T \in \mathbf{LI}$ such that S divides the semigroup $M \times T$. (See [12, Chapter V, Exercise 1.1].) We can prove the following adaptation to *ne*-varieties of the classical results about joins (see the appendix for the proof).

▶ **Proposition 2.** *Let $\mathbf{V}$ be a variety of monoids. Then $\langle \mathbf{V} \vee \mathbf{LI}\rangle_{\mathrm{ne}}$ is the inclusion-wise least* ne*-variety of stamps containing both $\langle \mathbf{V}\rangle_{\mathrm{all}}$ and $\langle \mathbf{LI}\rangle_{\mathrm{ne}}$. Moreover, $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ is the inclusion-wise least* ne*-variety of languages containing both $\mathcal{L}(\mathbf{V})$ and $\mathcal{L}(\mathbf{LI})$ and verifies that $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$ is the Boolean closure of $\mathcal{L}(\mathbf{V})(\Sigma) \cup \mathcal{L}(\mathbf{LI})(\Sigma)$ for each alphabet Σ.*

3 Essentially-$\mathbf{V}$ stamps

In this section, we give a characterisation of essentially-$\mathbf{V}$ stamps (first defined in [13]), for $\mathbf{V}$ a variety of monoids, in terms of identities. We first recall the definition.

▶ **Definition 3.** *Let $\mathbf{V}$ be a variety of monoids. Let $\varphi\colon \Sigma^* \to M$ be a stamp and let s be its stability index.*

We say that φ is essentially-$\mathbf{V}$ *whenever there exists a stamp $\mu\colon \Sigma^* \to N$ with $N \in \mathbf{V}$ such that for all $u, v \in \Sigma^*$, we have*

$$\mu(u) = \mu(v) \Rightarrow \big(\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s\big) .$$

We will denote by $\mathbf{EV}$ the class of all essentially-$\mathbf{V}$ stamps.[1]

Now, we give a characterisation for a stamp to be essentially-$\mathbf{V}$, based on a specific congruence depending on that stamp.

[1] Essentially-$\mathbf{V}$ stamps are called that way by analogy with quasi-$\mathbf{V}$ stamps and the class of essentially-$\mathbf{V}$ stamps is denoted by $\mathbf{EV}$ by analogy with $\mathbf{QV}$, the notation for the class of quasi-$\mathbf{V}$ stamps. This makes sense since the initial motivation for the definition of essentially-$\mathbf{V}$ stamps was to capture the class of stamps into monoids of $\mathbf{V}$ that have the additional ability to treat separately some constant-length beginning and ending of a word. This ability can indeed be seen as orthogonal to the additional ability of stamps into monoids in $\mathbf{V}$ to perform modular counting on the positions of letters in a word, which is often handled by considering quasi-$\mathbf{V}$ stamps. (See [13] for more.) Our definition of $\mathbf{EV}$ does unfortunately not coincide with the usual definition of $\mathbf{EV}$, that classically denotes the variety of monoids M such that the submonoid generated by the idempotents of M is in $\mathbf{V}$. (This comes, among others, from the fact that the obtained variety of monoids does always contain at least all finite groups.)

▶ **Definition 4.** *Let $\varphi\colon \Sigma^* \to M$ be a stamp and let s be its stability index. We define the equivalence relation $\equiv_\varphi$ on Σ^* by $u \equiv_\varphi v$ for $u, v \in \Sigma^*$ whenever $\varphi(xuy) = \varphi(xvy)$ for all $x, y \in \Sigma^{\geq s}$.*

▶ **Proposition 5.** *Let $\varphi\colon \Sigma^* \to M$ be a stamp. Then $\equiv_\varphi$ is a congruence of finite index and for any variety of monoids $\mathbf{V}$, we have $\varphi \in \mathbf{EV}$ if and only if $\Sigma^*/{\equiv_\varphi} \in \mathbf{V}$.*

Proof. Let us denote by s the stability index of φ.

The equivalence relation $\equiv_\varphi$ is a congruence because given $u, v \in \Sigma^*$ verifying $u \equiv_\varphi v$, for all $\alpha, \beta \in \Sigma^*$, we have $\alpha u\beta \equiv_\varphi \alpha v\beta$ since for any $x, y \in \Sigma^{\geq s}$, it holds that $\varphi(x\alpha u\beta y) = \varphi(x\alpha v\beta y)$ because $x\alpha, \beta y \in \Sigma^{\geq s}$. Furthermore, this congruence is of finite index because for all $u, v \in \Sigma^*$, we have that $\varphi(u) = \varphi(v)$ implies $u \equiv_\varphi v$.

Let now $\mathbf{V}$ be a variety of monoids. Assume first that $\Sigma^*/{\equiv_\varphi} \in \mathbf{V}$. It is quite direct to see that $\varphi \in \mathbf{EV}$, as the stamp $\mu\colon \Sigma^* \to \Sigma^*/{\equiv_\varphi}$ defined by $\mu(w) = [w]_{\equiv_\varphi}$ for all $w \in \Sigma^*$ witnesses this fact. Assume then that $\varphi \in \mathbf{EV}$. This means that there exists a stamp $\mu\colon \Sigma^* \to N$ with $N \in \mathbf{V}$ such that for all $u, v \in \Sigma^*$, we have

$$\mu(u) = \mu(v) \Rightarrow \big(\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s\big)\ .$$

Now consider $u, v \in \Sigma^*$ such that $\mu(u) = \mu(v)$. For any $x, y \in \Sigma^{\geq s}$, we have that $x = x_1x_2$ with $x_1 \in \Sigma^*$ and $x_2 \in \Sigma^s$ as well as $y = y_1y_2$ with $y_1 \in \Sigma^s$ and $y_2 \in \Sigma^*$, so that $\varphi(xuy) = \varphi(x_1)\varphi(x_2uy_1)\varphi(y_2) = \varphi(x_1)\varphi(x_2vy_1)\varphi(y_2) = \varphi(xvy)$. Hence, $u \equiv_\varphi v$. Therefore, for all $u, v \in \Sigma^*$, we have that $\mu(u) = \mu(v)$ implies $u \equiv_\varphi v$, so we can define the mapping $\alpha\colon N \to \Sigma^*/{\equiv_\varphi}$ such that $\alpha(\mu(w)) = [w]_{\equiv_\varphi}$ for all $w \in \Sigma^*$. It is easy to check that α is actually a surjective morphism. Thus, we can conclude that $\Sigma^*/{\equiv_\varphi}$, which divides N, belongs to $\mathbf{V}$. ◀

Using this characterisation, we prove that given a set of identities *ne*-defining $\langle \mathbf{V} \rangle_{all}$ for a variety of monoids $\mathbf{V}$, we get a set of identities *ne*-defining $\mathbf{EV}$.

▶ **Proposition 6.** *Let $\mathbf{V}$ be a variety of monoids and let E be a set of identities such that $\langle \mathbf{V} \rangle_{\mathrm{all}} = [\![E]\!]_{\mathrm{ne}}$. Then $\mathbf{EV}$ is an* ne*-variety of stamps and*

$$\mathbf{EV} = [\![x^\omega yuzt^\omega = x^\omega yvzt^\omega \mid u = v \in E, x, y, z, t \notin \mathrm{alph}(u) \cup \mathrm{alph}(v)]\!]_{\mathrm{ne}}\ .$$

Proof. Let

$$F = \{x^\omega yuzt^\omega = x^\omega yvzt^\omega \mid u = v \in E, x, y, z, t \notin \mathrm{alph}(u) \cup \mathrm{alph}(v)\}\ .$$

Central to the proof is the following claim.

▷ Claim 7. Let $\varphi\colon \Sigma^* \to M$ be a stamp. Consider the stamp $\mu\colon \Sigma^* \to \Sigma^*/{\equiv_\varphi}$ defined by $\mu(w) = [w]_{\equiv_\varphi}$ for all $w \in \Sigma^*$. It holds that for all $u, v \in \widehat{\Sigma^*}$,

$$\widehat{\mu}(u) = \widehat{\mu}(v) \Leftrightarrow \big(\widehat{\varphi}(\alpha^\omega\beta u\gamma\delta^\omega) = \widehat{\varphi}(\alpha^\omega\beta v\gamma\delta^\omega) \quad \forall \alpha, \beta, \gamma, \delta \in \Sigma^+\big)\ .$$

Before we prove Claim 7, we use it to prove that $\mathbf{EV} = [\![F]\!]_{ne}$.

Inclusion from left to right. Let $\varphi\colon \Sigma^* \to M$ be a stamp in $\mathbf{EV}$. Consider the stamp $\mu\colon \Sigma^* \to \Sigma^*/{\equiv_\varphi}$ defined by $\mu(w) = [w]_{\equiv_\varphi}$ for all $w \in \Sigma^*$. Since $\varphi \in \mathbf{EV}$, Proposition 5 tells us that $\Sigma^*/{\equiv_\varphi} \in \mathbf{V}$, hence $\mu \in \langle \mathbf{V} \rangle_{all}$.

Let us consider any identity $x^\omega yuzt^\omega = x^\omega yvzt^\omega \in F$. It is written on an alphabet B that is the union of the alphabet A on which $u = v \in E$ is written and of $x, y, z, t \in B \setminus A$. Let $f\colon B^* \to \Sigma^*$ be an *ne*-morphism. Since $\mu \in \langle \mathbf{V} \rangle_{all}$, we have that μ *ne*-satisfies the identity $u = v$, so that $\widehat{\mu}(\widehat{f}(u)) = \widehat{\mu}(\widehat{f}(v))$. Notice that we have that $\widehat{f}(x^\omega) = f(x)^\omega$ as well as $\widehat{f}(t^\omega) = f(t)^\omega$ and that $f(x), f(y), f(z), f(t) \in \Sigma^+$ because f is non-erasing. Therefore, we have

$$\begin{aligned}\widehat{\varphi}\big(\widehat{f}(x^\omega yuzt^\omega)\big) &= \widehat{\varphi}\big(f(x)^\omega f(y)\widehat{f}(u)f(z)f(t)^\omega\big)\\ &= \widehat{\varphi}\big(f(x)^\omega f(y)\widehat{f}(v)f(z)f(t)^\omega\big)\\ &= \widehat{\varphi}\big(\widehat{f}(x^\omega yvzt^\omega)\big)\end{aligned}$$

by Claim 7. As this holds for any ne-morphism $f\colon B^* \to \Sigma^*$, we can conclude that φ ne-satisfies the identity $x^\omega yuzt^\omega = x^\omega yvzt^\omega$.

This is true for any identity in F, so $\varphi \in [\![F]\!]_{ne}$. In conclusion, $\mathbf{EV} \subseteq [\![F]\!]_{ne}$.

Inclusion from right to left. Let $\varphi\colon \Sigma^* \to M$ be a stamp in $[\![F]\!]_{ne}$. Consider the stamp $\mu\colon \Sigma^* \to \Sigma^*/{\equiv_\varphi}$ defined by $\mu(w) = [w]_{\equiv_\varphi}$ for all $w \in \Sigma^*$. We are now going to show that $\mu \in \langle \mathbf{V} \rangle_{all}$.

Take any identity $u = v \in E$ written on an alphabet A. There exists an identity $x^\omega yuzt^\omega = x^\omega yvzt^\omega \in F$ written on an alphabet B such that $A \subseteq B$ and $x, y, z, t \in B \setminus A$. Let $f\colon A^* \to \Sigma^*$ be an ne-morphism.

Take any $\alpha, \beta, \gamma, \delta \in \Sigma^+$. Let us define the ne-morphism $g\colon B^* \to \Sigma^*$ as the unique one which extends f by letting $g(x) = \alpha$, $g(y) = \beta$, $g(z) = \gamma$ and $g(t) = \delta$. Observe in particular that $\widehat{g}(w) = \widehat{f}(w)$ for any $w \in \widehat{A^*}$ and that $\widehat{g}(x^\omega) = g(x)^\omega = \alpha^\omega$ as well as $\widehat{g}(t^\omega) = \delta^\omega$. Now, as φ ne-satisfies $x^\omega yuzt^\omega = x^\omega yvzt^\omega$, we have that

$$\widehat{\varphi}\big(\alpha^\omega\beta\widehat{f}(u)\gamma\delta^\omega\big) = \widehat{\varphi}\big(\widehat{g}(x^\omega yuzt^\omega)\big) = \widehat{\varphi}\big(\widehat{g}(x^\omega yvzt^\omega)\big) = \widehat{\varphi}\big(\alpha^\omega\beta\widehat{f}(v)\gamma\delta^\omega\big)\ .$$

Since this holds for any $\alpha, \beta, \gamma, \delta \in \Sigma^+$, by Claim 7, we have that $\widehat{\mu}(\widehat{f}(u)) = \widehat{\mu}(\widehat{f}(v))$.

Therefore, $\widehat{\mu}(\widehat{f}(u)) = \widehat{\mu}(\widehat{f}(v))$ for any ne-morphism $f\colon A^* \to \Sigma^*$, which means that μ ne-satisfies $u = v$.

Since this holds for any $u = v \in E$, we have that $\mu \in \langle \mathbf{V} \rangle_{all}$, which implies that $\Sigma^*/{\equiv_\varphi} \in \mathbf{V}$ and thus $\varphi \in \mathbf{EV}$ by Proposition 5. In conclusion, $[\![F]\!]_{ne} \subseteq \mathbf{EV}$.

The claim still needs to be proved.

Proof of Claim 7. Let $\varphi\colon \Sigma^* \to M$ be a stamp of stability index s. Consider the stamp $\mu\colon \Sigma^* \to \Sigma^*/{\equiv_\varphi}$ defined by $\mu(w) = [w]_{\equiv_\varphi}$ for all $w \in \Sigma^*$. We now want to show that for all $u, v \in \widehat{\Sigma^*}$,

$$\widehat{\mu}(u) = \widehat{\mu}(v) \Leftrightarrow \big(\widehat{\varphi}(\alpha^\omega\beta u\gamma\delta^\omega) = \widehat{\varphi}(\alpha^\omega\beta v\gamma\delta^\omega) \quad \forall \alpha, \beta, \gamma, \delta \in \Sigma^+\big)\ .$$

Let $u, v \in \widehat{\Sigma^*}$. There exist two Cauchy sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ in Σ^* such that $u = \lim_{n\to\infty} u_n$ and $v = \lim_{n\to\infty} v_n$. As $\Sigma^*/{\equiv_\varphi}$ and M are discrete, we have that all four Cauchy sequences $\big(\mu(u_n)\big)_{n\geq 0}$, $\big(\varphi(u_n)\big)_{n\geq 0}$, $\big(\mu(v_n)\big)_{n\geq 0}$ and $\big(\varphi(v_n)\big)_{n\geq 0}$ are ultimately constant. So there exists $k \in \mathbb{N}$ such that $\widehat{\mu}(u) = \mu(u_k)$, $\widehat{\varphi}(u) = \varphi(u_k)$, $\widehat{\mu}(v) = \mu(v_k)$ and $\widehat{\varphi}(v) = \varphi(v_k)$.

Assume first that $\widehat{\mu}(u) = \widehat{\mu}(v)$. Take any $\alpha, \beta, \gamma, \delta \in \Sigma^+$. Since M is discrete, both Cauchy sequences $\big(\varphi(\alpha^{n!})\big)_{n\geq 0}$ and $\big(\varphi(\delta^{n!})\big)_{n\geq 0}$ are ultimately constant. So there exists $l \in \mathbb{N}$ such that for all $m \in \mathbb{N}, m \geq l$, we have $\widehat{\varphi}(\alpha^\omega) = \varphi(\alpha^{m!})$ and $\widehat{\varphi}(\delta^\omega) = \varphi(\delta^{m!})$. Hence, taking $m \in \mathbb{N}, m \geq l$ such that $\left|\alpha^{m!}\beta\right| \geq s$ and $\left|\gamma\delta^{m!}\right| \geq s$, it follows that

$$\widehat{\varphi}(\alpha^\omega\beta u\gamma\delta^\omega) = \varphi(\alpha^{m!}\beta u_k\gamma\delta^{m!}) = \varphi(\alpha^{m!}\beta v_k\gamma\delta^{m!}) = \widehat{\varphi}(\alpha^\omega\beta v\gamma\delta^\omega)$$

because $[u_k]_{\equiv_\varphi} = \widehat{\mu}(u) = \widehat{\mu}(v) = [v_k]_{\equiv_\varphi}$. Thus, we have that

$$\widehat{\varphi}(\alpha^\omega \beta u \gamma \delta^\omega) = \widehat{\varphi}(\alpha^\omega \beta v \gamma \delta^\omega)$$

for all $\alpha, \beta, \gamma, \delta \in \Sigma^+$.

Assume then that $\widehat{\varphi}(\alpha^\omega \beta u \gamma \delta^\omega) = \widehat{\varphi}(\alpha^\omega \beta v \gamma \delta^\omega)$ for all $\alpha, \beta, \gamma, \delta \in \Sigma^+$. Take any $\alpha, \beta \in \Sigma^{\geq s}$. Since $\varphi(\Sigma^s)$ is a finite semigroup and verifies that $\varphi(\Sigma^s) = \varphi(\Sigma^s)^2$, by a classical result in finite semigroup theory (see e.g. [14, Chapter 1, Proposition 1.12]), we have that there exist $\alpha_1, e, f, \beta_2 \in \Sigma^s$ and $\alpha_2, \beta_1 \in \Sigma^{\geq s}$ such that $\varphi(\alpha_1 e \alpha_2) = \varphi(\alpha)$ and $\varphi(\beta_1 f \beta_2) = \varphi(\beta)$ with $\varphi(e)$ and $\varphi(f)$ idempotents. Now, since $\varphi(e)$ is idempotent, we have that

$$\widehat{\varphi}(e^\omega) = \widehat{\varphi}(\lim_{n\to\infty} e^{n!}) = \lim_{n\to\infty} \varphi(e^{n!}) = \lim_{n\to\infty} \varphi(e)^{n!} = \varphi(e)$$

and similarly, $\widehat{\varphi}(f^\omega) = \varphi(f)$. So it follows that

$$\begin{aligned}
\varphi(\alpha u_k \beta) &= \varphi(\alpha_1 e \alpha_2 u_k \beta_1 f \beta_2) \\
&= \widehat{\varphi}(\alpha_1 e^\omega \alpha_2 u \beta_1 f^\omega \beta_2) \\
&= \widehat{\varphi}(\alpha_1 e^\omega \alpha_2 v \beta_1 f^\omega \beta_2) \\
&= \varphi(\alpha_1 e \alpha_2 v_k \beta_1 f \beta_2) \\
&= \varphi(\alpha v_k \beta) \ .
\end{aligned}$$

As this is true for any $\alpha, \beta \in \Sigma^{\geq s}$, by definition it holds that $u_k \equiv_\varphi v_k$, hence $\widehat{\mu}(u) = \mu(u_k) = \mu(v_k) = \widehat{\mu}(v)$. ◁

This concludes the proof of the proposition. ◀

4 Essentially-V stamps and the join of V and LI

In this section, we establish the link between essentially-$\mathbf{V}$ stamps and $\mathbf{V} \vee \mathbf{LI}$ and give a criterion that characterises exactly when they do correspond.

More precisely, consider the following criterion for a variety of monoids $\mathbf{V}$.

▶ **Criterion (A).** For any $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ with Σ an alphabet, we have $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$ for all $x, y \in \Sigma^*$.

It is a kind of mild closure condition that appears to be a sufficient and necessary condition for $\mathbf{EV}$ and $\mathbf{V} \vee \mathbf{LI}$ to correspond.

▶ **Proposition 8.** *Let $\mathbf{V}$ be a variety of monoids. Then $\langle \mathbf{V} \vee \mathbf{LI} \rangle_{\mathrm{ne}} \subseteq \mathbf{EV}$ and equality holds if and only if $\mathbf{V}$ verifies criterion (A).*

Why this proposition is useful to give characterisations of $\mathbf{V} \vee \mathbf{LI}$ in terms of identities will become clear in the next section. For now, we focus on its proof, that entirely relies on the following characterisation of the languages recognised by essentially-$\mathbf{V}$ stamps.

▶ **Proposition 9.** *Let $\mathbf{V}$ be a variety of monoids. For any alphabet Σ, the set $\mathcal{L}(\mathbf{EV})(\Sigma)$ consists of all Boolean combinations of languages of the form xLy for $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ and $x, y \in \Sigma^*$.*

Proof. Let $\mathcal{C}$ be the class of languages such that for any alphabet Σ, the set $\mathcal{C}(\Sigma)$ consists of all Boolean combinations of languages of the form xLy for $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ and $x, y \in \Sigma^*$.

Let Σ be an alphabet. We need to show that $\mathcal{L}(\mathbf{EV})(\Sigma) = \mathcal{C}(\Sigma)$.

Inclusion from right to left. Let $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ and $x, y \in \Sigma^*$. Let $\mu\colon \Sigma^* \to N$ be the syntactic morphism of L: this implies that $N \in \mathbf{V}$ and that there exists $F \subseteq N$ such that $L = \mu^{-1}(F)$. Let also $\varphi\colon \Sigma^* \to M$ be the syntactic morphism of the language $xLy = x\Sigma^*y \cap \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|}$ and let s be its stability index. We then consider $u, v \in \Sigma^*$ such that $\mu(u) = \mu(v)$. Take any $x', y' \in \Sigma^*$ such that $|x'| \geq |x|$ and $|y'| \geq |y|$. We clearly have that $x'uy' \in x\Sigma^*y$ if and only if $x'vy' \in x\Sigma^*y$. Moreover, $x' = x'_1x'_2$ for $x'_1 \in \Sigma^{|x|}$ and $x'_2 \in \Sigma^*$ and $y' = y'_1y'_2$ for $y'_1 \in \Sigma^*$ and $y'_2 \in \Sigma^{|y|}$, so that

$$\begin{aligned} x'uy' \in \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|} &\Leftrightarrow \mu(x'_2uy'_1) \in F \\ &\Leftrightarrow \mu(x'_2vy'_1) \in F \\ &\Leftrightarrow x'vy' \in \Sigma^{|x|}\mu^{-1}(F)\Sigma^{|y|} . \end{aligned}$$

Hence, $x'uy' \in xLy$ if and only if $x'vy' \in xLy$ for all $x', y' \in \Sigma^*$ such that $|x'| \geq |x|$ and $|y'| \geq |y|$, so that, by definition of the stability index s of φ and as φ is the syntactic morphism of xLy, we have $\varphi(x'uy') = \varphi(x'vy')$ for all $x', y' \in \Sigma^s$. Thus, it follows that $\varphi \in \mathbf{EV}$.

This implies that $xLy \in \mathcal{L}(\mathbf{EV})(\Sigma)$. Therefore, since this is true for any $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ and $x, y \in \Sigma^*$ and since $\mathcal{L}(\mathbf{EV})(\Sigma)$ is closed under Boolean operations, we can conclude that $\mathcal{C}(\Sigma) \subseteq \mathcal{L}(\mathbf{EV})(\Sigma)$.

Inclusion from left to right. Let $L \in \mathcal{L}(\mathbf{EV})(\Sigma)$ and let $\varphi\colon \Sigma^* \to M$ be its syntactic morphism: it is an essentially-$\mathbf{V}$ stamp. Given s its stability index, this means there exists a stamp $\mu\colon \Sigma^* \to N$ with $N \in \mathbf{V}$ such that for all $u, v \in \Sigma^*$, we have

$$\mu(u) = \mu(v) \Rightarrow \big(\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s\big) .$$

For each $m \in N$ and $x, y \in \Sigma^s$ consider the language $x\mu^{-1}(m)y$. For any two words $w, w' \in x\mu^{-1}(m)y$, we have $w = xuy$ and $w' = xvy$ with $\mu(u) = \mu(v) = m$, so that $\varphi(w) = \varphi(w')$. By definition of the syntactic morphism, this means that for all $m \in N$ and $x, y \in \Sigma^s$, either $x\mu^{-1}(m)y \subseteq L$ or $x\mu^{-1}(m)y \cap L = \emptyset$. Therefore, there exists a set $E \subseteq N \times \Sigma^s \times \Sigma^s$ such that $L \cap \Sigma^{\geq 2s} = \bigcup_{(m,x,y) \in E} x\mu^{-1}(m)y$, hence

$$L = \bigcup_{(m,x,y) \in E} x\mu^{-1}(m)y \cup F$$

for a certain $F \subseteq \Sigma^{<2s}$.

Take $w \in F$. We have that $\{w\} = w\Sigma^* \cap \bigcap_{a \in \Sigma}(\Sigma^* \setminus wa\Sigma^*)$ with $\Sigma^* \in \mathcal{L}(\mathbf{V})(\Sigma)$. Thus, the singleton language $\{w\}$ belongs to $\mathcal{C}(\Sigma)$ and since this is true for any $w \in F$ and F is finite, we can deduce from this that F is in $\mathcal{C}(\Sigma)$, as the latter is trivially closed under Boolean operations.

Now, for all $m \in N$, the language $\mu^{-1}(m)$ belongs to $\mathcal{L}(\mathbf{V})(\Sigma)$, so we finally have $L \in \mathcal{C}(\Sigma)$. This is true for any $L \in \mathcal{L}(\mathbf{EV})(\Sigma)$, so in conclusion, $\mathcal{L}(\mathbf{EV})(\Sigma) \subseteq \mathcal{C}(\Sigma)$. ◀

Proposition 8 then follows from the two next lemmata, that are both easy consequences of Proposition 9. For completeness, we give the proofs in the appendix.

▶ **Lemma 10.** *Let* $\mathbf{V}$ *be a variety of monoids. Then* $\langle \mathbf{V} \vee \mathbf{LI} \rangle_{\mathrm{ne}} \subseteq \mathbf{EV}$.

▶ **Lemma 11.** *Let* $\mathbf{V}$ *be a variety of monoids. Then* $\mathbf{EV} \subseteq \langle \mathbf{V} \vee \mathbf{LI} \rangle_{\mathrm{ne}}$ *if and only if* $\mathbf{V}$ *verifies criterion (A).*

5 Applications

In this last section, we use the link between essentially-$\mathbf{V}$ stamps and $\mathbf{V} \vee \mathbf{LI}$ to reprove some characterisations of joins between $\mathbf{LI}$ and some well-known varieties of monoids in terms of identities.

One thing seems at first glance a bit problematic about proving that a variety of monoids $\mathbf{V}$ satisfies criterion (A). Indeed, to this end, one needs to prove that certain languages belong to $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$; however, this poses a problem when one's goal is precisely to characterise $\mathbf{V} \vee \mathbf{LI}$, because one shall a priori not know more about $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ than what is given by Proposition 2. Nevertheless, there is a natural sufficient condition for criterion (A) to hold that depends only on $\mathcal{L}(\mathbf{V})$: if given any language $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ and any $x, y \in \Sigma^*$ with Σ an alphabet, there exists a language $K \in \mathcal{L}(\mathbf{V})(\Sigma)$ such that L is equal to the quotient $x^{-1}Ky^{-1}$, then $\mathbf{V}$ verifies criterion (A). We don't know whether this quotient-expressibility condition that solely depends on the variety $\mathbf{V}$ (without explicit reference to $\mathbf{LI}$) is actually equivalent to it satisfying criterion (A), but we can prove such an equivalence for a weaker quotient-expressibility condition for $\mathbf{V}$. The proof is to be found in the appendix.

▶ **Proposition 12.** *Let $\mathbf{V}$ be a variety of monoids. Then $\mathbf{V}$ satisfies criterion (A) if and only if for any $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ and any $x, y \in \Sigma^*$ with Σ an alphabet, there exist $k, l \in \mathbb{N}$ such that for all $u \in \Sigma^k, v \in \Sigma^l$, there exists a language $K \in \mathcal{L}(\mathbf{V})(\Sigma)$ verifying $u^{-1}Lv^{-1} = (xu)^{-1}K(vy)^{-1}$.*

This quotient-expressibility condition appears to be particularly useful to prove that a variety of monoids $\mathbf{V}$ does not satisfy criterion (A) without needing to understand what $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ is. We demonstrate this for the variety of finite commutative and idempotent monoids $\mathbf{J_1}$.

▶ **Proposition 13.** $\mathbf{J_1}$ *does not satisfy criterion (A).*

Proof. Given an alphabet Σ, the set $\mathcal{L}(\mathbf{J_1})(\Sigma)$ consists of all Boolean combinations of languages of the form $\Sigma^* a \Sigma^*$ for $a \in \Sigma$ (see [14, Chapter 2, Proposition 3.10]).

Let $L = \{a, b\}^* b \{a, b\}^* \in \mathcal{L}(\mathbf{J_1})(\{a, b\})$ and $x = b, y = \varepsilon$. Take any $k, l \in \mathbb{N}$ and set $u = a^k$ and $v = a^l$. Consider a $K \in \mathcal{L}(\mathbf{J_1})(\{a, b\})$. We have that $xuavy \in K \Leftrightarrow xuabvy \in K$ so that $a \in (xu)^{-1}K(vy)^{-1} \Leftrightarrow ab \in (xu)^{-1}K(vy)^{-1}$. But $a \notin u^{-1}Lv^{-1}$ and $ab \in u^{-1}Lv^{-1}$, hence $u^{-1}Lv^{-1} \neq (xu)^{-1}K(vy)^{-1}$ and this holds for any choice of K. So for any $k, l \in \mathbb{N}$, there exists $u \in \Sigma^k, v \in \Sigma^l$ such that no $K \in \mathcal{L}(\mathbf{J_1})(\{a, b\})$ verifies $u^{-1}Lv^{-1} = (xu)^{-1}K(vy)^{-1}$.

In conclusion, by Proposition 12, $\mathbf{J_1}$ does not satisfy criterion (A). ◀

We now prove the announced characterisations of joins between $\mathbf{LI}$ and some well-known varieties of monoids in terms of identities.

▶ **Theorem 14.** *We have the following.*

1. $\langle \mathbf{R} \vee \mathbf{LI} \rangle_{\mathrm{ne}} = \mathbf{ER} = [\![x^\omega y(ab)^\omega azt^\omega = x^\omega y(ab)^\omega zt^\omega]\!]_{\mathrm{ne}}$.
2. $\langle \mathbf{L} \vee \mathbf{LI} \rangle_{\mathrm{ne}} = \mathbf{EL} = [\![x^\omega yb(ab)^\omega zt^\omega = x^\omega y(ab)^\omega zt^\omega]\!]_{\mathrm{ne}}$.
3. $\langle \mathbf{J} \vee \mathbf{LI} \rangle_{\mathrm{ne}} = \mathbf{EJ} = [\![x^\omega y(ab)^\omega azt^\omega = x^\omega y(ab)^\omega zt^\omega, x^\omega yb(ab)^\omega zt^\omega = x^\omega y(ab)^\omega zt^\omega]\!]_{\mathrm{ne}}$.
4. $\langle \mathbf{H} \vee \mathbf{LI} \rangle_{\mathrm{ne}} = \mathbf{EH}$ *for any variety of groups* $\mathbf{H}$.

Proof. In each case, we prove that the variety of monoids under consideration satisfies criterion (A) using Proposition 12. We then use Propositions 8 and 6.

Proof of 1. It is well-known that given an alphabet Σ, the set $\mathcal{L}(\mathbf{R})(\Sigma)$ consists of all languages that are disjoint unions of languages that are of the form $A_0^* a_1 A_1^* \cdots a_k A_k^*$ where $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \Sigma$, $A_0, A_1, \ldots, A_k \subseteq \Sigma$ and $a_i \notin A_{i-1}$ for all $i \in [k]$ (see [14, Chapter 4, Theorem 3.3]).

Let Σ be an alphabet and take a language $A_0^* a_1 A_1^* \cdots a_k A_k^*$ where $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \Sigma$, $A_0, A_1, \ldots, A_k \subseteq \Sigma$ and $a_i \notin A_{i-1}$ for all $i \in [k]$. Take $x, y \in \Sigma^*$. Observe that y can be uniquely written as $y = zt$ where $z \in A_k^*$ and $t \in \{\varepsilon\} \cup (\Sigma \setminus A_k)\Sigma^*$. We have

$$A_0^* a_1 A_1^* \cdots a_k A_k^* = x^{-1}\Big(x A_0^* a_1 A_1^* \cdots a_k A_k^* t \cap \bigcap_{v \in A_k^{<|z|}} (\Sigma^* \setminus x A_0^* a_1 A_1^* \cdots a_k v t)\Big) y^{-1}$$

using the convention that $x A_0^* a_1 A_1^* \cdots a_k v t = xvt$ for all $v \in A_k^{<|z|}$ when $k = 0$. The language $x A_0^* a_1 A_1^* \cdots a_k A_k^* t \cap \bigcap_{v \in A_k^{<|z|}} (\Sigma^* \setminus x A_0^* a_1 A_1^* \cdots a_k v t)$ does belong to the set $\mathcal{L}(\mathbf{R})(\Sigma)$ because the latter is closed under Boolean operations and by definition of z and t. Thus, we can conclude that for each $L \in \mathcal{L}(\mathbf{R})(\Sigma)$ and $x, y \in \Sigma^*$, there exists $K \in \mathcal{L}(\mathbf{R})(\Sigma)$ such that $L = x^{-1} K y^{-1}$ by using the characterisation of $\mathcal{L}(\mathbf{R})(\Sigma)$, the fact that quotients commute with unions [14, p. 20] and closure of $\mathcal{L}(\mathbf{R})(\Sigma)$ under unions.

Proof of 2. It is also well-known that given an alphabet Σ, the set $\mathcal{L}(\mathbf{L})(\Sigma)$ consists of all languages that are disjoint unions of languages that are of the form $A_0^* a_1 A_1^* \cdots a_k A_k^*$ where $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \Sigma$, $A_0, A_1, \ldots, A_k \subseteq \Sigma$ and $a_i \notin A_i$ for all $i \in [k]$ (see [14, Chapter 4, Theorem 3.4]). The proof is then dual to the previous case.

Proof of 3. Given an alphabet Σ, for each $k \in \mathbb{N}$, we define the equivalence relation $\sim_k$ on Σ^* by $u \sim_k v$ for $u, v \in \Sigma^*$ whenever u and v have the same set of subwords of length at most k. This relation is a congruence of finite index on Σ^*. Simon proved [19] that a language belongs to $\mathcal{L}(\mathbf{J})(\Sigma)$ if and only it is equal to a union of $\sim_k$-classes for a $k \in \mathbb{N}$.

Let Σ be an alphabet and take $L \in \mathcal{L}(\mathbf{J})(\Sigma)$ as well as $x, y \in \Sigma^*$. Thus, there exists $k \in \mathbb{N}$ such that L is a union of $\sim_k$-classes. Define the language $K = \bigcup_{w \in L} [xwy]_{\sim_{|xy|+k}}$: it belongs to $\mathcal{L}(\mathbf{J})(\Sigma)$ by construction. We now show that $L = x^{-1} K y^{-1}$, which concludes the proof. Let $w \in L$: we have that $xwy \in [xwy]_{\sim_{|xy|+k}} \subseteq K$, so that $w \in x^{-1} K y^{-1}$. Let conversely $w \in x^{-1} K y^{-1}$. This means that $xwy \in K$, which implies that there exists $w' \in L$ such that $xwy \sim_{|xy|+k} xw'y$. Actually, it holds that any $u \in \Sigma^*$ of length at most k is a subword of w if and only if it is a subword of w', because xuy is a subword of xwy if and only if it is a subword of $xw'y$. Hence, $w \sim_k w'$, which implies that $w \in L$.

Proof of 4. Consider any variety of groups $\mathbf{H}$. Take a language $L \in \mathcal{L}(\mathbf{H})(\Sigma)$ for an alphabet Σ and let $x, y \in \Sigma^*$. Consider the syntactic morphism $\eta\colon \Sigma^* \to M$ of L: we have that M is a group in $\mathbf{H}$. Define the language $K = \eta^{-1}\big(\eta(x)\eta(L)\eta(y)\big)$: it belongs to $\mathcal{L}(\mathbf{H})(\Sigma)$. We now show that $L = x^{-1} K y^{-1}$, which concludes the proof. Let $w \in L$: we have that $\eta(xwy) \in \eta(x)\eta(L)\eta(y)$, so that $w \in x^{-1} K y^{-1}$. Conversely, let $w \in x^{-1} K y^{-1}$. We have that $xwy \in K$, which means that $\eta(xwy) = \eta(x)\eta(w')\eta(y)$ for a $w' \in L$, so that $\eta(w) = \eta(w') \in \eta(L)$, as any element in M is invertible. Thus, $w \in L$. ◀

6 Conclusion

The general method presented in this paper actually allows to reprove in a straightforward language-theoretic way even more characterisations of the join of **LI** with some variety of finite monoids. This can for instance be done for the variety of finite commutative monoids **Com** or the variety of finite commutative aperiodic monoids **ACom**.

In fact, as already observed in some sense by Costa [9], many varieties of finite monoids seem to verify criterion (A). The main question left open by this present work is to understand better what exactly those varieties are. Another question left open is whether Proposition 12 can be refined by using the stronger quotient-expressibility condition alluded to before the statement of the proposition. The answers to both questions are unclear to the author, but making progress on them may also lead to a better understanding of joins of varieties of finite monoids with **LI**.

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A Missing proofs

Proof of Proposition 2. Let $\mathbf{W}$ be an *ne*-variety of stamps such that $\langle \mathbf{V} \rangle_{all} \cup \langle \mathbf{LI} \rangle_{ne} \subseteq \mathbf{W}$. There exists a variety of semigroups $\mathbf{W'}$ such that $\langle \mathbf{W'} \rangle_{ne} = \mathbf{W}$.

Let $S \in \mathbf{V} \cup \mathbf{LI}$. We denote by S^1 the monoid S if S is already a monoid and the monoid $S \cup \{1\}$ otherwise. Then the evaluation morphism $\eta_S \colon S^* \to S^1$ such that $\eta_S(s) = s$ for all $s \in S$ verifies $\eta_S(S^+) = S$ and additionally $S^1 = S$ when $S \in \mathbf{V}$. This implies that $\eta_S \in \langle \mathbf{V} \rangle_{all} \cup \langle \mathbf{LI} \rangle_{ne} \subseteq \mathbf{W}$. But by definition of $\mathbf{W'}$, it must be that $S = \eta_S(S^+) \in \mathbf{W'}$.

Therefore, $\mathbf{W'}$ contains both $\mathbf{V}$ and $\mathbf{LI}$, which implies that $\mathbf{V} \vee \mathbf{LI} \subseteq \mathbf{W'}$ by inclusion-wise minimality of $\mathbf{V} \vee \mathbf{LI}$. By definition, we can then conclude that $\langle \mathbf{V} \vee \mathbf{LI} \rangle_{ne} \subseteq \langle \mathbf{W'} \rangle_{ne} = \mathbf{W}$. So $\langle \mathbf{V} \vee \mathbf{LI} \rangle_{ne}$ is the inclusion-wise least *ne*-variety of stamps containing both $\langle \mathbf{V} \rangle_{all}$ and $\langle \mathbf{LI} \rangle_{ne}$.

Let now $\mathcal{W}$ be an *ne*-variety of languages such that $\mathcal{L}(\mathbf{V}) \cup \mathcal{L}(\mathbf{LI}) \subseteq \mathcal{W}$. It holds that $\mathcal{W} = \mathcal{L}(\mathbf{W})$ for an *ne*-variety of stamps $\mathbf{W}$. We have that $\langle \mathbf{V} \rangle_{all}$, which is in particular an *ne*-variety of stamps, is included in $\mathbf{W}$ because $\mathcal{L}(\langle \mathbf{V} \rangle_{all}) = \mathcal{L}(\mathbf{V}) \subseteq \mathcal{W} = \mathcal{L}(\mathbf{W})$, but also that $\langle \mathbf{LI} \rangle_{ne}$ is included in $\mathbf{W}$ because $\mathcal{L}(\langle \mathbf{LI} \rangle_{ne}) = \mathcal{L}(\mathbf{LI}) \subseteq \mathcal{W} = \mathcal{L}(\mathbf{W})$. By inclusion-wise minimality of $\langle \mathbf{V} \vee \mathbf{LI} \rangle_{ne}$, it follows that $\langle \mathbf{V} \vee \mathbf{LI} \rangle_{ne} \subseteq \mathbf{W}$. Hence, using again the above fact on the Eilenberg correspondence, we can conclude that $\mathcal{L}(\mathbf{V} \vee \mathbf{LI}) = \mathcal{L}(\langle \mathbf{V} \vee \mathbf{LI} \rangle_{ne}) \subseteq \mathcal{L}(\mathbf{W}) = \mathcal{W}$. So $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ is the inclusion-wise least *ne*-variety of languages containing both $\mathcal{L}(\mathbf{V})$ and $\mathcal{L}(\mathbf{LI})$.

Consider now the class of languages $\mathcal{C}$ such that $\mathcal{C}(\Sigma)$ is the Boolean closure of $\mathcal{L}(\mathbf{V})(\Sigma) \cup \mathcal{L}(\mathbf{LI})(\Sigma)$ for each alphabet Σ. By closure under Boolean operations of $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$, we have that $\mathcal{C} \subseteq \mathcal{L}(\mathbf{V} \vee \mathbf{LI})$. Now, as Boolean operations commute with both quotients [14, p. 20] and inverses of *ne*-morphisms [14, Proposition 0.4], by closure of $\mathcal{L}(\mathbf{V})$ and $\mathcal{L}(\mathbf{LI})$ under quotients and inverses of *ne*-morphisms, we actually have that $\mathcal{C}$ is an *ne*-variety of languages. Therefore, by inclusion-wise minimality of $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$, we can conclude that $\mathcal{L}(\mathbf{V} \vee \mathbf{LI}) = \mathcal{C}$. ◀

Proof of Lemma 10. We actually have that $\mathcal{L}(\mathbf{V}) \cup \mathcal{L}(\mathbf{LI}) \subseteq \mathcal{L}(\mathbf{EV})$, which allows us to conclude by inclusion-wise minimality of $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ (Proposition 2) and by the fact that $\mathcal{L}(\mathbf{EV})$ is an *ne*-variety of languages (Proposition 6).

Let Σ be an alphabet. The fact that $\mathcal{L}(\mathbf{V})(\Sigma) \subseteq \mathcal{L}(\mathbf{EV})(\Sigma)$ follows trivially from Proposition 9. Moreover, for all $u \in \Sigma^*$, since necessarily $\Sigma^* \in \mathcal{L}(\mathbf{V})(\Sigma)$, we have that both $u\Sigma^*$ and $\Sigma^* u$ belong to $\mathcal{L}(\mathbf{LI})(\Sigma)$. Thus, as $\mathcal{L}(\mathbf{EV})(\Sigma)$ is closed under Boolean operations, it follows that $\mathcal{L}(\mathbf{LI})(\Sigma) \subseteq \mathcal{L}(\mathbf{EV})(\Sigma)$.

This concludes the proof, since it holds for any alphabet Σ. ◀

Proof of Lemma 11. Assume that $\mathbf{EV} \subseteq \langle \mathbf{V} \vee \mathbf{LI} \rangle_{ne}$. For any $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ and any $x, y \in \Sigma^*$ with Σ an alphabet, by Proposition 9, we have that $xLy \in \mathcal{L}(\mathbf{EV})(\Sigma) \subseteq \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$. Hence, $\mathbf{V}$ verifies criterion (A).

Conversely, assume that $\mathbf{V}$ verifies criterion (A). For any alphabet Σ, the set $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$ contains all languages of the form xLy for $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ and $x, y \in \Sigma^*$, so it contains all Boolean combinations of languages of that form, since it is closed under Boolean operations. Therefore, by Proposition 9, we have $\mathcal{L}(\mathbf{EV}) \subseteq \mathcal{L}(\mathbf{V} \vee \mathbf{LI})$, so that $\mathbf{EV} \subseteq \langle \mathbf{V} \vee \mathbf{LI} \rangle_{ne}$. ◀

Proof of Proposition 12. Let us first observe that given any alphabet Σ, given any language K on that alphabet and given any two words $x, y \in \Sigma^*$, we have that $x(x^{-1}Ky^{-1})y = x\Sigma^*y \cap K$ and $x^{-1}(xKy)y^{-1} = K$.

Implication from right to left. Assume that for any $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ and any $x, y \in \Sigma^*$ with Σ an alphabet, there exist $k, l \in \mathbb{N}$ such that for all $u \in \Sigma^k, v \in \Sigma^l$, there exists a language $K \in \mathcal{L}(\mathbf{V})(\Sigma)$ verifying $u^{-1}Lv^{-1} = (xu)^{-1}K(vy)^{-1}$. Take $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ for an alphabet Σ and take $x, y \in \Sigma^*$. Consider also $k, l \in \mathbb{N}$ that are guaranteed to exist by the assumption we just made.

For all $u \in \Sigma^k, v \in \Sigma^l$, there exists a language $K \in \mathcal{L}(\mathbf{V})(\Sigma)$ verifying $u^{-1}Lv^{-1} = (xu)^{-1}K(vy)^{-1}$, so that by our observation at the beginning of the proof, we have

$$x(u\Sigma^*v \cap L)y = xu(u^{-1}Lv^{-1})vy = xu\big((xu)^{-1}K(vy)^{-1}\big)vy = xu\Sigma^*vy \cap K \ .$$

Using Proposition 2, we thus have that $x(u\Sigma^*v \cap L)y \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$ for all $u \in \Sigma^k, v \in \Sigma^l$. Moreover, since we have that the set of words of L of length at least $k + l$ is

$$\Sigma^{\geq k+l} \cap L = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} (u\Sigma^*v \cap L)$$

and since

$$L = (\Sigma^{\geq k+l} \cap L) \cup F$$

where F is a finite set of words on Σ of length less than $k + l$, we have that

$$xLy = x\big((\Sigma^{\geq k+l} \cap L) \cup F\big)y = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} x(u\Sigma^*v \cap L)y \cup xFy \ .$$

We can thus conclude that $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$ since $xFy \in \mathcal{L}(\mathbf{LI})(\Sigma)$ and because $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$ is closed under unions.

Implication from left to right. Assume that $\mathbf{V}$ satisfies criterion (A). Take $L \in \mathcal{L}(\mathbf{V})(\Sigma)$ for an alphabet Σ and take $x, y \in \Sigma^*$. By hypothesis, we know that $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})(\Sigma)$.

By Proposition 2, this means that xLy is a Boolean combination of languages in $\mathcal{L}(\mathbf{V})(\Sigma) \cup \mathcal{L}(\mathbf{LI})(\Sigma)$. Further, this implies that xLy can be written as the union of intersections of languages of $\mathcal{L}(\mathbf{V})(\Sigma)$ and $\mathcal{L}(\mathbf{LI})(\Sigma)$ or their complements, which in turn implies, by closure of $\mathcal{L}(\mathbf{V})(\Sigma)$ and $\mathcal{L}(\mathbf{LI})(\Sigma)$ under Boolean operations, that xLy can be written as a finite union of languages of the form $K \cap (U\Sigma^*V \cup W)$ with $K \in \mathcal{L}(\mathbf{V})(\Sigma)$ and $U, V, W \subseteq \Sigma^*$ finite. Since any word in xLy must be of length at least $|xy|$ and have x as a prefix and y as a suffix, we can assume that any language $K \cap (U\Sigma^*V \cup W)$ appearing in a finite union as described above verifies that $U \subseteq x\Sigma^*$, that $V \subseteq \Sigma^*y$ and that $W \subseteq x\Sigma^*y$. Now, if we take $k, l \in \mathbb{N}$ big enough, we thus have that

$$xLy = \bigcup_{u \in \Sigma^k, v \in \Sigma^l} (K_{u,v} \cap xu\Sigma^* vy) \cup F$$

where $K_{u,v} \in \mathcal{L}(\mathbf{V})(\Sigma)$ for all $u \in \Sigma^k, v \in \Sigma^l$ and $F \subseteq \Sigma^{<|xy|+k+l}$. Hence, for all $u \in \Sigma^k, v \in \Sigma^l$, we have

$$\begin{aligned}
u^{-1}Lv^{-1} &= u^{-1}\big(x^{-1}(xLy)y^{-1}\big)v^{-1} \\
&= (xu)^{-1}\Big(\bigcup_{u' \in \Sigma^k, v' \in \Sigma^l} (K_{u',v'} \cap xu'\Sigma^* v'y) \cup F\Big)(vy)^{-1} \\
&= \bigcup_{u' \in \Sigma^k, v' \in \Sigma^l} (xu)^{-1}\Big(xu'\big((xu')^{-1}K_{u',v'}(v'y)^{-1}\big)v'y\Big)(vy)^{-1} \cup \\
&\quad\ (xu)^{-1}F(vy)^{-1} \\
&= (xu)^{-1}K_{u,v}(vy)^{-1}\ ,
\end{aligned}$$

using classical formulae for quotients [14, p. 20] and observing that $(xu)^{-1}K(vy)^{-1} = \emptyset$ for any $K \subseteq \Sigma^*$ such that $K \cap xu\Sigma^* vy = \emptyset$. ◀