Perfect Forests in Graphs and Their Extensions

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Abstract

Let G be a graph on n vertices. For $i \in \{0,1\}$ and a connected graph G, a spanning forest F of G is called an i-perfect forest if every tree in F is an induced subgraph of G and exactly i vertices of F have even degree (including zero). An i-perfect forest of G is proper if it has no vertices of degree zero. Scott (2001) showed that every connected graph with even number of vertices contains a (proper) 0-perfect forest. We prove that one can find a 0-perfect forest with minimum number of edges in polynomial time, but it is NP-hard to obtain a 0-perfect forest with maximum number of edges. We also prove that for a prescribed edge e of G, it is NP-hard to obtain a 0-perfect forest containing e, but we can find a 0-perfect forest not containing e in polynomial time. It is easy to see that every graph with odd number of vertices has a 1-perfect forest. It is not the case for proper 1-perfect forests. We give a characterization of when a connected graph has a proper 1-perfect forest.

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1 Introduction

In this paper all graphs are finite, undirected, have no parallel edges or loops. We use standard terminology and notation, see e.g. [5]. The number of vertices (edges, respectively) of a graph G is called its *order* (*size*, respectively). The degree of a vertex x in a graph G is denoted by $d_G(x)$. A vertex x of a graph G is a *cut-vertex* if $G - x$ has more connected components than G. A maximal connected subgraph of a graph G without a cut-vertex is called a *block*. Thus, every block of G is either a maximal 2-connected subgraph or a bridge (including its vertices) or an isolated vertex, implying that a block of odd order in a connected graph of order at least 3, must be a maximal 2-connected subgraph.

A spanning forest F of G is called a *semiperfect forest* if every tree of F is an induced subgraph of G. Let G be a graph and let $f\colon V(G) \to \{0,1\}$ be a function such that $\sum_{v\in V(G)} f(v)$ is even (we will call such a function *even-sum*). A subgraph H in G where $d_H(x) \equiv f(x) \pmod 2$ for all $x \in V(G)$, is called an *f-parity subgraph*. Note that the requirement that f is even-sum is necessary as otherwise an f-parity subgraph does not exist. An f-parity subgraph H of G is called an *f-parity perfect forest* if H is a semiperfect forest.

For $i \in \{0,1\}$ and a graph G, an f-parity perfect forest is called an *i-perfect forest* if $f(x) = 1$ for all vertices of G for $i = 0$, and for all vertices of G apart from one for $i = 1$. An i-perfect forest of G is *proper* if it has no vertices of degree zero. Note that every 0-perfect forest (called a perfect forest in [3, 9] and a pseudo-matching in [18]) is proper. For examples of 0-perfect and 1-perfect forests, see Figures 1 and 2.

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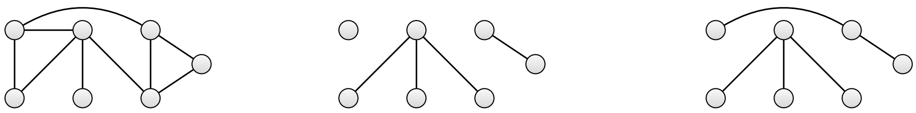
(a): G (b): A 0-perfect forest of G

Figure 1 A graph G is shown in (a) and a 0-perfect forest of G is shown in (b) (as all degrees are odd and the trees are induced in G).



(a): H (b): A 1-perfect forest of H (c): A proper 1-perfect forest of H

Figure 2 The graph H is shown in (a), a (non-proper) 1-perfect forest of H is shown in (b), and a proper 1-perfect forest of H is shown in (c).

Clearly, every connected graph with a 0-perfect forest is of even order. Scott [17] proved that somewhat surprisingly the opposite implication is also true.

▶ **Theorem 1.** *Every connected graph of even order contains a 0-perfect forest.*

The proof of Theorem 1 in [17] is graph-theoretical and relatively long. A short proof using basic linear algebra is obtained in [9] and two short graph-theoretical proofs are given in [3]. All the proofs of Theorem 1 are constructive and yield polynomial algorithms for finding 0-perfect forests. Intuitively, it is clear that a 0-perfect forest can provide a useful structure in a graph and, in particular, this notion was used by Sharan and Wigderson [18] to prove that the perfect matching problem for bipartite cubic graphs belongs to the complexity class $\mathcal{NC}$. Semiperfect forests were used in the proofs of three theorems in [7]. Gutin and Yeo [11] studied extensions of a 0-perfect forest to directed graphs.

Since a 0-perfect forest is a generalization of a matching, it is natural to study the following two problems for a connected graph G of even order n:

(1) Find a 0-perfect forest of G of minimum size. (Clearly, the minimum size is $n/2$ if and only if G has a perfect matching.)

(2) Find a 0-perfect forest of G of maximum size. (This is of interest in matching-like edge-decompositions of G.)

The following theorem solves the first problem.

▶ **Theorem 2.** *In polynomial time, we can find a* 0*-perfect forest of minimum size.*

Theorem 2 follows immediately from the next theorem by letting $f(x) = 1$ for all $x \in V(G)$. Theorem 3 shows usefulness of extending Problem 1 to f-parity perfect forests. Theorem 3 is proved in Section 2.

▶ **Theorem 3.** *Let G be a connected graph and let $f\colon V(G) \to \{0,1\}$ be an even-sum function. We can in polynomial time find an f-parity perfect forest H in G, such that $d_H(x) \equiv f(x) \pmod 2$ for all $x \in V(G)$ and $|E(H)|$ is minimized.*

As the following theorem shows, the second problem cannot be solved in polynomial time unless P=NP.

▶ **Theorem 4.** *It is* NP*-hard to find a* 0*-perfect forest of maximum size.*

Let $n = |V(G)|$. Theorem 4 follows from the next result proved in Section 3. Theorem 5 is optimal in the following sense. The problem of finding a 0-perfect forest of size at least $n-1$ is polynomial-time solvable because G has a 0-perfect forest of size at least $n-1$ if and only if G is a tree in which every vertex is of odd degree.

▶ **Theorem 5.** *It is* NP*-hard to decide whether a connected graph contains a* 0*-perfect forest with at least* $n-2$ *edges.*

It is easy to show that Theorem 5 holds if we replace $n-2$ by $n-k$ for any integer $k \geq 2$. Indeed, add two new vertices x and y to a graph G as well as two edges xy and yu, where u is any vertex in G. The resulting graph is denoted by G'. Observe that there is a 0-perfect forest of size $|V(G)| - k$ in G if and only if there is a 0-perfect forest of size $|V(G')| - (k+1)$ in G'.

Since the problem of finding a 0-perfect forest of maximum size is NP-hard, it is natural to study its parameterized complexity using appropriate parameterizations e.g. the parameterization below the tight upper bound $n-1$ and the parameterization above the tight upper bound $n/2$. In other words, we can ask whether there is a 0-perfect forest of size at least $n-k$ ($n/2+k$, respectively), where k is the parameter. (Above-tight-lower-bound and below-tight-upper-bound parameterizations were studied for many graph-theoretical and constraint satisfaction problems, see e.g. [1, 4, 10, 13, 14].) Theorem 5 shows that the parameterization $n-k$ is para-NP-complete (for an introduction to para-NP-completeness, see e.g. [6]). We do not know the answer to the following question. Is the parameterization $n/2+k$ fixed-parameter tractable?[1]

Here is another pair of natural problems on 0-perfect forests. They both are clearly polynomial-time solvable when restricted to perfect matchings. For a graph G of even order and an edge e in G,

(1′) find a 0-perfect forest containing e;
(2′) find a 0-perfect forest not containing e.

For Problem 1′, we prove the following result in Section 4.

▶ **Theorem 6.** *The following problem is* NP*-hard. Given a connected graph* G *and an edge* $e \in E(G)$*, decide whether* G *has a* 0*-perfect forest containing* e*.*

For Problem 2′, we have the next result, which follows immediately from Theorem 8, by letting $f(x) = 1$ for all x in G. Theorem 8 again demonstrates usefulness of f-parity perfect forests. It is proved in Section 5.

▶ **Theorem 7.** *Given a graph* G *and an edge* $e \in E(G)$ *we can in polynomial time decide whether* G *has a* 0*-perfect forest not containing* e*.*

▶ **Theorem 8.** *The following problem is polynomial time solvable. Given a graph* G*, an edge* $e \in E(G)$ *and an even-sum function* $f\colon V(G) \to \{0,1\}$*, decide whether* G *has an* f*-parity perfect forest not containing* e*.*

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[1] While working on the final version of this paper, we obtained a proof that the parameterized problem is W[1]-hard. We will include the proof in a journal version of the paper.

Since an odd order connected graph cannot have a 0-perfect forest, it is natural to ask whether every connected graph of odd order has a 1-perfect forest (recall that a 1-perfect forest has only one vertex of even degree). The answer is positive and the proof is trivial. In fact, it is not hard to show the following strengthening of this observation, which will be useful in the proof of Theorem 10.

▶ **Proposition 9.** *Let x be an arbitrary vertex of a connected graph G of odd order. Then G has a 1-perfect forest F such that $d_F(x)$ is even.*

Proof. Create a new graph H by adding a new vertex y to G and adding the edge xy. By Theorem 1, H has a 0-perfect forest, F_H. Deleting the vertex y from F_H, results in the desired 1-perfect forest of G where x is the only vertex of even degree. ◀

Note that not every connected graph of odd order has a proper 1-perfect forest. For example, no complete graph of odd order has such a forest. Thus, a more interesting question with a potentially more useful answer is when a connected graph of odd order has a proper 1-perfect forest? This question is answered in the following characterization proved in Section 6.

▶ **Theorem 10.** *Let $\mathcal{B}$ be the set of all connected graphs where every block is a complete graph of odd order. If G is a connected graph of odd order $n \geq 3$ then G contains a proper 1-perfect forest if and only if $G \notin \mathcal{B}$.*

Using this theorem and a linear-time algorithm for computing biconnected components in a graph [12], in polynomial time we can decide whether a connected graph G of odd order contains a proper 1-perfect forest. If $G \notin \mathcal{B}$, the proof by induction of Theorem 2 yields a polynomial-time recursive algorithm to construct a proper 1-perfect forest.

Our proof of Theorem 10 is graph-theoretical and so are the proofs of Theorem 1 in [17] and [3]. Recall that Gutin [9] gave a linear-algebraic proof of Theorem 1. It would interesting to see whether Theorem 10 can be proved using a linear-algebraic approach, too.

2 Proof of Theorem 3

▶ **Lemma 11.** *Let G be a connected graph and let $f\colon V(G) \to \{0,1\}$ be an even-sum function. If H is an f-parity subgraph of G of minimum size, then H is an f-parity perfect forest.*

Proof. Assume that H is an f-parity subgraph with minimum possible $|E(H)|$. Clearly H contains no cycles, as removing the edges of a cycle would contradict the minimality of $|E(H)|$. Assume that some tree T of H is not an induced tree in G. Let xy be an edge of G, not belonging to T but with $\{x,y\} \subseteq V(T)$. Remove the unique (x,y)-path in T from H and add the edge xy to H. This decreases the number of edges in H without changing the parity of the degree of any vertex, contradicting the minimality of $|E(H)|$. Therefore H is indeed an f-parity perfect forest. ◀

Lemma 11 implies the following:

▶ **Theorem 12.** *Let G be a connected graph and let $f\colon V(G) \to \{0,1\}$ be an even-sum function. Then there exists an f-parity perfect forest F in G.*

Proof. Let $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k$ be the vertices in G with f-value equal to one. Let P_i be any (x_i, y_i)-path in G for all $i = 1, 2, \dots, k$, which exists as G is connected. Let H be the spanning subgraph of G such that an edge $e \in E(G)$ belongs to H if and only if e

belongs to an odd number of paths in $P_1, P_2, \ldots, P_k$. Let $x \in V(G)$. Observe that $d_H(x)$ is odd if and only if x is incident with an odd number of edges in $\cup_{i=1}^k E(P_i)$, which is if and only if x is the endpoint of one of the paths i.e. $f(x) = 1$. Thus, H is an f-parity subgraph of G. Lemma 11 now implies that if H is the f-parity subgraph of G of minimum size, then H is an f-parity perfect forest. ◀

Note that Theorem 12 generalizes Theorem 1: set $f(x) = 1$ for all $x \in V(G)$. Thus, Theorem 12 provides an alternative proof of Theorem 1.

▶ **Theorem 3.** *Let G be a connected graph and let $f\colon V(G) \to \{0,1\}$ be an even-sum function. We can in polynomial time find an f-parity perfect forest H in G, such that $d_H(x) \equiv f(x) \pmod 2$ for all $x \in V(G)$ and $|E(H)|$ is minimized.*

Proof. Let G be a connected graph and let $f\colon V(G) \to \{0,1\}$ be an even-sum function. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. We will construct a weigthed auxillary graph H as follows. Let $V(H) = \cup_{i=1}^n X_i$, where for every $i \in [n]$, $|X_i| \in \{n-1, n\}$ and $|X_i| \equiv f(v_i) \pmod 2$. For all $1 \le i < j \le n$ and all $u \in X_i$ and $v \in X_j$, we let $uv \in E(H)$ if and only if $v_i v_j \in E(G)$. Finally add a matching $M_i = \{e_1^i, e_2^i, \ldots, e_{\lfloor |X_i|/2 \rfloor}^i\}$ to X_i for all $i \in [n]$. Let the weight of all the edges within each X_i (i.e. the edges in M_i) be zero and let all edges between different X_i's have weight one.

We first show that H contains a perfect matching. As $\sum_{v \in V(G)} f(v)$ is even we may assume that $\{v_1, v_2, \ldots, v_{2k}\}$ are the vertices of G with an f-value of one for some integer k with $0 \le k \le n/2$. Assume that $y_i \in X_i$ is the unique vertex in X_i that is not saturated by M_i for all $i \in [2k]$ and start of by letting M be the matching containing all M_i's.

Let $P_i = v_i v_{p_1^i} v_{p_2^i} \cdots v_{p_{l_i-1}^i} v_{i+k}$ be any path in G from v_i to v_{i+k} where $i \in [k]$. It is not difficult to see that there exists an M-augmenting path, Q_i, in H starting in y_i and ending in y_{i+k} and containing exactly the edges $e_i^{p_1^i}$, $e_i^{p_2^i}$, ..., $e_i^{p_{l_i-1}^i}$ from M. Also observe that $Q_1, Q_2, \ldots, Q_k$ are vertex disjoint, which implies that we can use all Q_i to increase the matching M thereby obtaining a perfect matching in H.

We will now show the following claim. The *size* of a multiset S is the total number of elements in S, where if an element $e \in S$ is of multiplicity r, then e is counted r times.

▷ Claim A.

(a) If there exists a perfect matching in H with weight w^* then there exists a multiset of edges E^* in G of size w^*, such that $d_{E^*}(x) \equiv f(x) \pmod 2$ for all $x \in V(G)$.

(b) Conversely if E^* is a multiset of edges in G of size w^*, such that $d_{E^*}(x) \equiv f(x) \pmod 2$ for all $x \in V(G)$, then there exists a perfect matching in H with weight at most w^*.

Proof of Claim A. First assume that we have a multiset of edges E^* in G of size $w^* \le W_{\max}$, such that $d_{E^*}(x) \equiv f(x) \pmod 2$ for all $x \in V(G)$. Let $M^* = \emptyset$. For every $v_i v_j \in E^*$ we will add edges between X_i and X_j to M^* as follows: if $v_i v_j$ is of multiplicity r in E^*, then we add an edge between X_i and X_j to M^* if and only if r is odd. Since we will add $2k_i + f(v_i)$ edges that are incident to X_i for each $i \in [n]$ (where k_i is some integer), we can add these edges such that their endvertices are $V(e_1^i) \cup V(e_2^i) \cup \cdots \cup V(e_{k_i}^i)$ if $f(v_i) = 0$ and $\{y_i\} \cup V(e_1^i) \cup V(e_2^i) \cup \cdots \cup V(e_{k_i}^i)$ if $f(v_i) = 1$ for each $i \in [n]$, where $V(e_j^i)$ denotes the pair of endvertices of e_j^i. We can now extend M^* to a perfect matching by adding $M_i \setminus \{e_1^i, e_2^i, \ldots, e_{k_i}^i\}$ for each $i \in [n]$. This gives us a perfect matching in H with weight at most $|E^*|$ as desired.

Conversely assume that there exists a perfect matching M^* in H with weight w^*. Initially let $E^* = \emptyset$. For every $xy \in M^*$ with weight one (i.e. $x \in X_i$ and $y \in X_j$ for some $i \neq j$), add $v_i v_j$ to E^*. This gives us the desired multiset E^*, thereby completing the proof of Claim A. ◁

We have proved that H has a perfect matching. Let $M_{\min}$ be a minimum weight perfect matching in H which can be determined in polynomial time using Edmonds' blossom algorithm as a subroutine, see e.g. [15]. Let $W_{\min}$ be the weight of $M_{\min}$. By Claim A(a), using $M_{\min}$, in polynomial time we can find a multiset of edges E^* in G of size $W_{\min}$, such that $d_{E^*}(x) \equiv f(x) \pmod 2$ for all $x \in V(G)$. By Claim A(b), since $W_{\min}$ is the minimum weight of a perfect matching in H, $W_{\min}$ is minimum size of a multiset of edges E^{**}, such that $d_{E^{**}}(x) \equiv f(x) \pmod 2$ for all $x \in V(G)$.

Note that no edge is in E^* more than once, since if some edge, e, appears twice, then we can delete two copies of e from E^*, thereby contradicting the minimality of $|E^*|$. Let F be the spanning subgraph of G with edge set E^*. By Lemma 11 we note that F is an f-parity perfect forest, which completes the proof of the theorem. ◀

3 Proof of Theorem 5

We will reduce from the *not-all-equal 3-SAT* problem, abbreviated to NAE-3-SAT, which is the problem of determining whether an instance of 3-SAT has a truth assignment to its variables such that every clause contains both a true and a false literal. If this is the case we say that the instance is *NAE-satisfied*. NAE-3-SAT is known to be NP-hard to solve [16]. Let I be an instance of NAE-3-SAT with clauses $C_1, C_2, \ldots, C_m$ and variables $v_1, v_2, \ldots, v_n$. We will construct a graph G such that G contains a 0-perfect forest with at least $n-2$ edges if and only if I is NAE-satisfied.

We first create a gadget H_i for each $i = 1, 2, \ldots, n$ as follows. Let

$$V(H_i) = \{x_1^i, z_1^i, y_1^i, x_2^i, z_2^i, y_2^i\}$$

and add all possible edges to H_i, except $x_1^i y_1^i$ and $x_2^i y_2^i$. For all $i = 1, 2, \ldots, n-1$ we then add all edges between $\{y_1^i, y_2^i\}$ and $\{x_1^{i+1}, x_2^{i+1}\}$. Now add a pendent edge to each vertex in $V(H_i) \setminus \{x_1^1, x_2^1, y_1^n, y_2^n\}$ for all $i = 1, 2, \ldots, n$. See Figure 3 for an illustration of this part of G, which is denoted by Q. We will now complete our construction of G.

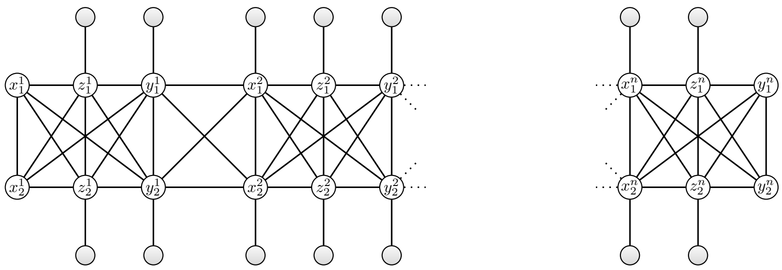

Figure 3 The gadgets $H_1, H_2, \ldots, H_n$ and the edges connecting these. The resulting graph is denoted by Q.

Let $V(G) = V(Q) \cup \{c_1, c_2, \ldots, c_m\} \cup \{c'_1, c'_2, \ldots, c'_m\}$. For each $j = 1, 2, \ldots, m$ we will add an edge from both c_j and c'_j to y_2^i if and only if v_i is a literal in the clause C_j. We will furthermore add an edge from both c_j and c'_j to y_1^i if and only $\overline{v_i}$ is a literal in the clause C_j. This completes the construction of G. See Figure 4 depicting G for $I = (v_1, v_2, \overline{v_3})$.

We will now show that G contains a 0-perfect forest of size at least $n-2$ if and only if I is NAE-satisfied. First assume that I is NAE-satisfied and consider a truth assignment τ NAE-satisfying I. We will construct two vertex-disjoint induced trees, T_1 and T_2, in G, such that all degrees in the trees T_i are odd for $i \in [2]$. If v_i is true in τ then add the vertices in

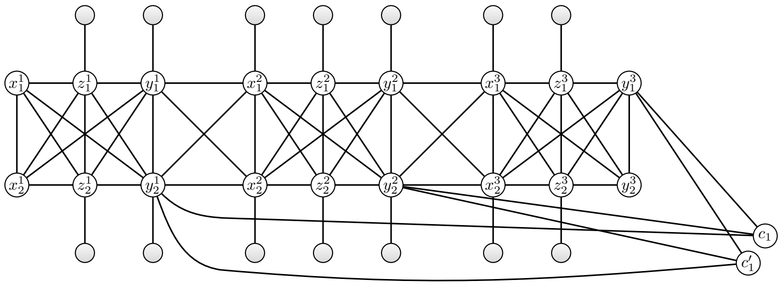

Figure 4 The graph G if $I = (v_1, v_2, \overline{v_3})$.

$\{x_1^i, z_1^i, y_1^i\}$ to T_1 and the vertices in $\{x_2^i, z_2^i, y_2^i\}$ to T_2. Conversely, if v_i is false in τ then add the vertices in $\{x_1^i, z_1^i, y_1^i\}$ to T_2 and the vertices in $\{x_2^i, z_2^i, y_2^i\}$ to T_1. We furthermore add all vertices of degree one to the same tree as their neighbour. Note that the vertices we have added so far to T_i (for $i \in [2]$) induce a tree in G, where every vertex has odd degree in T_i.

Finally as I is NAE-satisfied we note for $j \in [m]$, each of c_j and c_j' has one edge into one of the T_i's and two edges into the other T_i. Add each of c_j and c_j' to the T_i with which it is only connected by one edge. We note that after this operation the vertices we have added so far to T_i (for $i \in [2]$) still induces a tree in G where every vertex has odd degree in T_i. After doing the above operation for all $j \in [m]$ we have obtained the desired trees T_1 and T_2 whose union form a 0-perfect forest in G with $|V(G)| - 2$ edges. See Figure 5 for the found T_1 and T_2 if the instance of NAE-3-SAT is $I = (v_1, v_2, \overline{v_3})$ and the truth assignment is to set all variables equal to true.

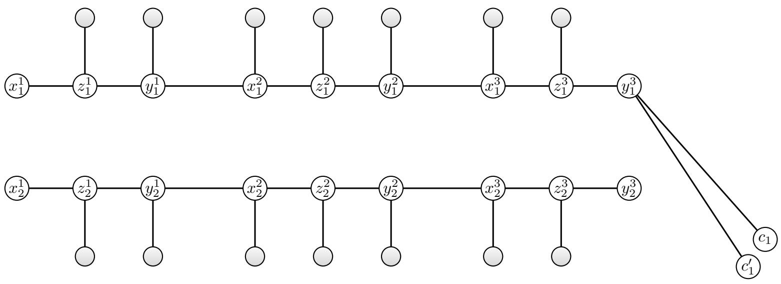

Figure 5 The trees T_1 and T_2 if $I = (v_1, v_2, \overline{v_3})$ and $v_1 = v_2 = v_3 = \mathit{true}$.

Conversely, assume that G contains a 0-perfect forest with at least $|V(G)| - 2$ edges. As G is not a tree this implies that G contain two vertex-disjoint trees T_1 and T_2 such that each T_i is an induced tree in G of order at least 2, all degrees in each T_i are odd, and $V(T_1)$ and $V(T_2)$ partition $V(G)$. We will now prove the following claims where Claim C completes the proof of the theorem.

▷ Claim A. For each $i \in [n]$ one of the following cases hold.

A.1: $\{x_1^i, z_1^i, y_1^i\} \in V(T_1)$ and $\{x_2^i, z_2^i, y_2^i\} \in V(T_2)$.

A.2: $\{x_1^i, z_2^i, y_1^i\} \in V(T_1)$ and $\{x_2^i, z_1^i, y_2^i\} \in V(T_2)$.

A.3: $\{x_1^i, z_1^i, y_1^i\} \in V(T_2)$ and $\{x_2^i, z_2^i, y_2^i\} \in V(T_1)$.

A.4: $\{x_1^i, z_2^i, y_1^i\} \in V(T_2)$ and $\{x_2^i, z_1^i, y_2^i\} \in V(T_1)$.

Proof of Claim A. As the only two non-edges in H_i are $x_1^i y_1^i$ and $x_2^i y_2^i$ we note that there exist a 4-cycle on every set of 4 vertices in H_i. Therefore $|V(T_j) \cap V(H_i)| \geq 4$ is not possible for any $j \in [2]$ and $i \in [n]$. So $|V(T_j) \cap V(H_i)| = 3$ for $j \in [2]$ and $i \in [n]$.

As there is no 3-cycle in $G[V(T_j)]$ for $j \in [2]$ we note that x_1^i and y_1^i must belong to one of the trees, say T_j, and x_2^i and y_2^i must belong to the other tree, T_{3-j}. So if $x_1^i \in V(T_1)$ then $y_1^i \in V(T_1)$ and $\{x_2^i, y_2^i\} \subseteq V(T_2)$ and we are in case A.1 or A.2. On the other hand if $x_1^i \in V(T_2)$ then $y_1^i \in V(T_2)$ and $\{x_2^i, y_2^i\} \subseteq V(T_1)$ and we are in case A.3 or A.4. This completes the proof of Claim A. ◁

▷ Claim B. For $i = 1, 2$, $G[V(Q) \cap V(T_i)]$ is a tree where all vertices have odd degree.

Proof of Claim B. Any vertex in G with degree one must belong to the same tree, T_j, as its neighbour, as both T_1 and T_2 have order at least two. By Claim A, we therefore note that $G[V(Q) \cap V(T_i)]$ is a path of length $3n$ with a pendent edge attached to each non-endpoint of the path. This implies that $G[V(Q) \cap V(T_i)]$ is a tree where all vertices have odd degree (as all degrees are either 1 or 3). This completes the proof of Claim B. ◁

▷ Claim C. The instance I is NAE-satisfiable.

Proof of Claim C. Assume that the vertex c_j belongs to T_1. First suppose that $|N_G(c_j) \cap V(T_1)| = 0$. In this case c_j has no neighbours in T_1, a contradiction, as T_1 is a tree with order at least two. So $|N_G(c_j) \cap V(T_1)| \geq 1$. Assume that $|N_G(c_j) \cap V(T_1)| \geq 2$. As T_1 is an induced tree in G, c_j must have at least two neighbours, say x and y, in T_1. However, by Claim B, there exists a (x, y)-path in T_1 using only vertices from $V(Q)$, which implies that there is a cycle in T_1, a contradiction. Therefore $|N_G(c_j) \cap V(T_1)| = 1$.

Analogously, we can show that $|N_G(c_j) \cap V(T_2)| = 1$, whenever $c_j \in V(T_2)$. So each clause C_j ($j \in [m]$) has either exactly one literal that is false (if $c_j \in V(T_1)$) or exactly one literal that is true (if $c_j \in V(T_2)$). This implies that I is NAE-satisfiable, which completes the proof of Claim C and the theorem. ◁

4 Proof of Theorem 6

To prove Theorem 6, we will use the following result. The proof of Theorem 4 follows the same approach as the proof that it is NP-hard to determine whether there is an induced cycle of odd length through a prescribed vertex, given in [2] by Bienstock. The proof is not given here but can be found in the appendix of [8].

▶ **Theorem 4.** *It is NP-hard to determine whether a graph contains an induced cycle through two given edges.*

▶ **Theorem 6.** *The following problem is NP-hard. Given a connected graph G and an edge $e \in E(G)$, decide whether G has a 0-perfect forest containing e.*

Proof. Let G be a graph and let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be distinct edges of G. We will construct an auxillary graph H with an edge $e'_2 \in E(H)$, such that H contains a 0-perfect forest containing e'_2 if and only if G contains an induced cycle, C, such that $e_1, e_2 \in E(C)$. This will complete the proof by Theorem 4.

Let H be obtained from G by adding a pendent edge to each vertex in $V(G) \setminus \{u_1, v_1\}$ and deleting the edge e_1. Let E_P denote the set of all the pendent edges we just added to G. Let $e'_2 = u_2v_2$ and note that $e'_2 \in E(H)$. This completes the construction of H.

Assume that there exists an induced cycle, C, in G such that $e_1, e_2 \in E(C)$. Let $E' = E_P \cup E(C) \setminus e_1$. Note that the edges in E' induce a 0-perfect forest in H containing the edge e'_2.

Conversely assume that there is a 0-perfect forest, F, in H containing e'_2. Clearly F contains all edges in E_p as each pendent edge is incident with a vertex of degree one. Let Q be the subgraph of H induced by the edges in $E(F) \setminus E_P$. Note that Q is a perfect forest where u_1 and v_1 have odd degree and all other vertices have even degree. As Q is a perfect forest all components are induced trees, and as u_1 and v_1 are the only vertices of odd degree, this implies that Q is an induced path between u_1 and v_1. Adding the edge e_1 to Q gives us an induced cycle in G containing both e_1 and e_2 (as $e_2' \in E(F)$).

Therefore we have proven that H contains a 0-perfect forest containing e_2' if and only if G contains an induced cycle, C, such that $e_1, e_2 \in E(C)$, as desired. ◀

5 Proof of Theorem 8

Let G be a graph and $e = uv$ an edge of G. Let $f\colon V(G) \to \{0,1\}$ be an even-sum function. Our polynomial-time algorithm will follow from the four claims proved below. At the end of the proof, we briefly discuss how the claims are used in the algorithm.

▷ Claim A. Suppose that G contains a cut-vertex x, which may or may not belong to $\{u, v\}$. Let C be the component in $G - x$ intersecting $\{u, v\}$ (there is exactly one such component as $uv \in E(G)$) and let $G' = G[V(C) \cup \{x\}]$. Let $f'(w) = f(w)$ for all $w \in V(C)$ and define $f'(x) \in \{0,1\}$ such that $\sum_{z \in V(G_i)} f'(z)$ is even. Then G has an f-parity perfect forest not containing e if and only if G' has an f'-parity perfect forest not containing e.

Proof of Claim A. Let G contain a cut-vertex x and let $C_1, C_2, \ldots, C_k$ be the components in $G - x$. Without loss of generality, assume that C_1 is the component intersecting $\{u, v\}$. Let $G_i = G[V(C_i) \cup \{x\}]$ for all $i \in [k]$.

For each $i \in [k]$ we will let $f_i : V(G_i) \to \{0,1\}$ be defined such that $f_i(w) = f(w)$ for all $w \in V(C_i)$ and $\sum_{z \in V(G_i)} f_i(z)$ is even (this defines the value of $f_i(x)$). We will show that G has an f-parity perfect forest not containing e if and only if G_1 has an f_1-parity perfect forest not containing e, which will complete the proof of Claim A.

First assume that G_1 has an f_1-parity perfect forest F_1 not containing e. By Theorem 12 there exists an f_i-parity perfect forest, F_i, in G_i for all $i = 2, 3, \ldots, k$. Now $F_1 \cup F_2 \cup \cdots \cup F_k$ is an f-parity perfect forest of G not containing e, as desired.

Conversely assume that G has an f-parity perfect forest F not containing e. If we restrict F to $V(G_1)$, then we obtain an f_1-parity perfect forest of G_1 not containing e. ◁

▷ Claim B. If G is 2-connected and $f(u) = 0$ or $f(v) = 0$ then G has an f-parity perfect forest not containing e.

Proof of Claim B. Assume without loss of generality that $f(u) = 0$. As G is 2-connected $G - u$ is connected and $\sum_{z \in V(G-u)} f(z)$ is even. Therefore, by Theorem 12, there exists an f-parity perfect forest in $G - u$, which is also an f-parity perfect forest in G not containing the edge e. ◁

▷ Claim C. If G is 2-connected and $f(u) = f(v) = 1$ then G has a f-parity perfect forest if and only if $\sum_{z \in V(G)} f(z) \geq 4$.

Proof of Claim C. Let $S = \sum_{z \in V(G)} f(z)$. As f is even-sum, S is even. Since $f(u) = f(v) = 1$, we have $S \geq 2$. If $S = 2$ and F is an f-parity perfect forest in G, then u and v must be leaves of the same tree in F (as they are the only vertices with an f-value of one). Therefore $e \in E(F)$, as otherwise the tree containing u and v is not induced in G. So, if $S = 2$ then G has no f-parity perfect forest F in G with $e \notin E(F)$.

We may therefore assume that $S \geq 4$ and let $w \in V(G) \setminus \{u, v\}$ have $f(w) = 1$. As G is 2-connected there exists a (u, v)-path, P, in G with $w \in V(P)$. (To see it, consider two internally disjoint paths from w to w' where w' is a new vertex added to G such that

$N(w') = \{u, v\}$.) We now create a spanning tree T in G, such that $E(P) \subseteq E(T)$ and $d_T(w) = 2$, as follows. Initially let $T = P$. While $V(T) \neq V(G)$ let $q \in V(G) \setminus V(T)$ be arbitrary such that q has an edge into $V(T) \setminus \{w\}$ (which exists as G is 2-connected). Add q and an edge from q into $V(T) \setminus \{w\}$ to T. When $V(T)$ becomes equal to $V(G)$ we have our desired tree T.

Let T_1 and T_2 be the two trees in $T - w$ (there are exactly two trees in $T - w$ as $d_T(w) = 2$). Let $S_1 = \sum_{z \in V(T_1)} f(z)$ and let $S_2 = \sum_{z \in V(T_2)} f(z)$. As $f(w) = 1$ and $V(T_1) \cup V(T_2) = V(G) \setminus \{w\}$, we note that $S_1 + S_2$ is odd. If S_i is odd then add w to T_i ($i \in [2]$), using the edge from w to $V(T_i)$ in T. This results in two trees, say T_1' and T_2', where $\sum_{z \in V(T_i')} f(z)$ is even for $i \in [2]$. Furthermore, as $w \in V(P)$ and $E(P) \subseteq E(T)$, we note that u and v do not belong to the same tree T_i'. By Theorem 12 there exists an f-parity perfect forest, F_i', of $G[V(T_i')]$ for $i \in [2]$ (as T_i' is a spanning tree in $G[V(T_i')]$, $G[V(T_i')]$ is connected). Now $F_1' \cup F_2'$ is an f-parity perfect forest of G not containing e. This completes the proof of Claim C. ◁

It is easy to see that the following algorithm is of polynomial time. Keep reducing the graph (see Claim A) as long as there exists a cut-vertex and when there are no more cut-vertices then the answer is "no" if the endpoints of e have an f-value of one and all other vertices have an f-value of zero and "yes", otherwise (see Claims B and C). See Figure 6 for an illustration of the algorithm.

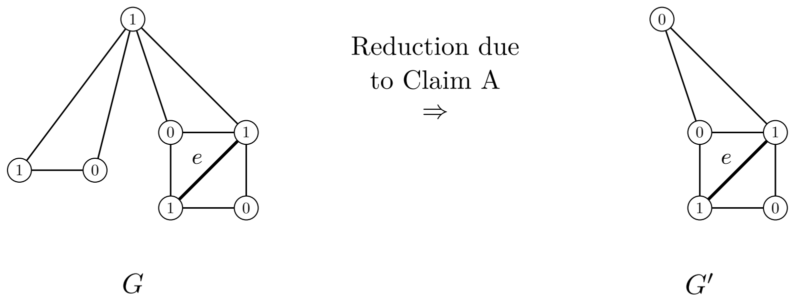

Figure 6 An illustration of the algorithm given in Theorem 8, where the values on the nodes indicate the f-values. As in the final graph the endpoints of e have an f-value of one and all other vertices have an f-value of zero there is no f-parity perfect forest in G' avoiding the edge e and therefore not in G either.

6 Proof of Theorem 10

Theorem 10 follows from Theorem 2 and Lemma 3 proved in this section. To prove Theorem 2, we will use the following:

▶ **Lemma 4.** *Let G be a connected graph of even order and let $xy \in E(G)$ such that $G - \{x, y\}$ is connected. If $G - x \in \mathcal{B}$ and $G - y \in \mathcal{B}$ then $N[x] = N[y]$.*

Proof. Let G be a connected graph of even order and let $xy \in E(G)$ be chosen such that $G - \{x, y\}$ is connected. Let $G_y = G - x$ and let $G_x = G - y$ and assume that $G_y \in \mathcal{B}$ and $G_x \in \mathcal{B}$. Let $C_1^x, C_2^x, \ldots, C_{l_x}^x$ be the blocks of G_x and without loss of generality assume that $x \in V(C_1^x)$. Analogously, let $C_1^y, C_2^y, \ldots C_{l_y}^y$ be the blocks of G_y and without loss of generality assume that $y \in V(C_1^y)$.

▷ Claim A. $N_{G_x}[x] = V(C_1^x)$ and C_1^x is a complete graph of odd order and $C_1^x - x$ is a block in $G - \{x, y\}$. Analogously, $N_{G_x}[y] = V(C_1^y)$ and C_1^y is a complete graph of odd order and $C_1^y - y$ is a block in $G - \{x, y\}$.

Proof of Claim A. For the sake of contradiction assume that $u_1, u_2 \in N_{G_x}(x)$ but u_1 and u_2 belong to different blocks of G_x. In this case there is a cut-vertex in G_x separating u_1 and u_2, which must be x (as $u_1 x u_2$ is a path in G_x). However x does not separate u_1 and u_2 as $G - \{x, y\}$ is connected. This contradiction implies that all vertices in $N_{G_x}(x)$ belong to the same block of G_x.

Therefore, $N_{G_x}[x] \subseteq V(C_1^x)$ as x is not a cut-vertex in G_x (as $G - \{x, y\}$ is connected) and hence x only belongs to one block of G_x. As $G_x \in \mathcal{B}$ we note that C_1^x is a complete graph of odd order. As $|V(C_1^x)| \geq 3$ (as all blocks contain at least two vertices, and $|V(C_1^x)|$ is odd) and x is not a cut-vertex in G_x we note that $C_1^x - x$ is a block in $G - \{x, y\}$. This completes the proof of Claim A. ◁

We now return to the proof of the lemma. By Claim A we note that $C_1^y - y$ is a block in $G - \{x, y\}$ which furthermore is a complete graph of even order. If $C_1^x - x$ and $C_1^y - y$ are different blocks in $G - \{x, y\}$, then $C_1^y - y$ is a block of even order in G_x, a contradiction to $G_x \in \mathcal{B}$. So, $C_1^x - x$ and $C_1^y - y$ are the same block in $G - \{x, y\}$. By Claim A, we have the following chain of equalities, which completes the proof of the lemma.

$$N_G[x] = V(C_1^x - x) \cup \{x, y\} = V(C_1^y - y) \cup \{x, y\} = N_G[y]$$ ◀

▶ **Theorem 2.** *Every connected graph, $G \notin \mathcal{B}$, of odd order $n \geq 3$ contains a proper 1-perfect forest.*

Proof. The proof is by induction over odd integers $n \geq 3$. For $n = 3$, we have $G \cong P_3$, the path of order 3, which is a proper 1-perfect forest. Now we assume that G is a connected graph of odd order $n \geq 5$ such that $G \notin \mathcal{B}$. Let us consider two cases.

Case 1: G is not 2-connected. Assume that G has a cut-vertex x such that $G - x$ has a component C_1 of even order. Let $G_1 = G[V(C_1) \cup \{x\}]$ and let $G_2 = G - V(C_1)$. Note that both G_1 and G_2 are connected graphs of odd order. Furthermore the set of blocks of G is exactly the union of the blocks in G_1 and G_2. As $G \notin \mathcal{B}$ (and therefore some block in G is not a complete graph of odd order) we note that either $G_1 \notin \mathcal{B}$ or $G_2 \notin \mathcal{B}$ (or both).

Let $i \in \{1, 2\}$ be defined such that $G_i \notin \mathcal{B}$ and let $j = 3 - i$. By induction hypothesis, there exists a proper 1-perfect forest F_i in G_i. By Theorem 9 there also exists a (not necessarily proper) 1-perfect forest, F_j, in G_j, where x is the vertex of even degree in F_j. We now note that $F_i \cup F_j$ is a proper 1-perfect forest of G, where the only vertex of even degree is the vertex of even degree in F_i. Thus, we may assume that G has no cut-vertex x such that some component in $G - x$ is of even order.

Now assume that G contains a cut-vertex x. By the previous assumption, all components in $G - x$ are of odd order, and let C_1 be a component of $G - x$. Let $G_1 = G[V(C_1) \cup \{x\}]$ and let $G_2 = G - V(C_1)$. Note that both G_1 and G_2 are connected graphs of even order. By Theorem 1 there exists a 0-perfect forest F_1 in G_1 and a 0-perfect forest F_2 in G_2. Note that $F_1 \cup F_2$ is now a proper 1-perfect forest of G, where the only vertex of even degree is x.

Case 2: G is 2-connected.

▶ **Definition A.** *As $G \notin \mathcal{B}$ and G has odd order, we note that G is not a complete graph. Therefore there exists an induced path $p_1p_2p_3$ in G (that is, $p_1p_2, p_2p_3 \in E(G)$ and $p_1p_3 \notin E(G)$). Let $C_1, C_2, \ldots, C_l$ be the components in $G - \{p_2, p_3\}$, such that $p_1 \in C_1$.*

Assume first that $|V(C_1)|$ is odd. By Theorem 9 there exists a 1-perfect forest F_1 in C_1, such that p_1 (see Definition A) is the vertex of even degree in F_1. Let $G' = G - V(C_1)$ and note that G' is connected and of even order. Therefore, by Theorem 1, there exists a 0-perfect forest, F', in G'.

If $d_{F_1}(p_1) > 0$ then $F_1 \cup F'$ is a proper 1-perfect forest in G. Now consider the case when $d_{F_1}(p_1) = 0$. As $N(p_1) \cap V(G') = \{p_2\}$ (as $p_1p_2p_3$ is an induced path in G) we note that adding the edge p_1p_2 to $F_1 \cup F'$ gives us a proper 1-perfect forest in G (where p_2 is the only vertex of even degree). Thus, in the rest of the proof, we may assume that $|V(C_1)|$ is even.

Let $G' = G[V(C_1) \cup \{p_2, p_3\}]$ and note that G is connected and of even order. Furthermore $G' - \{p_2, p_3\}$ is connected (as $G' - \{p_2, p_3\} = C_1$). As p_1 is adjacent to p_2 but not to p_3 we note that $N_{G'}[p_2] \neq N_{G'}[p_3]$. By Lemma 4 we must therefore have $G' - p_2 \notin \mathcal{B}$ or $G' - p_3 \notin \mathcal{B}$. Let $i \in \{2, 3\}$ be chosen such that $G' - p_i \notin \mathcal{B}$, which by induction hypothesis implies that there is a proper 1-perfect forest F_1 in $G' - p_i$.

As G is 2-connected, we note that p_{5-i} is not a cut-vertex of G. Therefore every component in $G - \{p_2, p_3\}$ has an edge to p_i, which implies that $G - V(F_1)$ is connected and of even order (as both G and F_1 are of odd order). By Theorem 1 there exists a 0-perfect forest, F_2, in $G - V(F_1)$. Now $F_1 \cup F_2$ is a proper 1-perfect forest in G. This completes the proof. ◀

A semiperfect forest F of G is called a *2-perfect forest* if exactly two vertices of F have even degree.

▶ **Lemma 3.** *If G is a connected graph of odd order and $G \in \mathcal{B}$ then G does not contain a proper 1-perfect forest.*

Proof. Let G be a connected graph of odd order and let $G \in \mathcal{B}$. We will prove that G contains no proper 1-perfect forest. We will prove this using induction on the number of blocks in G.

If G contains only one block then G is a complete graph of odd order. In this case, any forest where all trees are induced, can only contain trees of order 2 (and 1 if we allow isolated vertices). This implies that G cannot contain a proper 1-perfect forest as G has odd order. This completes the base case.

Now assume that G contains at least two blocks, which implies that G contains a cut-vertex, x. Let $C_1, C_2, \ldots, C_l$ be the components in $G - x$ and let $G_i = G[V(C_i) \cup \{x\}]$ for $i \in [l]$. For the sake of contradiction suppose that G contains a proper 1-perfect forest F and let F_i denote F restricted to G_i for $i \in [l]$. As F is a proper 1-perfect forest we note that $d_F(x) \geq 1$. Without loss of generality, assume that $d_{F_1}(x) \geq 1$. This implies that F_1 is a proper i-forest in G_1 where $i \in \{0, 1, 2\}$. However as $|V(G_1)|$ is odd (as $G \in \mathcal{B}$) this implies that F_1 is a proper 1-perfect forest in G_1. This is a contradiction to $G_1 \in \mathcal{B}$ (as $G \in \mathcal{B}$). ◀

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