Online Domination: The Value of Getting to Know All Your Neighbors

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Abstract

We study the dominating set problem in an online setting. An algorithm is required to guarantee competitiveness against an adversary that reveals the input graph one node at a time. When a node is revealed, the algorithm learns about the entire neighborhood of the node (including those nodes that have not yet been revealed). Furthermore, the adversary is required to keep the revealed portion of the graph connected at all times. We present an algorithm that achieves 2-competitiveness on trees. We also present algorithms that achieve 2.5-competitiveness on cactus graphs, $(t-1)$-competitiveness on $K_{1,t}$-free graphs, and $\Theta(\sqrt{\Delta})$ for maximum degree Δ graphs. We show that all of those competitive ratios are tight. Then, we study several more general classes of graphs, such as threshold, bipartite planar, and series-parallel graphs, and show that they do not admit competitive algorithms (i.e., when competitive ratio is independent of the input size). Previously, the dominating set problem was considered in a different input model (often together with the restriction of the input graph being always connected), where a vertex is revealed alongside its restricted neighborhood: those neighbors that are among already revealed vertices. Thus, conceptually, our results quantify the value of knowing the entire neighborhood at the time a vertex is revealed as compared to the restricted neighborhood. For instance, it was known in the restricted neighborhood model that 3-competitiveness is optimal for trees, whereas knowing the neighbors allows us to improve it to 2-competitiveness.

2012 ACM Subject Classification Theory of computation → Online algorithms

Keywords and phrases Dominating set, online algorithms, competitive ratio, trees, cactus graphs, bipartite planar graphs, series-parallel graphs, closed neighborhood

Digital Object Identifier 10.4230/LIPIcs.MFCS.2021.57

Related Version *Full Version*: https://arxiv.org/abs/2105.00299

Funding This research is supported NSERC.

1 Introduction

Given an undirected simple graph $G = (V, E)$, a subset of vertices $D \subseteq V$ is called *dominating* if every vertex of V is either in D or is adjacent to some vertex in D. In the well-known $\mathcal{NP}$-hard dominating set problem, the goal is to find a dominating set of minimum cardinality. We study this problem in the online setting, where a graph is revealed one node at a time. When a node is revealed its entire neighborhood is revealed as well. An algorithm is required to make an irrevocable decision on whether to include the newly revealed vertex into the dominating set the algorithm is constructing or not. This decision must be made before the next vertex is revealed. Performance of an online algorithm is measured against an optimal offline algorithm, i.e., an algorithm that knows the entire input in advance and has infinite computational resources. This measure is captured by the notion of competitive ratio and analysis, which is made precise below. For now, it suffices to note that competitive ratio is analogous to approximation ratio in the offline setting.

The dominating set problem has important practical and theoretical applications, such as establishing surveillance service ([1]), routing and transmission services in (wireless) networks ([5]), as well as broadcasting ([7, 8]). While the dominating set problem and its variants (connected dominating set, independent dominating set, weighted dominating set, etc.) have been extensively studied in the offline setting [1, 9, 11, 15, 16, 17], this problem has received little attention in the online algorithms community. The current paper attempts to fill in this gap, while making a quantitative comparison with another online model for dominating set.

Online dominating set problem has been studied in the vertex arrival model by Boyar et al. [3]. In that model, when a vertex is revealed only restricted neighborhood of that vertex is revealed as well, namely, those neighbors that appear among previously revealed vertices. Moreover, in the model considered by Boyar et al. decisions are only partially irrevocable, i.e., when a vertex arrives an algorithm may add this vertex together with *any of its neighbors from the restricted neighborhood* to the dominating set. Thus, the decision to include a vertex is irrevocable, while the decision not to include a vertex is only partially irrevocable – an algorithm has a chance to reconsider when any yet unrevealed neighbors arrive. The catch is that the algorithm does not know the input size and has to maintain a dominating set at all times. In the model considered in this paper, all decisions (to include or exclude a vertex from a dominating set) are irrevocable. Boyar et al. [3] considered the online dominating set problem in two settings, namely, with the restriction of an adversary being forced to maintain an always connected graph and without this restriction. For the fairness of comparison, when we talk about Boyar et al. results we refer to their results for the always-connected setting[1]. To summarize, on one hand, our model is stronger for the adversary since it forces the algorithm to make an irrevocable decision at each step. On another hand, our model is weaker for the adversary than the model of Boyar et al. in the aspect of the adversary being forced to reveal all neighbors of a newly revealed vertex at once. Thus, our results when compared to those of the vertex arrival model can be viewed as quantifying the value of getting to know all neighbors of a vertex at the time of its revelation.

Perhaps somewhat surprisingly, we discover in several results that the benefit of knowing all neighbors outweighs the drawbacks of fully irrevocable decisions. Our results are summarized below, but in particular we show that in our model Δ-bounded degree graphs admit $O(\sqrt{\Delta})$ online algorithms, while Boyar et al. show that $\Omega(\Delta)$ is necessary in their model. Similarly, we analyze a 2-competitive algorithm for trees, while Kobayashi [13] shows a lower bound of 3 in the vertex arrival model. Our degree upper bound implies that $O(\sqrt{n})$ competitive ratio is tight for general graphs, whereas Boyar et al. showed the lower bound of $\Omega(n)$ in the vertex arrival model. This paints a picture that knowing all the neighbors improves not only precise constants, when graph classes allow for small competitive ratio algorithms, but also give asymptotic improvements for more "challenging" graph classes for algorithms.

Prior to summarizing our results, we give a brief overview of competitive analysis framework. For more details, an interested reader should consult excellent books [2, 14] and references therein. Let *ALG* be an algorithm for the online dominating set problem.

[1] In our model, two natural definitions of always-connected restriction are possible: (i) with respect to all vertices that the algorithm is aware of at any particular moment (this includes vertices that have arrived and their neighbors that have not yet arrived), and (ii) with respect to only those vertices that have arrived. Our work is in setting (ii). This distinction is absent in the vertex arrival model.

Let $ALG(G, \sigma)$ denote the set of vertices that are selected by ALG on the input graph G with its vertices revealed according to the order σ. We sometimes abuse the notation and omit G or σ (or both) when they are clear from the context. Abusing notation even more, we sometimes write $ALG(G, \sigma)$ to mean $|ALG(G, \sigma)|$. Similar conventions apply to an offline optimal solution denoted by OPT. We say that ALG has *strict competitive ratio* c if $ALG \leq c \cdot OPT$ on all inputs. We say that ALG has *asymptotic competitive ratio* c (or, alternatively, that ALG is c-competitive) if $\limsup_{OPT \to \infty} \frac{ALG}{OPT} \leq c$. The *competitive ratio* of ALG is the infimum over all c such that ALG is c-competitive. When we simply write "competitive ratio" we typically mean "asymptotic competitive ratio" unless stated otherwise.

We shall consider performance of algorithms with respect to restricted inputs, specified by various graph classes, such as trees, cactus graphs, series-parallel, etc. The above definitions of competitive ratios can be modified by restricting them to inputs coming from certain graph classes. We denote the competitive ratio of an algorithm ALG with respect to the restricted graph class CLASS by $\rho(ALG,\text{CLASS})$.

The following is a summary of our contributions with the section numbers where the results appear. Due to space considerations some of our results have been moved to appendix:

- tight competitive ratio 2 on trees (Section 3.1);
- tight competitive ratio $\frac{5}{2}$ on cactus graphs (Section 3.2);
- tight competitive ratio $\Theta(\sqrt{\Delta})$ on maximum degree Δ graphs (Section 3.3);
- tight competitive ratio $t - 1$ on $K_{1,t}$-free graphs (Section 3.4);
- tight competitive ratio $\Theta(\sqrt{n})$ for threshold graphs (Section B.1), planar bipartite graphs (Section B.2), and series-parallel graphs (Section B.3).

We note that all our upper bounds are in terms of strict competitive ratios, and all our lower bounds, with the exception of $K_{1,t}$-free graphs, are in terms of asymptotic competitive ratios.[2] Most of our upper bounds are established by charging arguments. Our charging schemes are natural and to analyze them we establish several combinatorial properties of relevant graph classes. We suspect that these (or similar) techniques can be used to extend the results to other graph classes, such as almost-tree(k). Our main contribution is conceptual: we begin a systematic study of a well known $\mathcal{NP}$-hard problem in an online setting that hasn't been extensively considered before and which allows quantifying how much extra information about the neighborhood helps the competitive ratio.

2 Preliminaries

In this section we describe definitions and establish notations that will be used frequently in the rest of the paper. Let $G = (V, E)$ be a connected undirected graph on $n = |V| \geq 1$ vertices. The *closed neighborhood* of a subset of vertices $S \subseteq V$, denoted by $N[S]$, is defined as $S \cup \{v \in V \mid \exists u \in S, \{u, v\} \in E\}$.

The vertices V are revealed online in order $(v_1, ..., v_n)$. Since we consider the online input model where vertices are revealed alongside their neighbors, we distinguish between two notions: those vertices that are revealed by a certain time and those that are visible. More precisely, we have the following:

[2] With the small caveat that the performance ratio for threshold graphs is measured as a function of input size for reasons provided later.

▶ **Definition 1.**
- v_i *is* revealed by time j *if* $i \leq j$*.*
- v_j *is* visible at time i *if it is either revealed by time* i *or it is adjacent to some vertex revealed by time* i*.*
- R_i *denotes the set of all vertices revealed by time* i*.*
- V_i *denotes the vertices visible at time* i *(i.e.* $V_i = N[R_i]$*).*

The adversary chooses the graph G as well as the revelation order of vertices; however, the adversary is restricted to those revelation orders that guarantee that the induced subgraph on R_i is connected for all i. Thus, we observe that the process of revelation of a graph by the adversary is a natural generalization of the breadth-first search (BFS) and depth-first search (DFS) explorations of the graph. Thus, we can define the *revelation tree* analogous to BFS and DFS trees. We need the following observation first:

▶ **Observation 2.** *If* $v_j \in V_i \setminus V_{i-1}$ *with* $j > i \geq 2$ *then* v_i *is the unique neighbor of* v_j *at time* i*.*

In the preceding observation, we say that v_j is a *child* of v_i and that v_i is the *parent* of v_j. The edge $\{v_i, v_j\}$ is called a *tree edge*. The subgraph induced on the tree edges is the revelation tree. Any edge $\{u, v\}$ where u is not the parent of v nor v the parent of u is called a *cross edge*.

After the vertex v_i is revealed together with its closed neighborhood $N[v_i]$, an online algorithm ALG must make a decision $d_i \in \{0, 1\}$, which indicates whether the algorithm takes this vertex to be in the dominating set or not. For a given online algorithm ALG we define the following:

▶ **Definition 3.**
- $S_i = \{v_k \mid d_k = 1, 1 \leq k \leq i\}$ *is the set of revealed vertices selected by* ALG *after* i *decisions where* $S_0 = \emptyset$*.*
- $D_i = N[S_i]$ *is the set of visible vertices that are dominated after* i *decisions.*
- $U_i = V_i \setminus D_{i-1}$ *is the set of visible vertices undominated immediately before decision* d_i *where* $U_0 = \emptyset$*.*

A series of figures are provided below which illustrate the preceding definitions. For these figures, and all others in this paper, the convention is that vertices that are shaded in gray are those selected by ALG, vertices with thicker boundaries belong to OPT, an edge that is dashed is a cross edge, and all the solid edges are tree edges.

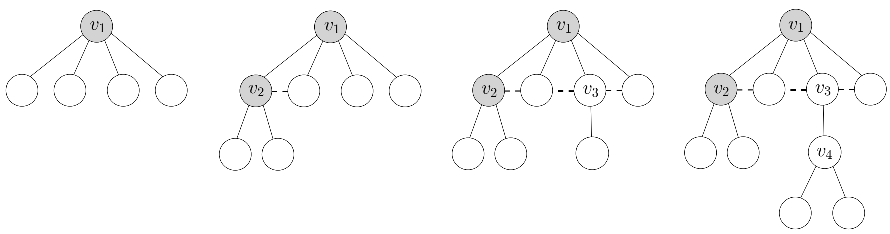


Figure 1 An example of vertices v_1, v_2, v_3, v_4 from some input graph being revealed in that order (from left to right). Empty vertices in this figure are visible but not yet revealed. The adversary must maintain the connectivity of revealed vertices (ignoring visible but not yet revealed vertices) at all times. The process continues until all vertices are revealed. An edge that is dashed is a cross edge and one that is solid is a tree edge.

Since an algorithm makes irrevocable decisions and must produce a feasible solution, there may be situations where an algorithm is forced to select a vertex v_j to be in the dominating set. This happens because v_j is the "last chance" to dominate some other vertex v_i. In this case, we say that v_j saves v_i or that v_j is the savior of v_i. Note that it is possible for a vertex v_j to save itself. The following definition makes the notion of "saving" precise.

▶ **Definition 4.** *A vertex v_j saves a vertex v_i if $j = \max\{k \mid v_k \in N[v_i]\}$ and $N[v_i] \setminus \{v_j\}$ contains no vertices from S_{j-1}. Let $s(v_j)$ denote the set of vertices that v_j saves.*

Observe that if a vertex is saved then it must be that every one of its neighbors (itself included) had a chance to dominate the said vertex.

▶ **Observation 5.** *If v_i is saved then $v_i \in N[v_j] \cap U_j$ for any $v_j \in N[v_i]$.*

All our upper bounds are established by either a GREEDY algorithm or a k-DOMINATE algorithm for some fixed integer value of parameter k:

- The algorithm GREEDY selects a newly revealed vertex if and only if the vertex is not currently dominated. Using the notation introduced above, GREEDY selects $v_i, i \geq 1$ if and only if $v_i \in U_i$.
- The algorithm k-DOMINATE (for some fixed integer parameter k) selects a newly revealed vertex if and only if either (1) the vertex has at least k undominated neighbors, or (2) the vertex saves at least one other vertex. Using the notation introduced before, v_i is selected if and only if either (1) $|N(v_i) \cap U_i| \geq k$, or (2) $|s(v_i)| \geq 1$.

Both GREEDY and k-DOMINATE give rise to rather efficient offline algorithms so that any of the positive results given in this paper may be realized as efficient offline approximation algorithms.

3 Competitive Graph Classes

3.1 Trees

In this section we establish the tight bound of 2 on the best competitive ratio when the input graph is restricted to be a tree. The upper bound is achieved by the 2-DOMINATE algorithm and is proved in Theorem 7 below. The lower bound on all online algorithms is established in Theorem 6. Within this section all of the formal statements implicitly assume that the input is a tree. We begin the section by proving the lower bound.

▶ **Theorem 6.** *$\rho(ALG, TREE) \geq 2$ for any algorithm ALG.*

Proof. Consider an arbitrary small $\epsilon > 0$. We will give an adversarial input that guarantees that $ALG \geq (2 - \epsilon)OPT$. Let $k = \lceil \frac{3}{\epsilon} \rceil \geq 4$. At the start, the adversary reveals v_1 with k children $\{c_1, \ldots, c_k\}$. Then we start the process described in the next paragraph at c_1. The process can terminate in two ways: (i) ALG stops selecting vertices to be in the dominating set, or (ii) ALG selects k vertices revealed after c_1 (inclusive). If the process terminates because of (i), then the adversary restarts the process at child c_2 of v_1. The process again terminates either with (i) or (ii) with respect to c_2. If it is due to (i), then the adversary restarts the process at c_3, and so on. If the process terminates with (ii) with respect to c_i then we reveal c_j for $j > i$ as leaves of v_1.

Next, we describe the process with respect to c_i. The adversary reveals c_i with 2 children and if ALG selects c_i then exactly one child of c_i is revealed with two additional children. If ALG selects the child then one of its children is revealed with two additional children, and so on. Let j_i be the number of these vertices that are selected by ALG. This process terminates only if ALG stops selecting these vertices with two children ($j_i < k$) or when ALG selects k of them ($j_i = k$). At this point the subtree grown at c_i has some revealed vertices as well as visible, but not yet revealed vertices. To finish revealing the entire subtree, the adversary proceeds as follows.

If $j_i < k$ then the two children on the (j_i+1)'st vertex are revealed to be leaves. Moreover, each of the j_i selected vertices have exactly one visible child that is not yet revealed. Reveal those j_i children, called *support vertices*, with an additional leaf child (i.e. the child is revealed to be a leaf after its parent is revealed). Including the 2 children of the (j_i+1)'st vertex ALG must select at least $j_i + 2$ additional vertices to dominate these leaves for a total of $j_i + (j_i + 2) = 2(j_i + 1)$ selected vertices in this subtree. In this case, OPT can select the support vertices together with the $(j_i + 1)$'st vertex for a total $j_i + 1$ vertices to dominate the entire subtree.

If $j_i = k$ the procedure to finish revealing the entire subtree at c_i is similar: the k'th vertex children are both revealed to be leaves and each of the other $k - 1$ selected vertices has the other child become a support vertex, i.e., revealed with an additional leaf child. The performance is similar here but ALG is not forced to select the two children of the k'th vertex so ALG selects at least $k + (k - 1) = 2k - 1$. In this case, OPT needs only select the k'th vertex together with the support vertices for a total of k vertices to dominate the subtree.

To finish the analysis, we consider the following two cases:

Case 1: for all i we have $j_i < k$. Then $ALG \geq 2(j_i+1)$ on each subtree whereas $OPT \leq j_i+1$ on each subtree. Summing over all subtrees and remarking that OPT might select v_1 we obtain that $ALG/OPT \geq \left(\sum 2(j_i + 1)\right) / \left(1 + \sum(j_i + 1)\right) \geq 2 - 2/k \geq 2 - \epsilon$.

Case 2: there exists ℓ such that $j_\ell = k$. Then OPT selects j_i+1 vertices for $i < \ell$, k vertices for $i = \ell$, 0 vertices for $i > \ell$ per subtree, plus v_1. Whereas ALG selects at least $2(j_i + 1)$ for $i < \ell$, $2k - 1$ for $i = \ell$, and 0 for $i > \ell$. By a similar calculation to **Case 1**, we obtain that $ALG/OPT \geq 2 - 3/k \geq 2 - \epsilon$. ◀

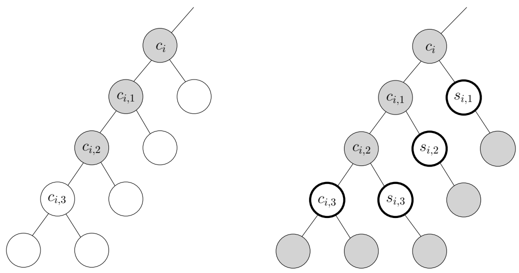


Figure 2 An example of the process described in Theorem 6 where ALG selects $j_i = 3$ vertices on the subtree rooted at c_i. The top depicts the subtree immediately after revealing $c_{i,3}$ whereas the bottom shows the entirely revealed subtree.

Now that we have established an asymptotic lower bound of 2 for any algorithm we show that 2-DOMINATE is 2-competitive.

▶ **Theorem 7.** *ρ(2-DOMINATE, TREE) $= 2$.*

High level overview of the proof. Consider an arbitrary input $T = (V, E)$ on $n \geq 3$ vertices and let OPT denote a minimum dominating set of T which contains no vertices of degree 1 (i.e. any such vertex can be exchanged for its only neighbor). Recall that S is the set of vertices selected by 2-DOMINATE. Initially, we assign charge 1 to each vertex v in S and charge 0 to each vertex v not in S. Thus, $|S| = \sum_{v \in S} ch_{init}(v)$ where $ch_{init}(v)$ denotes the initial charge of v. With a charging scheme described shortly, we spread the charge from the vertices in S to the vertices of V. Let $ch(v)$ denote the new charge associated with vertex v. We extend the functions ch_{init} and ch to subsets of vertices linearly, e.g., for $W \subseteq V$ we have $ch(W) = \sum_{v \in W} ch(v)$. We shall demonstrate that the procedure of spreading the charge satisfies two properties:

1. conservation property: $\sum_v ch_{init}(v) = \sum_v ch(v)$ meaning that the total charge is preserved; and
2. OPT-concentration property: for each $v \in OPT$ we have $ch(N[v]) \leq 2$.

With these two properties it follows that 2-DOMINATE $\leq \sum_v ch_{init}(v) = \sum_v ch(v) \leq \sum_{v \in OPT} ch(N[v]) \leq 2OPT$, so 2-DOMINATE is strictly 2-competitive.

Before we proceed with this plan, we make a couple of useful observations:

▶ **Lemma 8.** *There are no cross edges incident on any vertex v_i. In particular, any vertex v_i has at most one neighbor before it is revealed.*

▶ **Corollary 9.** *If $deg(v_i) \geq 3$ then $v_i \in S$.*

Now, we are ready to present formal details of the above plan. We spread the charges according to the following rule:

Consider any $v_i \in S$ with $X_i = N[v_i] \cap U_i$. Remarking that $X_i \neq \emptyset$ we then give each vertex in X_i an equal charge of $\frac{1}{|X_i|}$. That is, a vertex selected by 2-DOMINATE spreads its charge evenly to all the newly dominated vertices in its closed neighborhood. We say that each vertex in X_i is charged by v_i.

▶ **Observation 10.** *Every vertex is charged by exactly one vertex.*

The preceding observation immediately implies that any vertex has charge at most 1. This observation is tight in the sense that, on certain inputs, there are vertices with charge equal to 1. A vertex with charge 1 is a rather special case though. Suppose that a vertex v_i receives charge 1 from a vertex v_j where v_i and v_j are not necessarily distinct. Therefore we have that $|X_j| = |N[v_j] \cap U_j| = 1$ which implies that $|N(v_j) \cap U_j| \leq |N[v_j] \cap U_j| = 1 < 2$. Therefore when v_i receives charge it must be from a vertex $v_j \in S$ that was selected due to the "saviour" rule of 2-DOMINATE. Hence, v_j must have saved a vertex, and only one vertex since $|X_j| = 1$. Ultimately we conclude that if v_i has charge 1 then it must be saved by some vertex v_j where $X_j = \{v_i\}$ (this does not exclude the possibility that $v_i = v_j$). If v_i does not meet this condition then it must have charge at most $\frac{1}{2}$.

▶ **Lemma 11.** *If v_i and v_j both have charge equal to 1 then they share no common neighbors.*

Proof. Suppose for the sake of deriving a contradiction that $v_{i'}$ were a common neighbor of v_i and v_j. Since v_i is saved, by Observation 5 it must be that $v_i \in N(v_i') \cap U_{i'}$. Similarly, we have that $v_j \in N(v_i') \cap U_{i'}$. That is, $|N(v_i') \cap U_{i'}| \geq 2$ and thus $v_i' \in S$. Moreover, $X_{i'} = N[v_{i'}] \cap U_{i'}$ contains v_i and v_j. In particular, we have that $|X_{i'}| \geq 2$ with $v_i, v_j \in X_{i'}$ and therefore v_i and v_j receive charge no larger than $\frac{1}{2}$, a contradiction. ◀

▶ **Lemma 12.** *If v_i and v_j both have charge equal to 1 then they are not adjacent.*

Proof. It is easy to see that v_1 cannot have charge 1 on any input with at least 2 vertices. Therefore we safely assume that $1 < i < j$ such that both v_i and v_j have a parent. We assume for the sake of deriving a contradiction that v_i and v_j are adjacent.

Now, since both v_i and v_j have charge 1 it follows that they are both saved vertices. First we show that both $v_i, v_j \notin S$. Notice that any saved vertex v_k has the property that $|N[v_k] \cap S| = 1$. Therefore, if we assume by way of contradiction that $v_i \in S$ we obtain that $N[v_i] \cap S = N[v_j] \cap S = \{v_i\}$ and therefore v_i saves itself and v_j. This yields that $X_i = N[v_i] \cap U_i$ contains v_i and v_j. In particular, we have that $|X_i| \geq 2$ with $v_i, v_j \in X_i$ and therefore v_i and v_j receive charge no larger than $\frac{1}{2}$, a contradiction. An identical argument will yield that $v_j \notin S$.

Therefore it must be that v_i is saved by some vertex $v_{i'}$ with $i' \notin \{i, j\}$. Moreover, we must have $i < j < i'$ since $i < j$ by assumption and $i' = \max\{k \mid v_k \in N[v_i]\}$. This implies that both $v_j, v_{i'}$ are children of v_i by Observation 8 yielding that $|N(v_i) \cap U_i| \geq 2$ but v_i cannot be in S. ◀

From the two preceding lemmas we have the immediate corollary.

▶ **Corollary 13.** *For any vertex v_i, at most one vertex in $N[v_i]$ has charge* 1.

Now, we finish the proof of 2-competitiveness of 2-DOMINATE on trees.

Proof of Theorem 7. The lower bound follows from Theorem 6. Let $v_i \in OPT$ be an arbitrary vertex in OPT. We consider two cases **(1)** $deg(v_i) = 2$ or **(2)** $deg(v_i) \geq 3$.

Case 1: Suppose that $deg(v_i) = 2$ and hence $|N[v_i]| = 3$. By Corollary 13 it follows that at most one vertex in $N[v_i]$ has charge 1. If no vertices in $N[v_i]$ have charge 1 then $ch(x) \leq \frac{1}{2}$ for each $x \in N[v_i]$ and we obtain that $\sum_{x \in N[v_i]} ch(x) \leq 3\left(\frac{1}{2}\right) < 2$. If there is exactly one vertex $x' \in N[v_i]$ with charge 1 we therefore obtain that $\sum_{x \in N[v_i]} ch(x) = \sum_{x \in N[v_i] \setminus \{x'\}} ch(x) + ch(x') \leq \frac{2}{2} + 1 = 2$.

Case 2: Suppose that $deg(v_i) \geq 3$. By Corollary 9 it follows that $v_i \in S$ with at least 2 children. Let $C_i = V_i \setminus V_{i-1}$ denote the children of v_i and remark that $C_i \subseteq X_i$. That is, each child of v_i is charged by v_i and only v_i. Therefore the children of v_i can receive at most the full initial charge on v_i and thus attribute a charge of at most 1.

Now we claim that any vertex in $N[v_i] \setminus C_i$ has a charge of at most $\frac{1}{2}$. Indeed, suppose a vertex $v_{i'} \in N[v_i] \setminus C_i$ has charge 1 then it must be saved by v_i since $|N[v_{i'}] \cap S| = 1$ for any saved vertex $v_{i'}$. That is, there is exactly one vertex in its closed neighborhood that is selected and since v_i is selected it must be v_i. Thus, we must have that $v_{i'} \in X_i$ but since $C_i \subseteq X_i$ we know that $|X_i| \geq 2$ and thus $v_{i'}$ receives a charge of no more than $\frac{1}{2} < 1$, contradicting our assumption that $v_{i'}$ has charge 1.

Thus, by remarking that $|N[v_i] \setminus C_i| \leq 2$ we obtain that $\sum_{x \in N[v_i]} ch(x) = \sum_{v_j \in C_i} ch(v_j) + \sum_{v_{i'} \in N[v_i] \setminus C_i} ch(v_{i'}) \leq 1 + 2\left(\frac{1}{2}\right) = 2$ as desired. ◀

3.2 Cactus Graphs

A graph G is said to be a cactus graph if it is connected and every edge lies on at most one cycle. Hedetniemi, Laskar, and Pfaff [10] provide an exact offline algorithm that runs in linear time for finding a minimum dominating set of a cactus graph. Of course, an efficient offline algorithm does not guarantee that an online algorithm can perform well but fortunately, cactus graphs are a class of graphs for which an online algorithm can achieve constant competitive ratio. In this section, we show that 2-DOMINATE is $\frac{5}{2}$-competitive when inputs are restricted to cactus graphs, and that this is as well as any algorithm can perform. Within this section all of the formal statements implicitly assume that the input is a cactus graph.

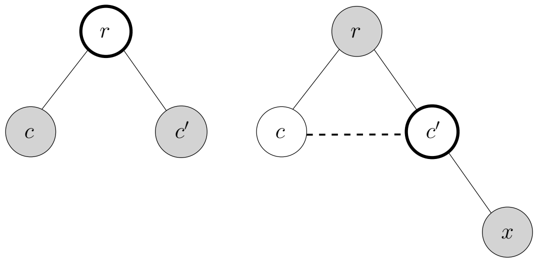


Figure 3 The cactus 2-gadget : The leftmost figure depicts the case where *ALG* does not select the root r and rightmost depicts the case where *ALG* selects r.

Before presenting a lower bound of $\frac{5}{2}$ on all online algorithms we describe a gadget that is used in the proof. The gadget itself is a cactus graph on $3 \leq n \leq 4$ vertices with the property that OPT selects exactly 1 vertex and any algorithm ALG selects at least 2 vertices. Consider revealing a root vertex r with 2 children c and c'. If ALG does not select r then both c, c' are revealed as only adjacent to r and ALG must select both whereas OPT selects only r. If ALG does select r then c is revealed as adjacent to c', and c' is revealed with an additional child x. The vertex x is adjacent only to c' and thus ALG must select at least one of c', x whereas OPT selects only c' (both cases are depicted in Figure 3). Given any input cactus graph with a visible vertex r not yet revealed this gadget can be constructed with r as the root. Within the proof of the lower bound we call this a 2-gadget.

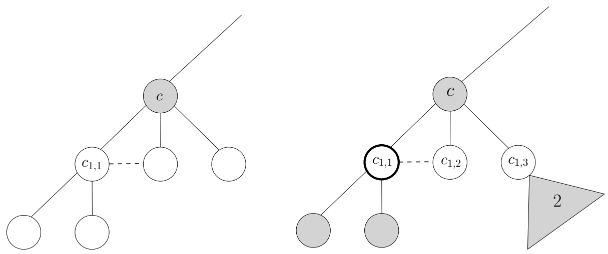


Figure 4 The case described in Theorem 14 where *ALG* does not select $c_{1,1}$.

▶ **Theorem 14.** *$\rho(ALG, CACTUS) \geq \frac{5}{2}$ for any algorithm ALG.*

Proof. Consider an arbitrary small $\epsilon > 0$ and let $k = \lceil \frac{4}{\epsilon} \rceil \geq 5$. We will give an adversarial input that guarantees that $OPT \geq k$ and $ALG \geq (\frac{5}{2} - \epsilon)OPT$. To begin the input, the adversary reveals v_1 with k children $\{c_1, \dots, c_k\}$. Then we run an adversarial process starting with the child c_1 of v_1. The process consists of rounds, where each round increases OPT by 2 while increasing ALG by 5. The process might terminate for one of two reasons: either (i) we guarantee strict competitive ratio at least $5/2$ on the subcactus rooted at c_1, or (ii) k rounds starting at c_1 elapse. If the process terminates because of (i), then the adversary restarts the process at child c_2 of v_1. The process again terminates either with (i) or (ii) with respect to c_2. If it is due to (i), then the adversary restarts the process at c_3, and so on. If the process terminates with (ii) with respect to c_i then we reveal c_j for $j > i$ as leaves. Below we describe the process starting at a child of v_1 although the first round of the process differs from the others that follow.

We now describe the first round starting at a child c of v_1. Initially, we reveal c with 3 children. If ALG does not select c then each child of c is revealed as leaf and ALG must select all 3 children whereas OPT selects c. Suppose then that ALG selects c and let $c_{1,1}, c_{1,2}, c_{1,3}$ be the three children of c. Reveal $c_{1,1}$ as adjacent to $c_{1,2}$ along with 2 additional children. If ALG does not select $c_{1,1}$ then the children of $c_{1,1}$ are revealed as leaves, forcing ALG to select them and $c_{1,3}$ is revealed as the root of a 2-gadget ($c_{1,2}$ is revealed with no additional neighbors). Thus, $\frac{ALG}{OPT} \geq \frac{5}{2}$ in this case (see Figure 4). If instead ALG selects $c_{1,1}$ then $c_{1,2}$ and $c_{1,3}$ are revealed as the roots of two distinct 2-gadgets and since c is dominated by v_1 (we assume that $v_1 \in OPT$) we have that $\frac{ALG}{OPT} \geq \frac{5}{2}$ on this subcactus (excluding $c_{1,1}$) thus far (see Figure 5a). At this point, $c_{1,1}$ is selected by ALG and we start the second round (which is described below) with $c_{1,1}$ as the root. Every round that follows will be the same as the second and requires a root selected by ALG which has two children.

The second round starts at a selected root $c_{1,1}$ and we let $c_{2,1}, c_{2,2}$ be the 2 children of $c_{1,1}$. We reveal $c_{2,1}$ as adjacent to $c_{2,2}$ with 2 children $c_{3,1}, c_{3,2}$. If ALG does not select $c_{2,1}$ then $c_{3,1}, c_{3,2}$ are revealed as leaves and ALG selects $c_{1,1}, c_{3,1}, c_{3,2}$ and OPT can select $c_{2,1}$ for a performance of 3 along with the running performance of $\frac{5}{2}$ (see Figure 5b). If ALG does select $c_{2,1}$ then $c_{3,1}$ is revealed as adjacent to $c_{3,2}$ with two children $c_{4,1}, c_{4,2}$. If ALG does not select $c_{3,1}$ then $c_{4,1}, c_{4,2}$ are revealed as leaves and $c_{2,2}$ is revealed with an additional leaf neighbor $l_{2,2}$ so that ALG must select at least one of $c_{2,2}, l_{2,2}$. Thus, ALG here selects $c_{1,1}, c_{2,1}, c_{4,1}, c_{4,2}$ and at least one of $c_{2,2}, l_{2,2}$ whereas OPT can select $c_{3,1}$ and $c_{2,2}$ for a performance of $\frac{5}{2}$ (see Figure 6a). If instead ALG selects $c_{3,1}$ (thus far $c_{1,1}, c_{2,1}$ and $c_{3,1}$ are all selected) then $c_{2,2}$ is revealed with an additional leaf neighbor $l_{2,2}$ so that ALG must select at least one of $c_{2,2}, l_{2,2}$, and $c_{3,2}$ is revealed as the root of a 2-gadget so that $\frac{ALG}{OPT} \geq \frac{5}{2}$ on the subcactus thus far (excluding $c_{3,1}$) and we repeat the trap with $c_{3,1}$ as the selected root (see Figure 6b).

Let $j_i \geq 1$ denote the number of rounds that passed in the adversarial process starting at the child c_i. To finish the analysis, we consider the following two cases:

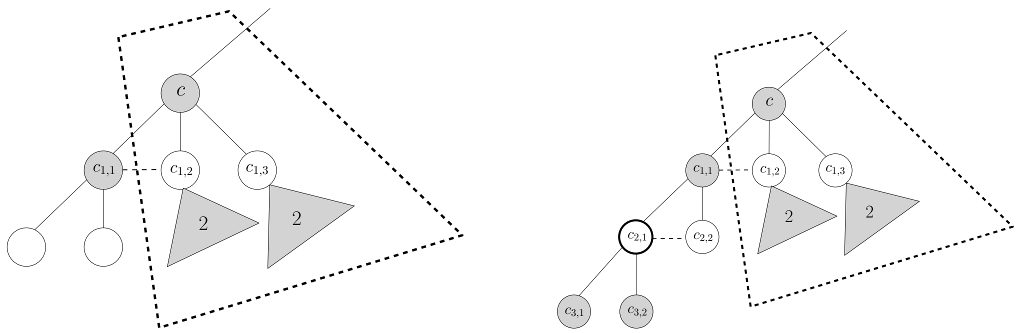
Case 1: For all i we have that $j_i < k$. Then $ALG \geq 5j_i$ on each subcactus whereas $OPT \leq 2j_i$ on each subcactus[3]. Summing over all subcacti and remarking that OPT selects v_1 we obtain that $ALG/OPT \geq \left(\sum 5j_i\right) / \left(1 + \sum 2j_i\right) \geq \frac{5}{2} - \frac{5}{2k} \geq \frac{5}{2} - \epsilon$.

Case 2: There exists ℓ such that $j_\ell = k$. In this case, there is an additional vertex $c_{j,1}$ with $j = 3(k-1)$ that was selected by ALG and must also be selected by OPT. (i.e. c_j is the root where a $(k+1)$'st round could start). Therefore, OPT selects $2j_i$ vertices for each process on child c_i with $i < \ell$, $2k+1$ vertices for $i = \ell$, 0 vertices for $i > \ell$ plus v_1. Whereas ALG selects at least $5j_i$ for $i < \ell$, $5k+1$ for $i = \ell$, and 0 for $i > \ell$. Ultimately, we obtain that $ALG/OPT \geq \left(\sum 5j_i + 5k + 1\right) / \left(\sum 2j_i + 2k + 2\right) \geq \frac{5}{2} - 4/k \geq \frac{5}{2} - \epsilon$. ◀

▶ **Theorem 15.** *ρ(2-DOMINATE, CACTUS) $= \frac{5}{2}$.*

The proof can be viewed as an adaptation of our proof for trees to cactus graphs. We use a charging argument similar to the one given in the section on trees. Initially, a charge of 1 is given for each $v \in S$, the charge on each vertex is then spread to certain neighbors, and we then show that $\sum_{x \in N[v_i]} ch(x) \leq \frac{5}{2}$ for each $v_i \in OPT$. We spread the charge according

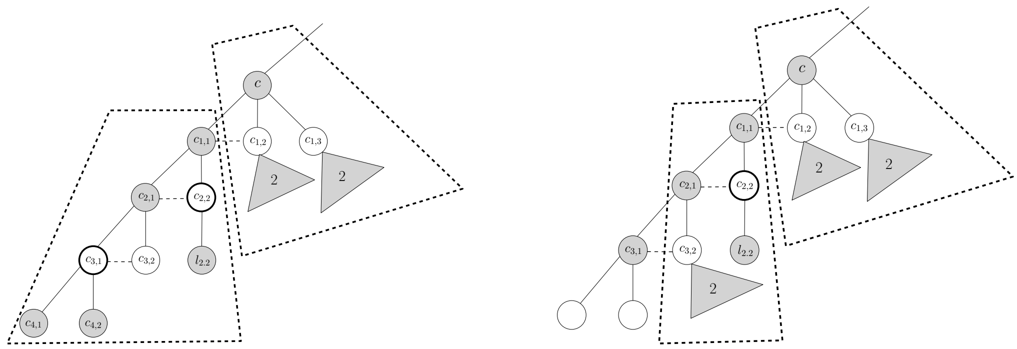
[3] We have omitted the cases where ALG does not select the root c_i. These cases result in ALG selecting 3 vertices on the subcacti with OPT selecting only 1 and the result clearly still holds in this case.



(a) *ALG* does select $c_{1,1}$. The enclosed region contributes a performance of $\frac{5}{2}$. A trap is continued in this case with the root $c_{1,1}$.

(b) *ALG* does not select $c_{2,1}$.

Figure 5 Two cases described in Theorem 14.



(a) *ALG* does not select $c_{3,1}$. The enclosed regions each contribute a performance of $\frac{5}{2}$.

(b) *ALG* does select $c_{3,1}$. The enclosed regions each contribute a performance of $\frac{5}{2}$. The trap that was used on a selected root $c_{1,1}$ is repeated with $c_{3,1}$ as the selected root.

Figure 6 Two more cases described in Theorem 14.

to the same rule given in the preceding section and recall that Observation 10 (each vertex receives a new charge from one other vertex) still holds. In the analysis of how the charge gets reallocated, the structure of the underlying graph is of paramount importance. We begin with an analogue to Lemma 8.

▶ **Lemma 16.** *There is at most one cross edge incident on any v_i. In particular, v_i has at most 2 neighbors before it is revealed.*

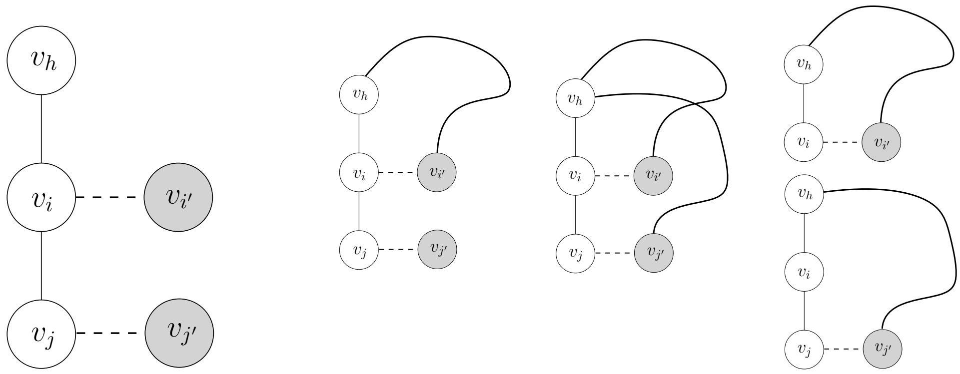
Proof. Suppose that $v_i \neq v_1$ since the statement is clearly true for $v_i = v_1$. Suppose for the sake of deriving a contradiction that, at time $i-1$, v_i has three neighbors v_h, v_{i_1}, v_{i_2} where v_h is the parent of v_i and $\{v_i, v_{i_1}\}, \{v_i, v_{i_2}\}$ are cross edges. Notice that v_{i_1} is visible at time $i-1$ as otherwise would imply that $\{v_i, v_{i_1}\}$ were a tree edge. Thus, at time $i-1$, v_{i_1} is visible and there is only one tree edge incident on v_i. In particular, this implies that there is a path consisting entirely of tree edges from v_{i_1} to v_h where said path does not contain the edge $\{v_h, v_i\}$ since it does not pass through v_i nor does it contain the edges $\{v_i, v_{i_1}\}, \{v_i, v_{i_2}\}$ since they are cross edges. Thus, by adding edges $\{v_h, v_i\}, \{v_i, v_{i_1}\}$ to this path we obtain a cycle (in the completely revealed input graph) that contains the edge $\{v_h, v_i\}$ but does not contain the edge $\{v_i, v_{i_2}\}$. A similar argument yields that there is a path consisting of tree edges from v_{i_2} to v_h that does not contain the edges $\{v_h, v_i\}, \{v_i, v_{i_1}\}, \{v_i, v_{i_2}\}$ and hence by adding edges $\{v_h, v_i\}, \{v_i, v_{i_2}\}$ we obtain a cycle which contains the edge $\{v_h, v_i\}$ but does not contain the edge $\{v_i, v_{i_1}\}$. That is, two distinct cycles that share the common edge $\{v_h, v_i\}$, a contradiction. ◀

Since v_i has at most 2 neighbors before it is revealed then it has at least $deg(v_i) - 2$ children. The following is analogous to Corollary 9 for trees.

▶ **Corollary 17.** *If $deg(v_i) \geq 4$ then $v_i \in S$.*

▶ **Lemma 18.**
1. *If v_i and v_j both have charge equal to 1 then they share no common neighbors.*
2. *If v_i and v_j both have charge equal to 1 then they are not adjacent.*
3. *For any vertex v_i, at most one vertex in $N[v_i]$ has charge 1.*

Proof.
1. Follows identically to the proof of Lemma 11.
2. First, note that v_1 cannot have charge 1 on any input with at least 2 vertices. Therefore we safely assume that $1 < i < j$ such that both v_i and v_j have a parent. We assume for the sake of deriving a contradiction that v_i and v_j are adjacent.
Now, since both v_i and v_j have charge 1 it follows that they are both saved vertices. We first argue that both $v_i, v_j \notin S$. Notice that any saved vertex v_k has the property that $|N[v_k] \cap S| = 1$. Therefore, if we assume by way of contradiction that $v_i \in S$ we obtain that $N[v_i] \cap S = N[v_j] \cap S = \{v_i\}$ and therefore v_i saves itself and v_j. This yields that $X_i = N[v_i] \cap U_i$ contains v_i and v_j. In particular, we have that $|X_i| \geq 2$ with $v_i, v_j \in X_i$ and therefore v_i and v_j receive charge no larger than $\frac{1}{2}$, a contradiction. An identical argument will yield that $v_j \notin S$.
Thus, we assume that v_i is saved by a neighbor $v_{i'}$ and v_j is saved by a neighbor $v_{j'}$ where $i', j' \notin \{i, j\}$. Moreover, $i' \neq j'$ since v_i and v_j can share no common neighbors by part 1. Thus, we have that i, j, i', j' are all distinct with $i < j < i'$ and $i < j < j'$ since $i' = \max\{k \mid v_k \in N[v_i]\}$ and $j' = \max\{k \mid v_k \in N[v_j]\}$. As mentioned above v_i must have a parent v_h where $h < i < j < i'$. Therefore, $deg(v_i) \geq 3$ and since $v_i \notin S$ it follows by Corollary 17 that $deg(v_i) = 3$.
We are now in the situation where $v_i, v_j \notin S$ and v_i is incident on exactly 3 edges $\{v_h, v_i\}$, $\{v_i, v_j\}$, $\{v_i, v_{i'}\}$ where exactly one of the edges $\{v_i, v_j\}, \{v_i, v_{i'}\}$ is a tree edge (and the other a cross edge). We finish the proof by examining the two cases where **(1) :** $\{v_i, v_{i'}\}$ is a tree edge or **(2) :** $\{v_i, v_j\}$ is a tree edge.
Case 1: Suppose $\{v_i, v_{i'}\}$ is a tree edge so that $v_{i'}$ is a child of v_i. Therefore, $v_{i'} \in C_i \subseteq N(v_i) \cap U_i$, that is, $v_{i'}$ is an undominated neighbor of v_i when v_i is revealed. Since v_j is saved then by Observation 5 it follows that $v_j \in N(v_i) \cap U_i$, that is, v_j is also an undominated neighbor of v_i when v_i is revealed. That is, both $v_{i'}, v_j \in N(v_i) \cap U_i$ implying that $|N(v_i) \cap U_i| \geq 2$ but $v_i \notin S$, a contradiction.
Case 2: Suppose $\{v_i, v_j\}$ is a tree edge so that v_j is a child of v_i. First notice that $\{v_i, v_j\}$ is the only tree edge incident on v_j. Indeed, if there were a tree edge $\{v_j, v_l\}$ then v_l would be the child of v_j. Since v_i is saved we have $v_i \in N(v_j) \cap U_j$ by Observation 5 implying that $|N(v_j) \cap U_j| \geq 2$ but $v_j \notin S$. Thus, we are in the situation depicted in Figure 7a where $\{v_i, v_j\}$ is the only tree edge incident on v_j and by assumption $\{v_h, v_i\}, \{v_i, v_j\}$ are the only two tree edges incident on v_i. Therefore we have a path from $v_{i'}$ to v_h consisting of tree edges where said path does not contain the edges



(a) Case 2 of the second part of Lemma 18.

(b) Resolution of the preceding case in Figure 7a. Two cycles sharing the common edge $\{v_h, v_i\}$.

Figure 7 Figures used in the proof of Lemma 18.

$\{v_h, v_i\}, \{v_i, v_{i'}\}, \{v_i, v_j\}, \{v_j, v_{j'}\}$. Thus, by adding edges $\{v_h, v_i\}, \{v_i, v_i'\}$ to this path we obtain a cycle (in the completely revealed input) that contains the edge $\{v_h, v_i\}$ but does not contain the edge $\{v_j, v_{j'}\}$. Similarly, there is a path from $v_{j'}$ to v_h consisting of tree edges where said path does not contain the edges $\{v_h, v_i\}, \{v_i, v_{i'}\}, \{v_i, v_j\}, \{v_j, v_{j'}\}$ and by adding edges $\{v_h, v_i\}, \{v_i, v_j\}, \{v_j, v_{j'}\}$ we obtain a cycle (in the completely revealed input) that contains the edge $\{v_h, v_i\}$ but does not contain the edge $\{v_i, v_{i'}\}$. That is, two distinct cycles that share the common edge $\{v_h, v_i\}$, a contradiction.

3. Follows immediately from the previous parts. ◀

Now, we are ready to prove the upper bound for Theorem 15.

Proof of Theorem 15. The lower bound follows from Theorem 14. Let $v_i \in OPT$ be an arbitrary vertex in OPT. We consider two cases **(1)** $deg(v_i) \leq 3$ or **(2)** $deg(v_i) \geq 4$.

Case 1: Suppose that $deg(v_i) \leq 3$ and hence $|N[v_i]| \leq 4$. By Lemma 18 part 3 it follows that at most one vertex in $N[v_i]$ has charge 1. If no vertices in $N[v_i]$ have charge 1 then $ch(x) \leq \frac{1}{2}$ for each $x \in N[v_i]$ and we obtain that $\sum_{x \in N[v_i]} ch(x) \leq 4\left(\frac{1}{2}\right) = 2 < \frac{5}{2}$. If there is exactly one vertex $x' \in N[v_i]$ with charge 1 we therefore obtain that $\sum_{x \in N[v_i]} ch(x) = \sum_{x \in N[v_i] \setminus \{x'\}} ch(x) + ch(x') \leq \frac{3}{2} + 1 = \frac{5}{2}$.

Case 2 : Suppose that $deg(v_i) \geq 4$. By Corollary 17 it follows that $v_i \in S$ with at least 2 children. Let $C_i = V_i \setminus V_{i-1}$ denote the children of v_i and remark that $C_i \subseteq X_i$. That is, each child of v_i is charged by v_i and only v_i. Therefore the children of v_i can receive at most the full initial charge on v_i and thus attribute a charge of at most 1.

Now we claim that any vertex in $N[v_i] \setminus C_i$ has a charge of at most $\frac{1}{2}$. Indeed, suppose a vertex $v_{i'} \in N[v_i] \setminus C_i$ has charge 1 then it must be saved by v_i since $|N[v_{i'}] \cap S| = 1$ for any saved vertex $v_{i'}$. That is, there is exactly one vertex in its closed neighborhood that is selected and since v_i is selected it must be v_i. Thus, we must have that $v_{i'} \in X_i$ but since $C_i \subseteq X_i$ we know that $|X_i| \geq 2$ and thus $v_{i'}$ receives a charge of no more than $\frac{1}{2} < 1$, contradicting our assumption that $v_{i'}$ has charge 1. Thus, by remarking that $|N[v_i] \setminus C_i| \leq 3$ we obtain that $\sum_{x \in N[v_i]} ch(x) = \sum_{v_j \in C_i} ch(v_j) + \sum_{v_{i'} \in N[v_i] \setminus C_i} ch(v_{i'}) \leq 1 + 3\left(\frac{1}{2}\right) = \frac{5}{2}$ as desired. ◀

3.3 Graphs of Bounded Degree

We study the problem when the inputs are restricted to graphs of bounded degree. That is, a positive integer $\Delta \geq 2$ is provided to the algorithm beforehand and the adversary is restricted to presenting graphs where every vertex has degree no larger than Δ. The problem of bounded degree graphs was explored in [3] although within the vertex arrival model described earlier. The authors show that a greedy strategy obtains a competitive ratio no larger than Δ and, when inputs are further restricted to be "always-connected" (i.e. each prefix of the input is connected) they provide a lower bound of $\Delta - 2$ for any algorithm.

By definition, any input belonging to our setting is "always-connected" yet the lower bound of $\Delta - 2$ does not apply. In particular, we show that $\lceil\sqrt{\Delta}\rceil$-DOMINATE is $3\sqrt{\Delta}$-competitive along with a lower bound of $\Omega(\sqrt{\Delta})$ for any online algorithm, essentially closing the problem in our setting. As previously mentioned, the authors in [12] consider a setting similar to ours where their adversary is not required to reveal visible vertices and they assume that an algorithm has additional knowledge of input size n. In this setting they provide an algorithm that achieves competitive ratio of $\Theta(\sqrt{n})$ for arbitrary graphs. For the upper bound below we follow a proof nearly identical to theirs modulo some minor details and definitions.

▶ **Definition 19.** *A vertex $v_i \in S$ is said to be heavy if $|N(v_i) \cap U_i| \geq \lceil\sqrt{\Delta}\rceil$ and light otherwise. We let H and L denote the set of heavy and light vertices in S so that $|S| = |H| + |L|$.*

To establish that $\lceil\sqrt{\Delta}\rceil$-DOMINATE is $3\sqrt{\Delta}$-competitive we use a charging argument, but it is quite different from the arguments in Sections 3.1 and 3.2. Initially, let $ch(v) = 1$ for each $v \in S$ so that $|S| = \sum_{v\in S} ch(v)$. Then spread the charge from S strictly to vertices in OPT so that $\sum_{v\in S} ch(v) = \sum_{v\in OPT} ch^*(v)$ where $ch^*(v)$ is the new charge on a vertex in OPT. We then show that $ch^*(v) \leq 2\sqrt{\Delta}$ for all $v \in OPT$ and thus $|S| = \sum_{v\in S} ch(v) = \sum_{v\in OPT} ch^*(v) \leq |OPT|2\sqrt{\Delta}$ and the result then follows. We spread the charge from S to OPT according to the following rules:

1. If $v_i \in S \cap OPT$ then v_i keeps its full initial charge.
2. If $v_i \in H \setminus OPT$ then its spread its initial charge evenly over all vertices in OPT. That is, each $v \in OPT$ obtains an additional charge of $\frac{1}{|OPT|}$ from v_i.
3. For each $v_i \in L \setminus OPT$, let $s(v_i)$ denote the set of vertices saved by v_i. Given a vertex $v_{i'} \in s(v_i)$ let *opt* be the mapping that maps $v_{i'}$ to itself if it is in OPT or to its earliest revealed neighbor in OPT otherwise. That is, $opt(v_{i'}) = v_{i'}$ if $v_{i'} \in OPT$ and $opt(v_{i'}) = v_k$ where $k = \min\{j \mid v_j \in N(v_{i'}) \cap OPT\}$ otherwise. For each $v_{i'} \in s(v_i)$, v_i spreads a charge of $\frac{1}{|s(v_i)|}$ to $opt(v_{i'})$.

▶ **Lemma 20.** *If $v_i \in OPT$ then it receives charge from at most $\lceil\sqrt{\Delta}\rceil$ light vertices.*

Proof. We consider two cases; **(1)** $v_i \in S$ or **(2)** $v_i \notin S$.

Case 1: Assume that $v_i \in S$, we show that v_i receives no charge from a distinct light vertex (therefore it receives charge from at most one light vertex, itself). Since $v_i \in S$ this implies that it is not saved by any $v_j, j \neq i$. Thus, if v_i were to receive charge from a light vertex it must be that $v_i = opt(v_{i'})$ for some $v_{i'}$ that is saved by some $v_k \in L$ different from v_i. More precisely, v_i must be adjacent to some $v_{i'}$ that is saved by some v_k with $k \neq i$. Yet, if $v_{i'} \in N(v_i)$ is saved then $N[v_{i'}] \cap S = \{v_i\}$ so this cannot be the case.

Case 2: Assume that $v_i \notin S$ and first remark that v_i is saved by at most one vertex so that it receives at most one charge from a light vertex in this way. If v_i receives charge from any other light vertex $v_k \in L$, it must be that v_i is adjacent to some vertex $v_{i'}$

that is saved by v_k. By Observation 5 it must be that $v_{i'} \in N(v_i) \cap U_i$, that is, is undominated when v_i is revealed. All this to say, that any light vertex that charges v_i determines at least one neighbor of v_i that is undominated at time i. Since $v_i \notin S$ we have $|N(v_i) \cap U_i| \le \lceil\sqrt{\Delta}\rceil - 1$ and thus accounting for possibly one light vertex that charges v_i there are at most $\lceil\sqrt{\Delta}\rceil$ light vertices that charge v_i. ◀

▶ **Lemma 21.** $\frac{|H|}{|OPT|} \le \sqrt{\Delta} + \frac{1}{\sqrt{\Delta}}$.

Proof. Since every vertex in H is selected because it dominated at least $\lceil\sqrt{\Delta}\rceil$ undominated vertices it follows that $|H| \le \left\lfloor \frac{n}{\lceil\sqrt{\Delta}\rceil} \right\rfloor$. Moreover, by a standard result, first proved by Berge [1], a lower bound on OPT is $|OPT| \ge \left\lceil \frac{n}{\Delta+1} \right\rceil$. Ultimately this yields that $\frac{|H|}{|OPT|} \le \frac{\left\lfloor \frac{n}{\lceil\sqrt{\Delta}\rceil} \right\rfloor}{\left\lceil \frac{n}{\Delta+1} \right\rceil} \le \frac{\frac{n}{\lceil\sqrt{\Delta}\rceil}}{\frac{n}{\Delta+1}} \le \frac{\frac{n}{\sqrt{\Delta}}}{\frac{n}{\Delta+1}} = \frac{\Delta+1}{\sqrt{\Delta}} = \sqrt{\Delta} + \frac{1}{\sqrt{\Delta}}$. ◀

▶ **Theorem 22.** $\rho(\lceil\sqrt{\Delta}\rceil\text{-}DOMINATE, \Delta\text{-}BOUNDED) \le 3\sqrt{\Delta}$.

Proof. Consider an arbitrary vertex $v_i \in OPT$. In light of Lemma 20 we see that it receives charge from at most $\lceil\sqrt{\Delta}\rceil$ light vertices, where each charge is no larger than 1. Moreover, by Lemma 21 the charge received by the heavy vertices is at most $\sqrt{\Delta} + \frac{1}{\sqrt{\Delta}}$ and v_i possibly receives charge from itself (it may be a heavy or light vertex). In particular we obtain that $ch(v_i) \le \frac{|H|}{|OPT|} + \lceil\sqrt{\Delta}\rceil + 1 \le \left(\sqrt{\Delta} + \frac{1}{\sqrt{\Delta}}\right) + \lceil\sqrt{\Delta}\rceil + 1 \le 3\sqrt{\Delta}$. ◀

We now prove a lower bound $\Omega(\sqrt{\Delta})$ for any online algorithm. We should note that the adversarial input is bounded in size by a function of Δ. Although we have omitted the details, it is straightforward to extend the input so that the lower bound is in fact an asymptotic one.

▶ **Theorem 23.** $\rho(ALG, \Delta\text{-}BOUNDED) = \Omega(\sqrt{\Delta})$.

Proof. For simplicity we assume that Δ is a perfect square. Reveal v_1 with Δ children and reveal each child of v_1 with an additional $\sqrt{\Delta}$ children. Of the Δ children of v_1, suppose that ALG selects exactly j where $0 \le j \le \Delta$. For the $\Delta - j$ vertices not selected, their $\sqrt{\Delta}$ neighbors are revealed to have degree 1 and ALG is forced to select each of these $(\Delta-j)(\sqrt{\Delta})$ vertices of degree 1.

Let S_j denote the set of the j selected vertices in $N(v_1)$ and $X = \bigcup_{v_i \in S_j} N(v_i)$. Since each vertex in S_j has $\sqrt{\Delta}$ children, it follows that $|X| = j\sqrt{\Delta}$. Partition the vertices of X into $\lceil \frac{j\sqrt{\Delta}}{\Delta} \rceil = \lceil \frac{j}{\sqrt{\Delta}} \rceil$ parts of size Δ (with at most one part having size $< \Delta$). Letting the parts be $X_1, X_2, ..., X_{\lceil \frac{j}{\sqrt{\Delta}} \rceil}$ we reveal each vertex in a given part to a common vertex y_i (see figure 8 for an example). ALG must select at least one vertex for each part to dominate y_i and therefore at least an additional $\lceil \frac{j}{\sqrt{\Delta}} \rceil$ vertices are selected.

In total, ALG selects at least $j + (\Delta - j)(\sqrt{\Delta}) + \frac{j}{\sqrt{\Delta}}$ whereas OPT simply selects v_1, the $\Delta - j$ vertices in $N(v_1) \setminus S_j$ and the $\frac{j}{\sqrt{\Delta}}$ vertices with labels y_i. Ultimately we have $\frac{ALG}{OPT} \ge \frac{j+(\Delta-j)(\sqrt{\Delta})+\frac{j}{\sqrt{\Delta}}}{1+(\Delta-j)+\frac{j}{\sqrt{\Delta}}} = \frac{j+j\sqrt{\Delta}+(\Delta-j)\Delta}{j+\sqrt{\Delta}+(\Delta-j)\sqrt{\Delta}} = \frac{\sqrt{\Delta}(j/\sqrt{\Delta}+j+(\Delta-j)\sqrt{\Delta})}{2(j/2+\sqrt{\Delta}/2+(\Delta-j)\sqrt{\Delta}/2)} \ge \frac{\sqrt{\Delta}}{2}$, where the last inequality follows from the fact that $j/2 + \sqrt{\Delta}/2 + (\Delta - j)\sqrt{\Delta}/2 \le j/\sqrt{\Delta} + j + (\Delta - j)\sqrt{\Delta}$, since $\sqrt{\Delta}/2 \le j/\sqrt{\Delta} + j/2 + (\Delta - j)\sqrt{\Delta}/2$, which can be seen since when $j < \Delta$ then the last term on the right hand side already is at least as large as the left hand side and when $j = \Delta$ then the middle term on the right hand side is at least the left hand side. ◀

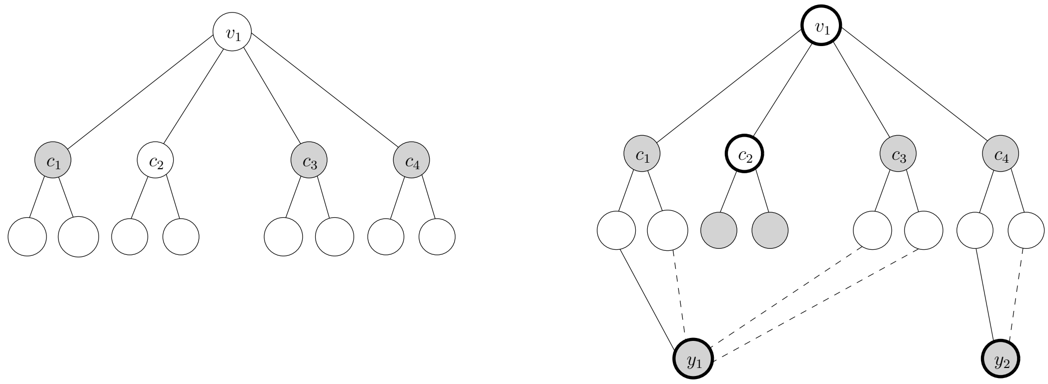


Figure 8 An instance described in the proof of Theorem 23 with $\Delta = 4$. The left depicts the graph after the children of v_1 have been revealed. Assuming that ALG selects $\{c_1, c_3, c_4\}$ above, the right depicts the completely revealed graph.

3.4 Graphs with Bounded Claws

In Appendix A we study $K_{1,t}$-free graphs, which we also refer to as graphs with bounded "claws" (for $t = 3$, this graph class is known as "claw-free graphs"). We show that the competitive ratio $t - 1$ is both necessary and sufficient for this class of graphs. The upper bounds that we have demonstrated so far were all based on the k-DOMINATE algorithm for a suitable choice of parameter k. Interestingly, our upper bound on $K_{1,t}$-free graphs is based on a conceptually simpler GREEDY algorithm. The analysis is no longer based on a charging scheme, but follows from combinatorial properties of graphs with bounded claws. For the details, one should consult the appendix.

4 Conclusions

In this paper we studied the minimum dominating set problem in an online setting where a vertex is revealed alongside all its neighbors. We also contrasted our results with those obtained by Boyar et al. [3] and Kobayashi [13] in a related vertex-arrival model. Dominating set is a difficult problem both offline and online. In our setting, the best achievable competitive ratio on general graphs is $O(\sqrt{n})$. This observation prompted us to study this problem with respect to more restrictive graph classes. Trees provide a natural graph class that usually allows for non-trivial competitive ratios. Indeed, we showed that in our model trees admit 2-competitive algorithms. There are several ways to try to extend this result to larger graph classes. We considered cactus graphs and showed that the optimal competitive ratio is 2.5 on them. Another way of generalizing trees is to consider graphs of higher treewidth. Unfortunately, once treewidth goes up to 2, competitive ratio jumps to $\Omega(\sqrt{n})$ (which is trivial in our setting due to $O(\sqrt{n})$ upper bound), as witnessed by series-parallel graphs. We also established non-trivial upper bounds on graphs of bounded degree, as well as graphs with bounded claws. When one moves to planar (even bipartite planar) graphs and threshold graphs, the competitive ratio jumps to $\Omega(\sqrt{n})$ again.

The above can be viewed as a larger program of developing a deeper understanding of the dominating set problem in an online setting. What are the main structural obstacles in graphs that prohibit online algorithms with small competitive ratios? Can one discover a family of graphs parameterized by some parameter t, which include cactus graphs, claw-free graphs, and bounded-degree graphs, such that the competitive ratio scales gracefully with t? Lastly, as another research direction, we mention that we have only considered the deterministic setting, so it would be of interest to extend our results to the randomized setting, as well as the setting of online algorithms with advice.

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A Tight Bound for Graphs with Bounded Claws

Let $t \geq 3$, a graph G is said to be $K_{1,t}$-free if it contains no induced subgraph isomorphic to $K_{1,t}$. When $t = 3$, this is the well-studied class of claw-free graphs. In this section we study $K_{1,t}$-free graphs, which we also refer to as graphs with bounded "claws".

From the preceding sections one might notice that the existence of an induced subgraph $K_{1,t}$ poses challenges for an algorithm. This section suggests that this intuition holds more than just a grain of truth. We show that, when inputs are restricted to $K_{1,t}$-free graphs, the competitive ratio of every algorithm is bounded below by $t - 1$ and there is an algorithm that achieves competitive ratio $t-1$. The upper bounds that we have demonstrated so far were all based on the k-DOMINATE algorithm for a suitable choice of parameter k. Interestingly, our upper bound on $K_{1,t}$-free graphs is based on a conceptually simpler GREEDY algorithm. The analysis is no longer based on a charging scheme, but follows from combinatorial properties of graphs with bounded claws.

▶ **Theorem 24.** $\rho(ALG, K_{1,t}\text{-}FREE) \geq t-1$.

Proof. Reveal v_1 with $t-1$ children. If ALG does not select v_1 then the input terminates as a star on t vertices (i.e. the $t-1$ neighbors of v_1 are revealed with no additional neighbors). Any feasible algorithm must select the $t-1$ neighbors of v_1 whereas OPT selects v_1 and the statement then follows. Suppose that ALG selects v_1 and let $c_i, 1 \leq i \leq t-1$ be the children of v_1. Reveal c_1 as adjacent to each child of v_1 and with an additional $t-2$ children. If ALG does not select c_1 then the children of c_1 are revealed as leaves whereas the rest of the input is revealed to be a clique. That is, $N[v_1]$ is a clique and only c_1 has children. ALG selected v_1 and is forced to select the $t-2$ children of c_1 whereas OPT selects only c_1 as a single dominating vertex. It is not hard to see that this input is $K_{1,t}$-free and the result then follows (see Figure 9 for an example).

Suppose that ALG selects c_1, the input then continues in the following way; For each $2 \leq j \leq t-2$, (as long as ALG is accepting c_j) we reveal c_j as adjacent to every visible vertex and with an additional $t-3$ children. That is, c_j is adjacent to each child $c_i, i \neq j$ of v_1 and the grandchildren of v_1 (i.e. the children of all the c_i with $1 \leq i \leq j$) so that c_j is a single dominating vertex of this prefix.

Case 1: If there is some $2 \leq j \leq t-2$ such that ALG does not select c_j then the $t-3$ children of c_j are revealed as leaves, $N[v_i]$ is revealed as a clique, and the $(t-2)+\sum_{i=2}^{j}(t-3) = j(t-3)+1$ grandchildren of v_1 are revealed to form a clique. At this point, ALG has selected $\{v_1, c_1, ..., c_{j-1}\}$ and is now forced to select the $t-3$ children of c_j for an output of at least $j+(t-3) \geq 2+(t-3) = t-1$ whereas OPT selects only c_j so that $\frac{ALG}{OPT} \geq \frac{t-1}{1}$. We now argue that this input is $K_{1,t}$-free. Notice that for all v in this input we have $N(v) \subseteq N(c_j)$ so that if there is a an induced $K_{1,t}$ with central vertex v then there is a claw with central vertex c_j. Therefore it is sufficient to show that is no claw with central vertex c_j to finish the claim. Suppose for contradiction's sake that there were an induced $K_{1,t}$ where c_j is the central vertex and the t neighbors of c_j are all pairwise non-adjacent. Let G denote the grandchildren of v_1 and remark that any neighbor of c_j is either a child of c_j, a grandchild of v_1, or a vertex from $N[v_1] \setminus \{c_j\}$. Since there are t vertices and c_j only has $t-3$ children by the pigeonhole principle we must have at least two vertices u, v that both are grandchildren of v_1 or both belong $N[v_1] \setminus \{c_j\}$. Yet, both the set of grandchildren of v_1 and $N[v_1] \setminus \{c_j\}$ are cliques. Therefore we have that u and v are adjacent, contradicting our assumption.

Case 2: If ALG selects each $c_i, 1 \leq i \leq t-2$ then the $(t-2)(t-3)+1$ grandchildren of v_1 are then revealed to form a clique ($N[v_1]$ has already been revealed as a clique). ALG has already selected $\{v_1, c_1, ..., c_{t-2}\}$ and therefore has an output of at least $t-1$ whereas OPT selects only c_{t-2}. An argument similar to the one above will yield that this input is $K_{1,t}$-free and the result then follows. ◀

When inputs are restricted to $K_{1,t}$-free graphs, we show that the online algorithm GREEDY is $(t-1)$-competitive. The crucial observation to make here is that the output of GREEDY is an independent set. We provide a result below that is a straightforward generalization of one given in [4]. The simplicity of the result suggests that it may have appeared in earlier work.

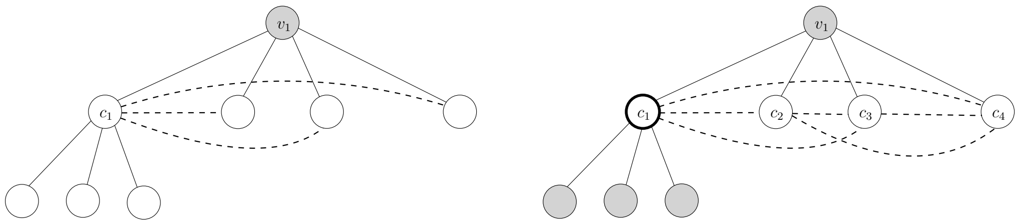

Figure 9 An instance described in Theorem 24 with $t = 5$ where ALG does not select c_1. The left depicts the graph at the moment c_1 was revealed and the right depicts the completely revealed graph.

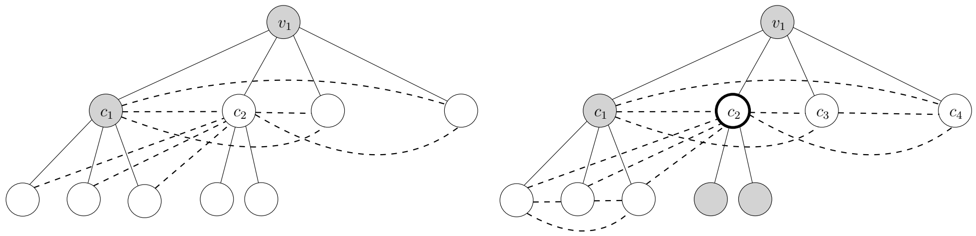

Figure 10 An instance described in Theorem 24 with $t = 5$ where ALG does not select c_2. The left depicts the graph at the moment c_1 was revealed and the right depicts the completely revealed graph.

▶ **Lemma 25.** *Let $t \geq 3$, $G = (V, E)$ be a $K_{1,t}$-free graph and I be any independent set in G. Then $|D| \geq \frac{|I|}{t-1}$ for any dominating set D in G.*

Proof. Suppose for the sake of deriving a contradiction that there is some dominating set D in G with $|D| < \frac{|I|}{t-1}$. Remarking that the vertices of D dominate the vertices of I as D is a dominating set we notice that there is some vertex $v \in D$ that dominates at least t vertices of I (i.e. if every vertex of D dominated at most $t - 1$ vertices then D would dominate at most $(t - 1)|D| < |I|$ vertices). Moreover, since v is adjacent to at least one of the $t \geq 3$ vertices of I it dominates, it cannot belong to I as I is independent. Therefore, the vertices of I dominated by $v \notin I$ are adjacent to v. In particular, at least t vertices of I, all pairwise non-adjacent, are neighbors of v and this induces $K_{1,t}$ in G. ◀

The preceding lemma shows that for any independent set I in a $K_{1,t}$-free graph G, $|I| \leq (t - 1)\gamma(G)$. Given that GREEDY outputs an independent set we obtain the following result which is of interest to us.

▶ **Theorem 26.** $\rho(GREEDY, K_{1,t}\text{-}FREE) = t - 1$.

Proof. The upper bound is a consequence of Proposition 25 and the remarks that follow. The lower bound follows from Theorem 24. ◀

B Noncompetitive Graph Classes

Recall that the setting defined in [12] is nearly identical to ours except that an algorithm knows the input size n beforehand and the induced subgraph on the revealed vertices is not necessarily connected. Within this setting the authors establish a lower bound of $\Omega(\sqrt{n})$ for arbitrary graphs. Their proof can be augmented to show a lower bound of $\Omega(\sqrt{n})$ in our model, which is tight by our upper bound of $O(\sqrt{\Delta})$ on degree at most Δ graphs (applied

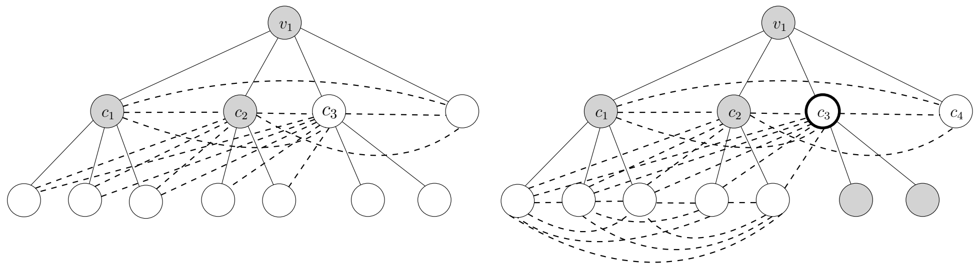

Figure 11 An instance described in Theorem 24 with $t = 5$ where ALG does not select c_3. The left depicts the graph at the moment c_3 was revealed and the right depicts the completely revealed graph.

to $\Delta = n - 1$). Instead, we strengthen such a result in several ways by showing that the lower bound of $\Omega(\sqrt{n})$ applies to several restricted classes such as threshold graphs[4], planar bipartite graphs, and series-parallel graphs. Some of the proofs are omitted in this section due to space limitations. These proofs can be found in the full version of the paper [6].

B.1 Threshold Graphs

The graph join operation applied to two graphs G_1 and G_2 takes the disjoint union of the two graphs and adds all possible edges between the two graphs to the result (in addition to retaining the edges of G_1 and G_2). The class of threshold graphs can be described recursively as follows:

1. K_1 (i.e. a single isolated vertex) is a threshold graph.
2. If G is a threshold graph then the disjoint union $G \cup K_1$ is a threshold graph.
3. If G is a threshold graph then the graph join $G \oplus K_1$ is a threshold graph.

It is not hard to see that any connected threshold graph has a dominating set of size 1. Since our setting only allows for connected graphs we instead measure ALG as a function of input size n since $OPT \leq 1$ on every input. In particular, we show that for any algorithm there is an infinite family of threshold graphs for which this algorithm selects $\Omega(\sqrt{n})$ vertices (where the input has n vertices). Although OPT does not tend towards infinity, we consider this to be an asymptotic lower bound, but with input size n tending to infinity. In a sense, this is a stronger lower bound since the algorithm is guaranteed an input graph with a single dominating vertex, yet it still selects more than $\Omega(\sqrt{n})$ vertices in the input.

▶ **Theorem 27.** *For infinitely many values of n there is a threshold graph G_n such that $ALG(G_n) = \Omega(\sqrt{n})$.*

B.2 Planar Bipartite Graphs

Below is a lower bound of $\Omega(\sqrt{n})$ for planar bipartite graphs. We should mention that is strikingly similar to the lower bound on general graphs given in [12]. We provide a simple augmentation of their lower bound so that it not only consists of inputs that are revealed according to our model but inputs that are also planar bipartite graphs.

▶ **Theorem 28.** $\rho(ALG, PLANAR\ BIPARTITE) = \Omega(\sqrt{n})$.

[4] With the caveat that, for threshold graphs, we instead consider the performance ratio as a function of input size.

B.3 Series-Parallel Graphs

In light of our 2-competitive algorithm for trees, it is natural to suppose that some class of graphs generalizing trees might admit competitive algorithms, that is, algorithms with bounded competitive ratio. One such generalization is graphs of bounded treewidth. Trees have treewidth 1, so the next step is to consider graphs of treewidth 2. Unfortunately, in this section we show that by increasing treewidth parameter from 1 to 2, the online dominating set problem becomes extremely hard for online algorithms. More specifically, we show that series-parallel graphs do not admit online algorithms with competitive ratio better than $\Omega(\sqrt{n})$. We remark that series-parallel graphs have treewidth at most 2.

We begin by recalling the definition of a series-parallel graph. It is defined with the help of the notion of a two-terminal graph (G, s, t), which is a graph G with two distinguished vertices s, called a source, and t, called a sink. For a pair of two-terminal graphs (G_1, s_1, t_1) and (G_2, s_2, t_2), there are two composition operations:

- *Parallel composition*: take a disjoint union of G_1 with G_2 and merge s_1 with s_2 to get the new source, as well as t_1 with t_2 to get the new sink.
- *Series composition*: take a disjoint union of G_1 with G_2 and merge t_1 with s_2, which now becomes an inner vertex of the resulting two-terminal graph; s_1 becomes the new source and t_2 becomes the new sink.

A two-terminal series-parallel graph is a two-terminal graph that can be obtained by starting with several copies of the K_2 graph and applying a sequence of parallel and series compositions. Lastly, a graph is called series-parallel if it is a two-terminal series-parallel graph for some choice of source and sink vertices. Observe that intermediate graphs resulting in the construction of a series-parallel graph may have multiple parallel edges, so they are multigraphs. This is permitted, as long as the resulting overall graph is a simple undirected graph at the end.

Now, we are ready to prove the main result of this section.

▶ **Theorem 29.** $\rho(ALG, SERIES\text{-}PARALLEL) = \Omega(\sqrt{n})$.

Proof. Let $k \geq 2$ be an integer. The adversary reveals s with k neighbors $c_1, \ldots, c_k$. Then $c_1, \ldots, c_k$ are revealed in this order with k new neighbors each. Let neighbors of c_i be $d_{i1}, \ldots, d_{ik}$. Let $S \subseteq \{c_1, \ldots, c_k\}$ be those vertices selected by ALG. For those $i \notin S$ we reveal their new neighbors in order $d_{i1}, \ldots, d_{ik}$. Each such d_{ij} is revealed with a single new neighbor f_{ij}. For $i \in S$ we reveal their new neighbors in order $d_{i1}, \ldots, d_{ik}$. Each such d_{ij} is revealed with a new neighbor t that is common to all these vertices. Then f_{ij} are revealed in arbitrary order with t as a new neighbor. Lastly t is revealed without any new neighbors.

Let $p = |S|$. Observe that in addition to these p vertices ALG must select at least one vertex from each of $\{d_{ij}, f_{ij}\}$ pairs for those $i \notin S$; otherwise, vertex d_{ij} would be undominated. Thus, $ALG \geq p + k(k-p)$. Also, observe that $\{s, t\} \cup \{c_i \mid i \notin S\}$ is a dominating set, so $OPT \leq k - p + 2$. The bound on the competitive ratio is $\frac{ALG}{OPT} \geq \frac{p+k(k-p)}{k-p+2} = k - \frac{2k-p}{k-p+2} \geq \frac{k}{2}$, where the last inequality is obtained as follows. For $k \geq 2$ we have $k^2 - kp \geq 2k - 2p$, which implies $k^2 - kp + 2k \geq 4k - 2p$. This in turn implies that $k(k-p+2) \geq 2(2k-p)$, hence $(2k-p)/(k-p+2) \leq k/2$. The quantitative part of the statement of this theorem follows from the fact that the total number of vertices is at most $2 + k + k^2 + k(k-p) = \Theta(k^2)$.

Lastly, we note that the adversarial graph thus constructed is, indeed, series-parallel. For each $i \notin S$ and $j \in \{1, \ldots, k\}$ the path $c_i \to d_{ij} \to f_{ij} \to t$ is a series-composition of 3 copies of K_2. These paths can be merged by a parallel composition to obtain the subgraph induced on $\{c_i, t\} \cup \{d_{ij}, f_{ij} \mid j \in \{1, \ldots, k\}\}$ for each $i \notin S$. Each of these subgraphs is composed at c_i with another copy of K_2 with the new vertex playing the role of s. Similar argument holds to show that the subgraph induced on $\{s, c_i, t\} \cup \{d_{ij} \mid j \in \{1, \ldots, k\}\}$ for $i \in S$ is a two-terminal series-parallel graph. Lastly, all these subgraphs are merged by a sequence of parallel compositions at s and t. ◀