The Gödel Fibration

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Abstract

We introduce the notion of a Gödel fibration, which is a fibration categorically embodying both the logical principles of traditional Skolemization (we can exchange the order of quantifiers paying the price of a functional) and the existence of a prenex normal form presentation for every logical formula. Building up from Hofstra's earlier fibrational characterization of de Paiva's categorical Dialectica construction, we show that a fibration is an instance of the Dialectica construction if and only if it is a Gödel fibration. This result establishes an intrinsic presentation of the Dialectica fibration, contributing to the understanding of the Dialectica construction itself and of its properties from a logical perspective.

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1 Introduction

Historically, the Dialectica interpretation was devised by Gödel [10] to prove the (relative) consistency of arithmetic. The interpretation allowed him to reduce the problem of proving the consistency of first-order arithmetic to the problem of proving the consistency of a simply-typed system of computable functionals, the well-known *System T*. The key feature of the translation is that it (mostly) constructively turns formulae of arbitrary quantifier complexity into formulae of the form $\exists x \forall y \alpha(x, y)$.

Over the years, several authors have explained the Dialectica interpretation in categorical terms. In particular, de Paiva [7] introduced the notion of *Dialectica categories* as an internal version of Gödel's Dialectica Interpretation. The idea is to construct a category $\mathfrak{Dial}(\mathsf{C})$ from a category C with finite limits. The main focus in de Paiva's original work is on the categorical structure of the category $\mathfrak{Dial}(\mathsf{C})$ obtained, as this notion of a Dialectica category turns out to be also a model of Girard's Linear Logic [9].

This construction was first generalized by Hyland, who investigated the Dialectica construction associated to a fibred preorder [12]. Later Biering in her PhD work [3] studied the Dialectica construction for an arbitrary cloven fibration.

Meanwhile Hofstra [11] wrote an exposition and interpretation of the Dialectica construction from a modern categorical perspective, emphasizing its universal properties. His work gives centre stage to the well-known concepts of pseudo-monads, simple products and co-products. We take Hofstra's work as the basis for our work here.

Hofstra shows that the original Dialectica construction $\mathfrak{Dial}$ can be seen as the composition of two free constructions $\mathfrak{Sum}$ and $\mathfrak{Prod}$, which are the simple sum (or co-product) and product completions, respectively. These completions are fully dual, so we only need to study one and can then deduce results for the other construction. However the whole Dialectica construction is not fully dual, as indicated by the order of the composition of the completions. Our work explains when the Dialectica construction can be performed, which hypotheses are necessary for the categorical construction, which properties of the construction are preserved and why. Most importantly we are able to connect these preservation properties to the logic of the original interpretation, leading up to the definition of what we call a Gödel fibration.

Our contributions

The main contributions of this paper are the following.

1. *We formalize the notion of fibrational quantifier-free formula.* Given Hofstra's characterization it is clear that instances of the Dialectica construction should have simple products and co-products, as the construction introduces completions under these. What else is necessary to get a Dialectica construction? The first novelty of this work is the characterization of "covering quantifier-free objects" of a fibration. These objects correspond to formulas in the logic system that are quantifier-free. As usually happens in a categorical framework, a syntactical notion of "being quantifier-free" needs to be formalized in terms of a universal property. The logical intuition behind our definition, is that an element α of a fibration p is called *quantifier-free* if it satisfies the following universal property, expressed in the internal language of p: if there is a proof π of a statement $\exists i \beta(i)$ assuming α, then there exists a *witness* t, which depends on the proof π, together with a proof of $\beta(t)$. Moreover, this must hold for every re-indexing $\alpha(f)$, because in logic if $\alpha(x)$ is quantifier-free then $\alpha(x)[f/x] = \alpha(f)$ is quantifier-free too. The covering requirement, as usual, means that, for every formula of the form $i : I \mid \alpha(i)$, there exists a formula $\beta(i, a, b)$ quantifier-free that is provably equivalent to it $\alpha(i) \dashv\vdash \exists a \forall b \beta(i, a, b)$.
   Notice that these requirements reflect Gödel's original translation and, at the same time, they recall standard conditions used in category theory to say that a category is *free* for a given structure. One could think for example about the condition of having enough projectives in the exact completion of Carboni [5].
2. *We introduce the notion of a Gödel fibration.* A Gödel fibration is a fibration with simple products and simple co-products, which, most importantly, admits a class of formal sub-objects which are *free from products and co-products* and *cover* all the elements of the fibre. Then we prove that a Gödel fibration is a fibration categorically embodying both the logical principle of traditional Skolemization and the existence of a prenex normal presentation for every logical formula.
3. *We provide an intrinsic presentation of the Dialectica fibration.* We prove that a given fibration is an instance of the Dialectica construction if and only if it is a Gödel fibration. This result helps understanding the existing notion of Dialectica fibration from a logical perspective because it shows which properties an arbitrary fibration should satisfy to be an instance of the Dialectica construction. In other words, given a fibration p there exists a fibration $\bar{\mathsf{p}}$ such that $\mathsf{p} \cong \mathfrak{Dial}(\bar{\mathsf{p}})$ if and only if p is a Gödel fibration. From a categorical perspective, we have classified the free-algebras for the Dialectica pseudo-monad.

4. *We prove that fibrations associated to the Dialectica construction satisfy a strong constructive feature in terms of witnesses.* We have shown that in the internal language of, say Hofstra's Dialectica fibration $\mathfrak{Dial}(\mathsf{p})$, i.e. in the logic theory that canonically corresponds to this categorical notion, if there is a proof π of a statement $\exists i\ \alpha(i)$, then there exists a *witness* t, which depends on the proof π, together with a proof of $\alpha(t)$. This principle is sometimes called the Rule of Choice. For example, Regular Logic (`https://ncatlab.org/nlab/show/regular+logic`) satisfies this principle, see [24].

Related work

In the present paper we provide an intrinsic characterization of the free algebras of the pseudo-monad $\mathfrak{Dial}(-)$ introduced by Hofstra in [11], i.e. we provide necessary and sufficient conditions for a (cloven and split) fibration to be of the form $\mathfrak{Dial}(\mathsf{p})$ for some fibration p. Hofstra's categorical presentation of the Dialectica construction generalizes to the fibrational setting the original construction introduced by de Paiva [7]. In particular, we recall the structural analysis due to Hyland [12] and Biering [2], where the first fibrational presentations of the Dialectica construction were introduced. For a complete presentation of the theory of fibrations and its connection to type theory, we refer the reader to Jacobs [13], and to [4] for an introduction to pseudomonads.

More recently, modern reformulations of the Dialectica interpretation based on the linearized version of de Paiva have been introduced, aiming to provide categorical models for type theory. A relevant example of this line of work is Moss and von Glehn [17], where the authors are interested not in the original construction, but in a modified version of Gödel's Dialectica interpretation for models of intensional Martin-Löf type-theory, using the notion of fibred display map category. Their work focus on the preservation of the type constructors, while they drop the layer of predicates from their Dialectica propositions, considering only those Dialectica propositions of the form $\exists x \forall y \top$. In fact, they call their construction the *polynomial model*, explaining that this name fits better, because they are considering the *predicate-free* Dialectica construction. On a similar line, we mention the work of Pédrot [18], investigating the validity of a Dialectica-like construction in a dependent setting. Different variations of the Dialectica interpretation have been devised for automata, e.g. the work or Pradic and Riba [19].

Finally, Topos-and tripos-theoretic versions of the Dialectica construction have been studied by Biering in [3], while the recent work of Shulman [20], describes a "polycategorical" version of a generalization of the Dialectica construction. Other applications of completions involving universal and existential quantifiers can be found in [24, 23, 8], where similar constructions are presented in the language of doctrines.

2 Revisiting categorical quantifiers

One of the pillars of categorical logic is Lawvere's crucial intuition which considers logical languages and theories as indexed categories and studies their 2-categorical properties. In this setting connectives and quantifiers are characterized in terms of adjointness relations [14, 15, 16].

In this fibrational setting, the intuition is that the base category B of a fibration $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ represents the category of (type-theoretical) contexts, a fibre E_I represents the propositions $\alpha(i)$ in the context I, and the morphisms are proofs. Cartesian morphisms of p induce a *re-indexing* or substitution operation. From this perspective, the simple form of quantification is described in terms of adjoints to weakening functors π^* along projections π. For example, existential quantification is given by an operation $\coprod_{\pi^*} : \mathsf{E}_{A\times B} \to \mathsf{E}_A$, which sends a proposition $\alpha(a,b)$ to $\exists b\ \alpha(a,b)$.

Now we briefly recall the formal definition of a fibration with simple products (or simple universal quantification) and coproducts (or simple existential quantification). For a complete presentation of the theory of fibrations and its connection to type theory, we refer the reader to Jacobs [13]. In this work, we will assume that a fibration p is always *cloven and split*, i.e. that the re-indexing operation is functorial (these definitions can be found in pages 47 and 49 of [13]).

▶ **Definition 2.1.** *We say a fibration* $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ *over a category* B *with finite products has* **simple coproducts** *when the weakening functors* π^* *have left adjoints* $\coprod_\pi$ *satisfying the* Beck-Chevalley Condition *(abbreviated as BCC), i.e. for every pullback square of the form*

$$\begin{array}{ccc} I\times X & \xrightarrow{\pi_I} & I \\ {\scriptstyle f\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} \\ J\times X & \xrightarrow[\pi_J]{} & J \end{array}$$

the canonical natural transformation $f^*\coprod_{\pi_J} \Rightarrow \coprod_{\pi_I}(f\times \mathrm{id})^*$ *is an isomorphism.*

Dually, we say that a fibration $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ *has* **simple products** *when the weakening functors* π^* *have have right adjoints* $\prod_\pi$ *satisfying BCC.*

For more details about the notion of fibration having simple coproducts (or simple products) we refer to [13, Def. 1.9.1].

When one deals with quantification, for example in first-order logic, it is very common to assert something like *a formula* α *is quantifier-free.* This assertion has a natural meaning from a syntactic point of view, but it is not clear how it should be presented from a categorical perspective. The aim of the following definitions, which are generalizations of definitions in [24] to the fibrational setting, is to capture the common property of those elements of a given fibration $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ which will appear as quantifier-free propositions in the internal language of the fibration p. We start by defining when an element of a fibre of p is free from the existential quantifier, and then we dualize the definition for the universal quantifier. (Recall that the symbols $\coprod$ and $\prod$ represent the logical quantifiers $\exists$ and $\forall$.)

The logical intuition behind the next definition is that an element α is *existential-free* if it satisfies the following universal property: if there is a proof π of a statement $\exists i\ \beta(i)$ assuming α, then there exists a *witness* t, which depends on the proof π, together with a proof of $\beta(t)$. Moreover, we require that this holds for every re-indexing $\alpha(f)$ because in logic quantifier-free propositions are stable under substitution, i.e. if $\alpha(x)$ is quantifier-free then $\alpha(f)$ is quantifier-free.

▶ **Definition 2.2.** *Let* $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ *be a fibration with simple coproducts. An object* α *of the fibre* E_I *is said to be* **$\coprod$-quantifier-free** *if it enjoys the following universal property. For every arrow* f *and every projection* π_A *in* B *as follows:*

$$A\times B \xrightarrow{\pi_A} A \xrightarrow{f} I$$

and every vertical arrow:

$$f^*\alpha \xrightarrow{h} \coprod_{\pi_A}\beta$$

of E_A*, where* β *is an object of the fibre* $\mathsf{E}_{A\times B}$*, there exist a unique arrow* $A \xrightarrow{g} B$ *of* B *and a unique vertical arrow* $f^*\alpha \xrightarrow{\overline{h}} \langle 1_A, g\rangle^*\beta$ *of* E_A *such that:*

$$h = \left(f^*\alpha \xrightarrow{\overline{h}} \langle 1_A, g\rangle^*\beta \xrightarrow{\langle 1_A,g\rangle^*\eta_\beta} \langle 1_A, g\rangle^*(\pi_A^* \coprod_{\pi_A} \beta) = \coprod_{\pi_A} \beta \right)$$

where $\beta \xrightarrow{\eta_\beta} \pi_A^* \coprod_{\pi_A} \beta$ *is the unit at* β *of the adjunction* $\coprod_{\pi_A} \dashv \pi_A^*$.

Clarifying the concrete meaning of Definition 2.2, the given object α of E_I represents a formula $\alpha(i)$. Given an arrow $A \xrightarrow{f} I$ a term $f(a) : I$ is in the context $a : A$, and it is the case that $f^*\alpha$ represents the corresponding formula $\alpha(f(a))$. The object β of $\mathsf{E}_{A\times B}$ corresponds to a formula $\beta(a, b)$, the object $\coprod_{\pi_A} \beta$ represents the formula $(\exists b)\beta(a, b)$, which is in the same context $a : A$ of $\alpha(fa)$. Meanwhile, the object $\langle 1_A, g\rangle^*\beta$ is again the re-indexing of $\beta(a, b)$ through an arrow $A \xrightarrow{\langle 1_A,g\rangle} A \times B$, hence it represents the formula $\beta(a, g(a))$, which is in the same context $a : A$ of $\alpha(f(a))$ and $(\exists b)\beta(a, b)$.

Thus the property we require of the formula $\alpha(i)$ is the following: whenever there is a proof (an arrow h of the fibre) of $(\exists b)\beta(a, b)$ from $\alpha(f(a))$ (for some term $f(a) : I$ in the context $a : A$), then there is a unique term $g(a) : B$ in the context $a : A$ together with a unique proof $\overline{h}$ of $\beta(a, g(a))$ from $\alpha(f(a))$, in such a way that, adding at the end of the proof $\overline{h}$ the canonical proof of $(\exists b)\beta(a, b)$ from $\beta(a, g(a))$ (which is represented by the re-indexing of the unit at β of the adjunction $\coprod_{\pi_A} \dashv \pi_A^*$), we get back to the proof h itself of $(\exists b)\beta(a, b)$ from $\alpha(f(a))$. The uniqueness requirement of the term and the proof is due to the proof-relevant nature of fibrations.

Observe that this is precisely the universal property, that we presented before Definition 2.2, enjoyed by a formula which is free from existential quantification.

▶ Remark 2.3. Notice that if we consider a fibration p with simple coproducts, then one can define a sub-fibration $\mathsf{p}' \to \mathsf{p}$ such that the fibres of p' are given by $\coprod$-quantifier-free objects, and the base category of p' is the same of p. This follows since $\coprod$-quantifier-free objects are stable under re-indexing by definition.

The next concept we are going to need in the categorical setting reminds us of the *existence of a prenex normal form* in logic. Recall, for example from [6], that in (classical) first-order logic (FOL) every formula is equivalent to some formula in prenex normal form.

▶ **Definition 2.4.** *We say that a fibration with simple coproducts* $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ *has* **enough $\coprod$-quantifier-free objects** *if, for every object* I *of* B *and for every element* $\alpha \in \mathsf{E}_I$*, there exist an object* A *and a* $\coprod$*-quantifier-free object* β *in* $\mathsf{E}_{I\times A}$ *such that* $\alpha \cong \coprod_{\pi_I} \beta$.

By duality we can define the same concept with respect to the universal quantifier.

▶ **Definition 2.5.** *Let* $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ *be a fibration with simple products. An object* α *of the fibre* E_I *is said to be* **$\prod$-quantifier-free** *if it enjoys the following universal property: for every arrow* f *and every projection* π_A *in* B *as follows:*

$$A \times B \xrightarrow{\pi_A} A \xrightarrow{f} I$$

and every vertical arrow:

$$\prod_{\pi_A} \beta \xrightarrow{h} f^*\alpha$$

of E_A*, where* β *is an object of the fibre* $\mathsf{E}_{A\times B}$*, there exist a unique arrow* $A \xrightarrow{g} B$ *of* B *and a unique vertical arrow* $\langle 1_A, g\rangle^*\beta \xrightarrow{\overline{h}} f^*\alpha$ *of* E_A *such that:*

$$h = \Big(\prod_{\pi_A} \beta = \langle 1_A, g\rangle^* (\pi_A^* \prod_{\pi_A} \beta) \xrightarrow{\langle 1_A, g\rangle^* \varepsilon_\beta} \langle 1_A, g\rangle^*\beta \xrightarrow{\overline{h}} f^*\alpha \Big)$$

where $\pi_A^* \prod_{\pi_A} \beta \xrightarrow{\varepsilon_\beta} \beta$ *is the counit at* β *of the adjunction* $\pi_A^* \dashv \prod_{\pi_A}$.

▶ **Definition 2.6.** *We say that a fibration with simple products* $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ *has* **enough-$\coprod$-quantifier-free objects** *if, for every object* I *of* B *and for every element* $\alpha \in \mathsf{E}_I$*, there exist an object* A *and a* $\coprod$*-quantifier-free object* β *in* $\mathsf{E}_{I\times A}$ *such that* $\alpha \cong \prod_{\pi_I}(\beta)$.

Now we can introduce a particular kind of fibration called a *Skolem fibration*. The name is chosen because these fibrations satisfy a version of the traditional principle of *Skolemization*, as presented in [10] and [11].

▶ **Definition 2.7.** *A fibration* $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ *is called a* **Skolem fibration** *if:*

- *its base category* B *is cartesian closed;*
- *the fibration* p *has simple products and simple coproducts;*
- *the fibration* p *has enough* $\coprod$*-quantifier-free objects.*
- $\coprod$*-quantifier-free objects are stable under simple products, i.e. if* $\alpha \in \mathsf{E}_I$ *is a* $\coprod$*-quantifier-free object, then* $\prod_\pi(\alpha)$ *is a* $\coprod$*-quantifier-free object for every projection* π *from* I.

Notice that the last point of Definition 2.7 implies that, given a Skolem fibration $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$, the sub-fibration $\mathsf{p}'\colon \mathsf{E}' \longrightarrow \mathsf{B}$ of $\coprod$-quantifier-free objects of p defined in Remark 2.3 has simple products.

▶ **Proposition 2.8** (Skolemization)**.** *Every Skolem fibration* p *validates the principle:*

$$\forall x \exists y \alpha(i, x, y) \cong \exists f \forall x \alpha(i, x, fx).$$

Proof. Let us consider an element $\alpha \in E_{A_1\times A_2\times B}$ and a $\coprod$-quantifier-free object $\gamma \in E_{A_1}$. Hence, for every arrow $\pi_1^*(\gamma) \xrightarrow{h} \coprod_{\langle \pi_1,\pi_2\rangle}(\alpha)$, there is a unique pair $(g, \overline{h})$ where $A_1 \times A_2 \xrightarrow{g} B$ and $\pi_1^*(\gamma) \xrightarrow{\overline{h}} \langle \pi_1, \pi_2, g\rangle^*(\alpha)$. Since B has exponents, then we have that g induces a unique arrow $A_1 \xrightarrow{m} B^{A_2}$ such $\langle \pi_1, \pi_2, g\rangle = \langle \pi_1, \pi_2, \mathrm{ev}\langle \pi_1, \pi_2\rangle\rangle\langle \pi_1, \pi_2, m\pi_1\rangle$. Therefore we have an arrow

$$\pi_1^*(\gamma) \xrightarrow{\overline{h}} \langle \pi_1, \pi_2, m\pi_1\rangle^*(\langle \pi_1, \pi_2, \mathrm{ev}\langle \pi_1, \pi_2\rangle\rangle(\alpha)).$$

Since p has simple products, $\overline{h}$ induces a unique arrow

$$\gamma \xrightarrow{\overline{\overline{h}}} \prod_{\pi_1} \langle \pi_1, \pi_2, m\pi_1\rangle^*(\langle \pi_1, \pi_2, \mathrm{ev}\langle \pi_1, \pi_2\rangle\rangle(\alpha)).$$

Notice that the following square

$$\begin{array}{ccc} A_1 \times A_2 & \xrightarrow{\pi_1} & A_1 \\ {\scriptstyle \langle \pi_1,\pi_2,m\pi_1\rangle}\downarrow & & \downarrow{\scriptstyle \langle id_{A_1}, m\rangle} \\ A_1 \times A_2 \times B^{A_2} & \xrightarrow[\langle \pi_1,\pi_3\rangle]{} & A_1 \times B^{A_2} \end{array}$$

is a pullback, hence by the BCC we have that $\prod_{\pi_1}\langle \pi_1, \pi_2, m\pi_1\rangle^* \cong \langle \mathrm{id}_{A_1}, m\rangle^* \prod_{\langle \pi_1,\pi_3\rangle}$. Thus, we get that an arrow f induces a unique pair of arrows $(m, \overline{\overline{h}})$, but again (since p has enough $\coprod$-quantifier-free objects) this pair represents a unique arrow of the fibre $\mathsf{E}_{A_1}(\gamma, \coprod_{\pi_3}\prod_{\langle \pi_1,\pi_1\rangle}(\langle \pi_1, \pi_2, \mathrm{ev}\langle \pi_1, \pi_2\rangle\rangle(\alpha))$, i.e. the fibre

$$\mathsf{E}_{A_1\times A_2}(\pi_1^*(\gamma), \coprod_{\langle \pi_1,\pi_2\rangle}(\alpha))$$

is isomorphic to

$$\mathsf{E}_{A_1}(\gamma, \coprod_{\pi_1}\prod_{\langle \pi_1,\pi_3\rangle}(\langle \pi_1, \pi_2, \mathrm{ev}\langle \pi_1, \pi_2\rangle\rangle(\alpha))$$

and this means exactly that

$$\prod_{\pi_1}\coprod_{\langle \pi_1,\pi_2\rangle}(\alpha) \cong \coprod_{\pi_1}\prod_{\langle \pi_1,\pi_3\rangle}(\langle \pi_1, \pi_2, \mathrm{ev}\langle \pi_1, \pi_2\rangle\rangle(\alpha).$$

The proof for the general case where γ is a generic element of the fibre and not a $\coprod$-quantifier-free object, follows by the observation that the arrows $\pi^*(\gamma) \to \beta$ are in bijection with those of the form $\pi_1^*(\gamma') \to \coprod_{\pi_2}\beta$ for appropriate projections, and where γ' is the $\coprod$-quantifier-free element which covers γ. ◀

Combining Definitions 2.4, 2.6 and 2.7, we introduce the notion of a *Gödel fibration*. The idea is that a Gödel fibration is a Skolem fibration, such that every formula $\alpha(i)$ is equivalent to a formula in *prenex normal form* with respect to p, i.e. there exists a formula $\beta(x, y, i)$ free from quantifiers, such that $\alpha(i) \cong \exists x\forall y\beta(x, y, i)$.

▶ **Definition 2.9.** *A Skolem fibration* $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ *is called a* **Gödel** *fibration if the sub-fibration* $\mathsf{p}'\colon \mathsf{E}' \longrightarrow \mathsf{B}$*, whose elements are* $\coprod$*-quantifier-free objects, has enough* $\prod$*-quantifier-free objects.*

▶ Remark 2.10. Observe that if we consider a Gödel fibration $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$, an element which is a $\prod$-quantifier-free object in the sub-fibration p' could not be $\prod$-quantifier-free object of the Gödel fibration. This because in Definition 2.9 of Gödel fibration, the universal property of being a $\prod$-quantifier-free object is required to hold only with respect to the $\coprod$-quantifier-free objects of p.

The following proposition is an immediate consequence of Definition 2.9.

▶ **Proposition 2.11** (Prenex normal form)**.** *In a Gödel fibration* $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$*, for every element* α *of a fibre* E_I *there exists an element* β *such that*

$$\alpha(i) \cong \exists x\forall y\beta(x, y, i)$$

and β *is* $\prod$*-quantifier-free in the sub-fibration* p' *of* $\coprod$*-quantifier-free objects of* p.

Proof. Let us consider an element α of the fibre E_I. Since p is a Gödel fibration, hence in particular a Skolem fibration, the fibration p has enough $\coprod$-quantifier-free objects, and hence there exists an element γ in the fibre $\mathsf{E}_{I\times X}$ such that $\alpha \cong \coprod_{\pi_I}(\gamma)$. Therefore, since the sub-fibration p' has enough $\prod$-quantifier-free objects, there exists a $\prod$-quantifier-free object β of p' in the fibre $\mathsf{E}_{I\times X\times Y}$ such that $\gamma \cong \prod_{\pi_X}(\beta)$, and hence $\alpha \cong \coprod_{\pi_I}\prod_{\pi_X}(\beta)$. ◀

3 The Dialectica monad

In this section we assume that $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ is a cloven and split fibration whose base category B has finite products. First we recall from Hofstra's [11] the free construction $\mathfrak{Sum}(-)$ which adds simple sums (or coproducts) to a fibration, and then the dual construction $\mathfrak{Prod}(-)$ which freely adds simple products. Then, we present the Dialectica construction $\mathfrak{Dial}(-)$ and its decomposition in terms of simple coproducts and products completions.

Simple coproducts completion. The category $\mathfrak{Sum}(\mathsf{p})$ has:

- as **objects** triples (I, X, α), where I and X are objects of the base category B and α is an object of the fibre $\mathsf{E}_{I\times X}$;
- as **morphisms** triples $(I, X, \alpha) \xrightarrow{(f_0,f_1,\phi)} (J, Y, \beta)$, where $I \xrightarrow{f_0} J$ and $I \times X \xrightarrow{f_1} Y$ are arrows of B and $\alpha(i, x) \xrightarrow{\phi} \beta(f_0(i), f_1(i, x))$ is a morphism of the fibre category $\mathsf{E}_{I\times X}$.

The category $\mathfrak{Sum}(\mathsf{p})$ is fibred over B via the first component projection and this fibration is denoted by $\mathfrak{Sum}(\mathsf{p})\colon \mathfrak{Sum}(\mathsf{p}) \longrightarrow \mathsf{B}$. This fibration is called the ***simple coproduct (or sum) completion*** of p. The intuition behind this definition is that an object (I, X, α) of the fibre category $\mathfrak{Sum}(\mathsf{p})_I$ represents a formula $(\exists x : X)\alpha(i, x)$. The assignment $\mathsf{p} \mapsto \mathfrak{Sum}(\mathsf{p})$ extends to a KZ pseudo-monad on the 2-category of cloven split fibrations, see [11, Theorem 3.9].

▶ Remark 3.1 (*A presentation of* $\mathfrak{Sum}(\mathsf{p})$ *reindexing functors*). Let $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ be a cloven and split fibration. Let $I \xrightarrow{f} J$ be an arrow of B and let (J, Y, β) be an object of $\mathfrak{Sum}(\mathsf{p})_J$. Then the triple:

$$(\ I \xrightarrow{f} J,\ I \times Y \xrightarrow{\pi_Y} Y,\ \langle f\pi_I, \pi_Y\rangle^*\beta \xrightarrow{1_{\langle f\pi_I,\pi_Y\rangle^*\beta}} \langle f\pi_I, \pi_Y\rangle^*\beta\ )$$

is $\mathfrak{Sum}(\mathsf{p})$-cartesian $(I, Y, \langle f\pi_I, \pi_Y\rangle^*\beta) \to (J, Y, \beta)$ over $I \xrightarrow{f} J$. In particular $\mathfrak{Sum}(\mathsf{p})$ is endowed with a cloven and split structure. If:

$$(J, Y, \beta) \xrightarrow{(J\times Y \xrightarrow{g} Y',\ \beta \xrightarrow{\gamma} \langle \pi_J, g\rangle^*\beta')} (J, Y', \beta')$$

is an arrow of $\mathfrak{Sum}(\mathsf{p})_J$ (observe the omission of the first component, as it is forced to be the identity arrow on J) then its f-reindexing is the pair:

$$(I, Y, \langle f\pi_I, \pi_Y\rangle^*\beta) \xrightarrow{(g\langle f\pi_I,\pi_Y\rangle,\ \langle f\pi_I,\pi_Y\rangle^*\gamma)} (I, Y', \langle f\pi_I, \pi_{Y'}\rangle^*\beta')$$

of $\mathfrak{Sum}(\mathsf{p})_I$, whose first component is the arrow $I \times Y \xrightarrow{g\langle f\pi_I,\pi_Y\rangle} Y'$ of B and whose second one is the arrow:

$$\langle f\pi_I, \pi_Y\rangle^*\beta \xrightarrow{\langle f\pi_I,\pi_Y\rangle^*\gamma} \langle f\pi_I, \pi_Y\rangle^*\langle \pi_J, g\rangle^*\beta' = \langle \pi_I, g\langle f\pi_I, \pi_Y\rangle\rangle^*\langle f\pi_I, \pi_{Y'}\rangle^*\beta'$$

of $\mathsf{E}_{I\times Y}$.

Now, let us assume that f is a projection $J \times K \xrightarrow{\pi_J} J$. In this particular case (in which we are mostly interested) such an annoying presentation *collapses* into the following easier one: the π_J-weakening of the arrow (g, γ) of $\mathfrak{Sum}(\mathsf{p})_J$ is the pair:

$$(J \times K, Y, \langle \pi_J, \pi_Y\rangle^*\beta) \xrightarrow{(g\langle \pi_J,\pi_Y\rangle,\langle \pi_J,\pi_Y\rangle^*\gamma)} (J \times K, Y', \langle \pi_J, \pi_{Y'}\rangle^*\beta')$$

of $\mathfrak{Sum}(p)_I$, whose first component is the arrow $J \times K \times Y \xrightarrow{g\langle \pi_J,\pi_Y\rangle} Y'$ of B and whose second one is the arrow:

$$\langle \pi_J, \pi_Y\rangle^*\beta \xrightarrow{\langle \pi_J,\pi_Y\rangle^*\gamma} \langle \pi_J, \pi_Y\rangle^*\langle \pi_J, g\rangle^*\beta' = \langle\langle \pi_J, \pi_K\rangle, g\langle \pi_J, \pi_Y\rangle\rangle^*\langle \pi_J, \pi_{Y'}\rangle^*\beta'$$

of $\mathsf{E}_{J\times K\times Y}$.

▶ Remark 3.2 ($\mathfrak{Sum}(\mathsf{p})$ *has simple coproducts*). Let p be a cloven and split fibration and let us consider a projection $J \times K \xrightarrow{\pi_J} J$ of B. The left adjoint $\coprod_{\pi_J}$ of the π_J-weakening π_J^* in $\mathfrak{Sum}(\mathsf{p})$ exists and sends an arrow:

$$(J \times K, Y, \beta) \xrightarrow{(J\times K\times Y \xrightarrow{g} Y',\ \beta \xrightarrow{\gamma} \langle\langle \pi_J, \pi_K\rangle, g\rangle^* \beta')} (J \times K, Y', \beta')$$

of $\mathfrak{Sum}(\mathsf{p})_{J\times K}$ to the arrow:

$$(J, K \times Y, \beta) \xrightarrow{(J\times K\times Y \xrightarrow{\langle \pi_K, g\rangle} K\times Y',\ \beta \xrightarrow{\gamma} \langle \pi_J, \langle \pi_K, g\rangle\rangle^* \beta')} (J, K \times Y', \beta')$$

of $\mathfrak{Sum}(\mathsf{p})_J$, which we also denote as:

$$\coprod_{\pi_J}(J \times K, Y, \beta) \xrightarrow{\coprod_{\pi_J}(g,\gamma)} \coprod_{\pi_J}(J \times K, Y', \beta').$$

▶ Remark 3.3. Let $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ be a fibration and consider its simple coproduct completion $\mathfrak{Sum}(\mathsf{p})\colon \mathfrak{Sum}(\mathsf{p}) \longrightarrow \mathsf{B}$. As a consequence of Remark 3.2, every element (I, A, α) of the fibre $\mathfrak{Sum}(\mathsf{p})_I$ isomorphic to an object of the form $\coprod_{\pi_I}(I \times A, 1, \alpha')$.

Notice that, by dualising the previous construction, one gets the notion of simple product completion together with its analogous version of the previous characterization.

Simple products completion. The category $\mathfrak{Prod}(\mathsf{p})$ is the one:

- whose **objects** are triples (I, X, α), where I and X are objects of the base category B and α is an object of the fibre $\mathsf{E}_{I\times X}$;
- whose **morphisms** are triples $(I, X, \alpha) \xrightarrow{(f_0, f_1, \phi)} (J, Y, \beta)$, where $I \xrightarrow{f_0} J$ and $I \times Y \xrightarrow{f_1} X$ are arrows of B and $\alpha(i, f_1(i, y)) \xrightarrow{\phi} \beta(f_0(i), y)$ is a morphism of the fibre category $\mathsf{E}_{I\times X}$.

Again, the category $\mathfrak{Prod}(\mathsf{p})$ is fibred over B via first component projection and this fibration is denoted by $\mathfrak{Prod}(\mathsf{p})\colon \mathfrak{Prod}(\mathsf{p}) \longrightarrow \mathsf{B}$ and called ***simple product completion*** of p. The intuition behind this definition is that an object (I, X, α) of the fibre category $\mathfrak{Prod}(\mathsf{p})_I$ represents a formula $(\forall x : X)\alpha(i, x)$.

▶ **Proposition 3.4** (Hofstra [11])**.** *There is an isomorphism of fibrations:*

$$\mathfrak{Prod}(\mathsf{p}) \cong \mathfrak{Sum}(\mathsf{p}^{\mathrm{op}})^{\mathrm{op}}$$

which is natural in p.

Here one has to recall that p^{op} stands for the *fibrewise opposite* of p, see [13] or [11, Def. 2.8].

Again, the assignment $\mathsf{p} \mapsto \mathfrak{Prod}(\mathsf{p})$ extends to a co-KZ pseudo-monad on the 2-category of cloven split fibrations, and its 2-category of pseudo-algebras is equivalent to the 2-category of fibrations with simple products, see [11, Theorem 3.12].

We conclude this section by recalling the presentation of the Dialectica construction and its presentation via the product-coproduct completions.

Dialectica construction. Let $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ be a fibration. Define a category $\mathfrak{Dial}(\mathsf{p})$ as follows:

- **objects** are quadruples (I, X, U, α) where I, X and U are objects of the base category B and $\alpha \in \mathsf{E}_{I\times X\times U}$ is an objects of the fibre of p over $I \times X \times U$;
- a **morphism** from (I, X, U, α) to (J, Y, V, β) is a quadruple (f, f_0, f_1, ϕ) where

1. $I \xrightarrow{f} J$ is a morphism in B;
2. $I \times X \xrightarrow{f_0} Y$ is a morphism in B;
3. $I \times X \times V \xrightarrow{f_1} U$ is a morphism in B;
4. $\alpha(i, x, f_1(i, x, v)) \xrightarrow{\phi} \beta(f(i), f_0(i, x), v)$ is an arrow in the fibre over $I \times X \times V$.

This is a fibration on B with the projection on the first component. Hofstra's key observation is that the construction of the fibration $\mathfrak{Dial}(\mathsf{p})$ can be decomposed in two steps.

▶ **Lemma 3.5** (Hofstra [11]). *There is an isomorphism of fibrations, natural in* p*:*

$$\mathfrak{Dial}(\mathsf{p}) \cong \mathfrak{Sum}(\mathfrak{Prod}(\mathsf{p})).$$

Notice that the pseudo-functor $\mathfrak{Sum}(\mathfrak{Prod}(-))$ is not a pseudo-monad in general, but, in the case the base category B of a fibration $\mathsf{p} \colon \mathsf{E} \longrightarrow \mathsf{B}$ is cartesian closed, one can show that there exists a pseudo-distributive law

$$\mathfrak{Prod}\mathfrak{Sum} \xrightarrow{\lambda} \mathfrak{Sum}\mathfrak{Prod}$$

of pseudo-monads, see [11, Theorem 4.4]. Therefore, by the known equivalence between liftings of pseudo-monads and pseudo-distributive laws, see for example [21, 22], in this case we have that $\mathfrak{Sum}(\mathfrak{Prod}(-))$ is a pseudo-monad.

A notably advantage of this algebraic presentation of the dialectica construction, is that the principle of Skolemisation is represented by the pseudo-distributive law λ.

▶ **Theorem 3.6** (Hofstra [11]). *When the base category* B *of a fibration* p *is cartesian closed, the fibration* $\mathfrak{Dial}(\mathsf{p})$ *satisfies the principle*

$$\forall x \exists y \alpha(i, x, y) \cong \exists f \forall x \alpha(i, x, fx)$$

for every α*.*

4 An intrinsic characterization of Dialectica fibrations

The main goal of this section is to connect the notion of Gödel fibration with that of Dialectica construction, proving that a given fibration p is an instance of the Dialectica construction, i.e. there exists a fibration p' such that $\mathsf{p} \cong \mathfrak{Dial}(\mathsf{p}')$, if and only if p is a Gödel fibration. This result allows us to give an intrinsic definition of a Dialectica fibration because it shows which *properties* an arbitrary fibration should satisfy to be an instance of the Dialectica construction. Moreover, our proof of this equivalence is constructive, in the sense that when p is a Gödel fibration, we are able to explicitly define and construct the fibration p' such that $\mathsf{p} \cong \mathfrak{Dial}(\mathsf{p}')$.

To show this, we take the advantage of Hofstra's decomposition $\mathfrak{Dial}(-) \cong \mathfrak{Sum}(\mathfrak{Prod}(-))$, and we start by showing how fibrations which are instances of the free construction $\mathfrak{Sum}(-)$ (and $\mathfrak{Prod}(-)$) can be described in terms of fibrations with $\coprod$-quantifier-free objects (and $\prod$-quantifier-free objects).

▶ **Proposition 4.1.** *Let* $\mathsf{p} \colon \mathsf{E} \longrightarrow \mathsf{B}$ *be a fibration, and let us consider the simple coproduct completion* $\mathfrak{Sum}(\mathsf{p})$*. Let* I *be an object of* B *and let* α *be an object of its fibre* E_I*. Then every object of the form* $(I, 1, \alpha)$ *in the fibre* $\mathfrak{Sum}(\mathsf{p})_I$ *is a* $\coprod$*-quantifier-free element of* $\mathfrak{Sum}(\mathsf{p})$*. Moreover, the* $\coprod$*-quantifier-free objects of* $\mathfrak{Sum}(\mathsf{p})$ *are up to isomorphism the elements of the form* $(I, 1, \alpha)$*.*

Proof. First we prove that every element of the form $(I,1,\alpha)$ is a $\coprod$-quantifier-free object. Let us consider an arrow

$$\eta_{\mathsf{p}}(\alpha) = (I,1,\alpha) \xrightarrow{(f,\phi)} \coprod_{\pi_I}(I \times A, B, \beta)$$

where $I \xrightarrow{f=\langle g_1,g_2\rangle} A \times B$. We are going to prove that

$$\eta_{\mathsf{p}}(\alpha) \xrightarrow{(g_2,\phi)} \langle 1_I, g_1\rangle^*(I \times A, B, \beta)$$

is an arrow of $\mathfrak{Sum}(\mathsf{p})_I$ and that $(f,\phi) = (\langle 1_I, g_1\rangle^* \eta_\beta)(g_2,\phi)$, where η_β is the unit at $(I \times A, B, \beta)$ of the adjunction $\coprod_{\pi_I} \dashv \pi_I^*$.

Moreover, we have to prove that such a choice of arrows $I \xrightarrow{g} A$ of B and $\eta_{\mathsf{p}}(\alpha) \xrightarrow{\overline{h}} \langle 1_I, h\rangle^*(I \times A, B, \beta)$ of $\mathfrak{Sum}(p)_I$ is unique. That is, whenever the equality:

$$(f,\phi) = (\langle 1_I, g\rangle^* \eta_\beta)\overline{h}$$

holds, it is the case that $g = g_1$ and $\overline{h} = (g_2,\phi)$.

By Remarks 3.1 and 3.2, it is the case that $\coprod_{\pi_I}(I \times A, B, \beta) = (I, A \times B, \beta)$, and that $\langle 1_I, g_1\rangle^*(I, A \times B, \beta) = (I, B, \langle \pi_I, g_1\pi_I, \pi_B\rangle^*\beta)$, where π_I and π_B are the projections from $I \times B$. Then:

$$\eta_{\mathsf{p}}(\alpha) \xrightarrow{(g_2,\phi)} \langle 1_I, g_1\rangle^*(I \times A, B, \beta) = (I, B, \langle \pi_I, g_1\pi_I, \pi_B\rangle^*\beta)$$

is a morphism of $\mathfrak{Sum}(\mathsf{p})_I$ since $I \xrightarrow{g_2} B$ is an arrow of B and:

$$\alpha \xrightarrow{\phi} \langle 1_I, g_2\rangle^* \langle \pi_I, g_1\pi_I, \pi_B\rangle^*\beta = \langle 1_I, g_1, g_2\rangle^*\beta = \langle 1_I, f\rangle^*\beta$$

is a vertical morphism of E_I. Observe that η_β is the transpose along the adjunction $\coprod_{\pi_I} \dashv \pi_I^*$ of the identity arrow of $(I, A \times B, \beta) = \coprod_{\pi_I}(I \times A, B, \beta)$. Hence η_β is the arrow:

$$(I \times A, B, \beta) \xrightarrow{(\pi_{A\times B}, 1_\beta)} (I \times A, A \times B, \langle \pi_I, \pi_{A\times B}\rangle^*\beta)$$

and its $\langle 1_I, g_1\rangle$-reindexing is the arrow:

$$(I, B, \langle \pi_I, g_1\pi_I, \pi_B\rangle^*\beta) \xrightarrow{(\langle g_1\pi_I, \pi_B\rangle,\ 1_{\langle \pi_I, g_1\pi_I, \pi_B\rangle^*\beta})} (I, A \times B, \beta)$$

whose precomposition by (g_2,ϕ) yields indeed the arrow (f,ϕ).

Let us assume that the equality:

$$(f,\phi) = (\langle 1_I, g\rangle^* \eta_\beta)\overline{h} \tag{1}$$

holds for some arrow $I \xrightarrow{g} A$ of B and $\eta_{\mathsf{p}}(\alpha) \xrightarrow{\overline{h}=(h_2,\psi)} \langle 1_I, h\rangle^*(I \times A, B, \beta)$ of $\mathfrak{Sum}(p)_I$. As it is the case that:

$$\langle 1_I, g\rangle^* \eta_\beta = (\langle g\pi_I, \pi_B\rangle,\ 1_{\langle \pi_I, g\pi_I, \pi_B\rangle^*\beta})$$

one might compute the right-hand side of the equality (1) and infer the equality:

$$(f,\phi) = (\langle g, h_2\rangle, \psi)$$

which implies that $g = g_1$ and $\overline{h} = (h_2,\psi) = (g_2,\phi)$.

Finally, notice that Whenever f is an arrow $A \to I$ of B, it is the case that the f-reindexing of $(I, 1, \alpha)$ is the triple $(A, 1, f^*\alpha)$, which is still a *quantifier-free formula*, that is, its second component is terminal in B.

Conversely, let us assume that the triple (I, A, α) is $\coprod$-quantifier-free and let us consider its identity arrow $(I, A, \alpha) \to (I, A, \alpha) = \coprod_{\pi_I}(I \times A, 1, \alpha)$. By $\coprod$-quantifier-freeness, there are an arrow $I \xrightarrow{g} A$ of B and an arrow:

$$(I, A, \alpha) \xrightarrow{(I\times A\xrightarrow{!}1,\ \alpha\xrightarrow{\phi}\pi_I^*\langle 1_I, g\rangle^*\alpha = \langle \pi_I, g\pi_I\rangle^*\alpha)} \langle 1_I, g\rangle^*(I \times A, 1, \alpha) = (I, 1, \langle 1_I, g\rangle^*\alpha)$$

of $\mathfrak{Sum}(\mathsf{p})_I$ such that the identity arrow $(\pi_A, 1_\alpha)$ of (I, A, α) equals the composition:

$$\Big(\ (I, A, \alpha) \xrightarrow{(I\times A\xrightarrow{!}1,\ \phi)} (I, 1, \langle 1_I, g\rangle^*\alpha) \xrightarrow{(g, 1_{\langle 1_I, g\rangle^*\alpha})} (I, A, \alpha)\ \Big)$$

where the pair $(g, 1_{\langle 1_I,g\rangle^*\alpha})$ is nothing but the $\langle 1_I, g\rangle$-reindexing of the unit at $(I \times A, 1, \alpha)$ of the adjunction $\coprod_{\pi_I} \dashv \pi_I^*$. We infer by this arrow equality that it needs to be the case that $(I \times A \xrightarrow{\pi_A} A) = (I \times A \xrightarrow{\pi_I} I \xrightarrow{g} A)$ and that:

$$(\ \alpha \xrightarrow{\phi} \langle \pi_I, g\pi_I\rangle^*\alpha = \langle \pi_I, \pi_A\rangle^*\alpha = \alpha \xrightarrow{1_\alpha} \alpha\ ) = 1_\alpha$$

which means that $\phi = 1_\alpha$. Finally we observe that the composition:

$$(I, 1, \langle 1_I, g\rangle^*\alpha) \xrightarrow{(g, 1_{\langle 1_I, g\rangle^*\alpha})} (I, A, \alpha) \xrightarrow{(I\times A\xrightarrow{!}1,\ \phi = 1_\alpha)} (I, 1, \langle 1_I, g\rangle^*\alpha)$$

equals the identity arrow $(I \times 1 \xrightarrow{!} 1, 1_{\langle 1_I,g\rangle^*\alpha})$. This concludes that the pair $(I \times A \xrightarrow{!} 1, 1_\alpha)$ is actually an isomorphism $(I, A, \alpha) \cong (I, 1, \langle 1_I, g\rangle^*\alpha)$. ◀

▶ Remark 4.2. Let $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ be a fibration and let I be an object of B. Let us consider an arrow $(I, A, \alpha) \xrightarrow{(f,\phi)} (I, B, \beta)$ of $\mathfrak{Sum}(\mathsf{p})_I$. W.l.o.f we can assume (I, A, α) to be of the form $\coprod_{\pi_I}(I \times A, 1, \alpha)$, see Remark 3.3, so we might consider its transpose $(I \times A, 1, \alpha) \xrightarrow{(1_{I\times A}, f, \phi)} \pi_I^*(I, B, \beta) = (I \times A, B, \langle \pi_I, \pi_B\rangle^*\beta)$, which is the unique arrow making the following diagram:

$$\begin{array}{ccc}
\coprod_{\pi_I}(I \times A, 1, \alpha) & \xrightarrow{(1_I, f, \phi)} & (I, B, \beta) \\
{\scriptstyle \coprod_{\pi_I}(1_{I\times A}, f, \phi)}\big\downarrow & \nearrow{\scriptstyle \varepsilon_{(I,B,\beta)}} & \\
\coprod_{\pi_I} \pi_I^*(I, B, \beta) & &
\end{array}$$

commute. Moreover, as $(I \times A, B, \langle \pi_I, \pi_B\rangle^*\beta) = \coprod_{\pi_{I\times A}}(I \times A \times B, 1, \langle \pi_I, \pi_B\rangle^*\beta)$, see Proposition 4.1, the arrow $(1_{I\times A}, f, \phi)$ factors uniquely as the arrow:

$$(I \times A, 1, \alpha) \xrightarrow{(!,\phi)} \langle 1_{I\times A}, f\rangle^*(I \times A \times B, 1, \langle \pi_I, \pi_B\rangle^*\beta) = (I \times A, 1, \langle \pi_I, f\rangle^*\beta)$$

(which can be uniquely expressed as the image $(\overline{\mathsf{p}} \hookrightarrow \mathfrak{Sum}(\overline{\mathsf{p}}))\phi$ of the arrow $\alpha \xrightarrow{\phi} \langle \pi_I, f\rangle^*\beta$ of $\overline{E}_{I\times A}$) followed by the arrow:

$$(I \times A, 1, \langle \pi_I, f\rangle^*\beta) \xrightarrow{\langle 1_{I\times A}, f\rangle^*\eta_{(I\times A\times B, 1, \langle \pi_I, \pi_B\rangle^*\beta)}} \coprod_{\pi_{I\times A}} (I \times A \times B, 1, \langle \pi_I, \pi_B\rangle^*\beta)$$

which is the $\langle 1_{I\times A}, f\rangle$-reindexing of the unit:

$$(I \times A \times B, 1, \langle \pi_I, \pi_B\rangle^*\beta) \xrightarrow{\eta_{(I\times A\times B, 1, \langle \pi_I, \pi_B\rangle^*\beta)}} (I \times A \times B, B, \langle \pi_I, \pi_B\rangle^*\beta)$$

of the adjunction $\coprod_{\pi_{I\times A}} \dashv \pi_{I\times A}^*$.

Notice that in Proposition 4.1 the elements of the form $(I,1,\alpha)$ represent propositions which are free from the existential quantifier.

▶ Remark 4.3. The analogous of Remark 4.2 can be proved for a fibration having enough $\coprod$-quantifier-free objects. In other words, in this kind of fibration the arrows of the fibres are completely described by arrows between quantifier-free objects, unit and counit of adjunctions given by coproducts.

Now we have all the tools to give an *intrinsic* description of the free-algebras for the pseudo-monad which adds the simple coproducts to a given fibration.

▶ **Theorem 4.4.** *A fibration* $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ *with simple coproducts is an instance of simple coproduct completion if and only if it has enough* $\coprod$*-quantifier-free objects. Moreover, in this case* $\mathsf{p} \cong \mathfrak{Sum}(\mathsf{p}')$ *where* p' *is the subfibration of* $\coprod$*-free-quantifers objects of* p*.*

Proof. We define $\mathsf{p}'\colon \mathsf{E}' \longrightarrow \mathsf{B}$ the full-subfibration of $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ such that the objects of E' are the $\coprod$-quantifier-free objects. By the universal property of the inclusion morphism $\mathsf{p}' \hookrightarrow \mathfrak{Sum}(\mathsf{p})$, there is unique a morphism of fibrations with simple coproducts $F\colon \mathfrak{Sum}(\mathsf{p}') \longrightarrow \mathsf{p}$ commuting with the inclusion morphisms $\mathsf{p}' \overset{\eta_{\mathsf{p}'}}{\hookrightarrow} \mathfrak{Sum}(\mathsf{p}')$ and $\mathsf{p}' \hookrightarrow \mathsf{p}$. We claim that F is an equivalence of fibrations. Firstly we observe that F is essentially surjective and then we show that it is fully faithful. From now on, whenever π is a projection in B, we indicate as $\coprod_\pi$ the left adjoint to the π-weakening w.r.t. $\mathfrak{Sum}(\mathsf{p}')$ and as $\sum_\pi$ the one w.r.t. p. Observe that:

$$F(I,1,\gamma) = F\big(\mathsf{p}' \overset{\eta_{\mathsf{p}'}}{\hookrightarrow} \mathfrak{Sum}(\mathsf{p}')\big)\gamma = (\mathsf{p}' \hookrightarrow \mathsf{p})\gamma = \gamma$$

for every I in B and every γ in E'_I.

Essential surjectivity. Let α be an object of E and let I be the object $\mathsf{p}\alpha$ of B. Since p has enough $\coprod$-quantifier-free objects, there are J in B and β in $\mathsf{E}_{I\times J}$ such that $\sum_{\pi_I}\beta \cong \alpha$. Since F preserves simple coproducts, it is the case that:

$$F(I,J,\beta) = F\coprod_{\pi_I}(I\times J,1,\beta) = \sum_{\pi_I} F(I\times J,1,\beta) = \sum_{\pi_I}\beta$$

and we are done. Observe that (I,J,β) is an object of E_I, hence the functor $\mathsf{E}_I \to \mathsf{E}'_I$ induced by F is essentially surjective as well.

Full faithfulness. It sufficies to prove that the morphism F of fibrations over B gives rise to an equivalence $\mathsf{E}_I \to \mathsf{E}'_I$, for any given object I of B (see [13]). As the essential surjectivity of $F\restriction_{\mathsf{E}_I}\colon \mathsf{E}_I \to \mathsf{E}'_I$ follows by the previous part, we only need to observe its full faithfulness.

By Remark 4.2 we write a given arrow $(I,A,\alpha) \xrightarrow{(f,\phi)} (I,B,\beta)$ of E_I as the composition:

$$\varepsilon_{(I,B,\beta)}\Big(\coprod_{\pi_I}\langle 1_{I\times A}, f\rangle^*\eta_{(I\times A\times B,1,\langle\pi_I,\pi_B\rangle^*\beta)}\Big)\Big(\coprod_{\pi_I}(\mathsf{p}' \hookrightarrow \mathfrak{Sum}(\mathsf{p}'))\phi\Big)$$

and this factorisation is unique, because of the uniqueness of adjoint transposition, because of the uniqueness-part of Proposition 4.1 and because of faithfulness of the functor $\mathsf{p}' \hookrightarrow \mathfrak{Sum}(\mathsf{p}')$. As F is forced to preserve simple coproducts and commutes with the inclusion morphisms $\mathsf{p}' \overset{\eta_{\mathsf{p}'}}{\hookrightarrow} \mathfrak{Sum}(\mathsf{p}')$ and $\mathsf{p}' \hookrightarrow \mathsf{p}$, the arrow $F(f,\phi)$ equals the composition:

$$\varepsilon_{(\sum_{\pi_I}\beta)}\Big(\sum_{\pi_I}\langle 1_{I\times A}, f\rangle^*\eta_{\langle\pi_I,\pi_B\rangle^*\beta}\Big)\Big(\sum_{\pi_I}\phi\Big)$$

which is indeed an arrow $\sum_{\pi_I} \alpha \to \sum_{\pi_I} \beta$. Observe that, analogously, every arrow $\sum_{\pi_I} \alpha \to \sum_{\pi_I} \beta$ of E'_I can be uniquely factored as such a composition, again by the existence and the uniqueness of the adjoint transposition, by Definition 2.2 (recall that p is assumed to have $\coprod$-quantifier-free objects) and by full faithfulness of $\mathsf{p}' \hookrightarrow \mathsf{p}$. Hence the function:

$$\mathsf{E}_I((I,A,\alpha),(I,B,\beta)) \to \mathsf{E}'_I\Big(\sum_{\pi_I} \alpha, \sum_{\pi_I} \beta\Big)$$

induced by $F\restriction_{\mathsf{E}_I}$ is bijective, i.e. $F\restriction_{\mathsf{E}_I}$ is fully faithful. ◀

Notice that the characterization of Theorem 4.4 can be obtained also for the simple product completion because of the equivalence the equivalence $\mathfrak{Prod}(\mathsf{p}) \cong \mathfrak{Sum}(\mathsf{p}^{\mathrm{op}})^{\mathrm{op}}$, see Proposition 3.4.

▶ **Theorem 4.5.** *A fibration $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ with simple products is an instance of simple product completion if and only if it has enough- $\prod$-quantifier-free objects. Moreover, in this case $\mathsf{p} \cong \mathfrak{Prod}(\mathsf{p}'')$ where p'' is the subfibration of $\prod$-free-quantifers objects of p.*

Proof. It follows by Theorem 4.4 and Proposition 3.4. ◀

Combining Lemma 3.5, Theorem 4.4 and Theorem 4.5 we can prove the main result of our work, which allows us to recognize if an arbitrary fibration p is an instance of the Dialectica construction or not, and if it is, we can construct the fibration $\bar{\mathsf{p}}$ such that $\mathfrak{Dial}(\bar{\mathsf{p}}) \cong \mathsf{p}$.

▶ **Theorem 4.6.** *Let $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ be a fibration with products, coproducts and such that B is cartesian closed. Then there exists a fibration $\bar{\mathsf{p}}$ such that for $\mathfrak{Dial}(\bar{\mathsf{p}}) \cong \mathsf{p}$ if and only if p is a Gödel fibration.*

Proof. By Lemma 3.5 we have that $\mathfrak{Dial}(-) \cong \mathfrak{Sum}(\mathfrak{Prod}(-))$. Therefore, the result follows from Theorem 4.5 and Theorem 4.4 by directly rephrasing the sequential application of these results. ◀

▶ Remark 4.7. Notice that from a pure categorical perspective Theorem 4.6 provides a characterization of the free-algebras of the pseudo-monad $\mathfrak{Dial}(-)$.

5 Conclusion

Our results develop the original Dialectica construction from both a categorical and logical perspectives, which contributes to a deeper understanding of the construction.

Our main result Theorem 4.6, provides an internal characterization of fibrations which are instances of the Dialectica construction, highlighting the key features a fibration should satisfy, namely it must be a Gödel fibration, to be an instance of the Dialectica construction.

Our presentation in terms of Gödel fibrations underlines a double nature of Dialectica fibrations: they satisfy principles which are typical of classical logic, such as the existence of a prenex normal form presentation for formulae, but they also satisfy principles normally associated to intuitionistic logic. For example, they satisfy the existence of terms witnessing a proof: for every proof of $\alpha \vdash \exists x \beta(x)$ where α is quantifier-free, we have a proof of $\alpha \vdash \beta(t)$ for some term t.

Dialectica-like constructions are pervasive in several areas of mathematics and computer science, and we briefly describe some future work, based on our previous analysis. We wonder if the decomposition introduced by Hofstra can be extended or modified to provide similar results for *cousins* of the Dialectica construction. In particular, we believe that this decomposition, combined with the results presented in [23], could be generalized to the context of dependent type theory.

There are two fibrations which seem to share common features with the Dialectica construction. In particular, we would like to investigate and compare the fibrations arising from work by Abramsky and Väänänen [1] on the Hodges semantics for independence-friendly logic and the Dialectica tripos, which is a model of separation logic [3].

Finally, the strong constructive features of Dialectica fibrations we have shown suggest that these kinds of fibrations could lead to interesting applications in constructive foundations for mathematics and proof theory.

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