Improved Bounds for Coloring Locally Sparse Hypergraphs

Fotis Iliopoulos
Institute for Advanced Study, Princeton, NJ, USA
Princeton University, NJ, USA

Abstract

We show that, for every $k \geq 2$, every k-uniform hypergaph of degree Δ and girth at least 5 is efficiently $(1 + o(1))(k-1)(\Delta/\ln\Delta)^{1/(k-1)}$-list colorable. As an application we obtain the currently best deterministic algorithm for list-coloring random hypergraphs of bounded average degree.

2012 ACM Subject Classification Theory of computation → Generating random combinatorial structures; Theory of computation → Pseudorandomness and derandomization; Theory of computation → Random search heuristics; Theory of computation → Random network models

Keywords and phrases hypergaph coloring, semi-random method, locally sparse, random hypergraphs

Digital Object Identifier 10.4230/LIPIcs.APPROX/RANDOM.2021.39

Category RANDOM

Related Version *Full Version*: `https://arxiv.org/pdf/2004.02066.pdf`

Funding This material is based upon work directly supported by the IAS Fund for Math and indirectly supported by the National Science Foundation Grant No. CCF-1900460. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation. This work is also supported by the Simons Collaboration on Algorithms and Geometry

Acknowledgements The author is grateful to Dimitris Achlioptas, Irit Dinur and anonymous reviewers for detailed comments and feedback.

1 Introduction

In hypergraph coloring one is given a hypergraph $H(V, E)$ and the goal is to find an assignment of one of q colors to each vertex $v \in V$ so that no hyperedge is monochromatic. In the more general *list-coloring* problem, a list of q allowed colors is specified for each vertex. A graph is q-list-colorable if it has a list-coloring no matter how the lists are assigned to each vertex. The *list chromatic number*, $\chi_\ell(H)$, is the smallest q for which H is q-list colorable.

Hypergraph coloring is a fundamental constraint satisfaction problem with several applications in computer science and combinatorics, that has been studied for over 60 years. In this paper we consider the task of coloring locally sparse hypergraphs and its connection to coloring sparse random hypegraphs.

A hypergraph is *k-uniform* if every hyperedge contains exactly k vertices. An i-cycle in a k-uniform hypergraph is a collection of i distinct hyperedges spanned by at most $i(k-1)$ vertices. We say that a k-uniform hypergraph has girth at least g if it contains no i-cycles for $2 \leq i < g$. Note that if a k-uniform hypergraph has girth at least 3 then every two of its hyperedges have at most one vertex in common.

The main contribution of this paper is to prove the following theorem.

Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2021).
Editors: Mary Wootters and Laura Sanità; Article No. 39; pp. 39:1–39:16
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

▶ **Theorem 1.** *Let H by any k-uniform hypergraph, $k \geq 2$, of maximum degree Δ and girth at least 5. For all $\epsilon > 0$, there exist a positive constant $\Delta_{\epsilon,k}$ such that if $\Delta \geq \Delta_{\epsilon,k}$, then*

$$\chi_\ell(H) \leq (1+\epsilon)(k-1)\left(\frac{\Delta}{\ln \Delta}\right)^{\frac{1}{k-1}}. \tag{1}$$

Furthermore, if H is a hypergraph on n vertices then there exists a deterministic algorithm that constructs such a coloring in time polynomial in n.

Theorem 1 is interesting for a number of reasons. First, it generalizes a well-known result of Kim [20] for coloring graphs of degree Δ and girth 5, and it implies the classical theorem of Ajtai, Komlós, Pintz, Spencer and Szemerédi [4] regarding the independence number of k-uniform hypergraphs of degree Δ and girth 5. The latter is a seminal result in combinatorics, with applications in geometry and coding theory [21, 22, 24]. Second, Theorem 1 is tight up to a constant [8]. Note also that, without the girth assumption, the best possible bound [11] on the chromatic number of k-uniform hypergraphs is $O(\Delta^{1/(k-1)})$, i.e., it is asymptotically worse than the one of Theorem 1. For example, there exist graphs of degree Δ whose chromatic number is exactly $\Delta + 1$. Third, when it applies, Theorem 1 improves upon a result of Frieze and Mubayi [14] regarding the chromatic number of simple hypergraphs, who showed (1) with an unspecified large leading constant (of order at least $\Omega(k^4)$). Finally, Theorem 1 can be used to provide the currently best *deterministic* algorithm for list-coloring random k-uniform hypergarphs of bounded average degree. We discuss the connection between locally sparse hypergraphs and sparse random hypergraphs with respect to the task of coloring in the following section.

1.1 Application to coloring pseudo-random hypergraphs

The random k-uniform hypergraph $H(k,n,p)$ is obtained by choosing each of the $\binom{n}{k}$ k-element subsets of a vertex set V ($|V| = n$) independently with probability p. The chosen subsets are the hyperedges of the hypergraph. Note that for $k = 2$ we have the usual definition of the random graph $G(n,p)$. We say that $H(k,n,p)$ has a certain property A *almost surely* or *with high probability*, if the probability that $H \in H(k,n,p)$ has A tends to 1 as $n \to \infty$.

In this paper we are interested in $H(k,n,d/\binom{n}{k-1})$, i.e., the family of random k-uniform hypergraphs of bounded average degree d. Specifically, we use Theorem 1 to prove the following theorem.

▶ **Theorem 2.** *For any constants $\delta \in (0,1)$, $k \geq 2$, there exists $d_{\delta,k} > 0$ such that for every constant $d \geq d_{\delta,k}$, almost surely, the random hypergraph $H(k,n,d/\binom{n}{k-1})$ can be $(1+\delta)(k-1)(d/\ln d)^{1/(k-1)}$-list-colored by a deterministic algorithm whose running time is polynomial in n.*

▶ Remark 3. Note that, for k,d constants, a very standard argument reveals that distribution $H(k,n,d/\binom{n}{k-1})$ is essentially equivalent to $\mathbb{H}(k,n,kdn)$, namely the uniform distribution over k-uniform hypergraphs with n vertices and exactly kdn hyperedges. Thus, Theorem 2 extends to that model as well.

We note that previous approaches [3, 23, 31] for list-coloring random k-uniform hypergraphs of bounded average degree d are either randomized, or require significantly larger lists of colors per vertex in order to succeed. Indeed, to the best of our knowledge, current deterministic approaches require lists of size at least $O(k^4(d/\ln d)^{1/(k-1)})$. Moreover, it is believed that *all* efficient algorithms (including randomized ones) require lists of size at least $(1+o(1))((k-1)d/\ln d)^{1/(k-1)}$, as this bound corresponds to the so-called *shattering* *threshold* [1, 7, 15] fo coloring sparse random hypergraphs, which is also often referred to as the "algorithmic barrier" [1]. This threshold arises in a plethora of random constraint satisfaction problems, and it corresponds to a precise phase transition in the geometry set of solutions. In all of these problems, we are not aware of any efficient algorithm that works beyond the algorithmic barrier, despite the fact that solutions exist for constraint-densities larger than the one in which the shattering phenomenon appears. We refer the reader to [1, 33] for further details.

In order to prove Theorem 2, we show that random k-uniform hypergraphs of bounded average degree d can essentially be treated as hypergraphs of girth 5 and maximum degree d for the purposes of list-coloring, and then apply Theorem 1. In particular, we identify a pseudo-random family of hypergraphs which we call *girth-reducible*, and show that almost all k-uniform hypergraphs of bounded average degree belong in this class. Then we show that girth-reducible hypergraphs can be colored efficiently using Theorem 1.

Formally, a k-uniform hypergraph H is *κ-degenerate* if the induced subhypergraph of all subsets of its vertex set has a vertex of degree at most κ. The *degeneracy* of a hypergraph H is the smallest value of κ for which H is κ-degenerate. Note that it is known that κ-degenerate hypergraphs are $(\kappa+1)$-list colorable and that the degeneracy of a hypergraph can be computed efficiently by an algorithm that repeatedly removes minimum degree vertices. Indeed, to list-color a κ-degenerate hypergraph we repeatedly find a vertex with (remaining) degree at most κ, assign to it a color that does not appear in any of its neighbors so far, and remove it from the hypergraph. Clearly, if the lists assigned to each vertex are of size at least $\kappa+1$ this procedure always terminates successfully.

▶ **Definition 4.** *For $\delta \in (0,1)$, we say that a k-uniform hypergraph $H(V,E)$ of average degree d is* δ-girth-reducible *if its vertex set can be partitioned in two sets, U and $V \setminus U$, such that:*
(a) *U contains all cycles of length at most 4, and all vertices of degree larger than $(1+\delta)d$;*
(b) *subhypergraph $H[U]$ is $\left(\frac{d}{\ln d}\right)^{\frac{1}{k-1}}$-degenerate;*
(c) *every vertex in $V \setminus U$ has at most $\delta\left(\frac{d}{\ln d}\right)^{\frac{1}{k-1}}$ neighbors in U.*

In words, a hypergraph is δ-girth-reducible if its vertex set can be seen as the union of two parts: A "low-degeneracy" part, which contains all vertices of degree more than $(1+\delta)d$ and all cycles of lengths at most 4, and a "high-girth" part, which induces a hypergraph of maximum degree at most $(1+\delta)d$ and girth 5. Moreover, each vertex in the "high-girth" part has only a few neighbors in the "low-degeneracy" part.

Note that given a δ-girth-reducible hypergraph we can efficiently find the promised partition $(U, V \setminus U)$ as follows. We start with $U := U_0$, where U_0 is the set of vertices that either have degree at least $(1+\delta)d$, or they are contained in a cycle of length at most 4. Let ∂U denote the vertices in $V \setminus U$ that violate property (c). While $\partial U \neq \emptyset$, update U as $U := U \cup \partial U$. The correctness of the process lies in the fact that in each step we add to the current U a set of vertices that must be in the low-degeneracy part of the hypergraph. Observe also that this process allows us to efficiently check whether a hypergraph is δ-girth-reducible.

We prove the following theorem regarding the list-chromatic number of girth-reducible hypergraphs.

▶ **Theorem 5.** *For any constants $\delta \in (0,1)$ and $k \geq 2$, there exists $d_{\delta,k} > 0$ such that if H is a δ-girth-reducible, k-uniform hypergraph of average degree $d \geq d_{\delta,k}$, then*

$$\chi_\ell(H) \leq (1+\epsilon)(k-1)\left(\frac{d}{\ln d}\right)^{\frac{1}{k-1}},$$

where $\epsilon = 4\delta = O(\delta)$. Furthermore, if H is a hypergraph on n vertices then there exists a deterministic algorithm that constructs such a coloring in time polynomial in n.

Proof of Theorem 5. Let $\epsilon = 4\delta$. Given lists of colors of size $(1+\epsilon)(k-1)\left(\frac{d}{\ln d}\right)^{\frac{1}{k-1}}$ for each vertex of H, we first color the vertices of U using the greedy algorithm which exploits the low degeneracy of $H[U]$. Now each vertex in $V-U$ has at most $\delta\left(\frac{d}{\ln d}\right)^{\frac{1}{k-1}}$ forbidden colors in its list as it has at most that many neighbors in U. We delete these colors from the list. Observe that if we manage to properly color the induced subgraph $H[V \setminus U]$ using colors from the updated lists, then we are done since every hyperedge with vertices both in U and $V \setminus U$ will be automatically "satisfied", i.e., it cannot be monochromatic. Notice now that the updated list of each vertex still contains at least $(1+3\delta)(k-1)\left(\frac{d}{\ln d}\right)^{\frac{1}{k-1}}$ colors, for sufficiently large d. Since the induced subgraph $H[V \setminus U]$ is of girth at least 5 and of maximum degree at most $(1+\delta)d$, it is efficiently $(1+\delta)(k-1)\left(\frac{(1+\delta)d}{\ln((1+\delta)d)}\right)^{\frac{1}{k-1}}$-list-colorable for sufficiently large d per Theorem 1. This concludes the proof since $(1+\delta)(1+\delta)^{\frac{1}{k-1}} < (1+3\delta)$. ◀

Moreover, we show that girth-reducibility is a pseudo-random property which is admitted by almost all sparse k-uniform hypregraphs.

▶ **Theorem 6.** *For any constants $\delta \in (0,1)$, $k \geq 2$, there exists $d_{\delta,k} > 0$ such that for every constant $d \geq d_{\delta,k}$, almost surely, the random hypergraph $H(k, n, d/\binom{n}{k-1})$ is δ-girth-reducible.*

Theorem 6 follows by simple, although somewhat technical, considerations on properties of sparse random hypergraphs, which are mainly inspired by the results of Alon, Krivelevich and Sudakov [6] and Łuczak [25]. Observe that combining Theorem 6 with Theorem 5 immediately implies Theorem 2.

Overall, the task of coloring locally sparse hypergraphs is inherently related to the average-case complexity of coloring. In particular, in this section we showed that Theorem 1 implies a *robust* algorithm for hypergraph coloring, namely a deterministic procedure that applies to worst-case k-uniform hypergraphs, while at the same using a number of colors that is only a $(k-1)$-factor away from the algorithmic barrier for random instances (matching it for $k = 2$). We remark that this application is inspired by recent results that study the connection between local sparsity and efficient randomized algorithms for coloring sparse regular random graphs [26, 2, 10].

1.2 Technical overview

The intuition behind the proof of Theorem 1 comes from the following observation, which we explain in terms of graph coloring for simplicity. Let G be a triangle-free graph of degree Δ, and assume that each of its vertices is assigned an arbitrary list of q colors. Fix a vertex v of G, and consider the random experiment in which the neighborhood of v is properly list-colored randomly. Since G contains no triangles, this amounts to assigning to each neighbor of v a color from its list randomly and independently. Assuming that $q \geq q^* := (1+\epsilon)\Delta/\ln\Delta$, the expected number of *available* colors for v, i.e., the colors from the list of v that do not appear in any of its neighbors, is at least $q(1-1/q)^{\Delta} = \omega(\Delta^{\epsilon/2})$. In fact, a simple concentration argument reveals that the number of available colors for v in the end of this experiment is at least $\Delta^{\epsilon/2}$ with probability that goes to 1 as Δ grows. To put it differently, as long as $q \geq q^*$, the vast majority of valid ways to list-color the neighborhood of v "leaves enough room" to color v without creating any monochromatic edges.

A completely analogous observation regarding the ways to properly color the neighborhood of a vertex can be made for k-uniform hypergraphs. In order to exploit it we employ the so-called *semi-random method*, which is the main tool behind some of the strongest graph coloring results, e.g., [16, 17, 18, 19, 27, 32], including the one of Kim [20]. The idea is to gradually color the hypergraph in iterations until we reach a point where we can finish the coloring with a simple, e.g., greedy, algorithm. In its most basic form, each iteration consists of the following simple procedure (using graph vertex coloring as a canonical example): Assign to each vertex a color chosen uniformly at random; then uncolor any vertex that receives the same color as one of its neighbors. Using the Lovász Local Lemma [11] and concentration inequalities, one typically shows that, with positive probability, the resulting partial coloring has useful properties that allow for the continuation of the argument in the next iteration. (In fact, using the Moser-Tardos algorithm [29] this approach yields efficient, and often times deterministic [9], algorithms.) Specifically, one keeps track of certain parameters of the current partial coloring and makes sure that, in each iteration, these parameters evolve almost as if the coloring was totally random. For example, recalling the heuristic experiment of the previous paragraph, one of the parameters we would like to keep track of in our case is a lower bound on the number of available colors of each vertex in the hypergraph: If this parameter evolves "randomly" throughout the process, then the vertices that remain uncolored in the end are guaranteed to have a non-trivial number of available colors.

Applications of the semi-random method tend to be technically intense and this is even more so in our case, where we have to deal with constraints of large arity. Large constraints introduce several difficulties, but the most important one is that our algorithm has to control many parameters that interact with each other. Roughly, in order to guarantee the properties that allow for the continuation of the argument in the next iteration, for each uncolored vertex v, each color c in the list of v, and each integer $r \in [k-1]$, we should keep track of a lower bound on the number of adjacent to v hyperedges that have r uncolored vertices and $k-1-r$ vertices colored c. Clearly, these parameters are not independent of each other throughout the process, and so the main challenge is to design and analyze a coloring procedure in which all of them, simultaneously, evolve essentially randomly.

1.3 Organization of the paper

The paper is organized as follows. In Section 2 we present the necessary background. In Section 3 we present the algorithm and state the key lemmas for the proof of Theorem 1. (The proofs of these lemmas can be found in the full version of our paper). In Section 4 we prove Theorem 6.

2 Background and preliminaries

In this section we give some background on the technical tools that we will use in our proofs.

2.1 The Lovász Local Lemma

We will find useful the so-called *lopsided* version of the Lovász Local Lemma [11, 12].

▶ **Theorem 7.** *Consider a set $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ of (bad) events. For each $B \in \mathcal{B}$, let $D(B) \subseteq \mathcal{B} \setminus \{B\}$ be such that $\Pr[B \mid \bigcap_{C \in S} \overline{C}] \le \Pr[B]$ for every $S \subseteq \mathcal{B} \setminus (D(B) \cup \{B\})$. If there is a function $x : \mathcal{B} \to (0,1)$ satisfying*

$$\Pr[B] \le x(B) \prod_{C \in D(B)} (1 - x(C)) \quad \text{for all } B \in \mathcal{B} \ , \tag{2}$$

then the probability that none of the events in $\mathcal{B}$ occurs is at least $\prod_{B \in \mathcal{B}} (1 - x(B)) > 0$.

In particular, we will need the following two corollaries of Theorem 7. For their proofs, the reader is referred to Chapter 19 in [28].

▶ **Corollary 8.** *Consider a set $\mathcal{B} = \{B_1, \ldots, B_m\}$ of (bad) events. For each $B \in \mathcal{B}$, let $D(B) \subseteq \mathcal{B} \setminus \{B\}$ be such that $\Pr[B \mid \bigcap_{C \in S} \overline{C}] \leq \Pr[B]$ for every $S \subseteq \mathcal{B} \setminus (D(B) \cup \{B\})$. If for every $B \in \mathcal{B}$:*
(a) $\Pr[B] \leq \frac{1}{4}$;
(b) $\sum_{C \in D(B)} \Pr[C] \leq \frac{1}{4}$,
then the probability that none of the events in $\mathcal{B}$ occurs is strictly positive.

▶ **Corollary 9.** *Consider a set $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ of (bad) events such that for each $B \in \mathcal{B}$:*
(a) $\Pr[B] \leq p < 1$;
(b) *B is mutually independent of a set of all but at most Δ of the other events.*
If $4p\Delta \leq 1$ then with positive probability, none of the events in $\mathcal{B}$ occur.

2.2 Talagrand's inequality

We will also need the following version of Talagrand's inequality [30] whose proof can be found in [28].

▶ **Theorem 10.** *Let X be a non-negative random variable, not identically 0, which is determined by n independent trials $T_1, \ldots, T_n$, and satisfying the following for some $c, r > 0$:*
1. *changing the outcome of any trial can affect X by at most c, and*
2. *for any s, if $X \geq s$ then there is a set of at most ws trials whose outcomes certify that $X \geq s$,*

then for any $0 \leq t \leq \mathbb{E}[X]$,

$$\Pr[|X - \mathbb{E}[X]| > t + 60c\sqrt{w\mathbb{E}[X]}] \leq 4\mathrm{e}^{-\frac{t^2}{8c^2 w\mathbb{E}[X]}} .$$

3 List-coloring high-girth hypergraphs

In this section we describe the algorithm of Theorem 1 and state the key lemmas behind its analysis. As we already explained, our approach is based on the semi-random method. For an excellent exposition both of the method and Kim's result the reader is referred to [28].

We assume without loss of generality that $\epsilon < \frac{1}{10}$. Also, it will be convenient to define the parameter $\delta := (1+\epsilon)(k-1) - 1$, so that the list of each vertex initially has at least $(1+\delta)(\frac{\Delta}{\ln \Delta})^{\frac{1}{k-1}}$ colors.

We analyze each iteration of our procedure using a probability distribution over the set of (possibly improper) colorings of the uncolored vertices of H where, additionally, each vertex is either activated or deactivated. We call a pair of coloring and activation bits assignments for the uncolored vertices of hypergraph H a *state*.

Let V_i denote the set of uncolored vertices in the beginning of the i-th iteration. (Initially, all vertices are uncolored.) For each $v \in V_i$ we denote by $L_v = L_v(i)$ the list of colors of v in the beginning of the i-th iteration. Further, we say that a color $c \in L_v$ is *available for* v in a state σ if assigning c to v does not cause any hyperedge whose initially uncolored vertices are all activated in σ to be monochromatic.

For each vertex v, color $c \in L_v$ and iteration i, we define a few quantities of interest that our algorithm will attempt to control. Let $\ell_i(v)$ be the size of L_v. Further, for each $r \in [k]$, let $D_{i,r}(v,c)$ denote the set of hyperedges h that contain v and (i) exactly r vertices $\{u_1, \ldots, u_r\} \subseteq h \setminus \{v\}$ are uncolored and $c \in L_{u_j}$ for every $j \in [r]$; (ii) the rest $k-1-r$ vertices other than v are colored c. We define $t_{i,r}(v,c) = |D_{i,r}(v,c)|$.

As it is common in the applications of the semi-random method, we will not attempt to keep track of the values of $\ell_i(v)$ and $t_{i,r}(v,c)$, $r \in [k-1]$, for every vertex v and color c but, rather, we will focus on their extreme values. In particular, we will define appropriate $L_i, T_{i,r}$ such that we can show that, for each i, the following property holds at the beginning of iteration i:

Property P(i): For each vertex $v \in V_i$, color $c \in L_v$ and $r \in [k-1]$,

$$\begin{aligned} \ell_i(v) &\geq L_i, \\ t_{i,r}(v,c) &\leq T_{i,r}. \end{aligned}$$

As a matter of fact, it would be helpful for our analysis (though not necessary) if the inequalities defined in $P(i)$ were actually tight. Given that $P(i)$ holds, we can always enforce this stronger property in a straightforward way as follows. First, for each vertex v such that $\ell_i(v) > L_i$ we choose arbitrarily $\ell_i(v) - L_i$ colors from its list and remove them. Then, for each vertex v and color $c \in L_i$ such that $t_{i,r}(v,c) < T_{i,r}$ we add to the hypergraph $T_{i,r} - t_{i,r}(v,c)$ new hyperedges of size $r+1$ that contain v and r new "dummy" vertices. (As it will be evident from the proof, we can always assume that $L_i, T_{i,r}$ are integers, since our analysis is robust to replacing $L_i, T_{i,r}$ with $\lfloor L_i \rfloor$ and $T_{i,r}$ with $\lceil T_{i,r} \rceil$.) We assign each dummy vertex a list of L_i colors: $L_i - 1$ of them are new and do not appear in the list of any other vertex, and the last one is c.

▶ Remark 11. Dummy vertices are only useful for the purposes of our analysis and can be removed at the end of the iteration. Indeed, one could use the technique of "equalizing coin flips" instead. For more details see e.g., [28].

Overall, without loss of generality, at each iteration i our goal will be to guarantee that $P(i+1)$ holds assuming $Q(i)$.

Property Q(i): For each vertex $v \in V_i$, color $c \in L_v$ and $r \in [k-1]$,

$$\begin{aligned} \ell_i(v) &= L_i, \\ t_{i,r}(v,c) &= T_{i,r}. \end{aligned}$$

An iteration. For the i-th iteration we will apply the Local Lemma with respect to the probability distribution induced by assigning to each vertex $v \in V_i$ a color chosen uniformly at random from L_v and activating v with probability $\alpha = \frac{K}{\ln \Delta}$, where $K = (100k^{3k})^{-1}$. That is, we will apply the Moser-Tardos algorithm in a configuration space consisting of $2|V_i|$ variables corresponding to the color and activation bit of each variable in V_i. (We will define the family of bad events for each iteration shortly.)

When the execution of the Moser-Tardos algorithm terminates, we will uncolor some of the vertices in V_i, to get a new partial coloring. In particular, the partial coloring of the hypergraph, set V_{i+1}, and the lists of colors for each uncolored vertex in the beginning of iteration $i+1$ are induced as follows. Let σ be the output state of the application of the Moser-Tardos algorithm in the i-th iteration. The list of each vertex v, $L_v(i+1)$, is induced from $L_v(i)$ by removing every non-available color $c \in L_v(i)$ for v in σ. We obtain the partial coloring ϕ for the hypergraph and set V_{i+1} for the beginning of iteration $i+1$ by removing the color from every vertex $v \in V_i$ which is either deactivated or is assigned a non-available for it color in σ.

Overall, the i-th iteration of our algorithm can be described at a high-level as follows:

1. Apply the Moser-Tardos algorithm to the probability space induced by assigning to each vertex $v \in V_i$ a color chosen uniformly at random from $L_v(i)$, and activating v with probability α.

2. Let σ be the output state of the Moser-Tardos algorithm.
3. For each vertex $v \in V_i$, remove any non-available color $c \in L_v(i)$ in σ to get a list $L_v(i+1)$.
4. Uncolor every vertex $v \in V_i$ that has either received a non-available color or is deactivated in σ, to get a new partial coloring ϕ.

Controlling the parameters of interest. Next we describe the recursive definitions for L_i and $T_{i,r}$ which, as we already explained, will determine the behavior of the parameters $\ell_i(v)$ and $t_{i,r}(v,c)$, respectively.

Initially, $L_1 = (1+\delta)\left(\frac{\Delta}{\ln \Delta}\right)^{\frac{1}{k-1}}$, $T_{1,k-1} = \Delta$ and $T_{1,r} = 0$ for every $r \in [k-2]$. Letting

$$\mathrm{Keep}_i = \prod_{r=1}^{k-1} \left(1 - \left(\frac{\alpha}{L_i}\right)^r\right)^{T_{i,r}}, \tag{3}$$

we define

$$L_{i+1} = L_i \cdot \mathrm{Keep}_i - L_i^{2/3}, \tag{4}$$

$$\begin{aligned} T_{i+1,r} = & \sum_{j=r}^{k-1} \left(T_{i,j} \cdot \binom{j}{r} \left(\mathrm{Keep}_i \left(1 - \alpha \mathrm{Keep}_i\right)\right)^r \left(\frac{\alpha \mathrm{Keep}_i}{L_i}\right)^{j-r} \right) \\ & + 3k^r \alpha^{-r+1} L_i^r \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell} (\ln \Delta)^{2\ell}} + \left(\sum_{j=r}^{k-1} \binom{j}{r} \alpha^{j-r} \frac{T_{i,j}}{L_i^{j-r}} \right)^{2/3}. \end{aligned} \tag{5}$$

To get some intuition for the recursive definitions (4), (5), observe that Keep_i is the probability that a color $c \in L_v(i)$ is present in $L_v(i+1)$ as well. Note further that this implies that the expected value of $\ell_{i+1}(v,c)$ is $L_i \cdot \mathrm{Keep}_i$, a fact which motivates (4). Calculations of similar flavor for $\mathbb{E}[t_{i+1,r}(v,c)]$ motivate (5).

The key lemmas. We are almost ready to state the main lemmas that will guarantee that our procedure eventually reaches a partial list-coloring of H with favorable properties that will allow us to extend it to a full list-coloring. Before doing so, we need to settle a subtle issue that has to do with the fact that $t_{i+1,r}(v,c)$ is not sufficiently concentrated around its expectation. To see this, notice for example that $t_{i+1,1}(v,c)$ drops to zero if v is assigned c. (Similarly, for $r \in \{2,\ldots,k-1\}$, if v is assigned c then $t_{i+1,r}(v,c)$ can be affected by a large amount.) To deal with this problem we will focus instead on variable $t'_{i+1,r}(v,c)$, i.e., the number of hyperedges h that contain v and (i) exactly $k-r-1$ vertices of $h \setminus \{v\}$ are colored c in the end of iteration i; (ii) the rest r vertices of $h \setminus \{v\}$ did not retain their color during iteration i and, further, c would be available for them if we ignored the color assigned to v. Observe that if c is not assigned to v then $t_{i+1,r}(v,c) = t'_{i+1,r}(v,c)$ and $t'_{i+1,r}(v,c) \geq t_{i+1,r}(v,c)$ otherwise.

The first lemma that we prove estimates the expected value of the parameters at the end of the i-th iteration. Its proof can be found in the full version of our paper.

▶ **Lemma 12.** *Let $S_i = \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell}(\ln\Delta)^{2\ell}}$ and $Y_{i,r} = \sum_{j=r}^{k-1} \frac{T_{i,j}}{L_i^j}$. If $Q(i)$ holds and for all $1 < j < i, r \in [k-1], L_j \geq (\ln \Delta)^{20(k-1)}, T_{i,r} \geq (\ln\Delta)^{20(k-1)}$, then, for every vertex $v \in V_{i+1}$ and color $c \in L_v$:*

(a) $\mathbb{E}[\ell_{i+1}(v)] = \ell_i(v) \cdot \mathrm{Keep}_i$;

(b)

$$\begin{aligned} \mathbb{E}[t'_{i+1,r}(v,c)] \leq & \sum_{j=r}^{k-1} \left(T_{i,j}(v,c) \cdot \binom{j}{r} \left(\mathrm{Keep}_i \left(1 - \alpha \mathrm{Keep}_i\right)\right)^r \left(\frac{\alpha \mathrm{Keep}_i}{L_i}\right)^{j-r} \right) \\ & + 3k^r \alpha^{-r+1} L_i^r S_i + O(Y_i). \end{aligned}$$

The next step is to prove strong concentration around the mean for our random variables per the following lemma. Its proof can be found in the full version of our paper.

▶ **Lemma 13.** *If $Q(i)$ holds and $L_i, T_{i,r} \geq (\ln \Delta)^{20(k-1)}$, $r \in [k-1]$, then for every vertex $v \in V_{i+1}$ and color $c \in L_v$,*
(a) $\Pr\left[|\ell_{i+1}(v) - \mathbb{E}[\ell_{i+1}(v)]| < L_i^{2/3}\right] < \Delta^{-\ln \Delta}$;
(b) $\Pr\left[t'_{i+1,r}(v,c) - \mathbb{E}[t'_{i+1,r}(v,c)] > \frac{1}{2}\left(\sum_{j=r}^{k-1}\binom{j}{r}\alpha^{j-r}\frac{T_{i,j}}{L_i^{j-r}}\right)^{2/3}\right] < \Delta^{-\ln \Delta}$.

Armed with Lemmas 12, 13, a straightforward application of the symmetric Local Lemma, i.e., Corollary 9, reveals the following.

▶ **Lemma 14.** *With positive probability, $P(i)$ holds for every i such that for all $1 < j < i$: $L_j, T_{j,r} \geq (\ln \Delta)^{20(k-1)}$ and $T_{j,k-1} \geq \frac{1}{10k^2} L_j^{k-1}$.*

The proof of Lemma 14 can be found in the full version of our paper.

In analyzing the recursive equations (4), (5), it would be helpful if we could ignore the "error terms". The next lemma shows that this is indeed possible. Its proof can be found in the full version of our paper.

▶ **Lemma 15.** *Define $L'_1 = (1+\delta)\left(\frac{\Delta}{\ln \Delta}\right)^{\frac{1}{k-1}}, T'_{1,k-1} = \Delta$, $T'_{1,r} = 0$ for $r \in [k-2]$, and recursively define*

$$\begin{aligned} L'_{i+1} &= L'_i \cdot \mathrm{Keep}_i, \\ T'_{i+1,r} &= \sum_{j=r}^{k-1}\left(T'_{i,j} \cdot \binom{j}{r}(\mathrm{Keep}_i \cdot (1-\alpha\mathrm{Keep}_i))^r\left(\frac{\alpha\mathrm{Keep}_i}{L'_i}\right)^{j-r}\right) \\ &\quad + 3k^r\alpha^{-r+1}L_i^r\sum_{\ell=1}^{k-1}\frac{T_{i,\ell}}{L_i^{2\ell}(\ln\Delta)^{2\ell}}. \end{aligned}$$

If for all $1 < j < i$, $L_j \geq (\ln \Delta)^{20(k-1)}$, $T_{j,r} \geq (\ln \Delta)^{20(k-1)}$ for every $r \in [k-1]$, and $T_{j,k-1} \geq \frac{L_j^{k-1}}{10k^2}$, then
(a) $|L_i - L'_i| \leq (L'_i)^{\frac{5}{6}}$;
(b) $|T_{i,r} - T'_{i,r}| \leq (T'_{i,r})^{\frac{100r}{100r+1}}$.

▶ Remark 16. Note that Keep_i in Lemma 15 is still defined in terms of $L_i, T_{i,r}$ and not $L'_i, T'_{i,r}$. Note also that in the definition of $T'_{i+1,r}$, the second summand is a function of $T_{i,\ell}, L_i$, $\ell \in [r-1]$, and not $T'_{i,\ell}, L'_i$.

Using Lemma 15 we are able to prove the following in the full version of our paper.

▶ **Lemma 17.** *There exists $i^* = O(\ln \Delta \ln\ln \Delta)$ such that*
(a) *For all $1 < i \leq i^*, T_{i,r} > (\ln \Delta)^{20(k-1)}, L_i \geq \Delta^{\frac{\epsilon/3}{(k-1)(1+\epsilon/2)}}$, and $T_{i,k-1} \geq \frac{1}{10k^2}L_i^{k-1}$;*
(b) *$T_{i^*+1,r} \leq \frac{1}{10k^2}L_{i^*+1}^r$, for every $r \in [k-1]$ and $L_{i^*+1} \geq \Delta^{\frac{\epsilon/3}{(k-1)(1+\epsilon/2)}}$.*

Lemmas 14, 17 and 18 imply Theorem 1.

▶ **Lemma 18.** *Let σ be the state promised by Lemma 17. Given σ, we can find a full list-coloring of H in polynomial time in the number of vertices of H.*

Proof of Theorem 1. We carry out i^* iterations of our procedure. If $P(i)$ fails to hold for any iteration i, then we halt. By Lemmas 14 and 17, $P(i)$ (and, therefore, $Q(i)$) holds with positive probability for each iteration and so it is possible to perform i^* iterations. Further, the fact that our LLL application is within the scope of the so-called *variable setting* [29] implies that the deterministic version of the Moser-Tardos algorithm [29, 9] applies and, thus, we can perform i^* iterations in polynomial time.

After i^* iterations we can apply the algorithm of Lemma 18 and complete the list-coloring of the input hypergraph. ◀

3.1 Proof of Lemma 18

Let $\mathcal{U}_\sigma$ denote the set of uncolored vertices in σ, and $\mathcal{U}_\sigma(h)$ the subset of $\mathcal{U}_\sigma$ that belongs to a hyperedge h. Our goal is to color the vertices in $\mathcal{U}_\sigma$ to get a full list-coloring.

Towards that end, let $L_v = L_v(\sigma)$ denote the list of colors for v at σ, and $D_r(v,c) := D_{i^*+1,r}(v,c)$ the set of hyperedges (of size $t_{i^*+1,r}(v,c)$) with r uncolored vertices in σ whose vertices "compete" for c with v, and recall the conclusion of Lemma 17. Let μ be the probability distribution induced by giving each vertex $v \in \mathcal{U}_\sigma$ a color from L_v uniformly at random. For every hyperedge h and color $c \in \bigcap_{u \in h} L_u$ we define $A_{h,c}$ to be the event that all vertices of h are colored c. Let $\mathcal{A}$ be the family of these (bad) events, and observe that for every $A_{h,c} \in \mathcal{A}$:

$$\mu\left(A_{h,c}\right) \leq \frac{1}{\prod_{v \in \mathcal{U}_\sigma(h)} |L_v(\sigma)|} < \frac{1}{4}$$

for large enough Δ, since $L_{i^*+1} = L_{i^*+1}(\Delta) \xrightarrow{\Delta \to \infty} \infty$.

Moreover, let $I(A_{h,c})$ denote the set of all bad events $A_{h',c'}$, where $h' \neq h$, such that either $\mathcal{U}_\sigma(h) \cap \mathcal{U}_\sigma(h') = \emptyset$, or c' is not in the list of colors of the (necessarily unique) uncolored vertex that h and h' share. Notice that conditioning on any the non-occurrence of any set $S \subseteq I(A_{h,c})$ does not increase the probability of $A_{h,c}$.

Let $D(A_{h,c}) := \mathcal{A} \setminus I(A_{h,c})$. Lemma 18 follows from Corollary 8 (and can be made constructive using the deterministic version of the Moser-Tardos algorithm [29, 9]) as, for every $A_{h,c} \in \mathcal{A}$:

$$\begin{aligned}
\sum_{A \in D(A_{h,c})} \mu(A) &\leq \sum_{v \in \mathcal{U}_\sigma(h)} \sum_{c' \in L_v} \sum_{r=1}^{k-1} \sum_{h' \in D_r(v,c')} \mu\left(A_{h',c'}\right) \\
&= \sum_{v \in \mathcal{U}_\sigma(h)} \sum_{c' \in L_v} \sum_{i=1}^{k-1} \sum_{h' \in D_r(v,c')} \frac{1}{\prod_{u \in \mathcal{U}_\sigma(h')} |L_u|} \\
&\leq \max_{v \in \mathcal{U}_\sigma(h)} \frac{k}{|L_v|} \sum_{c' \in L_v} \sum_{r=1}^{k-1} \frac{|D_r(v,c')|}{L_{i^*+1}^r} && (6) \\
&\leq \frac{k}{10k^2} \max_{v \in \mathcal{U}_\sigma(h)} \frac{L_{i^*+1}^r \cdot |L_v|}{|L_v| \cdot L_{i^*+1}^r} && (7) \\
&\leq \frac{1}{10} < \frac{1}{4}\ , && (8)
\end{aligned}$$

for large enough Δ, concluding the proof. Note that in (6) we used the facts that every hyperedge has at most k vertices and $L_{i^*+1} \geq \Delta^{\frac{\epsilon/3}{(k-1)(1+\epsilon/2)}}$, and in (7) we used the fact that $|D_r(v,c')| \leq T_{i^*+1}^r \leq \frac{1}{10k^2} L_{i^*+1}^r$.

4 A sufficient pseudo-random property for coloring

In this section we present the proof of Theorem 6. To do so, we build on ideas of Alon, Krivelevich and Sudakov [6] and show that the random hypergraph $H(k,n,d/\binom{n}{k-1})$ almost surely admits a few useful features.

The first lemma we prove states that all subgraphs of $H(k,n,d/\binom{n}{k-1})$ with not too many vertices are sparse and, therefore, of small degeneracy.

▶ **Lemma 19.** *For every constant $k \geq 2$, there exists $d_k > 0$ such that for any constant $d \geq d_k$, the random hypergraph $H(k,n,d/\binom{n}{k-1})$ has the following property almost surely: Every $s \leq nd^{-\frac{1}{k-1}}$ vertices of H span fewer than $s\left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}}$ hyperedges. Therefore, any subhypergraph of H induced by a subset $V_0 \subset V$ of size $|V_0| \leq nd^{-\frac{1}{k-1}}$, is $k\left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}}$-degenerate.*

Proof. Letting $r = \left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}}$, we see that the probability that there exists a subset $V_0 \subset V$ which violates the statement of the lemma is at most

$$\begin{aligned}
\sum_{i=r^{\frac{1}{k-1}}}^{nd^{-\frac{1}{k-1}}} \binom{n}{i}\binom{\binom{i}{k}}{ri}\left(\frac{d}{\binom{n}{k-1}}\right)^{ri} &\leq \sum_{i=r^{\frac{1}{k-1}}}^{nd^{-\frac{1}{k-1}}} \left[\frac{\mathrm{e}n}{i}\left(\frac{\mathrm{e}i^{k-1}}{r}\right)^{r}\left(\frac{d}{\binom{n}{k-1}}\right)^{r}\right]^{i} \qquad (9)\\
&\leq \sum_{i=r^{\frac{1}{k-1}}}^{nd^{-\frac{1}{k-1}}} \left[\mathrm{e}^{1+\frac{1}{k-1}}(k-1)\left(\frac{d}{r}\right)^{\frac{1}{k-1}}\left(\frac{\mathrm{e}i^{k-1}d}{r\binom{n}{k-1}}\right)^{r-\frac{1}{k-1}}\right]^{i}\\
&= o(1),
\end{aligned}$$

for sufficiently large d. Note that in the lefthand side of (9) we used the fact that any subset of vertices of size $s < r^{\frac{1}{k-1}}$ cannot violate the assertion of the lemma, since it can span at most $s^k < rs$ hyperedges. In deriving the final inequality we used that for any pair of integers α, β, we have that $\binom{\alpha}{\beta} \geq \left(\frac{\alpha}{\beta}\right)^{\beta}$. ◀

Next we show that, as far as the number of vertices of $H(k,n,d/\binom{n}{k-1})$ that have a constant degree c is concerned, the degree of each vertex of H is essentially a Poisson random variable with mean d.

▶ **Lemma 20.** *For constants $c \geq 1$, $k \geq 2$ and d, let X_c denote the number of vertices of degree c in $H(k,n,d/\binom{n}{k-1})$. Then, for $c = O(1)$, with high probability,*

$$X_c \leq \frac{d^c \mathrm{e}^{-d}}{c!} n\left(1 + O\left(\frac{\log n}{\sqrt{n}}\right)\right).$$

Proof. The lemma follows from standard ideas for estimation of the degree distribution of random graphs (see for example the proof of Theorem 3.3 in [13] for the case $k = 2$). In particular, assume that the vertices of $H(k, n, d/\binom{n}{k-1})$ are labeled $1, 2, \ldots, n$. Then,

$$\begin{aligned}
\mathbb{E}[X_c] &= n\Pr[\deg(1) = c] \\
&= n\binom{\binom{n-1}{k-1}}{c}\left(\frac{d}{\binom{n}{k-1}}\right)^c\left(1-\frac{d}{\binom{n}{k-1}}\right)^{\binom{n-1}{k-1}-c} \\
&\le n\frac{\left(\binom{n-1}{k-1}\right)^c}{c!}\left(1+O\left(\frac{c^2}{\binom{n-1}{k-1}}\right)\right)\left(\frac{d}{\binom{n}{k-1}}\right)^c\exp\left(-\left(\binom{n-1}{k-1}-c\right)\frac{d}{\binom{n}{k-1}}\right) \\
&\le n\frac{d^c e^{-d}}{c!}\left(1+O\left(\frac{1}{n^{k-1}}\right)\right).
\end{aligned}$$

To show concentration of X_c around its expectation, we will use Chebyshev's inequality. In order to do so, we need to estimate $\Pr[\deg(1) = \deg(2) = c]$. For $\ell \in \{0, \ldots, c\}$, let $E_{1,2}^{\ell}$ denote the event that there exist exactly ℓ hyperedges that contain both vertices 1 and 2. Then, letting $p = \frac{d}{\binom{n}{k-1}}$, we see that

$$\begin{aligned}
\Pr[\deg(1) = \deg(2) = c] &\le \sum_{\ell=0}^{c}\Pr\left[E_{1,2}^{\ell}\right]\left(\binom{\binom{n-1}{k-1}}{c-\ell}p^c(1-p)^{\binom{n-1}{k-1}-c}\right)^2 \\
&= \sum_{\ell=0}^{c}\binom{\binom{n-2}{k-2}}{\ell}p^{\ell}\,(1-p)^{\binom{n-2}{k-2}-\ell}\left(\binom{\binom{n-1}{k-1}}{c-\ell}p^c(1-p)^{\binom{n-1}{k-1}-c}\right)^2 \\
&= \Pr[\deg(1) = c]\cdot\Pr[\deg(2) = c]\left(1+O\left(\frac{1}{n^{k-1}}\right)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Var}[X_c] &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\Pr[\deg(i) = c, \deg(j) = c] - \Pr[\deg(1) = c]\Pr[\deg(2) = c]\right) \\
&\le \sum_{i\neq j=1} O\left(\frac{1}{n^{k-1}}\right) + \mathbb{E}[X_c] = An\ ,
\end{aligned}$$

for some constant $A = A(c, d)$.

Finally, applying the Chebyshev's inequality, we obtain that, for any $t > 0$,

$$\Pr\left[|X_c - \mathbb{E}[X_c]| \ge t\sqrt{n}\right] \le \frac{A}{t^2},$$

and, thus, the proof is concluded by choosing $t = \log n$. ◀

Lemma 20 implies the following useful corollary.

▶ **Corollary 21.** *For any constants $\delta \in (0, 1), k \ge 2, d > 0$, let $X = X(\delta, k, d)$ denote the random variable equal to the number of vertices in $H(k, n,\ d/\binom{n}{k-1})$ whose degree is in $[(1+\delta)d, 3(k-1)^{k-1}d]$. There exists a constant $d_\delta > 0$ such that if $d \ge d_\delta$ then, almost surely, $X \le \frac{n}{d^2}$.*

Proof. Let X_r denote the number of vertices of degree r in $H(k,n,d/\binom{n}{k-1})$. Since k,d are constants, using Lemma 20 and Stirling's approximation we see that, almost surely,

$$\sum_{r=(1+\delta)d}^{3(k-1)^{k-1}d} X_r \le n\left(1+O\left(\frac{\log n}{\sqrt{n}}\right)\right)\sum_{r=(1+\delta)d}^{3(k-1)^{k-1}d}\frac{d^r \mathrm{e}^{-d}}{r!}$$
$$\le n(1+\delta)\sum_{r=(1+\delta)d}^{3(k-1)^{k-1}d}\frac{d^r \mathrm{e}^{-d}}{\sqrt{2\pi r}\left(\frac{r}{\mathrm{e}}\right)^r} \le \frac{n}{d^2},$$

for sufficiently large d and n. ◀

Using Lemma 19 and Corollary 21 we show that, almost surely, only a small fraction of vertices of $H(k,n,d/\binom{n}{k-1})$ have degree that significantly exceeds its average degree.

▶ **Lemma 22.** *For every constants $k \ge 2$ and $\delta \in (0,1)$, there exists $d_{k,\delta} > 0$ such that for any constant $d \ge d_{k,\delta}$, all but at most $\frac{2n}{d^2}$ vertices of the random hypergraph $H(k,n,d/\binom{n}{k-1})$ have degree at most $(1+\delta)d$, almost surely.*

Proof. Corollary 21 implies that the number of vertices with degree in the interval $[(1+\delta)d, 3(k-1)^{k-1}d]$ is at most $\frac{n}{d^2}$, for sufficiently large d.

Suppose now there are more than $\frac{n}{d^2}$ vertices with degree at least $3(k-1)^{k-1}d$. Denote by S a set containing exactly $\frac{n}{d^2}$ such vertices. According to Lemma 19, almost surely, the induced subhypergraph $H[S]$ has at most

$$e(H[S]) \le \left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}}|S| = \frac{n}{d^{2-\frac{1}{k-1}}(\ln d)^{\frac{2}{k-1}}}$$

hyperedges. Therefore, the number of hyperedges between the sets of vertices S and $V \setminus S$ is at least

$$3(k-1)^{k-1}d|S| - ke(H[S]) \ge \frac{2.9(k-1)^{k-1}n}{d} =: N.$$

However, the probability that $H(k,n,d/\binom{n}{k-1})$ contains such a subhypergraph is at most

$$\binom{n}{\frac{n}{d^2}}\binom{\frac{n^k}{d^2}}{N}\left(\frac{d}{\binom{n}{k-1}}\right)^N \le \left(\mathrm{e}d^2\right)^{\frac{n}{d^2}}\left(\frac{n^k\mathrm{e}}{d^2N}\cdot\frac{d}{\binom{n}{k-1}}\right)^N = o(1),$$

for sufficiently large d. Note that in deriving the final equality we used that for any pair of integers α, β, we have that $\binom{\alpha}{\beta} \ge \left(\frac{\alpha}{\beta}\right)^\beta$. Therefore, almost surely there are at most $\frac{n}{d^2}$ vertices in G with degree greater than $3(k-1)^{k-1}d$, concluding the proof. ◀

Finally, we show that the neighborhood of a typical vertex of $H(k,n,d/\binom{n}{k-1})$ is locally tree-like.

▶ **Lemma 23.** *For every constants $k \ge 2, \delta \in (0,1)$, almost surely, the random hypergraph $H(k,n,d/\binom{n}{k-1})$ has a subset $U \subseteq V(H)$ of size at most $n^{1-\delta}$ such that the induced hypergraph $H[V \setminus U]$ is of girth at least 5.*

Proof. Let Y_2, Y_3, Y_4, denote the number of 2-, 3- and 4-cycles in $H(n,k,d/\binom{n}{k-1})$, respectively. A straightforward calculation reveals that for $i \in \{2,3,4\}$:

$$\mathbb{E}[Y_i] \le \sum_{s=1}^{i(k-1)} \binom{n}{s} \binom{\binom{s}{k-1}}{i} \left(\frac{d}{\binom{n}{k-1}} \right)^i$$
$$\le i(k-1)n^{i(k-1)} \left(\frac{(i(k-1))^{k-1} \mathrm{e}^2 (k-1)^{k-1}}{in^{k-1}} \right)^i = O(1).$$

By Markov's inequality this implies that $Y_2 + Y_3 + Y_4 \le n^{1-\sqrt{\delta}}$ almost surely. Denote by U the union of all 2-, 3- and 4- cycles in H. Then the induced subhypergraph $H[V \setminus U]$ has girth at least 5 and, almost surely, $|U| \le n^{1-\delta}$. ◀

We are now ready to prove Theorem 6.

Proof of Theorem 6. Our goal will be to find a subset $U \subset V$ of size $|U| \le nd^{-\frac{1}{k-1}}$ that (i) contains all cycles of length at most 4 and every vertex of degree more than $(1+\delta)d$; and (ii) such that, every vertex v in $V \setminus U$ has at most $9k^2 \left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}} = o\left(\left(\frac{d}{\ln d}\right)^{\frac{1}{k-1}}\right)$ neighbors in U. Note that in this case, according to Lemma 19, $H[U]$ is $k\left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}}$-degenerate, concluding the proof assuming d is sufficiently large. A similar idea has been used in [5, 6, 25].

Towards that end, let U_1 be the set of vertices of degree more than $(1+\delta)d$, and U_2 the set of vertices that are contained in a 2-,3- or a 4-cycle. Notice that U_1, U_2, can be found in polynomial time and, according to Lemmas 22 and 23, the size of $U_0 := |U_1 \cup U_2|$ is at most $\frac{3n}{d^2}$ for sufficiently large n and d.

We now start with $U := U_0$ and as long as there exists a vertex $v \in V \setminus U$ having at least $9k^2 \left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}}$ neighbors in U we do the following. Let $S_v = \{u_1, u_2, \dots, u_N\}$ be the neighbors of v in U. We choose an arbitrary hyperedge h that contains v and u_1 and update U and S_v by defining $U := U \cup h$ and $S_v := S_v \setminus h$. We keep repeating this operation until S_v is empty.

This process terminates with $|U| < nd^{-\frac{1}{k-1}}$ because, otherwise, we would get a subset $U \subset V$ of size $|U| = nd^{-\frac{1}{k-1}}$ spanning more than

$$\frac{1}{k}\left(\frac{n}{d^{\frac{1}{k-1}}} - |U_0|\right) \times 9k^2 \left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}} \times \frac{1}{k} > \frac{n}{d^{\frac{1}{k-1}}} \times \left(\frac{d}{(\ln d)^2}\right)^{\frac{1}{k-1}}$$

hyperedges, for sufficiently large d. According to Lemma 19 however, H does not contain any such set almost surely. ◀

References

1 Dimitris Achlioptas and Amin Coja-Oghlan. Algorithmic barriers from phase transitions. In *49th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2008, October 25-28, 2008, Philadelphia, PA, USA*, pages 793–802. IEEE Computer Society, 2008. doi:10.1109/FOCS.2008.11.

2 Dimitris Achlioptas, Fotis Iliopoulos, and Alistair Sinclair. Beyond the lovász local lemma: Point to set correlations and their algorithmic applications. In David Zuckerman, editor, *60th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2019, Baltimore, Maryland, USA, November 9-12, 2019*, pages 725–744. IEEE Computer Society, 2019. doi:10.1109/FOCS.2019.00049.

3 Dimitris Achlioptas and Michael Molloy. The analysis of a list-coloring algorithm on a random graph. In *Proceedings 38th Annual Symposium on Foundations of Computer Science*, pages 204–212. IEEE, 1997.

4 Miklós Ajtai, János Komlós, Janos Pintz, Joel Spencer, and Endre Szemerédi. Extremal uncrowded hypergraphs. *Journal of Combinatorial Theory, Series A*, 32(3):321–335, 1982.

5 Noga Alon and Michael Krivelevich. The concentration of the chromatic number of random graphs. *Combinatorica*, 17(3):303–313, 1997.

6 Noga Alon, Michael Krivelevich, and Benny Sudakov. List coloring of random and pseudo-random graphs. *Combinatorica*, 19(4):453–472, 1999.

7 Peter Ayre, Amin Coja-Oghlan, and Catherine Greenhill. Hypergraph coloring up to condensation. *Random Structures & Algorithms*, 54(4):615–652, 2019.

8 Tom Bohman, Alan Frieze, and Dhruv Mubayi. Coloring h-free hypergraphs. *Random Structures & Algorithms*, 36(1):11–25, 2010.

9 Karthekeyan Chandrasekaran, Navin Goyal, and Bernhard Haeupler. Deterministic algorithms for the Lovász local lemma. *SIAM J. Comput.*, 42(6):2132–2155, 2013. doi:10.1137/100799642.

10 Ewan Davies, Ross J Kang, François Pirot, and Jean-Sébastien Sereni. An algorithmic framework for coloring locally sparse graphs. *arXiv preprint arXiv:2004.07151*, 2020.

11 Paul Erdős and László Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, pages 609–627. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

12 Paul Erdös and Joel Spencer. Lopsided Lovász local lemma and latin transversals. *Discrete Applied Mathematics*, 30(2-3):151–154, 1991. doi:10.1016/0166-218X(91)90040-4.

13 Alan Frieze and Michał Karoński. *Introduction to random graphs*. Cambridge University Press, 2016.

14 Alan Frieze and Dhruv Mubayi. Coloring simple hypergraphs. *Journal of Combinatorial Theory, Series B*, 103(6):767–794, 2013.

15 Marylou Gabrié, Varsha Dani, Guilhem Semerjian, and Lenka Zdeborová. Phase transitions in the q-coloring of random hypergraphs. *Journal of Physics A: Mathematical and Theoretical*, 50(50):505002, 2017.

16 A. Johansson. Asympotic choice number for triangle free graphs, 1996.

17 A. Johansson. The choice number of sparse graphs. *Unpublished manuscript*, 1996.

18 Jeff Kahn. Asymptotics of the chromatic index for multigraphs. *Journal of Combinatorial Theory, Series B*, 68(2):233–254, 1996.

19 Jeff Kahn. Asymptotics of the list-chromatic index for multigraphs. *Random Structures & Algorithms*, 17(2):117–156, 2000.

20 Jeong Han Kim. On Brooks' theorem for sparse graphs. *Combinatorics, Probability and Computing*, 4(2):97–132, 1995.

21 János Komlós, János Pintz, and Endre Szemerédi. A lower bound for heilbronn's problem. *Journal of the London Mathematical Society*, 2(1):13–24, 1982.

22 Alexandr Kostochka, Dhruv Mubayi, Vojtěch Rödl, and Prasad Tetali. On the chromatic number of set systems. *Random Structures & Algorithms*, 19(2):87–98, 2001.

23 Michael Krivelevich and Benny Sudakov. The chromatic numbers of random hypergraphs. *Random Structures & Algorithms*, 12(4):381–403, 1998.

24 Hanno Lefmann. Sparse parity-check matrices over GF (q). *Combinatorics, Probability and Computing*, 14(1-2):147–169, 2005.

25 Tomasz Łuczak. The chromatic number of random graphs. *Combinatorica*, 11(1):45–54, 1991.

26 Michael Molloy. The list chromatic number of graphs with small clique number. *Journal of Combinatorial Theory, Series B*, 134:264–284, 2019.

27 Michael Molloy and Bruce Reed. A bound on the total chromatic number. *Combinatorica*, 18(2):241–280, 1998.

28 Michael Molloy and Bruce Reed. *Graph colouring and the probabilistic method*, volume 23 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2002.
29 Robin A. Moser and Gábor Tardos. A constructive proof of the general Lovász local lemma. *J. ACM*, 57(2):Art. 11, 15, 2010. doi:10.1145/1667053.1667060.
30 Michel Talagrand. Concentration of measure and isoperimetric inequalities in product spaces. *Publications Mathématiques de l'Institut des Hautes Etudes Scientifiques*, 81(1):73–205, 1995.
31 Van H Vu. On the choice number of random hypergraphs. *Combinatorics Probability and Computing*, 9(1):79–95, 2000.
32 Van H Vu. A general upper bound on the list chromatic number of locally sparse graphs. *Combinatorics, Probability and Computing*, 11(1):103–111, 2002.
33 Lenka Zdeborová and Florent Krzakala. Phase transitions in the coloring of random graphs. *Physical Review E*, 76(3):031131, 2007.