A Tight Local Algorithm for the Minimum Dominating Set Problem in Outerplanar Graphs

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Abstract

We show that there is a deterministic local algorithm (constant-time distributed graph algorithm) that finds a 5-approximation of a minimum dominating set on outerplanar graphs. We show there is no such algorithm that finds a $(5-\varepsilon)$-approximation, for any $\varepsilon > 0$. Our algorithm only requires knowledge of the degree of a vertex and of its neighbors, so that large messages and unique identifiers are not needed.

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1 Introduction

Given a sparse graph class, how well can we approximate the size of the minimum dominating set (MDS) in the graph using a constant number of rounds in the LOCAL model? A *dominating set* of a graph $G = (V, E)$ is a set $S \subseteq V$ such that every vertex in $V \setminus S$ has a neighbor in S. Given a graph G and an integer k, deciding whether G has a dominating set of size at most k is NP-complete even when restricting to planar graphs of maximum degree three [9]. Moreover, the size of the MDS is NP-hard to approximate within a constant factor (for general graphs) [16]. The practical applications of MDS are diverse but almost always involve large networks [3], and it is therefore natural to turn to the the distributed setting. No constant factor approximation of the MDS is possible using a sub-linear number of rounds in the LOCAL model [13], and so various structural restrictions have been considered on the graph classes with the hope of finding more positive results (see [8] for an overview).

Planar graphs are a hallmark case. For planar graphs, guaranteeing that some constant factor approximation can be achieved is already highly non-trivial [7, 14]. The current best known upper-bound is 52 [19], while the best lower-bound is 7 [11]. Substantial work has focused on generalizing the fact that some constant factor approximation is possible to more general classes of sparse graphs, like graphs that can be embedded on a given surface, or more recently graphs of bounded expansion [1, 2, 5, 12]. Tight bounds currently seem out of reach in those more general contexts.

In this paper we focus instead on restricted subclasses of planar graphs. Better approximation ratios can be obtained with additional structural assumptions: 32 if the planar graph contains no triangle [3] and 18 if the planar graph contains no cycle of length four [4]. These bounds are not tight, and in fact we expect they can be improved significantly. We are able to provide tight bounds for a different type of restriction: we consider planar graphs with no $K_{2,3}$-minor or K_4-minor[1], i.e. *outerplanar* graphs. Outerplanar graphs can alternatively be defined as planar graphs that can be embedded so that there is a special face which contains all vertices in its boundary.

Outerplanar graphs are a natural intermediary graph class between planar graphs and forests. A planar graph on n vertices contains at most $3n-6$ edges, and a forest on n vertices contains at most $n-1$ edges; an outerplanar graph on n vertices contains at most $2n-3$ edges. Every planar graph can be decomposed into three forests [15]; it can also be decomposed into two outerplanar graphs [10].

For planar graphs, as discussed above, we are far from a good understanding of how to optimally approximate Minimum Dominating Set in $O(1)$ rounds. Let us discuss the case of forests, as it is of very relevant to the outerplanar graph case. For forests, a trivial algorithm yields a 3-approximation: it suffices to take all vertices of degree at least 2 in the solution, as well as vertices with no neighbor of degree at least 2 (that is, isolated vertices and isolated edges). The output is clearly a dominating set, and the proof that it is at most three times as big as the optimal solution is rather straightforward. In fact, the trivial algorithm is tight because of the case of long paths. Indeed, no constant-time algorithm can avoid taking all but a sub-linear number of vertices of a long path, while there is a dominating set containing only a third of the vertices.

Our contribution

We prove that a similarly trivial algorithm (as the one described for forests above) works to obtain a 5-approximation of MDS for outerplanar graphs in the LOCAL model.

Algorithm 1 A local algorithm to compute a dominating set in outerplanar graphs.

Input: An outerplanar graph G
Result: A set $S \subseteq V(G)$ that dominates G
In the first round, every vertex computes its degree and sends it to its neighbors;
$S := \{\text{Vertices of degree} \geq 4\} \cup \{\text{Vertices with no neighbor of degree} \geq 4\}$;

It is easy to check that the algorithm indeed outputs a dominating set. It is significantly harder to argue that the resulting dominating set is at most 5 times as big as one of minimum size. To do that, we delve into a rather intricate analysis of the behavior of a hypothetical counterexample, borrowing tricks from structural graph theory (see Lemma 2).

The proof that the bound of 5 is tight for outerplanar graphs is similar to the proof that the bound of 3 is tight for trees. Every graph in the family depicted in Figure 1 is outerplanar, and every local algorithm that runs in a constant number of rounds selects all

[1] For any integer $n \geq 1$, K_n denotes the complete graph on n vertices. For integers $n, m \geq 1$, $K_{n,m}$ denotes the complete bipartite graph with partite classes of size n and m.

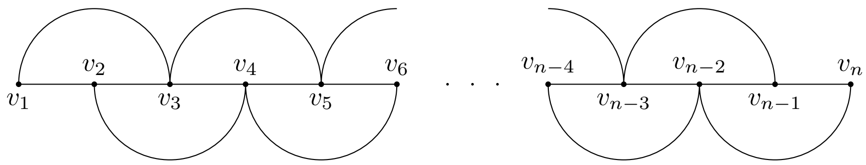

Figure 1 The graph G_n^- is a path $v_1, \ldots, v_n$ to which we add all edges between vertices of distance two. In this example n is even.

but a sub-linear number of vertices [7, pp. 87–88]. Informally, all but a sub-linear number of vertices "look the same" – see Section 3 for more details, and [17] for an excellent survey of lower bounds.

Our main result is the following.

▶ **Theorem 1.** *There is an algorithm that computes a* 5*-approximation of Minimum Dominating Set for outerplanar graphs in* $O(1)$ *rounds in the LOCAL model. This is tight, in the sense that no algorithm can compute a* $(5 - \varepsilon)$*-approximation with the same constraints, for any* $\varepsilon > 0$.

In other words, there is a trivial local algorithm for Minimum Dominating Set in outerplanar graphs that turns out to be tight. All the difficulty lies in arguing that the approximation factor is indeed correct.

We note that the algorithm is so trivial that every vertex only needs to send one bit of information to each of its neighbors ("I have degree at least 4" or "I have degree at most 3"). The network might be anonymous – names are not useful beyond being able to count the number of neighbors, and the solution is extremely easy to update when there is a change in network. For contrast, in anonymous planar graphs the best known approximation ratio is 636 [18].

It is important to note that there is no hope for such a trivial algorithm in the case of planar graphs. Indeed, in Figure 2, we can see that for any p, no algorithm taking all vertices of degree $\geq p$ in the solution can yield a constant-factor approximation in planar graphs. However, the case of outerplanar graphs shows that the road to a better bound for planar graphs might go through finer structural analysis rather than smarter algorithms.

Definitions and notation

For a vertex set $A \subseteq V$, let $G[A]$ denote the induced subgraph of G with vertex set A. Let $E(A)$ denote set of edges of $G[A]$. For vertex sets $A, B \subseteq G$, let $E(A, B)$ denote the set of edges in G with one end in A and the other end in B. We write $G \setminus e$ for the graph in which the edge e is removed from the edge set of G. For a set $P \subseteq V$ inducing a connected subgraph, we write G/P for the graph obtained by contracting the set P: we replace the vertices in P with a new vertex v_P, which is adjacent to $u \in V \setminus P$ if and only if u has some neighbor in P. For a set X of vertices, we let $N[X]$ denote the set $X \cup \cup_{x \in X} N(x)$ and we let $N(X)$ denote the set $N[X] \setminus X$. If $x_1, x_2, \ldots, x_k$ are the elements of X, we may also denote $N[X]$ and $N(X)$ as $N[x_1, x_2, \ldots, x_k]$ and $N(x_1, x_2, \ldots, x_k)$, respectively.

Given a graph G, let $V_{4+}(G)$ denote the set of vertices of degree at least 4 in G, and let $V^*(G)$ denote the set $V(G) \setminus N[V_{4+}(G)]$. In other words, $V^*(G)$ is the set of vertices of degree at most 3 in G which only have neighbors of degree at most 3. For a graph G and a dominating set S of G, we denote $V_{4^+}(G) \setminus S$ by $B_S(G)$ and we denote $V^*(G) \setminus S$ by $D_S(G)$. We additionally let $A_S(G)$ denote the set $V(G) \setminus (S \cup D_S(G) \cup B_S(G))$. In situations where

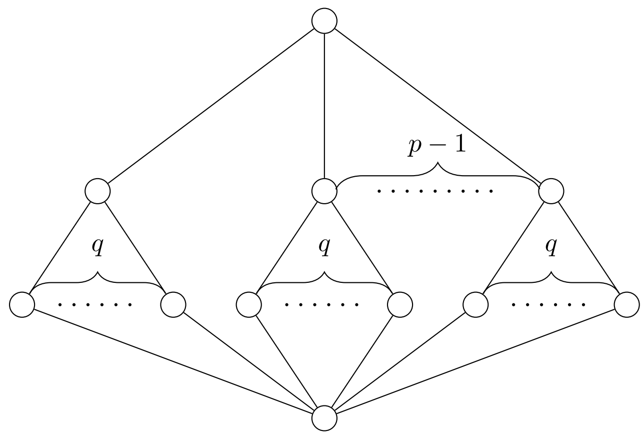

Figure 2 For any $p, q \in \mathbb{N}$, there is a planar graph $G_{p,q}$ which admits a dominating set of size 2 such that $|\{\text{Vertices with degree } \geq q\}| \geq p$.

our choice of G, S is not ambiguous we will simply write B, D, A for $B_S(G)$, $D_S(G)$ and $A_S(G)$, respectively. An overview of the notation is given in Table 1.

Table 1 An overview of the notation used in Section 2.

v is an element of	$\deg(v)$	degrees of neighbors of v	further restrictions
$V_{4^+}(G)$	≥ 4	arbitrary	-
$B_S(G)$	≥ 4	arbitrary	$v \notin S$
$V^*(G)$	≤ 3	≤ 3	-
$D_S(G)$	≤ 3	≤ 3	$v \notin S$
$A_S(G)$	≤ 3	at least one neighbor of degree ≥ 4	$v \notin S$

An outerplanar embedding of G is an embedding in which a special *outer face* contains all vertices in its boundary.

We denote by $H_G(S)$ the multigraph with vertex set S, obtained from G as follows. For every vertex u in $V(G) \setminus S$, we select a neighbor $s(u) \in N(u) \cap S$, and contract the edge $\{u, s(u)\}$. Contrary to the contraction operation mentioned earlier, this may create parallel edges, but we delete all self-loops. The resulting multigraph inherits the set S as its vertex set. We refer to Figure 3 for an example.

Note that $H_G(S)$ inherits an outerplanar embedding from G. If the graph G and the dominating set S are clear, we will write H for $H_G(S)$. Lemma 3 provides some intuition as to why the graph H is useful.

Properties of outerplanar graphs

Here we mention some standard but useful properties of outerplanar graphs. A graph H is a *minor* of a graph G if H can be obtained from G through a series of vertex or edge deletions and edge contractions. Alternatively, an H-minor of G consists of a connected set $X_h \subseteq V(G)$ for each $h \in V(H)$ and a set of paths $\{P_{hh'} \mid hh' \in E(H)\}$, where $P_{hh'}$ is a path in G between a vertex in X_h and a vertex in $X_{h'}$, all of which are pairwise vertex-disjoint except for possibly their ends. Note that any minor of an outerplanar graph is outerplanar. Neither K_4 nor $K_{2,3}$ can be drawn in the plane so that all vertices appear on the boundary of a special face. Therefore, outerplanar graphs are K_4-minor-free and $K_{2,3}$-minor-free.

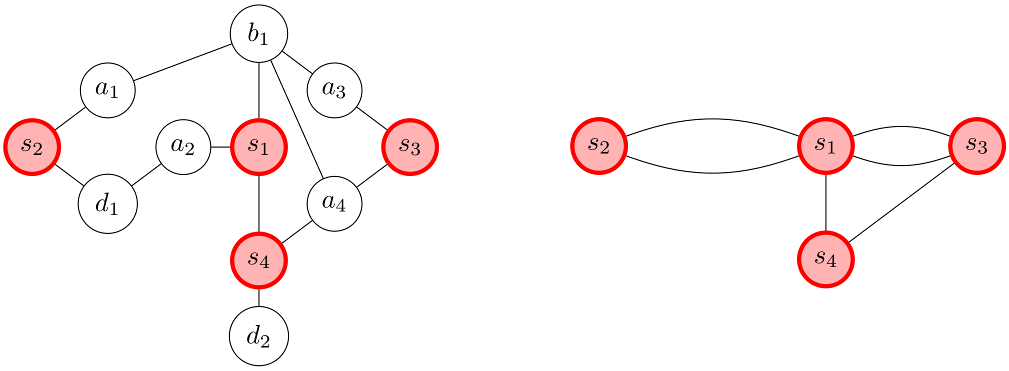

Figure 3 On the left is a graph G with dominating set $S = \{s_1, s_2, s_3, s_4\}$. The vertex $s(u)$ is uniquely determined for all $u \neq a_4$. On the right is the graph $H_G(S)$ for $s(a_4) = s_3$.

Any outerplanar graph G satisfies $|E(G)| \leq 2|V(G)| - 3$ by a simple application of Euler's formula. It follows immediately that every outerplanar graph contains a vertex of degree at most 3, but a standard structural analysis guarantees that every outerplanar graph contains a vertex of degree at most 2.

2 Analysis of the approximation factor

This section is devoted to proving the following result. (An overview of the relevant notation is given in Table 1.)

▶ **Lemma 2.** *For every outerplanar graph G, any dominating set S of G satisfies $|S| \geq \frac{1}{4}(|B_S(G)| + |D_S(G)|)$.*

We briefly argue that Lemma 2 yields the desired result. Given an outerplanar graph, Algorithm 1 outputs $V_{4^+}(G) \cup V^*(G)$ as a dominating set. To argue that it is a 5-approximation of the Minimum Dominating Set problem, it suffices to prove that any dominating set S of G satisfies $|S| \geq \frac{1}{5}(|V_{4^+}(G) \cup V^*(G)|)$. For technical reasons, it is easier to bound S as a function of the vertices in $V_{4^+}(G) \cup V^*(G)$ that are not in S, i.e. $|S| \geq \frac{1}{4}(|B_S(G)| + |D_S(G)|)$, which yields $|S| \geq \frac{1}{5}(|V_{4^+}(G) \cup V^*(G)|)$.

We prove the lemma by analyzing the structure of a "smallest" counterexample. A counterexample satisfies

$$|S| < \frac{1}{4}(|B_S(G)| + |D_S(G)|),$$

and we will choose one which minimizes $|S|$ and with respect to that maximizes $|B_S(G)| + |D_S(G)|$. For this, we need that $|B_S(G)| + |D_S(G)|$ is bounded in terms of $|S|$ by some constant, otherwise a counterexample maximizing $|B_S(G)| + |D_S(G)|$ might not exist since $|B_S(G)| + |D_S(G)|$ could be arbitrarily large. We therefore first prove the following much weaker result.

▶ **Lemma 3.** *For every outerplanar graph G, any dominating set S of G satisfies $|S| \geq \frac{1}{39}(|B_S(G)| + |D_S(G)|)$.*

We did not try to optimize the constant 39 and rather aim to get across some of the main ideas as clearly as possible. The proof shows the importance of the graph $H_G(S)$, which we will also use in the proof of Lemma 2.

Proof of Lemma 3. We may assume that the graph G is connected; otherwise, we can repeat the same argument for each connected component of G. We fix an outerplanar embedding of G. For each $u \in V(G) \setminus S$ we select an arbitrary neighbor $s(u) \in N(u) \cap S$ that we contract it with (keeping parallel edges but removing self-loops), resulting in the multigraph $H_G(S)$ on the vertex set S. The key step in our proof is showing that $H_G(S)$ has bounded edge multiplicity. Indeed, every edge s_1s_2 in $H_G(S)$ is obtained from G by contracting at least one vertex or from the edge s_1s_2 in G. For $i \in \{1,2\}$, let V_i be the set of vertices contracted to s_i that gave an edge between s_1 and s_2 in $H_G(S)$. Since there is no $K_{2,3}$-minor in G (as G is outerplanar), we find $|V_1| \leq 2$ and $|V_2| \leq 2$. Any edge between s_1 and s_2 in $H_G(S)$ can now be associated with an edge between $\{s_1\} \cup V_1$ and $\{s_2\} \cup V_2$ in G, and hence edges in $H_G(S)$ have multiplicity at most 9 (this is far from tight).

We derive that $|E(H_G(S))| \leq 9|E(H')|$, where H' is the simple graph underlying $H_G(S)$ (i.e. the simple graph obtained by letting $s_1, s_2 \in S$ be adjacent in H' if and only if there is an edge between them in $H_G(S)$). Note that H' is a minor of G. Since outerplanar graphs are closed under taking minors, the graph H' is an outerplanar graph. It follows that $|E(H')| \leq 2|S| - 3$. Combining both observations, we get $|E(H_G(S))| \leq 18|S|$.

By outerplanarity, we have $|E(H_G(S))| \geq \frac{1}{2}|B_S(G)|$. Indeed, each vertex $u \in B_S(G)$ has at most two common neighbors with $s(u)$ (otherwise there would be a $K_{2,3}$), hence u has at least one neighbor v such that $v \notin N[s(u)]$. The edge uv corresponds to an edge in $E(H_G(S))$, hence each $u \in B_S(G)$ contributes at least half an edge to $E(H_G(S))$ (as v could be also in $B_S(G)$). We derive $\frac{1}{2}|B_S(G)| \leq |E(H_G(S))| \leq 18|S|$. We observe that $|D_S(G)| \leq 3|S|$: indeed, each vertex from S is adjacent to at most 3 vertices from $D_S(G)$, since any vertex adjacent to a vertex in $D_S(G)$ has degree at most 3 by definition. We conclude that $|B_S(G)| + |D_S(G)| \leq (36+3)|S| = 39|S|$. ◀

In Lemma 3 we use that edges in H have low multiplicity, from which we then obtain a bound on the size of S. In order to improve the bound from Lemma 3, we dive into a deeper analysis of the graph H.

Proof of Lemma 2. We will consider a special counterexample (G, S) (satisfying $|S| < \frac{1}{4}(|B_S(G)| + |D_S(G)|)$) so that our counterexample has a structure we can deal with more easily than a general counterexample. In particular we will choose a counterexample (G, S) amongst those that minimize S and with respect to that maximize $B_S(G) \cup D_S(G)$ to maximize and minimize certain other graph parameters.

Namely, we assume that (G, S) in order: minimizes $|S|$; maximizes $|B_S(G) \cup D_S(G)|$; minimizes $|E(B_S(G))|$; minimizes $|V(G)|$; maximizes $|E(S, N(S))|$; minimizes $|E(G)|$. Note that this is well-defined since we established $|B_S(G) \cup D_S(G)| \leq 39|S|$, and clearly $|E(S, N(S))| \leq |E(G)| \leq 2|V(G)|$ by outerplanarity. Consequently, if a counterexample exists, then there exists one satisfying all of the above assumptions. More formally, we select a counterexample that is minimal for

$$(|S|, 39|S| - |B_S(G) \cup D_S(G)|, |E(B_S(G))|, |V(G)|, 2|V(G)| - |E(S, N(S))|, |E(G)|) \quad (\ddagger)$$

in the lexicographic order. Since all the elements in the sextuple are non-negative integers and their minimum is bounded below by zero, this is well-defined. (We remark that the parts indicated in gray were added to ensure the entries are non-negative; minimizing $39|S| - |B_S(G) \cup D_S(G)|$ comes down to maximizing $|B_S(G) \cup D_S(G)|$.)

While this approach is not entirely intuitive, the assumptions will prove to be extremely useful for simplifying the structure of G. For example, we can show that in a smallest counterexample that minimizes (‡), S is a stable set (Claim 5) and no two vertices in S have a common neighbor (Claim 7). In general, the Claims 4 to 14 show that such a minimal counterexample G has strong structural properties, by showing that otherwise we could delete some vertices and edges, or contract edges, and find a smaller counterexample.

Informally, for any vertex from S that we remove from the graph, we may decrease $|B_S(G) \cup D_S(G)|$ by 4 while maintaining $|S| < \frac{1}{4}(|B_S(G) \cup D_S(G)|)$. It is therefore natural to consider what happens when we reduce $|S|$ by one by contracting a connected subset containing two or more vertices from S. The result is again an outerplanar graph and we aim to show it is a smaller counterexample (unless the graph has some nice structure). Contracting an edge uv can affect the degrees of the remaining vertices in the graph G. Therefore $B_S(G)$ may "lose" additional vertices besides u and v if more than one of its neighbors are contracted and $D_S(G)$ may "lose" additional vertices if a neighbor got contracted, increasing the degree. We remark that vertices from $D_S(G)$ have no neighbors in $B_S(G)$, and therefore removing or contracting them does not affect the set $B_S(G)$.

We note that our minimal counterexample (G, S) is connected and again fix an outerplanar embedding of G. By definition, vertices in D can have no neighbors in B. In fact, the following stronger claim holds.

▷ Claim 4. Every vertex $d \in D$ satisfies $N(d) \subseteq S$.

Proof. Let $e = dv \in E(G)$ be such that $d \in D$. Suppose $v \notin S$. We consider the graph $G \setminus e$. Since $v \notin S$, S is a dominating set of $G \setminus e$. We find $|B_S(G \setminus e)| = |B_S(G)|$, since a vertex in D has no neighbor in B. Similarly, $|D_S(G \setminus e)| = |D_S(G)|$. Hence we also find that $|S| < \frac{1}{4}(|B_S(G \setminus e)| + |D_S(G \setminus e)|)$. Since $v \notin S$, the number of edges with one end incident to S is the same in G and $G \setminus e$. It follows that $(G \setminus e, S)$ is a counterexample to Lemma 2. Since $v \notin S$, G and $G \setminus e$ have the same number of edges with exactly one end in S. Hence since $|E(G \setminus e)| < |E(G)|$, the pair $(G \setminus e, S)$ is smaller with respect to ‡, contradicting our choice of (G, S). ◁

We are now ready to make more refined observations about the structure of (G, S). When considering a pair (G', S') that is smaller than (G, S) with respect to ‡ with $V(G') \subseteq V(G)$, it can be useful to refer informally to vertices that belong to $B_S(G)$ but not to $B_{S'}(G')$ as *lost* vertices (similarly for $D_S(G)$ and $D_{S'}(G')$). The number of lost vertices is an upper bound on $|B_S(G) \cup D_S(G)| - |B_{S'}(G') \cup D_{S'}(G')|$.

We need the following notation. Let $P \subseteq E(G)$. We denote the multigraph obtained from G by contracting every edge in P and deleting self-loops by G/P. Note G/P remains outerplanar and may contain parallel edges.

▷ Claim 5. The set S is a stable set.

Proof. Assume towards a contradiction that there are two vertices u and w in S that are adjacent.

Consider the outerplanar graph $G' = G/\{uw\}$ and let v_{uw} be the vertex resulting from the contraction of the edge uw. Let $S' = S \setminus \{u, w\} \cup \{v_{uw}\}$. Define $B' = B_{S'}(G')$, and $D' = D_{S'}(G')$. Note that S' dominates G'. Since we reduced the size of the dominating set by one, we are allowed to "lose up to four vertices from $B \cup D$", as then we get that

$$|B'| + |D'| \geq |B| + |D| - 4 > 4(|S| - 1) = 4|S'|.$$

We will now show the above inequality holds. If $v \in B \setminus B'$, then v is a common neighbor of u and w (and u, w have at most two such neighbors by outerplanarity). If $v \in D \setminus D'$, then v is a neighbor of u and/or w in G (since no vertex in $V(G) \cap V(G')$ has a higher degree in G' than G). We consider three cases, depending on the neighbors of u and of w in D.

- Suppose that u, w have no neighbors in D. Then $D' = D$ and we lose only vertices from B which are common neighbors of u, w, so at most 2.
- Suppose that u, w both have neighbors in D. Then u, w are of degree at most 3. Since they are adjacent to each other, they are adjacent to at most 4 other vertices in total. So $|B'| + |D'| \geq |B| + |D| - 4$.
- Suppose that u has a neighbor in D and w does not (the other case is analogous). Now $|B| + |D|$ decreases by at most 2, since any "lost" vertex is adjacent to u and distinct from w.

In all cases, S has decreased by 1 and $|B| + |D|$ by at most 4, so we indeed find $|B'| + |D'| > 4|S'|$. Since $|S'| < |S|$, this gives a contradiction with the minimality of our choice of (G, S). ◁

We remark that no vertex in S has only neighbors in D. Indeed, if some $s \in S$ has only neighbors in D then we remove s and its neighborhood from the graph and we have a smaller counterexample, as we reduced $|S|$ by one and $|D|$ by at most 3. In fact, we will show the following:

▷ Claim 6. If $d \in D$ and $s \in N(d) \cap S$, then s has a neighbor in B.

Proof. Suppose that $d \in D$ is adjacent to $s \in S$. Since $d \in D$, we find that s has degree at most 3. Say s has neighbors w_1 and w_2 (possibly equal, but both not equal to d). We argued above that s has a neighbor outside of D, so without loss of generality $w_1 \notin D$. Suppose towards a contradiction that $w_1, w_2 \notin B$. By Claim 5, we find $w_1, w_2 \notin S$, and so w_1, w_2 have degree at most 3. As $w_1 \notin D$, we find $w_1 \in A$.

Suppose first that w_1 and w_2 together have at most two neighbors outside of $\{w_1, w_2, s\}$. When we remove $N[s]$ from the graph, $|S|$ goes down by one and $|B \cup D|$ goes down by at most four ('counting' the two outside neighbors, w_2 and d), contradicting the minimality of our counterexample. So w_1, w_2 have at least three "outside" neighbors, which implies that w_2 exists and that w_1, w_2 are non-adjacent (see Figure 4). Moreover, w_1, w_2 together have

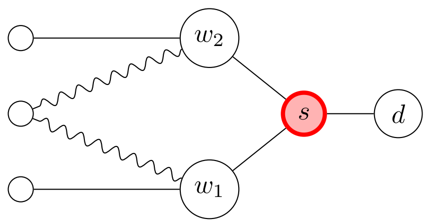

Figure 4 An illustration of the case when w_1, w_2 are not in B and together have three neighbors which are not s, w_1 or w_2. At least one of the wavy edges is present and at least one of w_1, w_2 has degree three in the picture. In particular, w_1 and w_2 are not adjacent.

at least three neighbors in B by using the same strategy (showing that deleting $N[s]$ would give a smaller counterexample). Since w_1 has degree at most 3 and is already adjacent to s, it has at most two neighbors in B. Thus w_2 is also not in D, since vertices in D have no neighbors in B.

Recalculating now that we know that $w_1, w_2 \notin B \cup D$, if $N(\{w_1, w_2\})$ contains at most three vertices of degree four in B, then $|B \cup D|$ goes down by at most four in $G \setminus N[s]$. This would be a contradiction as $(G, \setminus N[s], S \setminus \{s\})$ would be a smaller counterexample. Hence, w_1, w_2 have exactly four neighbors in B, all of which are of degree exactly 4.

We show that we may assume that d has degree 1. If d has another neighbor, it is in S by Claim 4. We delete the vertices s, w_1, w_2; since d is still in D and dominated, $|B \cup D|$ decreases by at most 4, and hence $(G \setminus \{s, w_1, w_2\}, S \setminus s)$ is a smaller counterexample. Hence we can assume d has degree one and therefore the edge ds is on the outer face.

Let G' be the graph obtained from G by adding the edge w_1w_2. Considering the local rotation of the three neighbors of s in an outerplanar embedding of G, we note that w_1, w_2 are consecutive neighbors of s. We can draw the edge w_1w_2 close to the path w_1-s-w_2 keeping the embedding outerplanar (note that the edge ds is still on the outer face). It follows that $w_1, w_2 \in B_S(G') \setminus B_S(G)$, so $|B_S(G) \cup D_S(G)| < |B_S(G') \cup D_S(G')|$. Thus, (G', S) is a smaller counterexample, a contradiction. ◁

▷ Claim 7. No two vertices in S have a common neighbor.

Proof. Assume towards a contradiction that there are two vertices s_1 and s_2 in S that have a common neighbor v. Since S is a stable set (Claim 5), we have $v \notin S$.

We will consider the outerplanar graph $G' = G/P$ obtained by contracting $P = \{s_1v, vs_2\}$ into a single vertex v_P. Let $S' = S \setminus \{s_1, s_2\} \cup \{v_P\}$. We use the abbreviations $B' = B_{S'}(G')$ and $D' = D_{S'}(G')$.

We will again do a case analysis, on the union of the neighbors of s_1 and the neighbors of s_2 in $D \setminus \{v\}$, to find a smaller counterexample. If $|B'| + |D'| \geq |B| + |D| - 4$, then (G', S') is a smaller counterexample. Note that vertices in $B \setminus B'$ have at least two neighbors in the set $\{s_1, v, s_2\}$.

- Suppose first that for some $i \in \{1, 2\}$, s_i is adjacent to at least two vertices in $D \setminus \{v\}$. Then v is the only other neighbor of s_i, so the graph G'' obtained from G by deleting s_i and its two neighbors in D, satisfies $|B(G'') \cup D(G'')| \geq |B \cup D| - 3$ whereas the set $S'' = S \setminus \{s_i\}$ is dominating. This gives a smaller counterexample.
- Suppose that both s_1, s_2 are adjacent to a single vertex in $D \setminus \{v\}$. Then both have degree at most 3. Let $d_1, d_2 \in D \setminus \{v\}$ be the neighbors of s_1, s_2 respectively (where d_1, d_2 might be equal). The graph G' is a smaller counterexample unless we lost two vertices from B besides possibly v, that is, $|B'| \leq |B \setminus \{v\}| - 2$. Any vertex lost from $B \setminus \{v\}$ must be adjacent to two vertices among $\{s_1, v, s_2\}$ (as otherwise its degree did not change), and since both s_1 and s_2 already have two named neighbors, G' is a counterexample unless there is, for each $i \in \{1, 2\}$, a common neighbor $b_i \in B$ of s_i and v, and all named vertices are distinct.

  Since $d_1 \in D$ and $b_1 \in B$, we find that b_1d_1 is not an edge of G. Since s_1 has three neighbors, b_1 and d_1 are consecutive neighbors and the edge b_1d_1 can be added without making the graph non-planar. Consider adding the edge b_1d_1 in G along the path $b_1s_1d_1$, such that there are no vertices in between the edge and the path. This may affect whether s_1 is on the outer face, but it does not affect whether s_2 is on the outer face. Therefore, after contracting this adjusted graph, the obtained graph G'' is still outerplanar. Moreover, b_1 has the same degree in G'' as in G, and so $|B_{S'}(G'') \cup D_{S'}(G'')| \geq |B| + |D| - 4$ and G'' is a smaller counterexample.
- Suppose that s_1 is adjacent to a vertex d_1 in $D \setminus \{v\}$ and s_2 is not (the symmetric case is analogous). There can be at most three vertices in $B \setminus \{v\}$ which are adjacent to two vertices in s_1, v, s_2 (as only one can be adjacent to s_1 and v, s_2 have at most two common neighbors since the graph is outerplanar). The only way in which G' is not a counterexample, is when there is a common neighbor b_1 of s_1 and v and two common neighbors b_2, b_3 of s_2 and v with all named vertices distinct. As before, we may now add the edge b_1d_1 in order to obtain a smaller counterexample G''.

- Finally, suppose that s_1 and s_2 have no neighbors in $D \setminus \{v\}$. By outerplanarity, there are at most four vertices with two neighbors among $\{s_1, v, s_2\}$. Hence G' is a counterexample unless there are exactly four (the only vertices "lost" from $B \cup D$ are either v or among such common neighbors, since s_1 and s_2 have no neighbors in $D \setminus \{v\}$). All four vertices are adjacent to v, because otherwise G contains a $K_{2,3}$-minor[2], a contradiction. In particular, G' is a counterexample unless there are two common neighbors of v and s_1 and two common neighbors of v and s_2 (and so $d(v) \geq 6$ and $v \in B$).
  Fix a clockwise order $w_1, w_2, \ldots, w_d$ on the neighbors of v such that the path $w_1 v w_d$ belongs to the boundary of the outer face. Let $i \neq j$ such that $w_i = s_1$ and $w_j = s_2$. After relabelling, we may assume $i < j$. Since s_1 and s_2 both have two common neighbors with v, we find $i > 1$, $j < d$ and $i + 1 < j - 1$. The vertices adjacent to multiple vertices in $\{s_1, v, s_2\}$ are $w_{i-1}, w_{i+1}, w_{j-1}$ and w_{j+1}. We create a new graph G'' by replacing v with two adjacent vertices v_1 and v_2, where v_1 is adjacent to $w_1, w_2, \ldots, w_{i+1}$ and v_2 to $w_{i+2}, \ldots, w_d$. This graph is outerplanar because both v_1 and v_2 have an edge incident to the outer face. Moreover, $d(v_1)$ and $d(v_2)$ are both at least 4, since they are adjacent to each other, to either s_1 or s_2 and to at least two vertices among $w_1, \ldots, w_d$. The set S is still a dominating set, but $|B(G'') \cup D(G'')| > |B \cup D|$ so this is a smaller counterexample.

In all cases, we found a smaller counterexample. This contradiction proves the claim. ◁

Since vertices in D only have neighbors in S, the claim implies in particular that each vertex of D has degree 1.

With the claims above in hand, we now analyze the structure of $H = H_G(S)$ as described in the notation section more closely. Note that the for each $u \in V(G) \setminus S$ the vertex $s(u)$ is uniquely defined by Claim 7.

Recall that H is outerplanar. It follows that there is a vertex $s_1 \in V(H)$ with at most 2 distinct neighbors in H.

We start with an easy observation.

▶ **Observation 8.** *Let $b \in B$ and $s(b)$ be its unique neighbor in S. Then there exists $w \in N(b) \setminus \{s(b)\}$, such that its unique neighbor $s(w) \in S$ is not equal to $s(b)$.*

Indeed, the vertex b can have at most two common neighbors with $s(b)$ (otherwise there would be a $K_{2,3}$, contradicting outerplanarity), and a vertex in B has degree at least 4 by definition.

Note that the vertex s_1 has at least one neighbor in H. Indeed, if s_1 has no neighbor in H, then $N[s_1]$ is a connected component in G. Since G is connected, $G = N[s_1]$. By Observation 8, we have $B = \emptyset$, so $|D \cup B| \leq 3$.

▷ Claim 9. The vertex s_1 has precisely two neighbors in H.

Proof. Assume towards a contradiction that s_1 has a single neighbor s_2 in H. Let $v_1, \ldots, v_k$ be the vertices in $N[s_1]$ that have a neighbor in $N[s_2]$, and conversely let $u_1, \ldots, u_\ell$ be the vertices in $N[s_2]$ that have a neighbor in $N[s_1]$. Note that by Claims 5 and 7, all of $\{v_1, \ldots, v_k, u_1, \ldots, u_\ell, s_1, s_2\}$ are pairwise distinct. If $\ell \geq 3$, then contracting the connected set $N[s_1]$ in G gives a $K_{2,3}$ on the contracted vertex and s_2 on one side and u_1, u_2, u_3 on the other. We derive that $\ell \leq 2$, and by symmetry, $k \leq 2$. By Observation 8, the only neighbors of s_1 that belong to B are in $\{v_1, v_2\}$. As we assumed that s_1 has degree 1 in H, we have $N[v_i] \subseteq N[s_1] \cup \{u_1, u_2\}$ for $i \in \{1, 2\}$. We will do a case distinction on $N[s_1] \cap D$.

[2] The vertices s_1, s_2 can have at most one further common neighbor v^* besides v. If v^* exists, we contract it with s_1 and s_2. We find a $K_{2,3}$ subgraph with v, v^* on one side and the three other common neighbors on the other side.

- If s_1 has no neighbor in D, we delete $N[s_1]$, and note that $D_S(G) = D_{S\setminus s_1}(G \setminus N[s_1])$, while $B_S(G) \setminus \{v_1, v_2, u_1, u_2\} \subseteq B_{S\setminus s_1}(G \setminus N[s_1])$. Therefore,

$$|S \setminus \{s_1\}| \geq \frac{1}{4} \cdot (|D_{S\setminus s_1}(G \setminus N[s_1])| + |B_{S\setminus s_1}(G \setminus N[s_1])|).$$

  So we have found a smaller counterexample.
- Suppose s_1 has two neighbors $d_1 \neq d_2$ in D. Then v_2 does not exist since $d(s_1) \leq 3$ and because v_1, v_2 are distinct from d_1, d_2 (vertices in D have degree 1). Suppose first that v_1 has degree at least 4. Let x be its neighbor distinct from u_1, u_2, s_1. By assumption on s_1, the vertex x has no neighbor in $S \setminus \{s_1\}$. Therefore, x is adjacent to s_1. However, x is distinct from d_1, d_2 and v_1, which contradicts $d(s_1) \leq 3$. This case is illustrated in Figure 5. Hence $v_1 \notin B$ and removing $N[s_1]$ now gives a smaller counterexample, a contradiction.

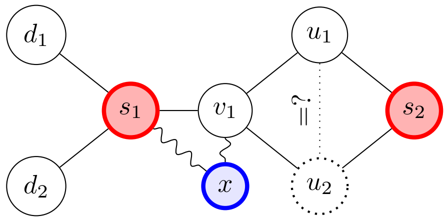

Figure 5 An illustration of the case where s_1 has degree one in H and two neighbors $d_1, d_2 \in D$ in G. If $v_1 \in B$, then some vertex x exists such that both wavy edges are present in G, a contradiction.

- Suppose that s_1 has a single neighbor d_1 in D. Removing $N[s_1]$ gives a smaller counterexample again, unless all of u_1, u_2, v_1, v_2 exist and belong to B. In particular, v_1, v_2 both have degree at least 4. Each of v_1 and v_2 can only have neighbors within $\{s_1, v_1, v_2, u_1, u_2\}$ because a neighbor x not within $\{s_1, v_1, v_2, u_1, u_2\}$ is a neighbor of s_1, but $d(s_1) \leq 3$. Therefore, both v_1 and v_2 are adjacent to u_1 and u_2. Together with s_1, this forms a $K_{2,3}$ subgraph (see Figure 6): a contradiction. $\triangleleft$

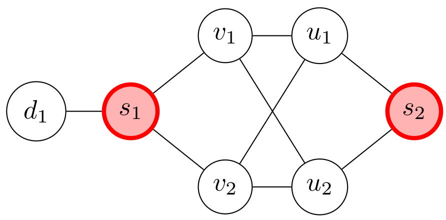

Figure 6 An illustration of the case where s_1 has degree one in H and has exactly one neighbor in D in G. We reduce to the case in which the depicted graph is a subgraph of G. We find a contradiction since the depicted graph contains a $K_{2,3}$.

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So s_1 has two neighbors in H. Let $s_2, s_3 \in V(H)$ be its neighbors. In G, let $w_1, \ldots, w_p$ be the vertices in $N[s_1]$ that have a neighbor in $N[s_3]$, and conversely let $x_1, \ldots, x_q$ be the vertices in $N[s_3]$ that have a neighbor in $N[s_1]$. By the same argument as before for s_1 and s_2, we obtain $p \leq 2$ and $q \leq 2$ and that all of $\{w_1, w_2, x_1, x_2, s_1, s_3\}$ are pairwise distinct. However, there may be a vertex in $\{w_1, w_2\} \cap \{v_1, v_2\}$; there may not be two such vertices since this would lead to a $K_{2,3}$-minor (with vertices $\{v_1, w_1\}$ and $\{v_2, w_2\}$ in one part, and $s_1, \{s_2, u_1, u_2\}, \{s_3, x_1, x_2\}$ in the other).

Our general approach is to delete $N[s_1]$ and add edges between $\{u_1, u_2\}$ and $\{x_1, x_2\}$ as appropriate so as to mitigate the impact on $|B \cup D|$. If this does not work, we obtain further structure on the graph which we exploit to create a different smaller counterexample. We will repeatedly apply the following Observation 10. Sometimes when deleting vertices and edges from the graph G, the result is a disconnected graph, so we can perform the "flipping" operation described below, and connect the different components to get a smaller counterexample (G', S').

▶ **Observation 10** (Flipping). *Let G be the disjoint union of two outerplanar graphs O_1 and O_2. Consider an outerplanar embedding of G, and let $(u_1, u_2, \ldots, u_q)$ denote the outer face of $G[O_2]$ in clockwise order. We can obtain a different outerplanar embedding of G by reversing the order of O_2 without modifying the embedding of O_1, so that the outer face of $G[O_2]$ is $(u_q, \ldots, u_2, u_1)$ in clockwise order.*

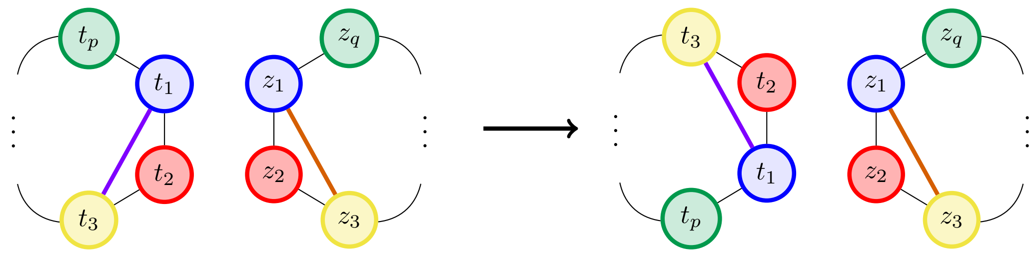

Figure 7 An illustration of Observation 10.

An example of the observation above is given in Figure 7. Beside Observations 8 and 10, the third useful observation is as follows.

▶ **Observation 11.** $N[v_1, v_2, w_1, w_2] \subseteq N[s_1] \cup \{u_1, u_2, x_1, x_2\}$*. Additionally, if* $\{v_1, v_2\} \cap \{w_1, w_2\} = \emptyset$*, then* $N[v_1, v_2] \subseteq N[s_1] \cup \{u_1, u_2\}$ *and* $N[w_1, w_2] \subseteq N[s_1] \cup \{x_1, x_2\}$*.*

This observation is argued similarly to Observation 8, we omit the argument.

Since s_1 is adjacent to s_2 and s_3 in H, all of u_1, x_1, v_1 and w_1 exist. We assume that either $\{v_1, v_2\} \cap \{w_1, w_2\} = \emptyset$ or $v_1 = w_1$. Note that $\{u_1, u_2\} \cap \{x_1, x_2\} = \emptyset$ since s_2 and s_3 do not have common neighbors by Claim 7. See Figure 8 for an illustration. For simplicity, when depicting which edges to add in which cases, we represent "u_2 does not exist" as "u_2 is possibly equal to u_1" (and variations). This means merely that if u_2 does not exist then the edges involving u_2 involve u_1 instead – multiple edges are ignored.

▷ Claim 12. One of w_2 and v_2 exists.

Proof. Suppose that neither w_2 nor v_2 exists. It is possible that $v_1 = w_1$, and that u_2 or x_2 do not exist. By Observation 11, if $v_1 \neq w_1$, then u_1, u_2 are not adjacent to w_1 and x_1, x_2 are not adjacent to v_1.

The degrees of x_1, x_2, u_1, u_2 in $G \setminus N[s_1]$ are at least one less than their degrees in G. Every vertex in $V(G) \setminus (N[s_1] \cup \{x_1, x_2, u_1, u_2\})$ has the same degree in G and in $G \setminus N[s_1]$. Let $S' = S \setminus \{s_1\}$, and note that S' dominates $G \setminus N[s_1]$.

- Suppose x_2, u_2 do not exist. If v_1 belongs to B, then it needs to have a neighbor which is not u_1, s_1 or one of x_1, w_1 (depending on whether $v_1 = w_1$), so it shares a neighbor with s_1 which is not in $B \cup D$. This implies that if $v_1 \in B$, then s_1 can have a neighbor

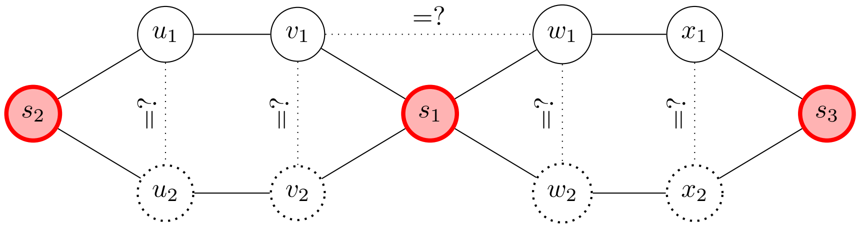

Figure 8 When s_1 has exactly two neighbors s_2, s_3 in H, each of s_2, s_3 has at most two neighbors with edges to vertices in $N[s_1]$. Moreover, s_2 and s_3 may have at most one common neighbor in $N[s_1]$. We draw vertices which may not exist in G as a dotted circle and connect vertices which may be equal with dotted edges. There may be more edges present in that are not drawn.

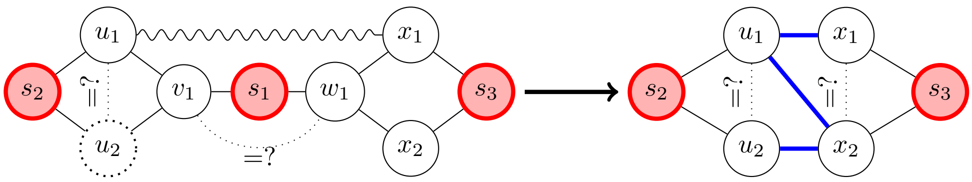

Figure 9 The case where w_2, v_2 do not exist. The original graph is drawn at the left and the modified graph is drawn at the right. The wavy line indicates there may be an edge between u_1 and x_1. Edges that may have been added are drawn in blue. Note that u_2 may not exist. There may be more edges which are not drawn (for instance v_1 might be adjacent to w_1) but these edges are not relevant to our argument.

$d_1 \in D$ or $w_1 \neq v_1$, but not both at the same time. It follows that regardless of whether $v_1 \in B$, we have $|N[s_1] \cap (B \cup D)| \leq 2$. But now $|(N[s_1] \cup \{x_1, u_1\}) \cap (B \cup D)| \leq 4$, so $(G \setminus N[s_1], S')$ is a smaller counterexample, a contradiction.

- By symmetry, we assume that x_2 exists. If u_1 and u_2 both exist, then they are not distinguishable at this point, which means we can swap their label. The same holds for x_1 and x_2. Hence we may assume that the vertices appear in the outer face in the order x_1, x_2, u_2, u_1, and that either u_1x_1 is an edge of G or there is no edge between $\{u_1, u_2\}$ and $\{x_1, x_2\}$. Let G' be the graph obtained from $G \setminus N[s_1]$ by adding the edges u_1x_1 (if it is not already present), u_1x_2 and (if u_2 exists) the edge u_2x_2 (see Figure 9). Note that G' is outerplanar and that S' dominates G'. Since G is outerplanar, if u_1x_1 is an edge in G, then neither u_1x_2 nor u_2x_2 is an edge in G. In G', the degrees of the vertices u_1, u_2, x_2 are at least as large as their respective degrees in G (the degree of x_1 might have dropped if the edge u_1x_1 was already present in G). Note that $|\{v_1, w_1, x_1\} \cup (N[s_1] \cap D)| \leq 4$, hence $|B_{S'}(G') \cup D_{S'}(G')| \geq |B \cup D| - 4$, a contradiction. ◁

▷ Claim 13. If $w_1 = v_1$, then v_2 and w_2 exist.

Proof. By Claim 12 we can assume v_2 exists. Suppose w_2 does not exist and $w_1 = v_1$. We remove $N[s_1]$ and add edges between $\{u_1, u_2\}$ and $\{x_1, x_2\}$ as above to ensure that for all but at most one of them, the degree does not decrease. To see an illustration of how the edges are added, see Figure 10. We suppose first that there are no edges between $\{u_1, u_2\}$ and $\{x_1, x_2\}$. The edges remedy the degree for x_1, x_2, since they only lost w_1, and for one of u_1, u_2; indeed, it is not possible that both u_1 and u_2 are adjacent to both v_1 and v_2 (since we would obtain a $K_{2,3}$ when considering s_1 as well).

By Observation 11, the degrees of other vertices are not affected by removing $N[s_1]$.

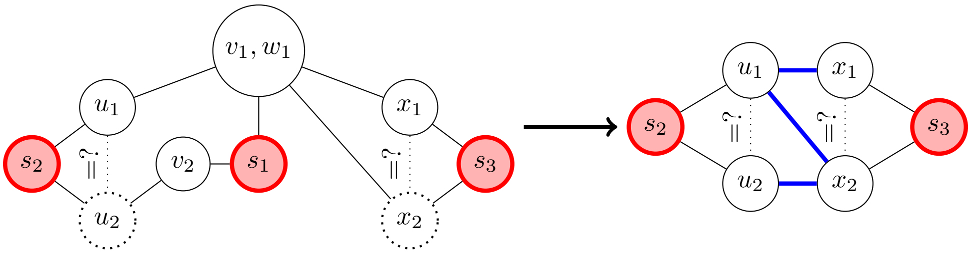

Figure 10 An example of the reduction for the case where v_1 is equal to w_1, v_2 exists and w_2 does not exist. We only draw edges which are relevant to our argument.

Again, since $|N[s_1] \cap (B \cup D)| \leq 3$, we have removed a vertex from S and at most 4 from $B \cup D$ so we have constructed a smaller counterexample.

We now assume u_1x_1 is an edge.

- Assume that u_2 does not exist. Now v_2 has degree at most 3 unless it has a common neighbor with s_1, but then s_1 has no neighbor in D and we lose only two vertices from $N[s_1]$ and possibly u_1, x_1.
- Assume now that u_2 exists. Both x_1 and x_2 lose at most one edge, and we can ensure both gain at least one edge. So if we lose only two vertices from $N[s_1]$, plus possibly u_1, u_2, then we lose at most four vertices from $B \cup D$ in total. If there are three vertices from $B \cup D$ in $N[s_1]$, then v_2 is adjacent to both u_1, u_2 and s_1 has a neighbor d in D. We know that v_1 is adjacent to u_1 or u_2, but since there is the path $v_1x_1u_1$, we know that if v_1 would be adjacent to u_2, then there would be a $K_{2,3}$-minor with v_1, v_2 in one part and $\{u_1, x_1\}, u_2, s_1$ in the other part. Since u_2 is not adjacent to v_1, we know that u_2 loses at most one edge when deleting $N[s_1]$ and gains an edge when we add the edge u_2x_2. This means we lose at most four vertices from $B \cup D$ (counting u_1, v_1, v_2 and d). ◁

We henceforth assume that $|\{v_1, v_2, w_1, w_2\}| \geq 3$. We can show that also in this case we can always reduce to a smaller counterexample.Since the details of the remaining casework are not particularly illuminating, we will omit them for brevity. Appendix A and the longer arXiv version [6] of our paper both contain the full details.

Since in all claims and cases we can show that there is a smaller counterexample, there can not be a counterexample to Lemma 2, which proves that Algorithm 1 computes a 5-approximation of Minimum Dominating Set for outerplanar graphs. ◀

3 Lower bound for outerplanar graphs

In this section we show that there is no deterministic local algorithm that finds a $(5-\epsilon)$-approximation of a minimum dominating set on outerplanar graphs using T rounds, for any $\epsilon > 0$ and $T \in \mathbb{N}$. To do so we use a result from Czygrinow, Hańćkowiak and Wawrzyniak [7, pp. 87–88] who gave a lower bound in the planar case. For $n \equiv 0 \mod 10$, they consider a graph G_n, which is a cycle $v_1, v_2, \ldots, v_n, v_1$ where edges between vertices of distance two are added. They showed that for every local distributed algorithm $\mathcal{A}$ and every $\delta > 0$ and $n_0 \in \mathbb{N}$ there exists $n \geq n_0$ for which the algorithm $\mathcal{A}$ outputs a dominating set for G_n that is not within a factor of $5 - \delta$ of the optimal dominating set for G_n. Their graph G_n is not outerplanar, but we can delete three of its edges to get an outerplanar graph G_n^-. The graph G_n^- is a path $v_1 \ldots v_n$ where all edges between vertices of distance two are added as in Figure 1. The argument of [7] builds on a lower bound for local algorithms computing a maximum independent set, which in turn depends on multiple applications of Ramsey's theorem. A similar approach is used by [11] to obtain the best-known lower-bound for planar graphs. Using the graph G_n^-, this approach can also be used to prove our result; the main idea is that since in the middle all the vertices "look the same", no local algorithm can do better than selecting almost all of them.

Alternatively, we can exploit the result of [7] as follows. For any bound $T \in \mathbb{N}$ on the number of rounds, any vertex in $M = \{v_{2T+1}, \ldots, v_{n-2T-1}\}$ has the same local neighborhood in G_n as in G_n^-. Since G_n is rotation symmetric, a potential local algorithm also finds a dominating set D for G_n (for $n \geq 4T + 2$), and with the result of [7] we obtain $|D| \geq (5 - \delta)\gamma(G_n)$. For n sufficiently large with respect to T, the set D is the same as the set D' that the algorithm would give for G_n up to at most $\delta n/10$. Since $n \equiv 0 \mod 10$, $\gamma(G_n) = \gamma(G_n^-) = \frac{n}{5}$ and we find the desired lower-bound $|D'| \geq \left(5 - \frac{\delta}{2}\right)\gamma(G_n^-) \geq (5-\epsilon)\gamma(G_n^-)$ for δ small enough.

4 Conclusion

Through a rather intricate analysis of the structure of outerplanar graphs, we were able to determine that a very naive algorithm gives a tight approximation for minimum dominating set in outerplanar graphs in $O(1)$ rounds. While there are some highly non-trivial obstacles to extending such work to planar graphs, we believe that similar techniques can be used to vastly improve the state of the art for triangle-free planar graphs and for C_4-free planar graphs. In the first case, recall that a 32-approximation is known [3], and there is a simple construction (a large 4-regular grid) showing that 5 is a lower bound. We believe that 5 is the right answer. In the second case, an 18-approximation is known [4], and there is no non-trivial lower bound. We refrain from conjecturing the right bound here – we simply point out that there is no reason yet to think 3 is out of reach. We believe that very similar techniques to the ones developed here can be used to obtain a 9-approximation, and possibly lower.

References

1 Saeed Akhoondian Amiri, Patrice Ossona de Mendez, Roman Rabinovich, and Sebastian Siebertz. Distributed domination on graph classes of bounded expansion. In *Proceedings of the 30th on Symposium on Parallelism in Algorithms and Architectures*, SPAA '18, pages 143–151, 2018.

2 Saeed Akhoondian Amiri, Stefan Schmid, and Sebastian Siebertz. A local constant factor mds approximation for bounded genus graphs. In *Proceedings of the 2016 ACM Symposium on Principles of Distributed Computing*, PODC '16, page 227–233, 2016.

3 Shrareh Alipour, Ehsan Futuhi, and Shayan Karimi. On distributed algorithms for minimum dominating set problem, from theory to application, 2021. `arXiv:2012.04883`.

4 Sharareh Alipour and Amir Jafari. A local constant approximation factor algorithm for minimum dominating set of certain planar graphs. In *Proceedings of the 32nd ACM Symposium on Parallelism in Algorithms and Architectures*, SPAA '20, pages 501–502, 2020.

5 Saeed Akhoondian Amiri, Stefan Schmid, and Sebastian Siebertz. Distributed dominating set approximations beyond planar graphs. *ACM Transactions on Algorithms (TALG)*, 15(3):1–18, 2019.

6 Marthe Bonamy, Linda Cook, Carla Groenland, and Alexandra Wesolek. A tight local algorithm for the minimum dominating set problem in outerplanar graphs, 2021. `arXiv:2108.02697`.

7 Andrzej Czygrinow, Michal Hańćkowiak, and Wojciech Wawrzyniak. Fast distributed approximations in planar graphs. In *International Symposium on Distributed Computing*, DISC '08, pages 78–92. Springer, 2008.

8 Laurent Feuilloley. Bibliography of distributed approximation beyond bounded degree, 2020. `arXiv:2001.08510`.

9 Michael R. Garey and David S. Johnson. *Computers and Intractability; A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., San Francisco, 1979.

10 Daniel Gonçalves. Edge partition of planar graphs into two outerplanar graphs. In *Proceedings of the thirty-seventh annual ACM Symposium on Theory of Computing*, STOC '05, pages 504–512, 2005.

11 Miikka Hilke, Christoph Lenzen, and Jukka Suomela. Brief announcement: local approximability of minimum dominating set on planar graphs. In *Proceedings of the 2014 ACM symposium on Principles of Distributed Computing*, PODC '14, pages 344–346, 2014.

12 Simeon Kublenz, Sebastian Siebertz, and Alexandre Vigny. Constant round distributed domination on graph classes with bounded expansion, 2021. `arXiv:2012.02701`.

13 Fabian Kuhn, Thomas Moscibroda, and Roger Wattenhofer. Local computation: Lower and upper bounds. *Journal of the ACM (JACM)*, 63(2):1–44, 2016.

14 Christoph Lenzen, Yvonne Anne Oswald, and Roger Wattenhofer. What can be approximated locally? case study: Dominating sets in planar graphs. In *Proceedings of the twentieth annual Symposium on Parallelism in Algorithms and Architectures*, SPAA '08, pages 46–54, 2008.

15 Crispin St. John Alvah. Nash-Williams. Decomposition of finite graphs into forests. *Journal of the London Mathematical Society (JLMS)*, s1-39(1):12–12, 1964.

16 Ran Raz and Shmuel Safra. A sub-constant error-probability low-degree test, and a sub-constant error-probability pcp characterization of np. In *Proceedings of the Twenty-Ninth Annual ACM Symposium on Theory of Computing*, STOC '97, page 475–484, 1997.

17 Jukka Suomela. Survey of local algorithms. *ACM Computing Surveys (CSUR)*, 45(2):1–40, 2013.

18 Wojciech Wawrzyniak. Brief announcement: a local approximation algorithm for mds problem in anonymous planar networks. In *Proceedings of the 2013 ACM Symposium on Principles of Distributed Computing*, PODC '13, pages 406–408, 2013.

19 Wojciech Wawrzyniak. A strengthened analysis of a local algorithm for the minimum dominating set problem in planar graphs. *Information Processing Letters (IPL)*, 114(3):94–98, 2014.

A Remaining casework

This appendix will provide the details of the case analysis from end of the proof of Lemma 2.

We are in the case in which $|\{v_1, v_2, w_1, w_2\}| \geq 3$. In particular, we may assume that s_1 has no neighbor in D.

▷ Claim 14. We have $v_1 \neq w_1$.

Proof. If not, then $v_1 = w_1$. By Claim 13, both v_2 and w_2 exist. Let G' be the outerplanar graph obtained from G by splitting the vertex v_1 into two vertices v_1' and w_1', both adjacent to s_1 and adjacent to each other, where v_1' is adjacent to $N[v_1] \cap N[s_2]$ and w_1' is adjacent to $N[v_1] \cap N[s_3]$. This gives three neighbors for both v_1' and w_1'. Since we can always add the edges $v_1'v_2$ and $w_1'w_2$ (which are chords of a cycle, using also that $N[s_1] \cap (N[s_2] \cup N[s_3]) = \emptyset$), we find $|B_S(G')| > |B_S(G)|$, whereas $D_S(G') = D_S(G)$, S is still dominating and G' is outerplanar. We find a contradiction with our assumption of the minimality of G. ◁

We henceforth assume that v_1, w_1 exist and are distinct and at least one of v_2, w_2 exists.

▷ Claim 15. The vertices v_2, w_2 exist.

Proof. By Claim 14 we can assume $v_1 \neq w_1$ and by Claim 12 we can assume v_2 exists. Suppose that w_2 does not exist. As in the previous case, $N[s_1]$ forms a vertex cut separating the component containing $N[s_2]$ from the component containing $N[s_3]$: if there was a path between $N[s_2]$ to $N[s_3]$ disjoint from $N[s_1]$, then we obtain a $K_{2,3}$-minor on vertex sets $N[s_2], \{s_1\}$ on one side and $\{v_1\}$, $\{v_2\}$, $N[s_3]$ on the other. When we delete $N[s_1]$, at most one of u_1, u_2 loses two neighbors, and the other (if it exists) loses only a single neighbor. The vertices x_1, x_2 can lose only a single neighbor. By Observation 10, after deleting $N[s_1]$ we can align the components of $N[s_2]$ and $N[s_3]$ in such a way that we can add the edges from $\{u_1x_1, u_2x_1, u_2x_2\}$. (For brevity, we handle the cases in which some of u_2, x_2 do not exist here as well, in which case we might add less edges.) We lost at most 4 vertices $B \cup D$, namely at most v_1, v_2, w_1 and one of the u_i (if one of them lost two neighbors). $\lhd$

We have one final case in which v_1, v_2, w_1, w_2 all exist and are all distinct. Note that, as in Claim 15 above, $N[s_1]$ forms a vertex cut separating $N[s_2]$ from $N[s_3]$. We break this problem into three subcases: The case where x_2, u_2 both exists, the case where exactly one of x_2 and u_2 exists and the case where neither x_2 nor u_2 exists.

A.1 The case where x_2 and u_2 both exist

Suppose that x_2 and u_2 both exist. Note that at most one of u_1, u_2 and one of x_1, x_2 is adjacent to two vertices in $N[s_1]$. Let us assume without loss of generality that x_1, u_2 have at most one neighbor in $N[s_1]$. By Observation 10, after deleting $N[s_1]$, we can re-embed the graph in a way that we can add the edges u_1x_1, u_1x_2, u_2x_2. This gives a smaller counterexample, since we have "fixed" the degrees of u_1, u_2, x_1, x_2 and only lost $N[s_1] \cap (B \cup D)$, which has size at most 4.

A.2 The case when exactly one of x_2, u_2 exists

Suppose now that only u_2 exists. (The case in which only x_2 exists is analogous.) Note that s_2 can have at most one neighbor in D. See also Figure 11.

- Suppose first that s_2 has a neighbor $d \in D$. We delete and add edges (if needed) and renumber such that u_1 is adjacent to v_1, u_2 to v_2 and u_1 to u_2, but no other edges among $\{u_1, u_2, v_1, v_2\}$ are present. Now we can add the chords u_1s_1, u_2s_1 to the cycle $s_1v_1u_1s_2u_2v_2s_1$. We delete s_2 and d. We have now lost at most four vertices from $B \cup D$: namely at most v_1, v_2, d and one of the u_i (if it was adjacent to v_1 and v_2 originally).

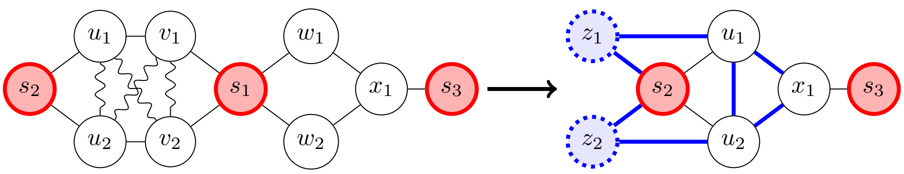

Figure 11 An illustration of the case v_1, v_2, w_1, w_2, u_2 all exist and are all distinct and x_2 does not exist. At top we show the case where s_2 has a neighbor $d \in D$. Some of the wavy edges may be present in G. As usual, there may be other edges present in G that have not been drawn, but they are not relevant to our argument. At the bottom we illustrate the case where s_2 has no neighbor in D. For $i \in \{1, 2\}$, we add the vertices z_i and edges z_is_2, z_iu_i if needed to make $\deg(u_i) \geq 4$ in G'.

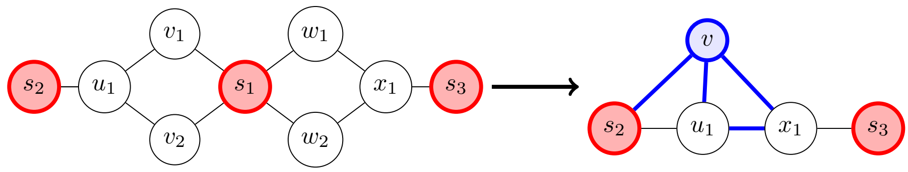

Figure 12 An illustration of the reduction for the case where u_2, w_2 do not exist and s_2 has no neighbors in D. We only draw edges that are relevant to our argument.

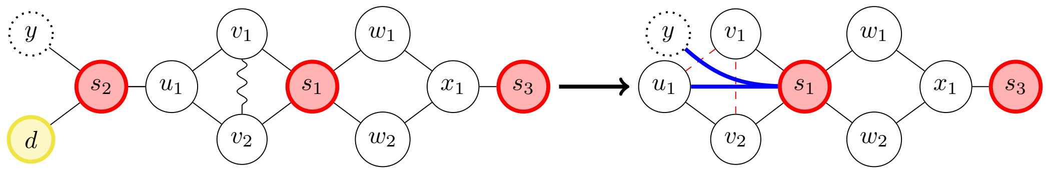

Figure 13 An illustration of the reduction for the case where u_2, w_2 do not exist and s_2 has exactly one neighbor $d \in D$. We use dashed red lines to emphasize that v_1 is not adjacent to u_1, u_2 in G' and a wavy line to show that v_1 may be adjacent to v_2 in G. As usual, we do not draw edges that are not relevant to our argument.

- Suppose now that s_2 has no neighbors in D. We remove $N[s_1]$ and add the edges u_1x_1,u_2x_1 and u_1u_2 if it is not already present. If u_1 has degree 3, then it has no neighbors outside of u_2, x_1, s_2 and so we may create a new vertex adjacent to both u_1 and s_2. Similarly, we can fix the degree of u_2 if needed. Note that since s_1 has degree at least four, $w_1, w_2 \notin D$. If we lost one of w_1, w_2 from B, then $x_1 \notin D$. In that case we lose at most v_1, v_2, w_1, w_2 from $B \cup D$.If $x_1 \in D$ we lose at most v_1, v_2, x_1 from $B \cup D$. In both cases we found a smaller counterexample.

A.3 The case when neither u_2 nor x_2 exists

If both u_1 and x_1 do not have degree exactly four, then we can remove $N[s_1]$ and add the edge u_1x_1; in this case we only lose a subset of $\{v_1, v_2, w_1, w_2\}$ from $B \cup D$. Hence we can assume by symmetry that u_1 has degree exactly four.

- We first handle the case in which s_2 has no neighbor in D. Since u_1 has degree exactly 4, after removing $N[s_1]$ we can create a new vertex v and add the edges u_1v, s_2v, x_1v, u_1x_1. As a result, we have lost at most v_1, v_2, w_1, w_2 from $B \cup D$ and found a smaller counterexample. See Figure 12.
- Suppose now that s_2 has only neighbors in $D \cup \{u_1\}$, which we name d_1, d_2 (where d_2 may or may not exist). We remove $N[s_2] \setminus \{u_1\}$ (at most three vertices), remove the edge v_1v_2 and add the edge u_1s_1. We again found a smaller counterexample as the only vertices we may have lost from $B \cup D$ are v_1, v_2, d_1, d_2.
- Suppose now that s_2 has exactly one neighbor $d \in D$. It may have another neighbor $y \neq u_1, d$, which if it exists, is not in D. We delete the vertices s_2, d as well as the edges u_1v_1 and v_1v_2 (if these exist). As u_1 was a cut-vertex previously, we can now add the edges u_1s_1 and ys_1 (say along the path $u_1v_2s_1$) to ensure that the size of the dominating set has dropped by one whereas we lost at most d, u_1, v_1, v_2 from $B \cup D$. See Figure 13.