Report

on the Dagstuhl Seminar

Discrete Tomography: Algorithms and Complexity

January 20 - 24, 1997

The workshop, organized by P. Gritzmann (Trier), and M. Nivat (Paris), was attended by 20 participants from 5 countries (7 nationalities). It was a workshop in the very sense of the word, without a fixed formal schedule, many of the talks were spontaneous and informal presentations at the black board, and the discussion of ideas and new approaches in discrete tomography was central.

The basic problem of discrete tomography is to reconstruct finite point sets that are accessable only through some of their discrete X-rays. In the simplest case, an X-ray of a finite set F in a direction u is a function giving the number of its points on each line parallel to u, effectively the projection, counted with multiplicity, of F on the subspace orthogonal to u.

The continuous analogue of this reconstruction problem is the classical task to invert X-ray or Radon-transformations, a problem of fundamental importance in computerized tomography. While the continuous problem is quite well understood (and the solution techniques are utilized in practise so prominently), the problem changes dramatically when turning to the discrete case. Questions of discrete tomography have long been studied in the context of image processing and data compression, and in the realm of data security; new motivation, however, comes from the need of practical reconstuction techniques in material sciences.

The talks discussed a broad range of aspects of discrete tomography. Some focussed on the real-world aspects of discrete tomography, others presented theoretical structural insight, partly with a view towards comparing discrete and continuous tomography. Some talks dealt with the computational complexity of various tasks relevant in this area while others focussed on algorithmic approaches using deterministic techniques from computer algebra or polyhedral combinatorics aiming at optimal solutions or randomized algorithms aiming at good approximations. Yet other presentations rounded off the area by giving insight in its connection to other problems and explaining related problems and results.

The workshop brought together scientists of various fields and with different scientific background. By way of exchanging ideas and problems, discussing possible approaches usually until late at night, presenting existing software and discussing encouraging new ideas and directions of possible further progress the workshop made a significant contribution to the solution of the important underlying real-world problems.

Elena Barcucci

X-rays characterizing some classes of digital sets

A *digital set* is a finite subset of the integer lattice $\mathbb{Z}^2$ defined up to a translation. An *X-ray* of a digital set in a direction u is a function giving the number of its points on each line parallel to u.

Reconstructing a digital set from its X-rays along a given set of directions is of primary importance in medical diagnostics (computer-aided tomography), pattern recognition, image processing and data compression. Several authors have been studying this theory and have proposed various algorithms for determining a digital set starting out from its X-rays in horizontal and vertical directions. One of the main difficulties involved in this reconstruction is the "ambiguity" deriving from the fact that, in some cases, many different digital sets have the same X-rays. In an effort to reduce this ambiguity and facilitate reconstruction, many authors suggest the following two methods:

- more than two X-rays are assigned,
- some of the properties of the digital set to be reconstructed are given "a priori" (for example: convexity, connection, symmetry) and the algorithms take advantage of this further information to reconstruct the set.

In this talk, we study the ambiguity problem with respect to some classes of digital sets on which some connection constraints are imposed. In particular, given a class $\mathcal{F}$ of digital sets, we want to know if a set U of directions exists such that among all of $\mathcal{F}$'s elements, each element in $\mathcal{F}$ is determined by its X-rays in U's directions. If the set U exists, we say that the class $\mathcal{F}$ is characterized by U. By extending the concept of switching component introduced by Chang and Ryser [1, 4], we prove that there are some classes of digital sets that cannot be characterized by any set of directions. One of these classes is the set of column-convex polyominoes (i.e., digital sets which are convex with respect to the vertical direction). Gardner and Gritzmann [3] studied the problem on the class of convex sets (i.e., sets which are convex with respect to all the directions). The authors show that if U is a set of four directions having cross ratio $\rho(U) \notin \{4/3, 3/2, 2, 3, 4\}$, then the class of convex sets is characterized by U. We prove that if the cross ratio $\rho(U) \in \{4/3, 3/2, 2, 3, 4\}$, then the convex sets cannot be characterized by U. We then try to find out whether these results can be extended to the class of convex polyominoes (i.e., sets which are only convex with respect to the horizontal and vertical directions). We prove that if the horizontal and vertical directions do not belong to U, Gardner and Gritzmann's result cannot be extended to convex polyominoes. If $U = \{(1,0), (0,1), u_3, u_4\}$, where $(1,0)$ and $(0,1)$ are the horizontal and vertical directions and the cross ratio $\rho(U) \notin \{4/3, 3/2, 2, 3, 4\}$, we believe that U can characterize the class of convex polyominoes. We wish to point out that, as shown in [2], there is an exponential number of convex polyominoes having the same horizontal and vertical X-rays. In order to give experimental evidence of this conjecture, we use an algorithm that reconstructs convex polyominoes from their discrete X-rays. Moreover, we prove that no number δ exists such that if $|U| \geq \delta$, then U characterizes the convex polyominoes. This number exists for convex sets and is equal to 7 (see [3]).

References

[1] S. K. Chang, The reconstruction of binary patterns from their projections *Comm. ACM*, 14 (1971) 21-25.

[2] A. Del Lungo, M. Nivat and R. Pinzani, The number of convex polyominoes reconstructable from their orthogonal projections, to appear in *Disc. Math.*

[3] R. J. Gardner and P. Gritzmann, Discrete tomography: determination of finite sets by X-rays, to appear *Trans. Amer. Math. Soc.*.

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Yves Caseau

Reconstruction of Polyominoes: A Contribution from a Constraint-Based Scheduling Perspective

We propose two approaches towards reconstructing a convex polyomino (i.e., h-convex and v-convex) from its two projections (horizontal and vertical). The first method uses constraint propagation techniques from employee scheduling to solve the exact problem. Using a straightforward model where a polyomino is seen as a set of bands representing different schedules, we obtain a simple-yet-efficient algorithm which can solve problems of sizes up to 100 in a few seconds. The second approach deals with the approximate problem, where the two projections are given as vectors of intervals (min-value-max). The question is to find a polyomino whose projections fall within the bounds and such that the sum of the absolute differences with the target value is minimal. We show that the previous naive model is not sufficient and we introduce a few techniques drawn from a specialized algorithm for the exact polyomino reconstruction. We show that although finding an approximate polyomino is easy, finding the best approximation is a hard combinatorial problem for which only problems of sizes up to 20 can be solved.

R.J. Gardner

Uniqueness issues in discrete tomography

In 1980, Peter McMullen and I proved that there is a set S of four directions in the plane that has the property that every planar convex body can be distinguished from any other by its X-rays in the directions in S. An X-ray gives the lengths of all chords of the body parallel to a particular direction. Larry Shepp asked whether a similar result holds for discrete X-rays of convex lattice sets when these are taken in lattice directions. A *convex lattice set* in $\mathcal{Z}^n$ is a finite subset of $\mathcal{Z}^n$ that is equal to the intersection of its convex hull with $\mathcal{Z}^n$. A *lattice direction* in $\mathcal{E}^n$ is a direction parallel to a line joining two different points of $\mathcal{Z}^n$. The *discrete X-ray* of a finite subset F of $\mathcal{E}^n$ in the direction u is the function on the hyperplane $u^\perp$ through the origin perpendicular to u giving the number of points in F lying on each line parallel parallel to u. Effectively, this is the projection (counted with multiplicity) of F on $u^\perp$.

In 1995, Peter Gritzmann and I obtained a positive answer to this question. We showed that a set S of lattice directions has this uniqueness property if the cross ratio of the slopes of some four directions in S, arranged in increasing order, is not $4/3$, $3/2$, 2, 3, or 4. The set $S = \{(1,0),(0,1),(2,1),(-1,2)\}$ is a specific example that has the uniqueness property. Moreover, any set of seven or more lattice directions has the uniqueness property. On the other hand, there is a set of six lattice directions that does not have the uniqueness property, and no set of three lattice directions does.

To establish the cross ratio condition for uniqueness, one has to find all solutions of a certain equation in which an expression involving four complex roots of unity is equal to a rational number. It turns out that the appropriate tool is p-adic analysis, in particular, p-adic valuations.

Yan Gerard

Properties of the Minkowski sum in discrete geometry

One can investigate the properties of the Minkowski sum in the framework of discrete geometry. We notice that connectivity and convexity are preserved by Minkowski addition. The convexity in the horizontal or (and) the vertical direction of the digital plane (Z^2), associated with 4-connectivity is also preserved by this sum. Minkowski addition commutes with the operator providing the convex hulls of sets. Moreover we have a geometric construction of the Minkowski sum of two discrete polygons (of Z^2).

Peter Gritzmann

On the computational complexity of reconstructing lattice sets from their X-rays

(joint work with R.J. Gardner and D. Prangenberg)

The talk discusses various inverse problems in discrete tomography. These questions are motivated by demands from material sciences for the reconstruction of crystalline structures from images produced by quantitative high resolution transmission electron microscopy.

In particular, we completely settle the complexity status of the basic problems of existence (data consistency), uniqueness (determination), and reconstruction of finite subsets of the d-dimensional integer lattice $\mathbb{Z}^d$ that are only accessible via their line sums (1-dimensional X-rays) in some prescribed finite set of lattice directions. Roughly speaking, it turns out that for all $d \geq 2$ and for a prescribed but arbitrary set of $m \geq 2$ pairwise noncollinear lattice directions, the problems are solvable in polynomial time if $m = 2$ and are $\mathcal{NP}$-complete (or $\mathcal{NP}$-equivalent) otherwise.

Paolo Gronchi

Reconstruction of finite convex sets

(joint paper with Maurizio Saroldi) Suppose we have a discrete convex set K contained in a given square $[0, a] \times [0, a]$ and we can check whether a considered straight line intersects K or not. The question is how many lines we have to consider in order to reconstruct the discrete set in an interactive way.

We prove that an upper bound is given by $4a + 2V_K$ where V_K is the number of vertices of the convex hull of K in $\mathbb{R}^2$. We find sharp bounds for V_a, the maximum of V_K among all sets contained in the given square. Furthermore we show that

$$\lim_{a \to \infty} \frac{V_a}{a^{\frac{2}{3}}} = 3 \left(\frac{4}{\pi} \right)^{\frac{2}{3}} .$$

Thomas Kasper

Reconstructing Polyominoes with Constraint Programming

(joint work with Alexander Bockmayr and Tomasz Zajac)
Constraint Programming is a new and promising methodology for tackling complex combinatorial problems. We propose to use constraint programming as a framework to model and solve discrete tomography problems. We exemplify the use of this framework by considering the polyomino reconstruction problem and the polyomino approximation problem. The goal of the reconstruction problem is the reconstruction of a binary pattern that is connected and convex in rows and columns from its vertical and horizontal projections. If the reconstruction fails, we turn to the approximation problem, where we compute an approximation pattern that is also connected and convex, but may have different projections. In this case the objective is to minimize the difference with respect to the projection numbers given in the original formulation. Using the extended modelling capabilities of constraint programming, like numeric and symbolic constraints, and the possibility to easy modify and extend the constraint model in an incremental way, we present models for both problems. For the reconstruction problem we give a pseudo-Boolean model and a finite domain model. For the approximation problem we use only a pseudo-Boolean model. Preliminary computational

tests indicate good results for the reconstruction problem. On the other hand, the results for the approximation problem leave much to be desired and call for an improved model. Further future work is to check whether constraint programming can be successfully applied to other discrete tomography problems.

Attila Kuba

Reconstruction of 3D Binary Matrices

The problem of reconstruction of binary matrices from their projections (row sums) is considered. This kind of reconstruction theory has been applied in combinatorics, graph theory, operations research, genetics, medicine, electron microscopy and image processing. A survey is given to show the differences between the 2D and 3D cases from the viewpoints of uniqueness, partial uniqueness, and reconstruction algorithm. It is proved that the existence of a mixed submatrix (containing only mixed rows, i.e. having both, 0 and 1 elements) is necessary and sufficient in 2D, however it is only necessary but not sufficient in 3D case for the non-uniqueness of the binary matrix. It is shown that the additivity property is suitable to decide the uniqueness if the projectional matrix is totally unimodular (as in the 2D case). Since the projectional matrix in 3D case is not totally unimodular, this theorem can not be applied here. A method is presented to find so-called invariant boxes (i.e. positions of invariant-1's and invariant-0's) in a 3D binary matrix. Finally, it is mentioned that the core-envelope algorithm can be used in the reconstruction of 3-directionally convex 3D binary matrices.

Alfred K. Louis

Discrete Versus Indiscrete Tomography

In this lecture we present results from conventional continuous tomography such as consistency conditions, derivation of inversion formulas and algorithms. In the continuous case the Radon transform of a density distribution f is defined for a direction ω and the distance s from the origin of the x–ray path as

$$Rf(\omega, s) = \int_{\mathbb{R}^2} f(x)\delta(s - x^\top \omega)dx$$

By a simple computation we find

$$\int_{\mathbb{R}} Rf(\omega, s)\psi(x)ds = \int_{\mathbb{R}^2} f(x)\psi(x^\top \omega)dx \ . \tag{1}$$

This relation serves as basis for deriving consistency conditions and inversion formulas. In the discrete case the distribution f is assumed to be a sum of point masses at integer points $k \in \mathbb{N}_0^2$ with weights 0 or 1:

$$f = \sum_k f_k \delta_k$$

where δ_k denotes the delta distribution at position $k \in \mathbb{N}_0^2$. The discrete x–ray transform is then defined as

$$Df(\omega, \ell) = \sum_k f_k \bar{\delta}(\ell - \omega^\top k)$$

where $\bar{\delta}$ is the Kronecker symbol with $\bar{\delta}(0) = 1$ and $\bar{\delta}(\ell) = 0$ for $\ell \neq 0$. The corresponding result to (1) for the discrete case and for fixed direction ω is

$$\sum_\ell Df(\omega, \ell)\psi(\ell) = \sum_k f_k \psi(\omega^\top k) \ . \tag{2}$$

Especially for $\psi = 1$ we see that the ray sum $\sum_\ell Df(\omega, \ell)$ is independent of the direction ω. The choice $\psi(\ell) = \exp(-2\pi\imath\ell n/q)$ leads to the discrete Fourier transforms of the data and the searched for distribution f.

In the lecture we further present a general tool to derive fast inversion formulas, the approximate inverse, by precomputing reconstruction kernels. The influence of the unavoidable data errors is demonstrated with reconstructions from real data. Finally a movie with reconstructions from real data in 3D X-ray computer tomography, measured at the Fraunhofer Institute for nondestructive testing in Saarbrücken, is shown.

Alberto Del Lungo

Reconstructing convex polyominoes from horizontal and vertical projections II

A *cell* is a unitary square $[i, i+1] \times [j, j+1]$ in which $i, j \in \mathbb{N}_0$. Let S be a finite set of cells. A *column* (*row*) of S is the intersection of S with an infinite vertical strip $[i, i+1] \times \mathbb{R}$ (horizontal $\mathbb{R} \times [i, i+1]$) in which $i \in \mathbb{N}_0$. The ith *row projection* and the j-th *column projection* of S are the number of cells in S's ith row and j-th column, respectively. We dealt with the reconstruction of objects from their projections: with regard to establishing the existence of a set S of cells in which the ith row projection and the j-th column projection are equal to h_i and v_j, respectively, and $H = (h_1, h_2, \dots, h_m) \in \mathbb{N}^m$ and $V = (v_1, v_2, \dots, v_n) \in \mathbb{N}^n$ are two assigned vectors. This problem is of primary importance in medical diagnostics (computer-aided tomography), pattern recognition, image processing and data compression and has been studied by various authors. In the paper [1], we studied the problem with respect to some classes of cell sets on which we imposed some connectivity constraints and devised an algorithm for convex polyomino reconstruction. This algorithm establishes the existence of a convex polyomino Λ having projections equal to (H, V). Moreover, if there is at least one convex polyomino having projection (H, V), the algorithm reconstructs one of them in a maximum of $O(n^4 m^4)$ time.

In this talk, we deduce some operations (called *partial sum operations*) for the reconstruction of convex polyominoes, from some properties of H's and V's partial sums. We use these operations to define a new algorithm in which it is not necessary to find feet's positions, whereas the "old" algorithm has to examine all of them (i.e. $O(n^2 m^2)$ positions). Since the computational cost of a partial sum operations is $O(nm)$, the new algorithm's complexity is less than $O(n^2 m^2)$ and is therefore smaller than that of the previous algorithm. At the moment, however, we only have experimental evidence to support the fact that our algorithm establishes the existence of a convex polyomino Λ whose projections are equal to (H, V), for all instances (H, V).

We wish to point out that Woeginger [2] proved that the reconstruction problem in the classes of horizontally and vertically convex sets (**h**,**v**) and polyominoes (**p**) is an NP-complete problem. In [1], we showed that the reconstruction is NP-complete in the classes (**p**,**h**), (**p**,**v**), (**h**) and (**v**). Therefore, the problem can be solved in polynomial time only if all three properties **p**, **h** and **v**, are shared by the cell set. This, in turn, means that the set is a convex polyomino.

References

[1] E. Barcucci, A. Del Lungo, M. Nivat and R. Pinzani, Reconstructing convex polyominoes from their horizontal and vertical projections, *Theor. Comp. Sci.* 155 (1996) 321-347.

[2] G. J. Woeginger, The reconstruction of polyominoes from their orthogonal projections, (1996) Preprint.

Maurice Nivat

Chords of a convex subset of $\mathbb{Z}^2$

The set of chords is the Minkowski sum $P + (-P)$ of P and its symmetric. The problem we attempt to solve is the following: given a centrally symetric subset of $\mathcal{Z}^2$, say Q, such that $Q = -Q$, construct P if it exists such that $Q = C(P)$ the set of chords of P. We use crucially the convex hull $\hat{P}$ of P and the property that $C(\hat{P}) = \hat{C(P)}$, building first the convex polygons such that $C(\hat{P}) = \hat{Q}$. The construction is well known and leads to a Knapsack problem with a pseudo polynomial algorithm (polynomial in the size of $\hat{Q}$, NP-complete in the minimal representation of $\hat{Q}$ as the sequence of its oriented edges) with unfortunately a possibly exponential number of solutions. To obtain the P's such that $Q = C(P)$ if any we have to "carve" the polygons $\hat{P}$, but there are too many.

Thus we show how to adapt the standard algorithm in order to check at each step that the $\hat{P}$ being built is compatible with Q: as a result we have a pseudo-polynomial algorithm producing P's such that all chords issuing from a vertex of its convex hull are in Q and, of course, $C(\hat{P}) = \hat{Q}$. There remains the problem of small chords, linking interior points of P, which may be elements of Q: it remains to prove that one can always delete points in P to remove them.

Renzo Pinzani

Reconstructing convex polyominoes from horizontal and vertical projections

A *cell* is an unitary square $[i, i+1] \times [j, j+1]$, in which $i, j \in \mathbb{N}_0$. Let S be a finite set of cells. The ith *row projection* and the jth *column projection* of S are the number of cells in the ith row and in the jth column of S, respectively. In this talk, we examine an aspect of the reconstruction of objects from their projections: that of establishing the existence of a set of cells S, in which the ith row projection and the jth column projection are equal to h_i and v_j, respectively, and $H = (h_1, h_2, \ldots, h_m) \in \mathbb{N}^m$ and $V = (v_1, v_2, \ldots, v_n) \in \mathbb{N}^n$ are two assigned vectors. Determining the existence of a set S having assigned projections (H, V) means establishing the existence of solutions to the following system in $n + m$ equations and in $n \times m$ binary variables $s_{i,j}$:

$$\sum_{j=1}^{n} s_{i,j} = h_i, \;\; 1 \le i \le m,$$

$$\sum_{i=1}^{m} s_{i,j} = v_j, \;\; 1 \le j \le n.$$

A set S of cells can be represented by a matrix of 0 and 1, that is, a binary pattern. First Ryser [5], and subsequently Chang [2] and Wang [6] studied the problem of proving the existence of solutions to this system and therefore of binary patterns S having assigned projections (H, V) and they showed that this decision problem can be solved in $O(nm)$ time. These authors also developed some algorithms that reconstruct S starting out from (H, V). The main problem met with in the reconstruction is the "ambiguity" involved because, in some cases, a great many sets have the same projections (H, V). For example, if $H = (h_1, h_2, \ldots, h_m) = (1, 1, \ldots, 1)$ and $V = (v_1, v_2, \ldots, v_n) = (1, 1, \ldots, 1)$, there are $n!$ different sets having these projections. The ambiguity is reduced if some of the properties of the set to be reconstructed are given "a priori" (for example: convexity, connection, symmetries). In this case, the algorithms take advantage of this further information to reconstruct the set. As far as this method is concerned, some properties imposed on the sets entirely eliminate all ambiguity (see [3]), while other properties only partially reduce it. It is shown in [4] that there is an exponential number of convex sets having the same projections. We also noted that, in some cases, ambiguity reduction does not facilitate the set's reconstruction. For instance, the problem of reconstructing a discrete set from its horizontal and vertical projections (H, V) is NP-complete for some classes of discrete sets on which some connection constraints are imposed (see [1, 7]).

In this talk, we consider the class of convex polyominoes. A *polyomino* is a connected finite set of adjacent cells lying two by two along a side and it is defined up to a translation. A polyomino is *convex* if all its columns and rows are connected. We define an algorithm that establishes the existence of a convex polyomino having horizontal and vertical projections equal to (H, V) in polynomial time. Moreover, if there is at least one convex polyomino having projection (H, V), the algorithm reconstructs one of them in a maximum of $O(n^4m^4)$ time.

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Dieter Prangenberg

Discrete Tomography: On the algorithmic complexity

(Joint work with R. Gardner and P. Gritzmann)

We focus on the algorithmic complexity of recovering discrete lattice sets from their discrete X-rays. This problem is of fundamental importance in many practical applications. In particular, improvements in the generation and interpretation of high resolution transmission microscopy images lead to the problem of recovering crystalline structures from the knowledge of X-ray information in certain directions.

While the complexity is completely settled for one-dimensional X-rays (line sums) the extension to higher dimensional X-rays (plane sums) is of interest. We are going to treat these problems from an algorithmic point of view.

We also consider the poly-atomic case, that is the problem of recovering crystalline structures composed of several types of atoms. This problem is related to the problem of reconstructing interfacial structures in material sciences. It is shown that reconstructing sets of several types of atoms is already $\mathcal{NP}$-complete for six types of atoms and two X-rays.

Jean - Pierre Reveilles

The MAT operator for edge detection
(abstract not available)

Peter Schwander

Application of Discrete Tomography to Electron Microscopy of Crystals
Recent progress in HRTEM (High-Resolution Transmission Electron Microscopy), allows us to probe microscopic properties of small crystals at the atomic level. This is important for the development of processing semiconductor materials for microelectronics.

In HRTEM an application of discrete tomography arises as follows: a parallel beam of electrons is directed at a small piece of a 3D crystal. After passage through the crystal and a high magnification lens system, the electrons form a 2D image. The microscope resolution is sufficient that individual atom-columns can be resolved, at least for some directions. A technique, named QUANTITEM, deduces a signal from the image that is directly proportional to the number of atoms contained in each atom-column. Therefore, for a small crystal, the measured values must be approximately integral multiples of a fixed quantity. Thus, line sums, each corresponding to the number of atoms contained in a single atom-column, can be obtained from the image.

For physics and materials science it is of great interest to reconstruct crystals consisting of about 10^6 atoms from the measured line sums. This reconstruction problem of discrete tomography brings up mathematical issues of practical relevance, such as uniqueness, computational complexity and algorithms. The mathematics is considerably different from continuous tomography so that the well known inversion techniques cannot be applied. The problem is finite but NP-complete when more than two projection directions are used. Therefore, enumeration techniques, such as Simulated Annealing, are of little practical use for the crystal sizes of interest. Recently LP methods on fuzzy sets have been considered and first results obtained from test sets were promising.

In the future, practical constraints of the measurement technique, such as number and type of projection directions, experimental noise and incomplete data due to limited field of view, must also be taken into account.

This presentation attempts to outline the unsolved aspects of the yet still young field of discrete tomography, rather than to give a closed mathematical description.

Sven de Vries

Polyhedral Methods for Discrete Tomography

(joint work with P. Gritzmann)

We focus on the algorithmic problem of recovering discrete lattice sets from their discrete X-rays. This problem is of fundamental importance in many practical applications. In particular, improvements in the interpretation and generation of high resolution transmission microscopy images lead to the problem of recovering crystalline structures with the knowledge of X-ray information for certain directions.

While the problem is known to be NP-complete if more than 2 X-rays are given, the algorithmic question, how to solve it in reasonable time, remains to be studied.

We introduce two linear programs to describe reconstruction-problems in discrete tomography:

$$\begin{aligned} Ax &= b, \\ x &\text{ binary;} \end{aligned}$$

and the (equivalent) maximization problem on its submissive:

$$\begin{aligned} \max \sum & x_i, \\ s.t. \quad & Ax \leq b, \\ & x \text{ binary.} \end{aligned}$$

The feasible region P of the second formulation turns out to be fulldimensional, while the decision problem, whether the first polytope is 0- or larger-dimensional given a single solution, is already NP-complete. We describe various families of facets of the *tomography polytopes P.*

To reduce the problem-size we propose an interior-point based strategy to fix variables.

For approximation results we use matroid based methods and a rounding strategy.

Markus Wiegelmann

Discrete Tomography: Switching Components and Primal Heuristics

Switching components are the combinatorial structures explaining the ambiguities in inversion problems that arise in Geometric Tomography. Using a natural formulation of the basic algorithmic problems in Discrete Tomography as $\{0,1\}$-integer programs we interpret switching components as moves in the $\{0,1\}$-fibers attached to the projection matrix. The Gröbner basis approach to integer programming allows to study the structural and algorithmic properties of these moves in a precise way. In particular, it provides systematic improvements of backprojection-like primal algorithms for NP-hard reconstruction problems. Implementations of these algorithms yield very high performances even for large reconstruction problems with up to 10^6 points.

Gerhard J. Woeginger

The Reconstruction of Polyominoes from their Orthogonal Projections

The reconstruction of discrete two-dimensional pictures from their projections is one of the central problems in the areas of medical diagnostics, computer-aided tomography, pattern recognition, image processing and data compression. In this talk, we show that it is NP-complete to reconstruct a two-dimensional connected pattern from its two orthogonal projections H and V.

Neal Young

Randomized Rounding for Discrete Tomography

Raghavan and Thompson's method of randomized rounding converts fractional solutions of linear programs to approximate, integer solutions. Discrete tomography is naturally described as a linear program whose integer solutions correspond to valid reconstructions of the hidden data from the given line sums. Randomized rounding yields an approximate solution in that each constraint with desired line sum b has line sum $b \pm O(\sqrt{b \log n})$ where n is the number of line sums.

Variants of the method yield deterministic algorithms and algorithms that obtain the same performance guarantee without solving the linear program first (the latter algorithms are somewhat more efficient). Also, variants can guarantee to meet the line sums in one direction (for a 2D problem) or two directions (for a 3D problem) exactly, while maintaining the same performance guarantees for the other line sums.

Open problems

During the workshop and particularly in an extended problem session various open problems were posed. The following is a list of those problems that we handed in in writing – not containing problems which were only presented orally.

Richard J. Gardner

Problem 1: Let $n \geq 3$. Is there a finite set S of lattice directions in general position in $\mathcal{E}^n$ with the property that every convex lattice set in $\mathcal{Z}^n$ can be distinguished from any other by its discrete X-rays in the directions in S?

The corresponding problem for $n = 2$ is completely solved by the results of R. J. Gardner and P. Gritzmann, Discrete tomography: Determination of finite sets by X-rays, Trans. Amer. Math. Soc., to appear, and E. Barcucci, A. Del Lungo, M. Nivat, and R. Pinzani, X-rays characterizing some classes of digital pictures, preprint. When $n = 3$, the set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, for example, does not have this property. To see this, color the vertices of the standard unit cube alternately black and white. Then the set of four black points and the set of four white points are different convex lattice sets with the same discrete X-rays in the directions in S. It is easy to see that this example is not optimal, however. The question corresponding to Problem 1 for continuous X-rays of convex bodies is also open and is posed in R. J. Gardner, Geometric Tomography, Cambridge University Press, New York, 1995, Problem 2.1 and R. J. Gardner, Geometric tomography, Notices Amer. Math. Soc., 42, 1995, 422–429. For the continuous version, there is an example of a set S with $|S| = 6$ that does not have the property, based on the polyhedron $(4, 6, 10)$; see R. J. Gardner, Geometric Tomography, Cambridge University Press, New York, 1995, Theorem 2.2.2 and Figure 2.1. I conjecture that there is an affirmative answer to Problem 1 for sets S with $|S| \geq 7$.

Problem 2: Is there a finite set S of lattice directions in $\mathcal{E}^2$, with the property that every convex lattice set in $\mathcal{Z}^2$ can be distinguished from any (arbitrary) lattice set in $\mathcal{Z}^2$ by its discrete X-rays in the directions in S?

It follows from R. J. Gardner and P. Gritzmann, Discrete tomography: Determination of finite sets by X-rays, Trans. Amer. Math. Soc., to appear that only sets S with $|S| \geq 4$ can have this property. Suppose that S is any set of lattice directions such that every convex lattice set in $\mathcal{Z}^2$ can be distinguished from any other by its discrete X-rays in the directions in S. Then it is possible that S also has the property required in Problem 2. In particular, it is not known whether the set $S = \{(1, 0), (0, 1), (2, 1), (-1, 2)\}$ has the required property, or whether any set S with $|S| \geq 7$ does. Again, the corresponding continuous version of Problem 2 is also open and is posed in R. J. Gardner, Geometric Tomography, Cambridge University Press, New York, 1995, Problem 2.8.

Let $\mathcal{F}^d$ denote the class of all finite subsets of $\mathbb{Z}^d$, let $\mathcal{G} \subset \mathcal{F}^d$ and let $\mathcal{L}_{1,d}$ denote the family of 1-dimensional subspaces of $\mathbb{E}^d$ that are spanned by a lattice vector $v \in \mathbb{Z}^d \setminus \{0\}$.

For given $S_1, \ldots, S_m \in \mathcal{L}_{1,d}$, $\textsc{Consistency}_{\mathcal{G}}(S_1, \ldots, S_m)$ asks whether given functions X_i for S_i, $i = 1, \ldots, m$, encode the X-rays of some set $F \in \mathcal{G}$, while $\textsc{Uniqueness}_{\mathcal{G}}(S_1, \ldots, S_m)$ asks whether there is a solution different from a given one. The precise definitions and relevant data structures of these problems are contained in [2], as are the complexity results: For $d \geq 2$ and $m \geq 3$ different lines $S_1, \ldots, S_m$ in $\mathcal{L}_{1,d}$, $\textsc{Consistency}_{\mathcal{F}^d}(S_1, \ldots, S_m)$ and $\textsc{Uniqueness}_{\mathcal{F}^d}(S_1, \ldots, S_m)$ are $\mathcal{NP}$-complete in the strong sense. The counting version $\#\big(\textsc{Consistency}_{\mathcal{F}^d}(S_1, \ldots, S_m)\big)$ that asks for the number of solutions is $\#\mathcal{P}$-complete for $m \geq 3$.

Problem 1: Determine the computational complexity of $\#\big(\textsc{Consistency}_{\mathcal{F}^d}(S_1, S_2)\big)$?

Let $\mathcal{C}^d$ denote the class of convex lattice sets, where a set $F \in \mathcal{F}^d$ is a convex lattice set if $F = \mathbb{Z}^d \cap \operatorname{conv} F$.

Problem 2: Characterize the computational complexity of $\textsc{Consistency}_{\mathcal{C}^d}(S_1, \ldots, S_m)$.

This problem is particularly interesting for $d = 2$, and certain sets of four and arbitrary sets of seven lattice lines since a result in [1] shows that X-rays in some 4 and any 7 mutually noncollinear lattice directions determine planar convex lattice sets uniquely.

This result is also relevant for the following problem. For a class $\mathcal{G} \subset \mathcal{F}^d$, define the Helly number $H(\mathcal{G})$ for consistency to be the least integer h such that there is a solution for any instance $\mathcal{I}$ of $\textsc{Consistency}_{\mathcal{G}}(S_1, \ldots, S_m)$ whenever there are solutions for all corresponding consistency problems obtained by considering h of the m X-sets in $\mathcal{I}$. If $U(\mathcal{G})$ denotes the least integer such that sets in the class $\mathcal{G}$ are uniquely determined by X-rays in any set of U mutually nonparallel lattice directions (if such an integer exists, ∞ otherwise) then $H(\mathcal{G}) \leq U(\mathcal{G}) + 1$. For convex lattice sets it follows that $H(\mathcal{C}^2) \leq 8$.

Problem 3: Is it true that for every $\mathcal{G} \subset \mathcal{F}^d$ a finite value of $H(\mathcal{G})$ implies that $U(\mathcal{G})$ is also finite?

While the first three problem are based on [2], the last problem is from [3], where similar problems are studied in the "polyatomic case" where $c \geq 2$ disjoint lattice sets have to be reconstructed simultaneously. The complexity for $m \geq 3$ is of course clear, but how difficult is $\textsc{Poly}_c$-$\textsc{Consistency}_{\mathcal{F}^d}(S_1, S_2)$ (where for each $l = 1, \ldots, c$ and $i = 1, 2$, a function $X_{i,l}$ is given and the question is whether there exist disjoint sets $F_l \in \mathcal{F}^d$ such that $X_{S_i} F_l \equiv X_{i,l}$ for $l = 1, \ldots, c$ and $i = 1, 2$)?

While the problem is easy for $c = 1$, it is shown to be $\mathcal{NP}$-complete for $c \geq 6$.

Problem 4: What is the computational complexity of
$\textsc{Poly}_c\textsc{-Consistency}_{\mathcal{F}^d}(S_1, S_2)$ for $c = 2, 3, 4, 5$?

It is conjectured in [3] that these problems are $\mathcal{NP}$-complete at least for $c \geq 3$, and there is an unpublished work of Picouleau on the case $c = 2$.

References

[1] R. J. Gardner and P. Gritzmann, Discrete tomography: Determination of finite sets by X-rays, to appear in Trans. Amer. Math. Soc.

[2] R. J. Gardner, P. Gritzmann, and D. Prangenberg, On the computational complexity of reconstructing lattice sets from their X-rays, Preprint.

[3] R. J. Gardner, P. Gritzmann, and D. Prangenberg, On the computational complexity of determining polyatomic structures by X-rays, Preprint.

A. Kuba

Problem: Find a 3D binary matrix which is unique (w.r.t. the given three projections taken in rows), but which is non-additive.

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Problem 1: Does there exist a polynomial time algorithm that takes as input a vertical projection vector $V \in R^n$ and a horizontal projection vector $H \in R^m$, and that behaves as follows: Either the algorithm outputs a polyomino with projections V^* and H^* such that every component of (V, H) differs by at most one from the corresponding component of (V^*, H^*), or otherwise the algorithm outputs "No" and in this case there does not exist a polyomino with projections V and H. (Note that this does not mean that the algorithm has to recognize *all* the non-polyomino projections).

Alternatively, one may ask that the algorithm outputs a polyomino with projections V^* and H^* such that the Manhattan distance between (V, H) and (V^*, H^*) is bounded e.g. by $O(\sqrt{n+m})$ or even by $O(1)$.

Problem 2: What is the computational complexity of recognizing the horizontal and vertical projections of the following class of almost convex patterns: The pattern is connected, and the intersection of every vertical and of every horizontal line with the pattern consists of at most two intervals.