Topology in Computer Science: Constructivity; Asymmetry and Partiality; Digitization

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organized by

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Topological notions and methods have successfully been applied in various areas of computer science. The seminar concentrated on the following aspects: constructivity, asymmetry and partiality, and digitization. These key words not only describe a central theme of present-day research in computer science oriented topology, but also reflect the various features in which topological spaces used in computer science applications differ from those classically studied in mathematics.

• Constructivity

In great part, topological research in computer science originates from work on the formal description of programming language semantics and on automated program verification. Because of this connection constructive (effective) approaches to the theory of semantic domains and the relation of such domains to logic have been considered from the very beginning.
Nowadays this work is continued in research on Effective Topologies, Locale Theory and Formal Topologies. In studies on Formal Topology the theories of Scott domains and of metric spaces (especially the real numbers) are developed in the framework of higher-order formal intuitionistic logic.

• Asymmetry and Partiality

Both notions are characteristic for topological spaces used in computer science, as opposed to those considered in classical mathematics. Typically, a space considered in theory of computation is a space of partially defined objects which represent stages of some computation process. As a result of this, given two points, in general only one of them is separable from the other by an observable (or positive) property: whenever a (partial) object satisfies such a property, then also every more fully defined object must satisfy it. Thus a topology based on such properties can satisfy only very weak separation axioms. In the traditional spaces coming from applications in analysis or physics, however, two points can symmetrically be separated by a pair of disjoint properties. In other words, the spaces are at least Hausdorff. In recent years, early ideas of Scott and Weihrauch/Schreiber to embed the real numbers or, more generally, metric spaces in suitable Scott domains formed by intervals or spheres, respectively, have been taken up again. The program is to develop an important part of analysis starting with domains which contain not only the real numbers, but also their "unsharp" approximations. In this way one will come up with data structures and algorithms

for numerical computations that are superior to those based on floating point representation and arithmetic.
Another line of research which is characterized by the above notions is the work on generalized metrics and quasi uniform spaces. By relaxing the classical requirements for a metric space categories of spaces are defined that contain Scott domains as well as metric spaces. Both types of spaces are used in giving meaning to certain programming language constructs. As has been shown this approach not only leads to a unified theory to be used in programming language semantics, but also allows the study of quantitative aspects of computations, which is impossible when dealing merely with Scott domains.

• Digitization

This draws attention to the digital nature of most computer applications. Starting from classical applications in physics, most topological notions have been developed by having the continuum in mind. In computer science applications they very often turn out to be no longer applicable in their classical form, or even to be meaningless, which means that they have to be redefined. Work done here is also basic to the other subfields of topology in computer science.
Obviously, all aspects mentioned above are strongly intertwined.

The aim of the workshop was to bring together computer scientists and those mathematicians who work on similar problems but from a different perspective and who, often, are not aware of the computer science motivations, and to create a common forum for the exchange of ideas and results. 56 top scientists and promising young researchers accepted the invitation to participate in the challenging experience. They came from 15 countries, mostly European countries and the USA, but also Canada, New Zealand, Russia, South Africa and Turkey. The 41 talks covered all of the areas mentioned above.
The atmoshere was very friendly, but the discussions were most lively. During the breaks and till late in the night, participants also gathered in smaller groups for continuing discussions, communicating new results and exchanging ideas. Some participants gave additional evening lectures to discuss their results in more detail. Moreover, there was a lively discussion on the relationship between the (theoretical) results presented in many talks and applications in computer science.
The success of the workshop exceeded our expectation. The particpants expressed high appreciation of this gathering and praised the extraordinary Dagstuhl atmosphere which made this success possible.
As organizers of the Dagstuhl seminar on Topology in Computer Science and on behalf of the particpants we want to thank the institute and its staff, both in Saarbrücken and in Dagstuhl, for the excellent work they did to make it all run smoothly in an efficient but always pleasant and friendly manner.

Contents

1 Recursion on the Continuous Functionals

Ulrich Berger

The hierarchy of total continuous (or countable) functionals introduced independently by Kleene and Kreisel in 1959, and partial versions thereof developed later by Ershov and Scott in a topological and domain-theoretical framework, are nowadays considered as the right model for extending recursion theory to higher types. In this talk we discuss three rival notions of computability on the continuous functionals, (a) domain theoretic computability (corresponding to recursive continuity or countability à la Kleene/Kreisel), (b) definability in (variants of) the idealized functional language PCF (this roughly corresponds to Kleene's schemata (S1-S9)), (c) computability via effective operations on codes. We survey classical and recent results, mainly due to Ershov, Plotkin, Normann and Escardó, clarifying the relationship between these three notions. We emphasise that besides classical recursion theoretic machinery topological notions and methods are crucial for obtaining these results. This is exemplified by a new topological density theorem for the total continuous functionals unifying previous results by Ershov, Normann and the author.

2 Recursive Quasi-Metric Spaces

Vasco Brattka

In computable analysis recursive metric spaces play an important role, since these are, roughly speaking, spaces with computable metric and limit operation. Unfortunately, the concept of a metric space is not powerful enough to capture all interesting phenomena which occur in computable analysis. Some computable objects are naturally considered as elements of asymmetric spaces which are not metrizable. Nevertheless, most of these spaces are T_0–spaces with countable bases and thus at least quasi-metrizable. We introduce a definition of recursive quasi-metric spaces in analogy to recursive metric spaces. We show that this concept leads to similar results as in the metric case and we prove that the most important spaces of computable analysis can be naturally considered as recursive quasi-metric spaces. Especially, we discuss some hyper and function spaces.

3 Ditopological Texture Spaces

Lawrence M. Brown

By a *texturing* of a set S is meant a subset $\mathcal{S}$ of the power set $\mathcal{P}(S)$ which is a complete, completely distributive lattice with respect to inclusion $\subseteq$, which contains S and $\emptyset$, separates the points of S and for which meet coincides with intersection and finite join with union. The pair $(S, \mathcal{S})$ is then called a *texture space*. These spaces were introduced as a point-set based setting for the study of fuzzy sets of various kinds and may therefore be of interest where partial or inexact knowledge is involved. They are also a fruitful setting for the development of (set) complement free concepts. In this context a *ditopology* consists of two, not necessarily related, families $\tau \subseteq \mathcal{S}$, $\kappa \subseteq \mathcal{S}$ of open and closed sets, respectively. Generally the study of ditopological texture spaces proceeds by considering pairs of dual concepts such as compact, co-compact; stable, co-stable and full advantage is taken of the conceptual relation with bitopology.

In this paper we review some ideas involved in the study of ditopological texture spaces. In particular we discuss binary relations in this context and show that this leads to an interesting form of symmetry. Difunctions are also mentioned and their use illustrated with an application to Urysohn's Lemma.

4 Canonical Extensions of the T_0 Topological Spaces via the Quasi-uniform Bicompletion

Guillaume C. L. Brümmer

Quasi–uniformities, like uniformities, are a setting for the study of uniform continuity and other uniform concepts such as completeness and total boundedness. Being asymmetric, they also carry order–theoretic information. Thus the category $\mathbf{QU}_{\mathrm{o}}$ of T_0 quasi–uniform spaces has a forgetful functor to the category of completely regular partially ordered spaces, and has been considered as a setting for the theory of computation by several authors, e.g. J. D. Lawson, M. B. Smyth and P. Sünderhauf.
The canonical completeness concept in $\mathbf{QU}_{\mathrm{o}}$, that is, injectivity with respect to epimorphic quasi–uniform embeddings, is called *bicompleteness*. We denote the well known bicompletion functor by K: $\mathbf{QU}_{\mathrm{o}} \longrightarrow \mathbf{QU}_{\mathrm{o}}$ and the corresponding epimorphic quasi–uniform embeddings by k_Y: $Y \longrightarrow KY$ ($Y \in \mathbf{QU}_{\mathrm{o}}$). Our purpose is to show how this reflection gives easy access to a wide range of canonical extensions of the T_0 topological spaces, including compactifications. To this end we consider the usual forgetful functor T: $\mathbf{QU}_{\mathrm{o}} \longrightarrow \mathbf{Top}_{\mathrm{o}}$, where $\mathbf{Top}_{\mathrm{o}}$ is the category of T_0 spaces and continuous maps. This functor has a large–complete lattice of *sections*, i.e. functors F: $\mathbf{Top}_{\mathrm{o}} \longrightarrow \mathbf{QU}_{\mathrm{o}}$ with $TF = id$. Fixing any one such F, we assign to each $X \in \mathbf{Top}_{\mathrm{o}}$ the extension $Tk_{FX} : X \longrightarrow TKFX$, denoted more briefly by $r_X : X \longrightarrow RX$ with the endofunctor $R = TKF : \mathbf{Top}_{\mathrm{o}} \longrightarrow \mathbf{Top}_{\mathrm{o}}$. The section F is called *lower* (resp. *upper*) *K–true* if (for each X) KFX is coarser (resp. finer) than FRX. One says that F is *K–true* if $KF = FR$.
The embedding $r_X : X \longrightarrow RX$ is dense but not necessarily $\mathbf{Top}_{\mathrm{o}}$–epi. It will be $\mathbf{Top}_{\mathrm{o}}$–epi for all X iff F is upper K–true, iff $\mathrm{Fix}(R, r)$ is epireflective in $\mathbf{Top}_{\mathrm{o}}$, iff F is finer than the well–monotone T–section (denoted by W). The extension $Tk_{WX} : X \longrightarrow TKWX$ is the sobrification, a reflection to $\mathbf{Sob}$, the smallest embedding–reflective subcategory of $\mathbf{Top}_o$. The largest embedding–reflective subcategory of $\mathbf{Top}_{\mathrm{o}}$ with reflections of the form

$X \longrightarrow TKFX$ is that of the topologically bicomplete spaces, with F the finest section of T. Every epireflective subcategory between **Sob** and the latter extreme is represented by some K–true T–section F.
If F is lower K–true, there exists a natural transformation $\mu : R^2 \longrightarrow R$, easily calculated from F, such that (R, r, μ) is a monad in $\mathbf{Top_o}$. Thus when F is the coarsest T–section (the Császár–Pervin quasi–uniformity) one gets the prime open filter monad in $\mathbf{Top_o}$, whose Eilenberg–Moore algebras are realized in $\mathbf{Top_o}$ as the stably compact (alias coherent) spaces, morphisms being the perfect continuous maps. The above monad dominates a certain type of T_0 compactifications.
Some of the above results are due to H. P. A. Künzi or come from current joint work with him. An overview by the author of results until 1997 is given in *Rend. Istit. Mat. Univ. Trieste* **30** Suppl. (1999), 45–74.

5 Entailment Relations and Formal Topology

Thierry Coquand

We show how the notion of entailment relations, introduced by D. Scott as an abstract version of Gentzen multi-conclusion sequent calculus, can be used to give a simple representation of some spectral spaces that occur naturally in algebra. Another intuition is from domain theory, where entailment relation may be seen as a multi-conclusion generalisation of information systems (joint work with Guo-Qiang Zhang). The examples are the following

1. If A is a commutative group, then $X \vdash Y$ iff a non empty sum of elements of X is equal to a sum of elements of Y
2. if A is a commutative ring, then $X \vdash Y$ iff the ideal generated by X meets the multiplicative monoid generated by Y
3. if K is a field and the set of token is the set of invertible elements then $X \vdash y_1, \ldots, y_n$ iff there is a relation $1 = \Sigma y_i^{-1} q_i$ where q_i is a polynomial in $X, y_1^{-1}, \ldots, y_n^{-1}$

In each case, the proof that we have an entailment relation is a rather direct purely algebraic argument.
In the second case, we have a concrete description of the *spectrum* of A and in the third case, of the *space of valuation* of K.
Using these descriptions, we can achieve a partial realisation of Hilbert's program for algebra giving a direct computational interpretation of abstract arguments. We illustrate this on an abstract proof of a theorem of Kronecker: if $c_k = \Sigma a_i b_j$ then each $a_i b_j$ is integral on the c_k.

6 Apartness Spaces as a Foundation for Constructive Topology

Luminita Simona Dediu

Errett Bishop, thanks to whom we now know that a broad spectrum of deep mathematics can be developed constructively [1], originally dismissed topology with the following remark:

> Very little is left of general topology after that vehicle of classical mathematics has been taken apart and reassembled constructively. With some regret, plus a large measure of relief, we see this flamboyant engine collapse to constructive size ([1], page 63).

Perhaps not surprisingly, general topology has since been marginalised in constructive mathematics.

In this talk I present some recent joint work with Douglas Bridges on the constructive theory of apartness spaces, which hold considerable promise as a framework for constructive topology. The theory is based on (i) five axioms for a relation apart (x, A) of apartness and (ii) a second–order definition of the corresponding notion of nearness between points x and subsets A of a set X with a nontrivial inequality relation. After a number of elementary deductions from the axioms, I introduce the topology associated with an apartness structure, and three classically equivalent, but constructively inequivalent, notions of continuity for a mapping between apartness spaces. When the codomain of such a mapping is completely regular in a natural sense, the two strongest of the three continuity notions turn out to be equivalent. An extension of some of these results enables us to show that the final axiom for apartness, an axiom that is a triviality with classical logic, is independent of the first four. The talk concludes with a discussion of the product of two apartness structures.

References

[1] Errett Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, New York 1967.

7 The Patch Topology

M. H. Escardó

In order to transform a stably compact space into a compact order-Hausdorff space in the sense of Nachbin, one uses the patch topology. An important example is the patch of the Scott topology of a continuous Scott domain, which coincides with Lawson topology of the domain. In particular, the patch of the topology of lower semi-continuity of the real line is the Euclidean topology.

Categorically, the patch construction gives one direction of the equivalence of the category of perfect maps of stably compact spaces with the category of continuous monotone maps of compact order-Hausdorff spaces. We observe that, alternatively and without reference to order, the patch construction arises as the right adjoint to the full inclusion of the category of continuous maps of compact Hausdorff spaces into the category of perfect maps of stably compact spaces. Thus, it produces the compact-Hausdorff coreflection of a stably compact space.

We present a localic version of the patch topology. In summary, the compact-regular coreflection of a stably compact locale is given by the locale of Scott continuous nuclei. (As opposed to the topological situation, the proof of the localic version doesn't rely on excluded-middle or choice principles.) For frames, this is of course a reflection. It can be interpreted as saying that the frame of Scott continuous nuclei is the universal solution to the problem of transforming the way-below relation into the well-inside relation (a fact that emphasizes the rôle of perfect maps). This is related to the well-known fact that the frame of *all* nuclei is the universal solution to the problem of adding boolean complements to the opens of the locale.

(This, a particular case and a generalization will appear in the *Journal of Pure and Applied Algebra* as "The regular-locally-compact coreflection of a stably locally compact locale". Meanwhile, it is available in electronic form at

`http://www.elsevier.nl/inca/publications/store/5/0/5/6/1/4/`

in the subpage "Author Versions of Accepted Papers".)

8 Time and Change in the Digital World

Antony Galton

Continuity is an important regulative principle for reasoning about many types of motion, but it is not always clear what notion of continuity is appropriate for a given model of the space and time with respect to which the motion is defined. We investigate three distinct models, both of space and time, giving rise to nine prima facie possible combinations. The three models are all variants of closure spaces, which generalise the topological notion of closure by omitting the requirement of idempotence. Two of the models are discrete in the sense that each point has a unique minimal neighbourhood; the third is continuous, being the standard Euclidean space. In the discrete spaces the closure operation in defined in terms of an irreflexive neighbourhood relation which may be either symmetric (in 'adjacency spaces') or transitive (in 'incidence spaces'), the latter being T_0 topologies which are not T_1, for example the Khalimsky spaces or, more generally, cellular complexes. Allowing time to be represented by one-dimensional spaces of each of these three types, and space likewise, we note that only four of the nine prima facie possible combinations are likely to be of interest in the context of spatio-temporal knowledge representation. The case where time is continuous but space is an incidence space is of particular interest as providing a useful mathematical model for a type of qualitative state-space much used by the knowledge representation community, as exemplified by the calculus of qualitative spatial relations introduced by

Randell, Cui, and Cohn. The topological representation of these state-spaces allows for a more accurate account of the possible patterns of temporal incidence than is afforded by the conceptual neighbourhood diagrams normally used for this purpose, and also allows one to derive product spaces with the correct neighbourhood relation, which is not possible from a consideration of the conceptual neighbourhood relation alone.

9 Continuity and Convergence in the Basic Picture

Silvia Gebellato, Giovanni Sambin

The same ideas and inspiring principles used by Sambin to develop the theory of basic pairs, that is the most basic notion of (nondistributive) topological space, (see the abstract of his talk *Formal topology, hows and whys* in this volume) are here applied to study the notions of continuity and convergence, and to introduce the notion of morphisms between basic pairs. We see how, once continuity is freed from its link with functions and it is formulated for relations, and it is made symmetric, it can be expressed in a fully structural way by commutative squares. These last are then assumed to define morphisms between basic pairs, here called *relation-pairs*. A relation-pair between the basic pairs $\mathcal{X} \equiv (X, \vdash_1, S)$ and $\mathcal{Y} \equiv (Y, \vdash_2, T)$, is a pair of relations (r, s), where $r : X \to Y$ and $s : S \to T$, which makes the following square

$$\begin{array}{ccc} X & \xrightarrow{\vdash_1} & S \\ r\downarrow & & \downarrow s \\ Y & \xrightarrow{\vdash_2} & T \end{array}$$

to commute.

The essence of continuity is then characterized in terms of relation-pairs by defining a notion of equality which identifies those relation-pairs which behave in the same way topologically. Two pairs of relations (r, s) and (r_1, s_1) identify the same relation-pair if, whatever among (r, s) or (r_1, s_1), we consider in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\vdash_1} & S \\ r\downarrow\downarrow r_1 & & s_1\downarrow\downarrow s \\ Y & \xrightarrow{\vdash_2} & T \end{array}$$

we obtain equivalent squares; that is $\vdash_2 \circ r = \vdash_2 \circ r_1$ and $s \circ \vdash_1 = s_1 \circ \vdash_1$. Basic pairs and relation-pairs form a self-dual category, called **BP**. It is the new discovery that any basic pair can be interpreted as a nondistributive topological space which on one hand allows the interpretation of morphisms in terms of continuity, and on the other hand motivates the new definition of equality between morphisms as a way to express that they are topologically indistinguishable.

Also the notions of continuity and convergence are compared. By the notion of nondistributive topological space we can see that continuity and convergence are independent notions (even when relations are graphs of continuous functions). Finally, a formal treatment of continuous and convergent relations is obtained by characterizing relations on the formal side of convergent relation-pairs. Such relations map formal points, which are the formal reconstructions of the notion of concrete points, into formal points. This reason made us to conclude that continuity *and* convergence actually characterize "usual" continuous functions.

10 Topology, Ditopology and Concurrency

Eric Goubault

Starting from old "continuous" models for concurrency, progress graphs, which were introduced by E. W. Dijkstra in 1968, we present the current development of the geometric methods it induced. It appears that this first model is a particular case of a compact-hausdorff topological space. Homotopy of "directed" paths (or dipaths, i.e. paths that are increasing in each coordinate) represents the equivalence of schedules. Going to the corresponding stably compact topological space, one can make a first construction of a "directed" homotopy theory that describes the essential schedules of a concurrent system. We give some geometric examples of the new phenomena (with respect to ordinary homotopy theory) that appear given the constraint of "directedness" i.e. given that time is a privileged coordinate, that one cannot reverse. Then we go to the more general situation of locally partially-ordered topological spaces, in which we can express loops, branchings, forks etc. We give again some flavour of what can happen, develop the first parts of the theory and derive some algorithms for checking deadlock and unreachable states, from the geometry. Finally we give some ideas of applications, among which problems concerning serialisability conditions in concurrent databases, fault-tolerant protocols for distributed systems and static analysis by abstract interpretation of concurrent programs.

11 A Non-Topological View of Dcpo's as Convergence Spaces

Reinhold Heckmann

The category TOP of topological spaces is not cartesian closed, but can be embedded into the cartesian closed category CONV of *convergence spaces* (aka *filter spaces*), whose objects are characterised by the convergence properties of filters. It is well-known that the category DCPO of dcpo's and Scott continuous functions can be embedded into TOP, and so into CONV, by considering the Scott topology.
We propose a different, non-topological embedding of DCPO into CONV, which, in contrast to the topological embedding, preserves products. We call this new embedding "*cotopological*" because of its behaviour in the function space construction: if X is a cotopological dcpo, i.e. a dcpo with the cotopological CONV-structure, and Y is a topological space,

then $[X \to Y]$ is again topological, and conversely, if X is a topological space, and Y a cotopological complete lattice, then $[X \to Y]$ is again a cotopological complete lattice.
For a dcpo D, the topological and the cotopological convergence structures coincide if and only if D is a continuous dcpo. Moreover, cotopological dcpo's still enjoy some of the properties which characterise continuous dcpo's if attention is restricted to TOP. For instance, all cotopological complete lattices are injective spaces (in CONV) w.r.t. topological subspace embeddings.

12 Effectivity versus Effective Continuity on Computable Metric Spaces

Peter Hertling

Based on the Turing machine model there are mainly two different notions of a computable function over the real numbers or on a metric space. The "effective" functions are defined only on computable elements, and effective with respect to a Gödel numbering of the computable elements. The other notion can be described as effective continuity. We characterize the effectively continuous functions on the computable elements as those effective functions which satisfy an additional condition which is expressed using a certain domain associated with the metric space. Our result is a sharp strengthening of a result by Tseitin and by Kreisel/Lacombe/Shoenfield.

13 Generalized Metric Spaces in Logic Programming Semantics

Pascal Hitzler, Anthony Karel Seda

The Denotational Semantics of Logic Programs is often characterized via fixed points of associated operators. The use of negation in extending syntax and expressibility renders some of these operators to be nonmonotonic. So, in contrast to other programming paradigms, fixed-point theorems such as the Tarski theorem are often not applicable in logic programming. We show, however, that generalizations and general variants of the Banach contraction mapping theorem can be applied successfully. In particular, we will employ the Priess-Crampe and Ribenboim theorem on generalized ultrametrics and a theorem of Matthews on dislocated metrics in order to semantically analyse some classes of logic programs. We will also be able to apply a theorem which merges the two concepts of generalized ultrametric and dislocated metric.

14 Khalimsky Topologies are the Only Simply-Connected Topologies on $\mathbb{Z}^n$ Whose Connected Sets Include All $2n$-Connected Sets but No (3^n-1)-Disconnected Sets

T. Yung Kong

We give the main steps of a proof of the result stated in the title. Here the concepts of $2n$- and (3^n-1)-(dis)connected sets are the natural generalizations to $\mathbb{Z}^n$ of the standard concepts of 4- and 8-(dis)connected sets in 2D digital topology.
Suppose we have an n-dimensional scanner that digitizes n-dimensional objects into subsets of $\mathbb{Z}^n$. Are there topological spaces $(\mathbb{Z}^n, \tau)$ that would allow standard concepts and methods of general topology to be directly and usefully applied to good digitizations produced by the scanner? Our result suggests that no topological space $(\mathbb{Z}^n, \tau)$ that is not a Khalimsky space can satisfy this condition.
Our proof involves some purely discrete arguments and a fact about simply connected polyhedra that is a well known consequence of the Simplicial Approximation Theorem, but also uses the following fact (which was established in an earlier paper by the author and Khalimsky): For any T_0 topological space in which each point lies in a finite open set and a finite closed set there exists a polyhedron, whose vertices are in $1-1$ correspondence with the points of the space, such that the homotopy classes of continuous maps into the topological space from any metric space are in $1-1$ correspondence with the homotopy classes of continuous maps from that metric space into the polyhedron.

15 Approximation of Compact Hausdorff Spaces by Finite T_0-Spaces

Ralph Kopperman (joint with Richard G. Wilson)

Theorem: A Hausdorff space is compact if and only if it is the Hausdorff reflection of an inverse limit of finite T_0-spaces and continuous maps.
It is easy to see that an inverse limit of finite spaces and continuous maps is compact, so its Hausdorff reflection is also compact.
The construction leading to the converse begins by considering for each finite set F of open subsets of the original space, the nonempty elements of the (finite) Boolean algebra it generates. The atoms of this Boolean algebra form a partition of the original space, let X_F be the natural quotient set, given the quotient topology from the original space; this turns out to be T_0. If F, G are two such sets and $F \subseteq G$, then the natural map $p_{GF} : X_G \to X_F$ is an onto quotient. These spaces and maps give us our inverse sequence. (Details in "Finite approximation of compact Hausdorff spaces". Topology Proceedings, 22 (1997), 175-200.)

It turns out that these maps are more than continuous: for each F there is a $G \supseteq F$ such that whenever $C, D \subseteq X_F$ are disjoint closed sets, then there are disjoint open $T, U \subseteq X_G$ such that $p_{GF}^{-1}[C] \subseteq T$ and $p_{GF}^{-1}[D] \subseteq U$.

16 Incidence Structure: A New Means for Topological Investigations by Means of Computer and for Encoding of 3D Images

Vladimir Kovalevsky

The paper presents a new means - the incidence structure - for investigating topological properties of sets by means of computer. Topological spaces are represented as finite cell complexes. The paper contains definitions and a theorem necessary to transfer some basic knowledge of the classical topology to finite topological spaces. The computer representation is based on splitting the given set into blocks of mutually simple cells in such a way, that a k-dimensional block be homeomorphic to a k-dimensional ball. The block complex is described by the data structure known as "cell list" which is generalized here for the multidimensional case. Experimental results are presented.

17 Interlaced Spheres and Multidimensional Tunnels

Vladimir Kovalevsky

The paper presents some theorems about interlaced spheres of different dimensions in multidimensional spaces. Two spheres S^k and S^m are called interlaced with each other if their intersection is empty, however, one of them crosses each topological ball whose boundary is the other sphere. We describe a method of simulating interlaced spheres in computer. We demonstrate the connection between the notion of a tunnel in a multidimensional "body", i.e. in a connected subset of a multidimensional space, and that of interlaced spheres. Examples of multidimensional tunnels in four- and five-dimensional spaces are demonstrated.

18 The Topology of Mazurkiewicz Traces

Ralph Kummetz (joint work with Dietrich Kuske)

Mazurkiewicz traces occur in the theory of concurrent systems and serve as an abstract description of processes. The basic idea is to view a process as being built up by atomic actions. One requires that either two actions depend on each other or they may be executed independently. Then a (possibly infinite) process is described by a *real trace*. This is a particular (isomorphism class of an) acyclic graph whose vertices are labelled with atomic actions and whose edges are exactly between dependent vertices. For a survey on traces the reader is referred to [V. Diekert and G. Rozenberg, *The Book of Traces*, World Scientific, Singapore, 1995].

Let (Σ, D) be a finite dependence alphabet. That is, Σ is a finite set of atomic actions and D is a reflexive and symmetric binary relation on Σ that specifies when two actions are dependent. The property of a real trace to be a downwards closed subgraph of another one yields a partial order called the prefix order. It is well-known that the set of real traces $\mathbb{R}(\Sigma, D)$ over (Σ, D) becomes an ω-algebraic Scott-domain with its compact elements being precisely the finite traces over (Σ, D). Furthermore, there is a canonical ultrametric turning $\mathbb{R}(\Sigma, D)$ into a compact space whose topology coincides with the Lawson topology of the domain.

In the talk we present a characterization of this topology in terms of properties of the underlying dependence alphabet. Given two dependence alphabets (Σ_1, D_1) and (Σ_2, D_2), we show that $\mathbb{R}(\Sigma_1, D_1)$ is homeomorphic to $\mathbb{R}(\Sigma_2, D_2)$ if and only if the two alphabets have the same number of atomic actions that are independent of all other actions and either both alphabets have a pair of dependent actions or both of them do not have such a pair. Moreover, we prove that the space $\mathbb{R}(\Sigma, D)$ is homeomorphic to a product space whose factors are (at most) the Cantor tree and a finite power of the Alexandroff one point compactification of the non-negative integers.

19 Integration via Domain Theory

Jimmie Lawson

The main result of this paper is that the domain-theoretic approach to the generalized Riemann integral first introduced by A. Edalat extends to a large class of spaces that can be realized as the set of maximal points of domains.

The approach is based on the theory of a Riemann-Stieltjes type integral on a topological space with respect to a finitely additive measure. We develop the theory of this integral for a bounded function f defined on the maximal points of a continuous domain and show that it gives an alternate approach to the Edalat integral.

20 An Algorithm for Zero Finding Based on the Informatic Derivative

Keye Martin

We say that a monotone mapping $f : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ on the interval domain $\mathbf{I}\mathbb{R}$ is *convex* if there is a continuous map $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(a) \cdot \mu x \leq \mu f(x) \leq g(b) \cdot \mu x$$

for all $x = [a, b] \in \mathbf{I}\mathbb{R}$, where $\mu[a, b] = b - a$ is the usual measurement on $\mathbf{I}\mathbb{R}$. The informatic derivative

$$df_\mu(p) := \lim_{x \to p} \frac{\mu f(x)}{\mu x}$$

of a convex map f at $p \in \max \mathbf{I}\mathbb{R} = \mathbb{R}$ exists and equals $g(p)$.

A mapping of the real line $f : \mathbb{R} \to \mathbb{R}$ is *informatically convex* if it has a convex extension to $\mathbf{I}\mathbb{R}$. One example is a mapping with nonnegative first and second derivatives on a compact interval. However, a Lipschitz map on a compact interval also qualifies.

In this talk we show that an informatically convex map on the real line whose extension has a positive informatic derivative admits an efficient algorithm for zero finding which beats the bisection method at every iteration. For Lipschitz mappings, the algorithm given is known to be optimal in the sense of the numerical accuracy minimax problem. For convex mappings, it provides an efficient and superior alternative to the bisection for obtaining close initial guesses for Newton's method.

21 An Algebraic Analog to the Probabilistic Power Domain

Michael W. Mislove

In this talk I describe how to construct an algebraic analog to the probabilistic power domain of Jones and Plotkin. The construction relies on an embedding-projection pair between the Cantor set, $\mathbb{C}$ viewed as the ideal completion of the rational numbers strictly less than 1 in the unit interval, and the unit interval itself. This object, $\mathcal{P}_{\mathbb{C}}(D)$ gives rise to a class of *rational probabilistic algebras*, and it is shown that the construction is the object level of a left adjoint to the forgetful functor from the category of such objects and Scott continuous mappings that preserve the rational probabilistic algebra structure, into the category of algebraic domains and Scott continuous maps.

The probabilistic power domain $\mathcal{P}_{Pr}(D)$ over a domain is also shown to be a retract of $\mathcal{P}_{\mathbb{C}}(D)$, and it is then easy to show that the probabilistic power domain is continuous. This approach avoids the complications of the Splitting Lemma, although its full power is not retrieved – the order on simple measures is not as clearly delineated under this approach as is possible using the Splitting Lemma. A hoped-for application of the construction was to resolve the question of whether the category RB of retracts of bifinite domains, or the category FS of FS-domains is closed under the probabilistic power domain construction, but this approach does not appear to shed much light on this question. The reason is that the rational probabilistic power domain of a coherent domain – in fact, of a simple five-point poset – is not coherent.

22 Strictly Completely Regular Bitopological Spaces and Admissible Quasi-uniformities

K. Rufus Nailana

We introduce the notion of a strictly completely regular bitopological space and show that the category of strictly completely regular bitopological spaces is isomorphic to the category of strictly completely regular ordered spaces. It is shown by means of examples that the category of strictly completely regular bitopological spaces is more flexible than its counterpart in ordered spaces. We show that the Stone Cech bicompactification of a strictly completely regular bitopological space can be obtained from the Stone Cech ordered compactification of the corresponding ordered space. We also determine quasi-uniformities that induce strictly completely regular bitopological spaces.

23 Topological Foundations for Geometric Modeling

T. J. Peters, A. C. Russell

For practical computing algorithms in geometric modeling, it is often desirable to create piecewise linear approximations to curvilinear geometric representations. We propose that any such approximation should be topologically equivalent to the original object by means of an ambient isotopy having compact support. We present a counterexample on a 1-manifold (without boundary), where the failure of the piecewise linear approximant to be ambiently isotopic has undesirable consequences. The counterexample motivated other authors to create an algorithm which guarantees ambient isotopic piecewise linear approximations over a broad class of 1-manifolds. Here, we present a proof of a piecewise linear approximation for any compact, C^2, 2-manifold (without boundary), which is embedded in R^3, so that the approximation is ambient isotopic to the original manifold and so that the isotopy has compact support. We relate this work to the contemporary unsolved problem of the stick number of a knot. The proof presented here directly generalizes work of Amenta and Bern, 1998.

24 Measurements on Domains and Topology

G. Michael Reed

The concept of a measurement on a continuous partial order has been recently introduced by Keye Martin. Measurement provides a uniform degree of approximation for elements of the kernal, i.e., those elements with measure zero. Measurement also induces a Scott topology for elements "near" the kernal.
The results below are joint work with Keye Martin.

1. | | | |
   |---|---|---|
   | X is developable and T_1 | iff | it is the kernel of a measurement on a continuous poset |
   | X is developable T_1 and choquet complete | iff | it is the kernal of a measurement on a continuous cpo. |

2. For each developable T_1-space X, there exists a developable T_1-space $M(X)$ with a poset order $<$ such that
   (a) X is the kernal of a measurement
   (b) the topology on $M(X)$ is exactly the topology induced by the measurement
   (c) if X is T_2, then $M(X)$ is a Moore space
   (d) if X is a complete (or even semi-complete) Moore space then $(M(X), <)$ is a cpo
   (e) if X is the real line, then $M(X)$ is a non-normal Moore space
   (f) (MA + not CH) if X is a subspace of the real line and $\omega < |X| < c$, then $M(X)$ is a normal, nonmetrizable Moore space
3. The countable ordinals with the order topology is the top of a continuous cpo, and is a G_δ set with respect to the Scott topology on the domain. [This answers several questions open questions in the area.]
4. Finally, we give a new recursion induction theorem for cpo's using measurement theory.

25 Noncommutative Topology in Computer Science

Pedro Resende

The Gelfand-Neumark representation theorem establishes that every commutative C^*-alge/-bra is, up to isomorphism, an algebra of continuous functions on a locally compact Hausdorff space. Noncommutative topology is the name usually given to the study of the topological properties of not necessarily commutative C^*-algebras according to the metaphor that they, too, are algebras of "functions" on "noncommutative spaces". This is motivated for instance by quantum mechanics, where noncommutative algebras arise naturally, and is at the basis of the noncommutative geometry of Connes, where several examples, such as spaces of tilings, spaces of orbits, locally compact grupoids, etc., are studied. The theory of quantales was originally motivated by these ideas. It adopts the view that quantales are the "topologies" of noncommutative spaces, providing a kind of noncommutative topology that generalizes locale theory. For instance, there is a notion of point of a quantale, due to Pelletier and Rosicky in the case of involutive quantales, and later adapted by Paseka for arbitrary quantales, which generalizes the notion of point of a locale, and with which it is possible to characterize the usual points of C^*-algebras, i.e., their irreducible representations, as shown by Mulvey and Pelletier. The purpose of this talk is to give examples of quantales, seen as noncommutative spaces, taken from work in computer science. Specifically, in a paper of Abramsky and Vickers, and followed by myself, the view was adopted that quantales are algebras of "finite observations" on concurrent systems, and left quantale modules are algebras of "observable properties". A notion of "tropological system" then exists, consisting of a set of states on which the quantale and the module act appropriately, in particular inducing a natural notion of behavioral equivalence on the set of states that was used in order to characterize various notions of process semantics. In this talk I recall these ideas and show that any tropological system can be identified with a "relational point" of a suitable quantale. In doing so I also show that the behavioral equivalences are preserved in the translation from systems to points of quantales, and thus systems "are" points, at least in so far as the only invariant the translation is required to preserve is behavioral equivalence.

26 Finite Metric Spaces and Data Analysis

Michael D. Rice

For the last year, I have been studying the use of distance functions in a variety of areas that involve the presentation and analysis of data. These areas include cluster analysis, multi-dimensional scaling, concept analysis, and computational biology.
The lecture will present ideas in several of these fields that involve finite metric spaces and suggest how the study of finite metric spaces can be useful. A good deal of the presentation will be simply a restatement of ideas and results from the mathematics literature and the literature in the various fields. However, I hope that the "repackaging" may be of some interest.

27 Application of Simplicial Complexes in Digital Topology

Gerhard X. Ritter

In order to analyze, synthesize, and manipulate geometric configurations by means of a digital computer, the need arises for precise and efficient methods of describing these configurations. One of the simplest and most useful methods which permits the encoding of arbitrary plane curves is the grid-intersection scheme for octagonal chain coding. In this discussion we describe a procedure which which employs basic aspects simplicial topology for the description and encoding of arbitrary digital surfaces. The coding method we describe can be viewed as the 2-dimensional analogue of the octagonal chain code. More precisely, the octagonal chain code for curves can be viewed as an oriented simplicial 1-complex, while the code we describe corresponds to as an oriented simplicial 2-complex. Being the 2-dimensional topological analogue of the octagonal chain code, the simplicial code exhibits many of the manipulative properties of the chain code.
We conclude our discussion with several open problems in simplicial coding theory.

28 Full Normality of Products of the Bitopological Khalimski Line

Salvador Romaguera, Sergio Salbany

It is well known that pseudometric spaces are fully normal. There is a natural generalization of full normality to bitopological spaces, however when d is a quasi–pseudometric and $P = T(d), Q = T(d^{-1})$, it is not necessarily true that the quasi–metrizable bispace (X, P, Q) is fully normal. It is straightforward to verify that the topology T of the Khalimski line, $\mathbb{K}$, and its natural dual $T^{\triangle}$ are induced by a quasi–pseudometric d and its conjugate, respectively. It is also fairly clear that finite products $(\mathbb{K}, T, T^{\triangle})^n$, $n = 1, 2, \ldots$, are fully normal bitopological spaces. We prove that a countable product $(\mathbb{K}, T, T^{\triangle})^w$ is also fully normal and observe that not all products are fully normal.

29 Formal Topology: Hows and Whys

Giovanni Sambin

The usual characterization of formal topology is: topology as developed in a constructive type theory. Here we present a more "structural" approach, and some general motivations and comments coming with it.
Given any pair of sets X and S, a binary relation $r : X \to S$, that is a propositional function $xra\ prop$ $(x \in X, a \in S)$, induces four operators $\Diamond, \Box : \mathcal{P}X \to \mathcal{P}S$ and $\mathsf{ext}, \mathsf{rest} : \mathcal{P}S \to \mathcal{P}X$, of which $\Diamond$, ext are existential (weak images of a subset, defined through $\exists$) and $\Box$, rest are universal. Moreover, $\Diamond$ is symmetric of ext, and $\Box$ of rest. Thinking of X as concrete points and S as observables, the discovery is that the composition $\mathsf{ext}\Box$ is nothing but the usual operator int of interior of a subset, and similarly $\mathsf{rest}\Diamond$ is closure cl. A second discovery is that symmetrically $\Box\mathsf{ext}$ is nothing but the closure operator $\mathcal{A}$ giving the formal open $\mathcal{A}U$ associated with a given subset U of S. So, this leads to the novelty of this approach for formal topology, namely the introduction of a new interior operator $\mathcal{J} \equiv \Diamond\mathsf{rest}$, dual of $\mathcal{A}$, and the definition of $F \subseteq S$ as formal closed if $F = \mathcal{J}F$. To confirm such definitions, one can prove that concrete and formal open subsets form isomorphic complete lattices, and similarly for closed.
All this new way of looking at just a pair of sets, which I have called the basic picture, leads to the new definition of formal basic topology, namely a triple $(S, \mathcal{A}, \mathcal{J})$ where $\mathcal{A}$ is a closure operator, $\mathcal{J}$ is an interior operator and they are linked by compatibility: $\mathcal{A}U \between \mathcal{J}F$ iff $U \between \mathcal{J}F$ (for any two subsets A, B, $A \between B$ by definition is $\exists x(x \epsilon A \ \&\ x \epsilon B)$). Breaking the symmetry, one adds $\mathcal{A}U \cap \mathcal{A}V \equiv \mathcal{A}(U \downarrow V)$, where $U \downarrow V \equiv \cup_{a\epsilon U, b\epsilon V} a \downarrow b$ and $a \downarrow b \equiv \mathcal{A}\{a\} \cap \mathcal{A}\{b\}$, and obtains a definition of formal topology with the novelty of a binary positivity predicate $\mathsf{Pos}(a, F) \equiv a \epsilon \mathcal{J}F$. This new formulation contains both locales and previous formal topologies as special cases.

30 On the Computational Content of the Lawson Topology

Frédéric De Jaeger, Martín Escardó, Gabriele Santini

We investigate a new notion of computability that we refer to as Lawson computability. It is introduced effectivizing the Lawson topology over domains via Weihrauch's theory of effective topological spaces. We compare it with that induced by the Scott topology, that coincides with that arising from effectively given domains theory. As the applications to standard domains show, these notions are both sound and extend the distinction between recursively enumerable and recursive sets of natural numbers. Applying the definitions to the domain of the upper space of an Euclidean space, we find computability notions equivalent to the most natural ones given in literature.

31 Semivaluation Spaces as a Minimal Foundation for Quantitative Domain Theory

Michel P. Schellekens

"Quantitative Domain Theory" emerged from attempts to reconcile metric approaches and order theoretic approaches to traditional Domain Theory.
The field typically concerns models involving "quantitative measurements" as opposed to "qualitative (= order-)comparisons". Examples of such models include: partial metric spaces (Matthews: Dataflow Networks), totally bounded Scott Domains (Smyth, Seda: Logic Programming) and complexity spaces (Schellekens).
Prior work in this field aimed at providing most general frameworks to support the applications as for instance Topological Quasi-Uniform Spaces (Smyth, Sünderhauf), Continuity Spaces (Flagg, Kopperman) and Enriched Categories (Wagner). We focus on determining a *minimal* extra notion required to extend classical Domain Theory with the notion of a "quantity". This is achieved by the introduction of the new notion of a *semivaluation*, which generalizes the fruitful notion of a valuation on a lattice, to the context of semilattices.
The proof of this result is obtained in the context of the theory of quasi-uniform semilattices. This involves the solution of an open problem in Non Symmetric Topology (Künzi), which arose independently in Computer Science (Bukatin).
An interesting aspect of semivaluations is that they allow for a unified approach to Quantitative Domain Theory in a broader sense, since models based on valuations can be incorporated in this context. This includes the probabilistic power domain (Jones, Plotkin), semantics for non deterministic programs (Heckmann) and the domain theoretic treatment of integration (Edalat).

32 Admissible Representations of Limit Spaces

Matthias Schröder

We define and investigate the class of admissible representations which allow to handle limit spaces in the Type–2 model of computation appropriately.
An *admissible* representation of a limit space X is a continuous surjection $\delta_X :\subseteq \Sigma^\omega \to X$ such that for every continuous $\phi :\subseteq \Sigma^\omega \to X$ there is some continuous $G :\subseteq \Sigma^\omega \to \Sigma^\omega$ with $\phi = \delta_X \circ G$. We show that admissible representations δ_X, δ_Y of limit spaces X and Y have the property that any partial function $f :\subseteq X \to Y$ is *continuously realizable* w.r.t. δ_X and δ_Y if and only if f is continuous w.r.t. the limit relations of X and Y. The former means that there is a continuous function $\Gamma :\subseteq \Sigma^\omega \to \Sigma^\omega$ with $\delta_Y \circ \Gamma = f \circ \delta_X$.
Furthermore, the class of those limit spaces that have an admissible representation is proven to equal the class of those limit spaces that have a countable limit base and satisfy the T_0–property. A *limit base* of a space X is a set $\mathcal{B} \subseteq 2^X$ such that for every $x \in X$, every

sequence (y_n) that converges to x and every sequence (z_n) that does not converge to x there is a set $B \in \mathcal{B}$ such that x is in B, (y_n) is eventually in B and (z_n) is not eventually in B. Many interesting operators creating new limit spaces from old ones are shown to preserve the existence of an admissible representation. In particular the class of limit spaces having an admissible representation turns out to be cartesian–closed.
Thus, a reasonable computability theory is possible on a wide class of important limit spaces which includes all topological countably–based T_0–spaces and contains important non countably-based topological ones as well as some non–topological ones.

33 Formal Zariski Topology

Peter M. Schuster

We indicate how formal topology could be used for building a constructive and predicative algebraic geometry that appears to be conceptually simpler than its classical predecessor. An appropriate covering relation due to Persson (and, in the case of monoids, to Battilotti-Sambin) gives rise to a formal version of the Zariski spectrum of any commutative ring R with unit. The formal points turn out to be the prime coideals of R, whereas the formal opens are just the radical ideals of R. Roughly the same theory results from inverting the entailment relation on R given by Cederquist-Coquand.

34 A Jordan Curve Theorem for n-ary Digital Plane

Josef Šlapal

For every natural number $n > 1$ we define a closure operator u_n on $\mathbb{Z}^2$, which is associated with an n-ary relation on $\mathbb{Z}^2$ (closure operators are assumed to be only grounded, extensive and monotone). The pair $(\mathbb{Z}^2, u_n)$ is then called the n-ary digital plane, and for $n = 2$ it coincides with the known Khalimsky plane. We study connectedness in $(\mathbb{Z}^2, u_n)$ and, as the main result, we prove an analogy of the Jordan curve theorem. While for $n = 2$ we get some known facts about the Khalimsky topology, for $n > 2$ our results are completely new and show that the closure operators u_n provide $\mathbb{Z}^2$ with topological structures suitable for solving problems of computer graphics and digital image processing. We also demonstrate some advantages of these structures in comparison with the commonly used Khalimsky topology.

35 The Convexity Quantale and the Connection Predicate

Michael B. Smyth

The aim of this work is to develop an axiomatic (synthetic) theory of discrete, or digital, geometry; more precisely, a theory which admits both discrete and continuous (say, Euclidean) spaces as models, such that the continuous ones occur as limiting cases.
A space is thought of as a set of regions (rather than of points). The main geometric primitive is a binary operation (the product) which, in effect, gives the region lying between two given regions. The other key primitive is the "connection predicate" (as in mereotopology); the join, or fusion, of regions is characterized via this predicate. With respect to the mentioned product and join, a space is a (commutative, non-unital) quantale.
After reviewing the development of convex and linear geometry in this setting (cf. [1]), we consider the topological (in the usual sense) aspect of our spaces. In the case of the appropriate discrete spaces, the topology agrees with that of Khalimsky.
Finally, we consider some problems (with their possible solution) concerning the connection predicate. The main reason that we work with this predicate, rather than having an explicit intersection of regions, is that we are thereby enabled to have non-trivial finite models of the theory. It can be argued that a multiary connection predicate would be more expressively adequate than a binary predicate. This , however, would be more awkward to work with, as well as appearing (even) more odd in the quantale context.

References

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36 Inductive Generation of Formal Topologies with a Binary Positivity Predicate with Proper Axioms

Silvio Valentini

Formal topologies are today an established topic in the development of constructive mathematics and many classical results of general topology have been already brought into the realm of constructive mathematics using this approach.
One of the main tools in formal topology is inductive generation and the problem of inductively generate formal topologies with a unitary positivity predicate has been completely solved in a recent paper by T. Coquand, G. Sambin, J. Smith and myself.
Anyhow, in order to deal both with open and closed subsets, a binary positivity predicate is necessary. That the reason why, after a short introduction to formal topologies with

a binary positivity predicate, we propose one possible way to generate inductively formal topologies with a cover relation and a binary positivity predicate which both have proper axioms.

37 Towards Quantitative Verification of Systems: a Metric Approach

Franck van Breugel

The majority of the verification methods for software systems only produce qualitative information. Questions like "Does the system satisfy the specification?" and "Are the systems semantically equivalent?" are answered. However, this information is often too restrictive in practice and a (complementary) quantitative approach to verification is needed. For example, answers to questions like "What is the probability that the system satisfies its specification?" and "Do the systems behave almost (up to some small time fluctuations, say of one millisecond) the same?" provide us with (often more useful) quantitative information about the systems.

Metric spaces (and generalizations thereof) seem a good candidate for measuring the difference in behaviour of systems. The behaviour of many software systems can be described by means of coalgebras (of an endofunctor on the category of sets). For most systems, the endofunctor (on sets) associated to the coalgebra can be naturally extended to an endofunctor on metric spaces. This extended endofunctor having a terminal coalgebra is the key to the success of my approach to quantitative verification. The approach will be illustrated by considering restricted classes of real-time and probabilistic systems.

38 Geometric Constructivism and Topology

Steve Vickers

The connection between so-called "geometric" theories and toposes (their classifying toposes) has long been known, but it is not so easy to understand Grothendieck's insight that these capture "generalized topological spaces".

The generalization is that open sets are not enough to determine the topological structure, and the sheaves must be used instead. An example is the space of groups: the sheaves are the filtered-colimit-preserving functors from the category of groups to that of sets (the object part being the stalk function), while the opens, the subsheaves of 1, are constant and hence describe an indiscrete topology. In addition, Grothendieck discovered useful generalized spaces for which the sheaves were evident but the points were obscure. Consequently, the technical expression of generalized spaces has been austerely categorical and point-free.

This is a shame, for the use of toposes can be a good way of expressing notions of topology and continuity such as, for example, the continuity of domain constructors used in solving domain equations.
I give an overview of geometric constructivism, which uses those constructions preserved by inverse image functors of geometric morphisms between toposes: most importantly, colimits, finite limits, finite powersets and free algebra constructions. It is a stringent constructivism, valid in toposes and hence free of the axiom of choice, but also predicative, lacking full power sets and even sets of functions. It leads to an impoverished set theory, but compensates by the ease with which it handles spaces. The models of a geometric theory form a space (possibly generalized) with an intrinsic topology, and geometrically constructed transformations are automatically continuous.
Characteristically, geometric reasoning manipulates presentations of theories, such as information sytems for domains, entailment systems for spectral or stably compact spaces, and structures like formal topologies.

39 Continuous and Discrete Geometric Spaces as Partial Vector Spaces

Julian Webster

We present a theory of "partial" vector spaces (in which scalar multiplication is a partial rather than total operation) as an axiomatic framework for discrete and computational geometry. The work should be regarded as an attempt to extend digital topology to "digital geometry". Partial vector spaces over $\mathbb{R}$ include the "discrete" spaces $\mathbb{Z}^n$ as well as the "continuous" vector spaces $\mathbb{R}^n$. For each n, $\mathbb{R}^n$ is the *free* vector space over $\mathbb{Z}^n$, and this construction is fundamental in the understanding of discrete space. We consider *two* types of geometry on a partial space: its point-set geometry and its *cellular geometry* (using a cellular complex, as in the work of Kovalevsky). It is in the interaction of these geometries that we obtain versions of Radon's theorem and Helly's theorem.

40 On Computable Metric Spaces Tietze-Urysohn Extension is Computable

Klaus Weihrauch

We prove computable versions of Urysohn's lemma and the Tietze-Urysohn extension theorem for computable metric spaces. We use the TTE approach to computable analysis where objects are represented by finite or infinite sequences of symbols and computations transform sequences of symbols to sequences of symbols. The theorems hold for standard representations of the metric space, the set of real numbers, the set of closed subsets and the set of continuous functions. We show that there are *computable procedures* determining the continuous functions from the initial data (closed sets, continuous functions). The paper generalizes results by Yasugi, Mori and Tsujii in two ways: (1) The Tietze-Urysohn extension applies not only to "strictly effectively σ-compact co-r.e." sets but to all co-r.e. closed sets. (2) Not only computable functions exist for computable sets and functions, respectively, but there are *computable procedures* which determine continuous functions from arbitrary closed sets and continuous functions, respectively.

41 Sequents, Frames, and Completeness

Guo-Qiang Zhang

Entailment relations, originated from Scott, have been used for describing mathematical concepts constructively and for representing categories of domains. This paper gives an analysis of the freely generated frames from entailment relations. This way, we obtain completeness results under the unifying principle of the spatiality of coherence logic. In particular, the domain of disjunctive states, derived from the hyperresolution rule as used in disjunctive logic programs, can be seen as the frame freely generated from the *opposite* of a sequent structure. At the categorical level, we present equivalences among the categories of sequent structures, distributive lattices, and spectral locales using appropriate morphisms. This talk is based on joint work with Thierry Coquand.