Preprocessing for Outerplanar Vertex Deletion: An Elementary Kernel of Quartic Size

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Abstract

In the $\mathcal{F}$-Minor-Free Deletion problem one is given an undirected graph G, an integer k, and the task is to determine whether there exists a vertex set S of size at most k, so that $G - S$ contains no graph from the finite family $\mathcal{F}$ as a minor. It is known that whenever $\mathcal{F}$ contains at least one planar graph, then $\mathcal{F}$-Minor-Free Deletion admits a polynomial kernel, that is, there is a polynomial-time algorithm that outputs an equivalent instance of size $k^{\mathcal{O}(1)}$ [Fomin, Lokshtanov, Misra, Saurabh; FOCS 2012]. However, this result relies on non-constructive arguments based on well-quasi-ordering and does not provide a concrete bound on the kernel size.

We study the Outerplanar Deletion problem, in which we want to remove at most k vertices from a graph to make it outerplanar. This is a special case of $\mathcal{F}$-Minor-Free Deletion for the family $\mathcal{F} = \{K_4, K_{2,3}\}$. The class of outerplanar graphs is arguably the simplest class of graphs for which no explicit kernelization size bounds are known. By exploiting the combinatorial properties of outerplanar graphs we present elementary reduction rules decreasing the size of a graph. This yields a constructive kernel with $\mathcal{O}(k^4)$ vertices and edges. As a corollary, we derive that any minor-minimal obstruction to having an outerplanar deletion set of size k has $\mathcal{O}(k^4)$ vertices and edges.

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1 Introduction

Background and Motivation. Kernelization [19] is a subfield of parameterized complexity [7, 15] that investigates the complexity of preprocessing NP-hard problems. A parameterized problem includes in its input an integer k which we call the parameter. This parameter can be seen as a measure of complexity of the problem input. A common choice is to treat the size of the desired solution as the parameter. A kernelization is a polynomial-time preprocessing algorithm that converts a problem instance with parameter k into an equivalent parameterized instance of the same problem such that both the size and the parameter value of the new instance are bounded by a function f of k. The function f is called the size of the kernel. It is known that a decidable parameterized problem has a kernel if and only if it is fixed-parameter tractable [7, Lemma 2.2]. A major challenge is to determine which parameterized problems admit a kernel of polynomial size.

One class of problems that received much attention [17, 18, 23, 26, 28] is $\mathcal{F}$-Minor-Free Deletion. For a fixed finite family of graphs $\mathcal{F}$, the $\mathcal{F}$-Minor-Free Deletion problem asks, given a graph G and parameter k, whether a vertex set $S \subseteq V(G)$ of size k exists such that the graph $G - S$, obtained from G by removing the vertices in S, does not contain any graph $F \in \mathcal{F}$ as a minor. This class of problems includes a large variety of well-studied problems such as Vertex Cover, Feedback Vertex Set, and Planarization, which are obtained by taking $\mathcal{F}$ equal to (respectively) $\{K_2\}$, $\{K_3\}$, and $\{K_5, K_{3,3}\}$. All of the $\mathcal{F}$-Minor-Free Deletion problems are fixed-parameter tractable [38], but it is unknown whether they all admit a polynomial kernel [18]. If each graph in $\mathcal{F}$ contains at least one edge, it follows from the general results of Lewis and Yannakakis [30] that $\mathcal{F}$-Minor-Free Deletion is NP-hard.

If $\mathcal{F}$ is restricted to only families containing a planar graph we speak of Planar-$\mathcal{F}$ Deletion. Since the family of $\mathcal{F}$-minor-free graphs has bounded treewidth if and only if $\mathcal{F}$ includes a planar graph [36], this restriction ensures that removing a solution to the problem yields a graph of constant treewidth. Hence any solution is a treewidth-η modulator for some $\eta \in \mathbb{N}$ depending on $\mathcal{F}$. For this more restricted class Fomin et al. [18] have shown that polynomial kernels exist for each choice of $\mathcal{F}$. However, the running time of this kernelization algorithm is described by the authors as "horrendous" and regarding the size the authors state the following:

> The size of the kernel, however, is not explicit. Several of the constants that go into the proof of Lemma 29 depend on the size of the largest graph in certain antichains in a well-quasi-order and thus we don't know what the (constant) exponent bounding the size of the kernel is. We leave it to future work to make also the size of the kernel explicit.

For some specific Planar-$\mathcal{F}$ Deletion problems kernels with explicit size are known. Most famous are Vertex Cover and Feedback Vertex Set which admit kernels with respectively a linear and quadratic number of vertices [5, 25, 42]. Additionally, if θ_c denotes the graph with two vertices and $c \geq 1$ parallel edges, then $\{\theta_c\}$-Minor-Free Deletion admits a kernel with $\mathcal{O}(k^2 \log^{3/2} k)$ vertices and edges [17, Theorem 1.2]; note that the cases $c = 1$ and $c = 2$ correspond to Vertex Cover and Feedback Vertex Set. Another problem for which an explicit kernel size bound is known is Pathwidth-one Deletion, where the goal is to obtain a graph of pathwidth one, i.e, each connected component is a caterpillar. First a kernel of quartic size was given [34] which was later improved to a quadratic kernel [8]. If we want to remove at most k vertices to reduce the treedepth to at most η, we obtain the Treedepth-η Deletion problem. Since this property can be characterized by forbidden minors and bounded treedepth implies bounded treewidth, this problem is also a special case of Planar-$\mathcal{F}$ Deletion. Giannopoulou et al. [23] have shown that for every η, there is a kernel with $2^{\mathcal{O}(\eta^2)} \cdot k^6$ vertices for Treedepth-η Deletion. They have also proven that in general there is no hope for a universal constant in the kernel exponent and the degree of the polynomial which bounds the kernel size must increase as a function of $\mathcal{F}$ unless $\mathsf{NP} \not\subseteq \mathsf{coNP/poly}$.

In this paper we investigate Outerplanar Deletion, which asks for a graph G and parameter k whether a set $S \subseteq V(G)$ of size k exists such that $G - S$ is outerplanar. A graph is outerplanar if it admits a planar embedding for which all vertices lie on the outer face, or equivalently, if it does not contain K_4 or $K_{2,3}$ as a minor. Outerplanar graphs form a rich superclass of forests and are frequently studied in graph theory [4, 6, 10, 16, 41], graph drawing [1, 21, 32], and optimization [22, 31, 33, 35].

Since outerplanarity can be characterized as being $\{K_4, K_{2,3}\}$-minor-free [4], the problem belongs to the class of Planar-$\mathcal{F}$ Deletion problems. It is arguably the easiest problem in the class for which no explicit polynomial kernel is known. This makes Outerplanar Deletion a well-suited starting point to deepen our understanding of Planar-$\mathcal{F}$ Deletion problems in the search for explicit kernelization bounds.

Results. Let $\mathsf{opd}(G)$ denote the minimum size of a vertex set $S \subseteq V(G)$ such that $G - S$ is outerplanar. Our main result is the following theorem:

▶ **Theorem 1.1.** *The Outerplanar Deletion problem admits a polynomial-time kernelization algorithm that, given an instance (G, k), outputs an equivalent instance (G', k'), such that $k' \leq k$, graph G' is a minor of G, and G' has $\mathcal{O}(k^4)$ vertices and edges. Furthermore, if $\mathsf{opd}(G) \leq k$, then $\mathsf{opd}(G') = \mathsf{opd}(G) - (k - k')$.*

The algorithm behind Theorem 1.1 is elementary, consisting of a subroutine to build a decomposition of the input graph G using marking procedures in a tree decomposition, together with a series of explicit reduction rules. In particular, we avoid the use of protrusion replacement (summarized below). Concrete bounds on the hidden constant in the $\mathcal{O}$-notation follow from our arguments. The size bound depends on the approximation ratio of an approximation algorithm that bootstraps the decomposition phase, for which the current state-of-the-art is 40. We will therefore present a formula to obtain a concrete bound on the kernel size, rather than its value using the current-best approximation (which would exceed 10^5).

Theorem 1.1 presents the first concrete upper bound on the degree of the polynomial that bounds the size of kernels for Outerplanar Deletion. We hope that it will pave the way towards obtaining explicit size bounds for all Planar-$\mathcal{F}$ Deletion problems and give an impetus for research on the kernelization complexity of the Planar Deletion problem, which is one of the major open problems in kernelization today [40, 4:28],[19, Appendix A].

Via known connections [18] between kernelizations that reduce to a minor of the input graph and bounds on the sizes of obstruction sets, we obtain the following corollary.

▶ **Corollary 1.2.** *If G is a graph such that $\mathsf{opd}(G) > k$ but each proper minor G' of G satisfies $\mathsf{opd}(G') \leq k$, then G has $\mathcal{O}(k^4)$ vertices and edges.*

The existence of a polynomial bound with unknown degree follows from the work of Fomin et al. [18]; Corollary 1.2 gives the first explicit size bounds and contributes to a large body of research on minor-order obstructions (e.g. [3, 11, 12, 13, 29, 37, 39]).

Techniques. The known kernelization algorithms [17, 18] for Planar-$\mathcal{F}$ Deletion make use of (near-)protrusions. A protrusion is a vertex set that induces a subgraph of constant treewidth and boundary size. Protrusion replacement is a technique where sufficiently large protrusions are replaced by smaller ones without changing the answer. Protrusion techniques were first used to obtain kernels for problems on planar and other topologically-defined graph classes [2]. Later Fomin at al. [17] described how to use protrusion techniques for problems on general graphs. They proved [17, Lemma 3.3] that any graph G, which contains a modulator X to constant treewidth such that $|X|$ and the size of its neighborhood can be bounded by a polynomial in k, contains a protrusion of size $|V(G)|/k^{\mathcal{O}(1)}$ that can be found efficiently. For any fixed $\mathcal{F}$ containing a planar graph, they present a method to obtain a small modulator to an $\mathcal{F}$-minor-free graph, which has constant treewidth. This leads to a polynomial kernel for Planar-$\mathcal{F}$ Deletion on graphs with bounded degree since the size of the neighborhood of the modulator can be bounded so protrusion replacement can be used to obtain a polynomial kernel. Specifically for $\{\theta_c\}$-Minor-Free Deletion they give reduction rules to reduce the maximum degree in a general graph, which leads to a polynomial kernel on general graphs.

The kernel for Planar-$\mathcal{F}$ Deletion given by Fomin et al. [18] does not rely on bounding the size of the neighborhood of the modulator followed by protrusion replacement. Instead they present the notion of a near-protrusion: a vertex set that will become a protrusion after removing any size-k solution from the graph. With an argument based on well-quasi-ordering they determine that if such near-protrusions are large enough one can, in polynomial time, reduce to a proper minor of the graph without changing the answer.

In this paper we present a method for Outerplanar Deletion to decrease the size of the neighborhood of a modulator to outerplanarity. This relies on a process that was called "tidying the modulator" in earlier work [43] and also used in the kernelization for Chordal Vertex Deletion [27]. The result is a larger modulator $X \subseteq V(G)$ but with the additional feature that it retains its modulator properties when omitting any single vertex, that is, $G - (X \setminus \{x\})$ is outerplanar for each $x \in X$. We proceed by decomposing the graph into near-protrusions, following along similar lines as the decomposition by Fomin et al. [17] but exploiting the structure of outerplanar graphs at several steps to obtain such a decomposition with respect to our larger tidied modulator, without leading to worse bounds. With the additional properties of the modulator X obtained from tidying we no longer need to rely on well-quasi-ordering, but instead are able to reduce the size of the neighborhood of the modulator in two steps. The first reduces the number of connected components of $G - X$ which are adjacent to any particular modulator vertex $x \in X$. In the case of $\{\theta_c\}$-minor-free graphs, if $G - (X \setminus \{x\})$ is $\{\theta_c\}$-minor-free then bounding the number of components of $G - X$ adjacent to each $x \in X$ this is sufficient to bound $|N_G(X)|$, since any $x \in X$ has less than c neighbors in any component of $G - (X \setminus \{x\})$. One of the major difficulties we face when working with $\{K_{2,3}\}$-minor-free graphs is that in such a graph there can be arbitrarily many edges between a vertex x and a connected component of $G - (X \setminus \{x\})$. Therefore we present an additional reduction rule that reduces, in a second step, the number of edges between a vertex and a connected component. After these two steps we obtain a bound on the size of the neighborhood of the modulator. At this point, standard protrusion replacement could be applied to prove the *existence* of a kernel for Outerplanar Deletion with $\mathcal{O}(k^4)$ vertices. In order to give an explicit kernelization algorithm we present a number of additional reduction rules to avoid the generic protrusion replacement technique. This eventually leads to a kernel with at most $c \cdot k^4$ vertices and edges for Outerplanar Deletion. It is conceptually simple (yet tedious) to extract the explicit value of c from the algorithm description.

Organization. In the next section we give basic definitions and notation we use throughout the rest of the paper, together with structural observations for outerplanar graphs. Section 3 describes how we obtain small modulators to outerplanarity with progressively stronger properties, and finally we obtain a modulator of size $\mathcal{O}(k^4)$ such that each remaining component has only 4 neighbors in the modulator, effectively forming a decomposition into protrusions. The second stage of the kernelization reduces the size of the connected components outside the modulator. These reduction rules are described in Section 4. In Section 5 we finally tie everything together to obtain a kernel with $\mathcal{O}(k^4)$ vertices and edges. Due to space restrictions, all proofs (except Lemma 3.17) have been deferred to the full version [14].

2 Preliminaries

Kernelization. A parameterized problem is a decision problem in which every input has an associated positive integer parameter that captures its complexity in some well-defined way. For a parameterized problem $A \subseteq \Sigma^* \times \mathbb{N}$ and a function $f\colon \mathbb{N} \to \mathbb{N}$, a kernelization for A of size f is an algorithm that, on input $(x, k) \in \Sigma^* \times \mathbb{N}$, takes time polynomial in $|x| + k$ and outputs $(x', k') \in \Sigma^* \times \mathbb{N}$ such that the following holds:

1. $(x, k) \in A$ if and only if $(x', k') \in A$, and
2. both $|x'|$ and k' are bounded by $f(k)$.

Graph theory. The set $\{1, \ldots, p\}$ is denoted by $[p]$. We consider simple undirected graphs without self-loops. A graph G has vertex set $V(G)$ and edge set $E(G)$. We use shorthand $n = |V(G)|$ and $m = |E(G)|$. For (not necessarily disjoint) $A, B \subseteq V(G)$, we define $E_G(A, B) = \{uv \mid u \in A, v \in B, uv \in E(G)\}$. The open neighborhood of $v \in V(G)$ is $N_G(v) := \{u \mid uv \in E(G)\}$, where we omit the subscript G if it is clear from context. For a vertex set $S \subseteq V(G)$ the open neighborhood of S, denoted $N_G(S)$, is defined as $\bigcup_{v \in S} N_G(v) \setminus S$. The closed neighborhood of a single vertex v is $N_G[v] := N_G(v) \cup \{v\}$, and the closed neighborhood of a vertex set S is $N_G[S] := N_G(S) \cup S$. The boundary of a vertex set $S \subseteq V(G)$ is the set $\partial_G(S) = N_G(V(G) \setminus S)$. For $A \subseteq V(G)$, the graph induced by A is denoted by $G[A]$ and we say that the vertex set A is connected if the graph $G[A]$ is connected. We use notation $G\langle A \rangle = G[N_G[A]]$ and, when H is an induced subgraph of G, we write briefly $G\langle H \rangle = G\langle V(H) \rangle$ or $\partial_G(H) = \partial_G(V(H))$. We use shorthand $G - A$ for the graph $G[V(G) \setminus A]$. For $v \in V(G)$, we write $G - v$ instead of $G - \{v\}$. For $A \subseteq E(G)$ we denote by $G \setminus A$ the graph with vertex set $V(G)$ and edge set $E(G) \setminus A$. For $e \in E(G)$ we write $G \setminus e$ instead of $G \setminus \{e\}$. If $e = uv$, then $V(e) = \{u, v\}$.

A vertex $v \in V(G)$ is an articulation point in a connected graph G if $G - v$ is not connected. A graph is called biconnected if it has no articulation points. A biconnected component in G is an inclusion-wise maximal subgraph which is biconnected. A vertex set $A \subseteq V(G)$ is an independent set in G if $E_G(A, A) = \emptyset$. A graph G is bipartite if there is a partition of $V(G)$ into two independent sets A, B. We write shortly $G = (A \cup B, E)$ to specify a bipartite graph on vertex set $E = E(G)$ admitting this partition.

▶ **Definition 2.1.** *For a vertex set $X \subseteq V(G)$ the* component graph $\mathcal{C}(G, X)$ *is a bipartite graph $(X \cup Y, E)$, where Y is the set of connected components of $G - X$, and $(v, C) \in E$ if there is at least one edge between $v \in X$ and the component $C \in Y$.*

For an integer q, the graph K_q is the complete graph on q vertices. For integers p, q, the graph $K_{p,q}$ is the bipartite graph $(A \cup B, E)$, where $|A| = p$, $|B| = q$, and $uv \in E$ whenever $u \in A, v \in B$.

Minors. A contraction of $uv \in E(G)$ introduces a new vertex adjacent to all of $N_G(\{u, v\})$, after which u and v are deleted. The result of contracting $uv \in E(G)$ is denoted G/uv. For $A \subseteq V(G)$ such that $G[A]$ is connected, we say we contract A if we simultaneously contract all edges in $G[A]$ and introduce a single new vertex. We say that H is a minor of G, if we can turn G into H by a (possibly empty) series of edge contractions, edge deletions, and vertex deletions. If this series is non-empty, then H is called a proper minor of G.

Planar and outerplanar graphs. A graph is called planar if it admits a plane embedding. By Wagner's theorem, a graph G is planar if and only if G contains neither K_5 nor $K_{3,3}$ as a minor. A graph is called outerplanar if it admits a plane embedding with all vertices lying on the outer face. A graph G is outerplanar if and only if G contains neither K_4 nor $K_{2,3}$ as a minor [4]. A graph G is planar (resp. outerplanar) if and only if every biconnected component in G induces a planar (resp. outerplanar) graph. Recall that $G\langle C\rangle = G[N_G(V(C))]$.

▶ **Observation 2.2.** *Let $v \in V(G)$. The graph G is outerplanar if and only if for each connected component C of $G - v$ the graph $G\langle C\rangle$ is outerplanar.*

For a graph G we call $S \subseteq V(G)$ an outerplanar deletion set if $G - S$ is outerplanar. The outerplanar deletion number of G, denoted $\mathsf{opd}(G)$, is the size of a smallest outerplanar deletion set in G.

Structural properties of outerplanar graphs. We present a number of structural observations of outerplanar graphs which will be useful in our later argumentation. The first is a characterization of outerplanar graphs similar to Observation 2.2. Rather than looking at the components of a graph with one vertex removed, it considers the components of a graph with both endpoints of an edge removed. This allows us for example to easily argue about outerplanarity of graphs obtained from "gluing" two outerplanar graphs on two adjacent vertices.

▶ **Lemma 2.3.** *Let G be a graph and $e \in E(G)$. Then G is outerplanar if and only if both of the following conditions hold:*
1. *for each connected component C of $G - V(e)$ the graph $G\langle C\rangle$ is outerplanar, and*
2. *the graph $G \setminus e$ does not have three induced internally vertex-disjoint paths connecting the endpoints of e.*

In order to more easily apply Lemma 2.3, we show that no two induced paths as referred to in Lemma 2.3(2) can lie in the same connected component C as referred to in Lemma 2.3(1).

▶ **Lemma 2.4.** *Suppose G is outerplanar with an edge $uv \in E(G)$. If P_1, P_2 are internally vertex-disjoint (u, v)-paths in $G \setminus uv$, then the interiors of P_1 and P_2 lie in different connected components of $G - \{u, v\}$.*

We now give a condition under which an edge can be added to an outerplanar without violating outerplanarity. Intuitively, this corresponds to adding an edge between two vertices that lie on the same interior face.

▶ **Lemma 2.5.** *Suppose G is outerplanar and vertices x, y lie on an induced cycle D with $xy \notin E(G)$. Then adding the edge xy to G preserves outerplanarity.*

Finally, we observe that if an outerplanar graph G has a cycle C, then any component of $G - V(C)$ is adjacent to at most two vertices of the cycle (else there would be a K_4 minor), and these must be consecutive on the cycle (else there would be a $K_{2,3}$ minor).

▶ **Lemma 2.6.** *If C is a cycle in an outerplanar graph G, then each connected component of $G - V(C)$ has at most two neighbors in C, and they must be consecutive along the cycle.*

3 Splitting the graph into pieces

In this section we show how to reduce any input of Outerplanar Deletion to an equivalent instance which admits a decomposition into a modulator of bounded size along with a bounded number of outerplanar components containing at most four neighbors of the modulator.

3.1 The augmented modulator

The starting point for both our kernelization algorithm and the one from Fomin et al. [18] is to employ a constant-factor approximation algorithm. We however begin with a different approximation algorithm, which has two advantages. First, the algorithm is constructive: it relies only on separating properties of bounded-treewidth graphs and rounding a fractional solution from a linear programming relaxation. Second, the approximation factor can be pinned down to a concrete value. In the full version, we show how the general theorem by Gupta et al. [24] implies the following.

▶ **Theorem 3.1** ([24]). *There is a polynomial-time deterministic 40-approximation algorithm for Outerplanar Deletion.*

In our setting, for a given graph G and integer k, we want to determine whether G admits an outerplanar deletion set of size at most k. Thanks to the theorem above, we can assume that we are given an outerplanar deletion set X (also called a modulator to outerplanarity) of size at most $40 \cdot k$. As a next step, we would like to augment this set to satisfy a stronger property. This step is inspired by the technique of tidying the modulator from van Bevern, Moser, and Niedermeier [43]. For each vertex $v \in X$ we would like to be able to "put it back" into $G - X$ while maintaining outerplanarity. In order to do so, we look for a set of vertices from $V(G) \setminus X$ that needs to be removed if v is put back. Since $G - X$ is outerplanar and hence has treewidth at most two, we can construct such a set of moderate size by a greedy approach. We scan a tree decomposition in a bottom-up manner and look for maximal subgraphs that are outerplanar when considered together with v. When such a subgraph cannot be further extended we mark one bag of a decomposition, which gives 3 vertices to be removed. We show that this idea leads to a 3-approximation algorithm. While this approach based on covering/packing duality is well-known, for completeness we include the proof in the full version.

▶ **Lemma 3.2.** *There is a polynomial-time algorithm that, given a graph G, an integer k, and a vertex v such that $G - v$ is outerplanar, either finds an outerplanar deletion set $S \subseteq V(G) \setminus \{v\}$ in G of size of most $3k$ or correctly concludes that there is no outerplanar deletion set $S \subseteq V(G) \setminus \{v\}$ in G of size of most k.*

Observe that if it is impossible to remove k vertices from $G - (X \setminus \{v\})$ to make it outerplanar, then any outerplanar deletion set in G of size at most k must contain v. In this situation it suffices to solve the problem on $G - v$. Otherwise, we identify a set $R(v)$ of at most $3k$ vertices whose removal allows v to be put back in $G - X$ without spoiling outerplanarity. After inserting $R(v)$ into the set X, we could put v back "for free". Let us formalize this idea of augmenting the modulator.

▶ **Definition 3.3.** *A* (k, c)-augmented modulator *in graph G is a pair of disjoint sets $X_0, X_1 \subseteq V(G)$ such that:*

1. *$G - X_0$ is outerplanar,*
2. *for each $v \in X_0$, there is a set $R(v) \subseteq X_1$, such that $|R(v)| \le 3k$ and $G - ((X_0 \setminus \{v\}) \cup R(v))$ is outerplanar, and*
3. *$|X_0| \le c \cdot k$, $X_1 = \bigcup_{v \in X_0} R(v)$, which implies $|X_1| \le 3c \cdot k^2$.*

We classify the pairs of vertices within $X_0 \cup X_1$. A pair (u, v): $u, v \in X_0 \cup X_1$ is of type:
A: if $u, v \in X_0$ or $(u \in X_0, v \in R(u))$ or $(v \in X_0, u \in R(v))$,
B: if (u, v) is not of type A and $\{u, v\} \cap X_0 \neq \emptyset$,
C: if $u, v \in X_1$.
We note that the number of type-A pairs is at most $c(3 + c) \cdot k^2$, the number of type-B pairs is at most $3c^2 \cdot k^3$, and the number of type-C pairs is at most $9c^2 \cdot k^4$.

The downside of the augmented modulator is that its size can be as large as $\mathcal{O}(k^2)$. However, in return we obtain an even stronger property than previously sketched. For most of the pairs of vertices u, v from the augmented modulator (X_0, X_1), putting them back into $G - (X_0 \cup X_1)$ at the same time still does not break outerplanarity. This property will come in useful for bounding the size of the kernel.

▶ **Observation 3.4.** *Let (X_0, X_1) be a (k, c)-augmented modulator in a graph G. Then for each $v \in X_0 \cup X_1$, the graph $G - (X_0 \cup X_1 \setminus \{v\})$ is outerplanar. Furthermore, if $u, v \in X_0 \cup X_1$ and the pair (u, v) is of type B or C, then the graph $G - (X_0 \cup X_1 \setminus \{u, v\})$ is outerplanar.*

Let us summarize what we can compute so far. We say that instances (G, k) and (G', k') are equivalent if $\mathsf{opd}(G) \leq k \Leftrightarrow \mathsf{opd}(G') \leq k'$.

▶ **Lemma 3.5.** *There is a polynomial-time algorithm that, given an instance (G, k), either correctly concludes that $\mathsf{opd}(G) > k$ or outputs an equivalent instance (G', k'), where $k' \leq k$ and G' is a subgraph of G, along with a $(k', 40)$-augmented modulator in G'. If $\mathsf{opd}(G) \leq k$ then it holds that $\mathsf{opd}(G') = \mathsf{opd}(G) - (k - k')$. Moreover, if for every vertex $v \in V(G)$ there is an outerplanar deletion set $S \subseteq V(G) \setminus \{v\}$ in G of size at most k, then $k' = k$.*

The reduction step above is the only one in our algorithm that may decrease the value of k. Moreover, no further reduction will modify the outerplanar deletion number as long as $\mathsf{opd}(G) \leq k$. This observation will come in useful for bounding the size of minimal minor obstructions to having an outerplanar deletion set of size k.

As the next step, we would like to bound the number of connected components in $G - (X_0 \cup X_1)$ and the number of connections between the components and the modulator vertices. We show that if vertices $u, v \in X_0 \cup X_1$ are adjacent to sufficiently many components, then at least one of u, v must be removed in any solution of size at most k. Together with the "putting back" property of the augmented modulator, this allows us to forget some of the edges without modifying the space of solutions of size at most k. We formalize this idea with the following marking scheme.

▶ **Reduction Rule 1.** *Let G be a graph, $k \in \mathbb{N}$, and (X_0, X_1) be a (k, c)-augmented modulator in G. Consider the component graph $\mathcal{C}(G, X_0 \cup X_1)$. For each pair $u, v \in X_0 \cup X_1$ choose up to $k + 3$ components C_i with edges to both u and v, and mark the edges $(u, C_i), (v, C_i)$ in $\mathcal{C}(G, X_0 \cup X_1)$. If an edge (v, C) is unmarked in the end, remove all the edges between v and C in G. If some component C of $G - (X_0 \cup X_1)$ or a vertex $v \in X_0 \cup X_1$ becomes isolated, remove it from G.*

▶ **Lemma 3.6** (Safeness). *Let G be a graph, $k \in \mathbb{N}$, and (X_0, X_1) be a (k, c)-augmented modulator in G. Let G' be obtained from G by applying Reduction Rule 1 with respect to (X_0, X_1, k). If $\mathsf{opd}(G) > k$ then $\mathsf{opd}(G') > k$ and if $\mathsf{opd}(G) \leq k$ then $\mathsf{opd}(G') = \mathsf{opd}(G)$.*

We show that after application of Reduction Rule 1 the component graph $\mathcal{C}(G, X_0 \cup X_1)$ cannot be too large. This will come in useful for proving further upper bounds. We could trivially bound the number of its edges by $|X_0 \cup X_1|^2 \cdot (k + 3) = \mathcal{O}(k^5)$ but, thanks to the properties of the augmented modulator, we can be more economical.

Recall the types of pairs from Definition 3.3 and their properties from Observation 3.4. We know that the number of type-A pairs is at most $c(3 + c) \cdot k^2$ and the number of type-B pairs is at most $3c^2 \cdot k^3$. The pairs of type B can be inserted back into $G - (X_0 \cup X_1)$ without affecting its outerplanarity. This implies that each type-B pair is responsible for marking at most 2 edges. Finally, the total number of edges marked due to type-C pairs is $\mathcal{O}(k^2)$.

▶ **Lemma 3.7.** *After the application of Rule 1 with respect to a (k,c)-augmented modulator (X_0, X_1), the number of vertices and edges in $\mathcal{C}(G, X_0 \cup X_1)$ is at most $f_1(c) \cdot (k+3)^3$, where $f_1(c) = 14c^2 + 60c$.*

3.2 The outerplanar decomposition

We proceed by enriching the augmented modulator further. We would like to provide additional properties at the expense of growing the modulator size to $\mathcal{O}(k^3)$. For two vertices u, v in an augmented modulator (X_0, X_1) ideally we would like to ensure that no two components of $G - (X_0 \cup X_1 \cup Z)$ are adjacent to both u and v, where Z is some vertex set of size $\mathcal{O}(k^3)$. This is not always possible, but we will guarantee that in such a case any outerplanar deletion set of size at most k must contain either u or v.

▶ **Definition 3.8.** *Let $Y \subseteq V(G)$ be a vertex subset in a graph G. We say that $u, v \in Y$ are Y-separated if no connected component of $G - Y$ is adjacent to both u and v.*

In Lemma 3.9 we are going to show that when G is outerplanar and $X \subseteq V(G)$, then there always exists a small set $Y \subseteq V(G)$ so that every pair from X is $(X \cup Y)$-separated.

▶ **Lemma 3.9.** *There is a polynomial-time algorithm that, given a vertex set $X \subseteq V(G)$ in an outerplanar graph G, finds a vertex set $Y \subseteq V(G) \setminus X$ of size at most $4 \cdot |X|$, so that every pair $u, v \in X$ with $u \neq v$ is $(X \cup Y)$-separated.*

Given an augmented modulator (X_0, X_1), we would like to find a set Z of moderate size so that for each pair (u, v) from $X_0 \cup X_1$ either u, v are $(X_0 \cup X_1 \cup Z)$-separated or there exist $k+4$ internally vertex-disjoint paths, with non-empty interior, connecting u and v in G. If the latter case occurs, then any outerplanar deletion set of size bounded by k, can intersect at most k of these paths' interiors. Therefore, this solution must remove either u or v in order to get rid of all $K_{2,3}$-minors. We remark that this property already holds if we request $k+3$ disjoint (u, v)-paths, but in this stronger form it also holds for a graph obtained from G by an edge removal. This fact will be crucial for the safeness proof for Reduction Rule 3.

In order to find the set Z, we could consider all pairs (u, v) from $X_0 \cup X_1$ and, if there exists an (u, v)-separator of size at most $k+3$, add it to Z. This however would make Z as large as $\mathcal{O}(k^5)$. We can make this process more economical by analyzing what happens for different types of pairs from Definition 3.3, similarly as in Lemma 3.7.

▶ **Lemma 3.10.** *There is a polynomial-time algorithm that, given an instance (G, k) with (k, c)-augmented modulator (X_0, X_1), returns a set $Z \subseteq V(G) \setminus (X_0 \cup X_1)$ of size at most $f_2(c) \cdot (k+3)^3$, where $f_2(c) = 4c^2 + 15c$, such that for each pair $u, v \in X_0 \cup X_1$ of distinct vertices one of the following holds:*

1. *vertices u, v are $(X_0 \cup X_1 \cup Z)$-separated, or*
2. *there are $k+4$ vertex-disjoint paths, with non-empty interior, connecting u and v in G.*

We would like to simplify the interface between a connected component C of $G - (X_0 \cup X_1 \cup Z)$ and the rest of the graph. Since $G - X_0$ is outerplanar, it has treewidth at most two, which implies there is a tree decomposition in which each pair of distinct bags intersects in at most 2 vertices. When constructing a separator $Z' \supseteq Z$ via the lowest common ancestor closure (see [20, §9.3.3]), the neighborhood of each connected component C of $G - Z'$ within the set Z' is contained in at most two bags of the decomposition. This allows us to guarantee that $|N_G(C) \cap Z'| \leq 4$.

▶ **Lemma 3.11.** *There is a polynomial-time algorithm that, given an outerplanar graph G and $Z \subseteq V(G)$, returns a set $Z' \supseteq Z$ of size at most $6 \cdot |Z|$ such that each connected component of $G - Z'$ has at most four neighbors in Z'.*

In order to keep the kernel size in check, we need to analyze the number of connected components of $G - (X_0 \cup X_1 \cup Z)$. We have managed to bound the size of Z by $\mathcal{O}(k^3)$ and, in Lemma 3.7, we have also bounded by $\mathcal{O}(k^3)$ the number of edges in the component graph $\mathcal{C}(G, X_0 \cup X_1)$. These two properties suffice to also bound the number of connected components of $G - (X_0 \cup X_1 \cup Z)$ that have at least two neighbors in $X_0 \cup X_1 \cup Z$. It will be easier to deal with the remaining ones later.

▶ **Lemma 3.12.** *Let (X_0, X_1) be a (k, c)-augmented modulator in G, so that the component graph $\mathcal{C}(G, X_0 \cup X_1)$ has at most s vertices and s edges, and let $Z \subseteq V(G) \setminus (X_0 \cup X_1)$. Then there are at most $3 \cdot s + 4 \cdot |Z|$ components of $G - (X_0 \cup X_1 \cup Z)$ that have two or more neighbors in $X_0 \cup X_1 \cup Z$.*

The previous lemma gives us a bound on the number of components outside the modulator with at least two neighbors. To bound the total number of components outside the modulator, we employ the following reduction rule to remove the remaining components with at most one neighbor.

▶ **Reduction Rule 2.** *If for some $C \subseteq V(G)$ the graph $G\langle C \rangle$ is outerplanar and it holds that $|N_G(C)| \le 1$, then remove the vertex set C.*

Safeness of this rule follows from Observation 2.2, which implies $\mathsf{opd}(G - C) = \mathsf{opd}(G)$.

With these properties at hand, we are able to construct the desired extension of the augmented modulator. The decomposition below is inspired by the notion of a near-protrusion [18], combined with the idea of the augmented modulator, and with an $\mathcal{O}(k^3)$ bound on the number of leftover connected components.

▶ **Definition 3.13.** *For $k, c, d \in \mathbb{N}$ a (k, c, d)-outerplanar decomposition of a graph G is a triple (X_0, X_1, Z) of disjoint vertex sets in G, such that:*

1. *(X_0, X_1) is a (k, c)-augmented modulator for (G, k),*
2. *for each pair $u, v \in X_0 \cup X_1$ of distinct vertices one of the following holds:*
   a. *vertices u, v are $(X_0 \cup X_1 \cup Z)$-separated, or*
   b. *there are $k + 4$ vertex-disjoint (u, v)-paths in G, each with non-empty interior.*
3. *for each connected component C of $G - (X_0 \cup X_1 \cup Z)$ it holds that $|N_G(C) \cap Z| \le 4$,*
4. *$|Z| \le d \cdot (k+3)^3$ and there are at most $d \cdot (k+3)^3$ connected components in $G - (X_0 \cup X_1 \cup Z)$.*

▶ **Lemma 3.14.** *There is a constant c, a function $f_3 \colon \mathbb{N} \to \mathbb{N}$, and a polynomial-time algorithm that, given an instance (G, k), either returns an equivalent instance (G', k'), where $k' \le k$ and G' is subgraph of G, along with a $(k', c, f_3(c))$-outerplanar decomposition of G', or concludes that $\mathsf{opd}(G) > k$. If $\mathsf{opd}(G) \le k$ then it holds that $\mathsf{opd}(G') = \mathsf{opd}(G) - (k - k')$. Furthermore, $c = 40$ and $f_3(c) = 3 \cdot f_1(c) + 24 \cdot f_2(c)$ (see Lemmas 3.7 and 3.10).*

As the last property of the (k, c, d)-outerplanar decomposition, we formulate the bound on the total number of connections between $X_0 \cup X_1 \cup Z$ and the leftover components, which will lead to the total kernel size $\mathcal{O}(k^4)$.

▶ **Lemma 3.15.** *Let (X_0, X_1, Z) be a (k, c, d)-outerplanar decomposition of a graph G. Then the number of edges in the component graph $\mathcal{C}(G, X_0 \cup X_1 \cup Z)$ is at most $f_4(c, d) \cdot (k + 3)^4$, where $f_4(c, d) = cd + 6c + 4d$.*

3.3 Reducing the size of the neighborhood

Given a (k, c, d)-outerplanar decomposition (X_0, X_1, Z), we will now present the final reduction rule to reduce the size of the neighborhood $N_G(X_0 \cup X_1)$ to $\mathcal{O}(k^4)$. As the size of Z is already bounded by $\mathcal{O}(k^3)$ we focus on reducing the size of $N_G(X_0 \cup X_1) \setminus Z$. We have already shown the number of edges in the component graph $\mathcal{C}(G, X_0 \cup X_1 \cup Z)$ is bounded by $\mathcal{O}(k^4)$, so it suffices to reduce the number of edges between a single modulator vertex $x \in X_0 \cup X_1$ and a connected component C of $G - (X_0 \cup X_1 \cup Z)$ to a constant. For this, we first show in Lemma 3.16 where the neighbors of x occur in C.

In the following lemma, we consider an outerplanar graph G containing a vertex x. When omitting vertex x from a drawing of G, the vertices of $N_G(x)$ remain on the outer face of the graph. If $G - x$ is still connected, then there is a subpath P of the outer face which visits all of $N_G(x)$. The outerplanarity of G ensures that P can be chosen to be induced and to contain at most two vertices from each biconnected component of $G - x$. Furthermore, it follows from Lemma 2.6 that the latter must be consecutive. This is formalized as follows.

▶ **Lemma 3.16.** *Suppose G is outerplanar, $x \in V(G)$, and $G - x$ is connected. Then the vertices from $N_G(x)$ lie on an induced path P in $G - x$ such that for each biconnected component B of $G - x$ and each pair of distinct vertices $u, v \in V(P) \cap V(B)$ we have that $uv \in E(G - x)$. We can find such a path in polynomial time.*

We now investigate what happens when a modulator vertex $x \in X_0 \cup X_1$ is the only vertex in $X_0 \cup X_1$ that is adjacent to a connected component C of $G - (X_0 \cup X_1 \cup Z)$. If x has sufficiently many edges to a part of C that is not adjacent to Z, then one of these edges can be removed without affecting the outerplanar deletion number $\mathsf{opd}(G)$. We will also exploit this property for a reduction rule later in this paper when we reduce the number of edges within a connected component of $G - (X_0 \cup X_1 \cup Z)$ (see Figure 3, left side). Since the following lemma is the key ingredient in our algorithm, we include its full proof.

▶ **Lemma 3.17.** *Suppose we are given a graph G, a vertex $x \in V(G)$, and five vertices $v_1, \ldots, v_5 \in N_G(x)$ that lie, in order of increasing index, on an induced path P in $G - x$ from v_1 to v_5, such that $N_G(x) \cap V(P) = \{v_1, \ldots, v_5\}$. Let C be the component of $G - \{v_1, v_5, x\}$ containing $P - \{v_1, v_5\}$. If $G\langle C \rangle$ is outerplanar, then $\mathsf{opd}(G) = \mathsf{opd}(G \setminus xv_3)$.*

Proof. Clearly for any $S \subseteq V(G)$ if $G - S$ is outerplanar, then $G \setminus xv_3 - S$ is also outerplanar, hence $\mathsf{opd}(G) \geq \mathsf{opd}(G \setminus xv_3)$. To show $\mathsf{opd}(G) \leq \mathsf{opd}(G \setminus xv_3)$, suppose $G \setminus xv_3 - S$ is outerplanar for some arbitrary $S \subseteq V(G)$. If $x \in S$ or $v_3 \in S$ then clearly $G - S$ is outerplanar, so suppose $x, v_3 \notin S$. We show $G - S'$ is outerplanar for some $S' \subseteq V(G)$ with $|S'| \leq |S|$. Consider the following cases:

1. If $|S \cap V(P)| = 0$ then $G \setminus xv_3 - S$ contains an induced cycle formed by x together with the subpath of P from v_2 to v_4. This cycle includes x and v_3, so by Lemma 2.5 the graph $G \setminus xv_3 - S$ remains outerplanar after adding the edge xv_3, hence $G - S$ is outerplanar.
2. If $|S \cap V(P)| \geq 2$ then let $S' := \{v_1, v_5\} \cup (S \setminus V(C))$. Since $|S'| \leq |S|$, showing that $G - S'$ is outerplanar proves the claim. Let $\overline{C} := G - V(C)$ and note that $\overline{C} - S'$ is outerplanar since it is a subgraph of $G \setminus xv_3 - S$. Also note that $G[V(C) \cup \{x\}]$ is outerplanar since it is a subgraph of $G[V(C) \cup \{v_1, v_5, x\}] = G\langle C \rangle$. Since for any connected component H of $G - S' - x$ the graph $(G - S')\langle H \rangle$ is a subgraph of $\overline{C} - S'$ or $G[V(C) \cup \{x\}]$ we have that $(G - S')\langle H \rangle$ is outerplanar. Then by Observation 2.2 the graph $G - S'$ is outerplanar.
3. If $|S \cap V(P)| = 1$ then let $u \in S \cap V(P)$ and assume without loss of generality that u lies on the subpath of P from v_3 to v_5, so the subpath of P from v_1 to v_3 does not contain

vertices of S (recall that $v_3 \notin S$). Let $S' := \{v_5\} \cup (S \setminus V(C))$ and note that $|S'| \leq |S|$. We shall show that $G - S'$ is outerplanar. Since $x, v_1 \notin S$, we have that also $x, v_1 \notin S'$, so $xv_1 \in E(G - S')$. In order to apply Lemma 2.3 to $G - S'$ and xv_1 we have to show that

- for each connected component C' of $G - S' - \{v_1, x\}$ the graph $(G - S')\langle C' \rangle$ is outerplanar, and
- there are at most two induced internally vertex-disjoint (v_1, x)-paths in $(G - S') \setminus v_1x$.

Because $v_5 \in S'$ we have $G - S' - \{v_1, x\} = G - \{v_1, v_5, x\} - S'$ and since C is a connected component of $G - \{v_1, v_5, x\}$ we have that all connected components of $G - S' - \{v_1, x\}$ are either a connected component of $C - S' = C$ or of $G - S' - \{v_1, x\} - V(C)$. It is given that C is connected and $G[V(C) \cup \{v_1, v_5, x\}]$ is outerplanar so then $G[V(C) \cup \{v_1, x\}] = (G - S')\langle C \rangle$ is also outerplanar. Any other connected component C' is a connected component of $G - S' - \{v_1, x\} - V(C)$, so we have that $(G - S')\langle C' \rangle$ is a subgraph of $G - S' - V(C)$. This is in turn, a subgraph of $G \setminus xv_3 - S$ which is outerplanar. Hence $(G - S')\langle C' \rangle$ is outerplanar.

It remains to show that there are at most two induced internally vertex-disjoint (v_1, x)-paths in $(G - S') \setminus v_1x$. Suppose for contradiction that $(G - S') \setminus v_1x$ contains three induced vertex-disjoint (v_1, x)-paths. As shown before, C is a connected component of $G - S' - \{v_1, x\}$ adjacent to v_1 and x, so there exists an induced (v_1, x)-path P_1 in $G - S' \setminus v_1x$ whose internal vertices all lie in C. Since $G\langle C \rangle$ is outerplanar and C is connected, by Lemma 2.4 the graph $G\langle C \rangle$ does not contain two vertex-disjoint (v_1, x)-paths with nonempty interiors. Hence there are two induced internally vertex-disjoint (v_1, x)-paths P_2, P_3 in $(G - S' \setminus v_1x) - V(C)$. Observe that P_2 and P_3 are then disjoint from $S \setminus S' \subseteq V(C)$ and do not contain xv_3. It follows that P_1, P_2 and P_3 are three induced internally vertex-disjoint (v_1, x)-paths in $G \setminus xv_3 - S$, contradicting its outerplanarity by Lemma 2.3. We conclude also the second condition of Lemma 2.3 holds for $G - S'$ and the edge v_1x, hence $G - S'$ is outerplanar. ◀

Next, we use the properties of the (k, c, d)-outerplanar decomposition to show that any solution of size at most k contains all but possibly one vertex from $(X_0 \cup X_1) \cap N_G(C)$, where C is a connected component from $G - (X_0 \cup X_1 \cup Z)$. Adding this vertex to C preserves its outerplanarity because (X_0, X_1) is a (k, c)-augmented modulator. We use this fact together with the result from Lemma 3.17 to identify an irrelevant edge, which leads to the following reduction rule:

▶ **Reduction Rule 3.** *Given a (k, c, d)-outerplanar decomposition (X_0, X_1, Z) of a graph G, a vertex $x \in X_0 \cup X_1$, and five vertices $v_1, \ldots, v_5 \in N_G(x) \setminus (X_0 \cup X_1)$ that lie, in order of increasing index, on an induced path P in $G - (X_0 \cup X_1)$ from v_1 to v_5, such that $N_G(x) \cap V(P) = \{v_1, \ldots, v_5\}$. Let C be the component of $G - (X_0 \cup X_1) - \{v_1, v_5\}$ containing $P - \{v_1, v_5\}$. If $V(C) \cap Z = \emptyset$ remove the edge xv_3.*

▶ **Lemma 3.18** (Safeness). *Suppose that Reduction Rule 3 removes the edge $e = xv_3$ from a graph G. If $\mathsf{opd}(G) > k$ then $\mathsf{opd}(G \setminus e) > k$ and if $\mathsf{opd}(G) \leq k$ then $\mathsf{opd}(G \setminus e) = \mathsf{opd}(G)$.*

We show how this reduction rule can be applied to reduce the number of edges between a vertex $x \in X_0 \cup X_1$ and a connected component in $G - (X_0 \cup X_1 \cup Z)$ to a constant, as depicted on Figure 1. This leads to an $\mathcal{O}(k^4)$ bound on $N_G(X_0 \cup X_1)$.

▶ **Lemma 3.19.** *There is a polynomial-time algorithm that, given a (k, c, d)-outerplanar decomposition (X_0, X_1, Z) of a graph G, a vertex $x \in X_0 \cup X_1$ and a component C of $G - (X_0 \cup X_1 \cup Z)$, either applies Reduction Rule 3 or concludes that $|N_G(x) \cap V(C)| \leq 20$.*

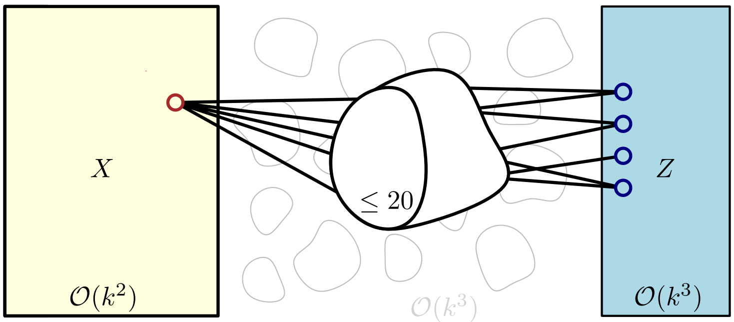

Figure 1 An illustration of Lemma 3.19. Given a (k, c, d)-outerplanar decomposition (X_0, X_1, Z) of a graph G, a vertex $x \in X = X_0 \cup X_1$ and a component C of $G - (X_0 \cup X_1 \cup Z)$, we are guaranteed that $|N(C) \cap Z| \leq 4$ and we can apply Reduction Rule 3 until $|N(x) \cap V(C)| \leq 20$. The expressions at the bottom bound the size of X, the number of components of $G - (X \cup Z)$, and size of Z.

We are going to apply Lemma 3.19 to a computed outerplanar decomposition in order to reduce the total neighborhood size of $X_0 \cup X_1$. This allows us to construct a final modulator L of size $\mathcal{O}(k^4)$ with a structure referred to in previous works as a protrusion decomposition. We can now proceed to stating a lemma that encapsulates application of Reduction Rule 3.

▶ **Lemma 3.20.** *There exists a function $f_5 \colon \mathbb{N}^2 \to \mathbb{N}$ and a polynomial-time algorithm that, given a (k, c, d)-outerplanar decomposition (X_0, X_1, Z) of a graph G, either applies Reduction Rule 2 or Reduction Rule 3, or outputs a set $L \subseteq V(G)$ such that*
1. $|L| \leq f_5(c, d) \cdot (k + 3)^4$,
2. $|E_G(L, L)| \leq f_5(c, d) \cdot (k + 3)^4$,
3. *there are at most $f_5(c, d) \cdot (k + 3)^4$ connected components in $G - L$, and*
4. *for each connected component C of $G - L$ the graph $G\langle C\rangle$ is outerplanar and $|N_G(C)| \leq 4$.*

Furthermore, $f_5(c, d) = 24 \cdot (20 \cdot f_4(c, d) + d + c + c^2)$ (see Lemma 3.15).

4 Compressing the outerplanar subgraphs

After the decomposition of Lemma 3.20, it suffices to apply four reduction rules which shrink outerplanar graphs which connect to the rest of the graph through at most four vertices. We present these rules below. Their correctness proofs are deferred to the full version.

The two rules below shrink outerplanar subgraphs which connect to the rest of the graph via at most two vertices. Both rules yield a minor of the original graph; see Figure 2.

▶ **Reduction Rule 4.** *Consider a graph G and vertex set $C \subseteq V(G)$ such that $N_G(C) = \{x, y\}$, $xy \notin E(G)$, $G[C]$ is connected, and $G\langle C\rangle$ is outerplanar. Let $P = (u_1, u_2, \ldots, u_m)$, $u_1 = x$, $u_m = y$ be any shortest path connecting x and y in $G\langle C\rangle$ and $D_1, D_2, \ldots, D_\ell$ be the connected components of $G\langle C\rangle - V(P)$. We consider 3 cases:*
1. *if there is a component D_i, for which $N_G(D_i)$ includes two non-consecutive elements of P, replace C with two vertices c_1, c_2, each adjacent to both x and y,*
2. *if there are two distinct components D_i, D_j, for which $|N_G(D_i) \cap N_G(D_j)| \geq 2$, replace C with two vertices c_1, c_2, each adjacent to both x and y,*
3. *otherwise replace C with one vertex c_1 adjacent to both x and y.*

▶ **Reduction Rule 5.** *Suppose that there is an edge $e = uv$ in a graph G such that $G - V(e)$ has a connected component C such that $G\langle C\rangle$ is outerplanar. Then contract C into a single vertex.*

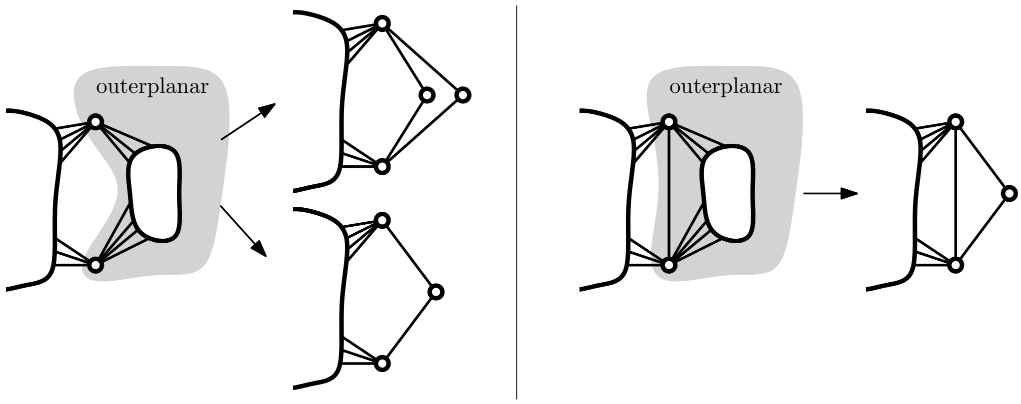

Figure 2 On the left a depiction of Reduction Rule 4, which reduces a connected subgraph to one or two vertices depending on its internal structure. On the right a depiction of Reduction Rule 5 which contracts a connected subgraph to a single vertex if it is outerplanar together with the two adjacent vertices that form its neighborhood.

The following reduction rule targets fan structures in outerplanar subgraphs. Its safeness follows directly from Lemma 3.17.

▶ **Reduction Rule 6.** *Suppose we are given a graph G, a vertex $x \in V(G)$, and five vertices $v_1, \dots, v_5 \in N_G(x)$ that lie, in order of increasing index, on an induced path P in $G - x$ from v_1 to v_5, such that $N_G(x) \cap V(P) = \{v_1, \dots, v_5\}$. Let C be the component of $G - \{v_1, v_5, x\}$ containing $P - \{v_1, v_5\}$. If $G\langle C\rangle$ is outerplanar, then remove the edge xv_3.*

The final reduction rule reduces ladder structures in biconnected outerplanar graphs. For its statement, we need the following terminology.

▶ **Definition 4.1.** *For a graph G, a sequence of edges $e_1, \dots, e_\ell \in E(G)$ is an* order-respecting matching *if the set of edges is a matching and if for all $1 \leq i < j < k \leq \ell$ we have that e_i and e_k are in different connected components of $G - V(e_j)$.*

▶ **Reduction Rule 7.** *Let G be a graph, $e_1, \dots, e_7$ be a matching in G, and let C be a connected component of $G - (V(e_1) \cup V(e_7))$. If $\{e_2, \dots, e_6\} \subseteq E(C, C)$, $N_G(C) = V(e_1) \cup V(e_7)$, $G\langle C\rangle$ is biconnected and outerplanar, and $e_1, \dots, e_7$ is an order-respecting matching in $G\langle C\rangle$, then remove e_4.*

The last two reduction rules are depicted on Figure 3. Intuitively, Reduction Rule 4 and Reduction Rule 5 together with the earlier stated Reduction Rule 2 reduce the number of biconnected components in an outerplanar graph $G\langle C\rangle$ with $|N_G(C)| \leq 4$ to a constant number (26). We show that if any biconnected component is large, then either it contains a large face so that Reduction Rule 4 can be applied to two vertices along the face that cut off a large outerplanar subgraph, it contains a large outerplanar subgraph attached onto an edge so that Reduction Rule 5 can be applied, or it contains a large fan (Reduction Rule 6) or ladder (Reduction Rule 7) structure that contains an irrelevant edge.

5 Wrapping up

The following lemma summarizes the effect of the four reduction rules described in Section 4.

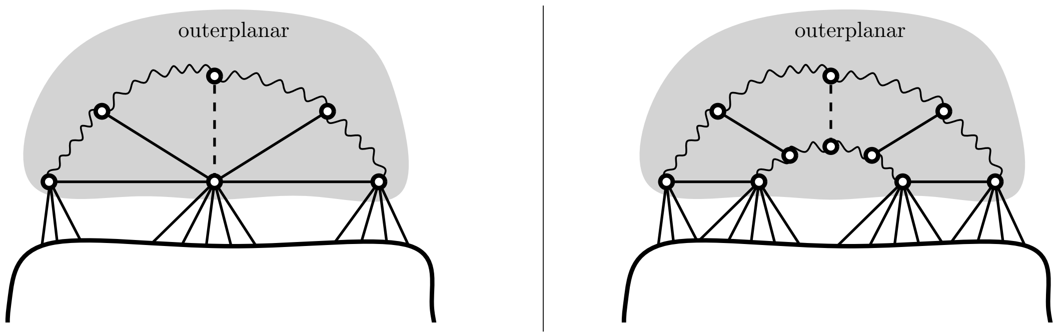

Figure 3 On the left a depiction of Reduction Rule 6 which is able to remove the middle edge of a fan structure in an outerplanar subgraph that is sufficiently isolated from the rest of the graph. On the right a depiction of Reduction Rule 7, which removes the middle edge of an order-respecting matching in an outerplanar subgraph that is sufficiently isolated from the rest of the graph.

▶ **Lemma 5.1.** *Consider a graph G and a vertex set $A \subseteq V(G)$, such that $|A| > 25 \cdot 6288$, $|N_G(A)| \leq 4$, $G[A]$ is connected, and $G\langle A\rangle$ is outerplanar. There is a polynomial-time algorithm that, given G and A satisfying the conditions above, outputs a proper minor G' of G, so that $\mathsf{opd}(G') = \mathsf{opd}(G)$.*

We repeatedly apply this reduction using the decomposition given by the set $L \subseteq V(G)$ of Lemma 3.20. It is important that the graph is guaranteed to shrink at each step, so after polynomially many invocations of Lemma 5.1 we must arrive at an irreducible instance. We now state the main theorem with the final bound on the size of compressed graph $2 \cdot (25 \cdot 6288 + 5) \cdot f_5(c, f_3(c)) \cdot (k_2 + 3)^4$ (see Lemmas 3.14 and 3.20), where $c = 40$. Recall that instances (G, k) and (G', k') are equivalent if $\mathsf{opd}(G) \leq k \Leftrightarrow \mathsf{opd}(G') \leq k'$.

▶ **Theorem** (1.1, restated)**.** *The* Outerplanar Deletion *problem admits a polynomial-time kernelization algorithm that, given an instance (G, k), outputs an equivalent instance (G', k'), such that $k' \leq k$, graph G' is a minor of G, and G' has $\mathcal{O}(k^4)$ vertices and edges. Furthermore, if $\mathsf{opd}(G) \leq k$, then $\mathsf{opd}(G') = \mathsf{opd}(G) - (k - k')$.*

As a consequence of the theorem above, we obtain the first concrete bounds on the sizes of minor-minimal obstructions to having an outerplanar vertex deletion set of size k.

▶ **Corollary** (1.2, restated)**.** *If G is a graph such that $\mathsf{opd}(G) > k$ but each proper minor G' of G satisfies $\mathsf{opd}(G') \leq k$, then G has $\mathcal{O}(k^4)$ vertices and edges.*

6 Conclusion

We presented a number of elementary reduction rules for Outerplanar Deletion that can be applied in polynomial time to obtain a kernel of $\mathcal{O}(k^4)$ vertices and edges. This kernel does not use protrusion replacement and the constants hidden by the $\mathcal{O}$-notation can be derived easily. This is the first concrete kernel for Outerplanar Deletion, and a step towards more concrete kernelization bounds for Planar-$\mathcal{F}$ Deletion. We hope it inspires new kernelization bounds for Planar Deletion.

In earlier work Dell and Van Melkebeek [9, Theorem 3] have shown that there is no kernel for Outerplanar Deletion of bitsize $\mathcal{O}(k^{2-\varepsilon})$ unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$. This naturally leads to the question, can these two bounds be brought closer together?

Another interesting direction for further research is to obtain concrete kernelization bounds for other Planar-$\mathcal{F}$ Deletion problems. Our work exploits the fact that $K_{2,3}$-minor-free graphs cannot have many disjoint paths between two vertices. Previous work [17] used a similar observation to derive a kernel for θ_c-Minor-Free Deletion. An interesting next case would be a Planar-$\mathcal{F}$ Deletion problem where $\mathcal{F}$ does not contain $K_{2,c}$ or θ_c for some c, for example 2-Transversal which asks whether a graph of treewidth at most 2 can be obtained by deleting k vertices.

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