A Polynomial Kernel for Bipartite Permutation Vertex Deletion

Lawqueen Kanesh ✉
National University of Singapore, Singapore

Jayakrishnan Madathil ✉
Chennai Mathematical Institute, Chennai, India

Abhishek Sahu ✉
The Institute of Mathematical Sciences, HBNI, Chennai, India

Saket Saurabh ✉
The Institute of Mathematical Sciences, HBNI, Chennai, India
University of Bergen, Norway

Shaily Verma ✉
The Institute of Mathematical Sciences, HBNI, Chennai, India

Abstract

In a permutation graph, vertices represent the elements of a permutation, and edges represent pairs of elements that are reversed by the permutation. In the Permutation Vertex Deletion problem, given an undirected graph G and an integer k, the objective is to test whether there exists a vertex subset $S \subseteq V(G)$ such that $|S| \leq k$ and $G - S$ is a permutation graph. The parameterized complexity of Permutation Vertex Deletion is a well-known open problem. Bożyk et al. [IPEC 2020] initiated a study towards this problem by requiring that $G - S$ be a bipartite permutation graph (a permutation graph that is bipartite). They called this the Bipartite Permutation Vertex Deletion (BPVD) problem. They showed that the problem admits a factor 9-approximation algorithm as well as a fixed parameter tractable (FPT) algorithm running in time $\mathcal{O}(9^k|V(G)|^9)$. And they posed the question *whether* BPVD *admits a polynomial kernel.*

We resolve this question in the affirmative by designing a polynomial kernel for BPVD. In particular, we obtain the following: Given an instance (G, k) of BPVD, in polynomial time we obtain an equivalent instance (G', k') of BPVD such that $k' \leq k$, and $|V(G')| + |E(G')| \leq k^{\mathcal{O}(1)}$.

2012 ACM Subject Classification Theory of computation → Fixed parameter tractability

Keywords and phrases kernelization, bipartite permutation graph, bicliques

Digital Object Identifier 10.4230/LIPIcs.IPEC.2021.23

Funding *Lawqueen Kanesh*: Supported in part by NRF Fellowship for AI grant [R-252-000-B14-281] and by Defense Service Organization, Singapore.
Jayakrishnan Madathil: Supported by the Chennai Mathematical Institute and the Infosys Foundation.
Saket Saurabh: Supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 819416), and Swarnajayanti Fellowship (no. DST/SJF/MSA01/2017-18).

1 Introduction

In a *graph modification problem*, the input consists of an n-vertex graph G and an integer k. The objective is to determine whether k *modification operations* – such as vertex deletions, or edge deletions, insertions or contractions – are sufficient to obtain a graph with prescribed structural properties such as being planar, bipartite, chordal, interval, acyclic or edgeless. Graph modification problems include some of the most basic problems in graph theory and

16th International Symposium on Parameterized and Exact Computation (IPEC 2021).
Editors: Petr A. Golovach and Meirav Zehavi; Article No. 23; pp. 23:1–23:18
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

graph algorithms. Unfortunately, most of these problems are NP-complete [25, 33]. Therefore, they have been studied intensively within various algorithmic paradigms for coping with NP-completeness [14, 17, 27], including approximation algorithms, parameterized complexity, and algorithms for restricted input classes.

Graph modification problems have played a central role in the development of parameterized complexity. Here, the number of allowed modifications, k, is considered a *parameter*. With respect to k, we seek a *fixed parameter tractable* (FPT) algorithm, namely, an algorithm whose running time has the form $f(k)n^{\mathcal{O}(1)}$ for some computable function f. One way to obtain such an algorithm is to exhibit a *kernelization algorithm*, (or *kernel*, for short). A kernel for a graph problem Π is an algorithm that given an instance (G, k) of Π, runs in polynomial time and outputs an equivalent instance (G', k') of Π such that $|V(G')|$ and k' are upper bounded by $f(k)$ for some computable function f. The function f is called the *size* of the kernel, and if f is a polynomial function, then we say that the kernel is a *polynomial kernel*. A kernel for a problem immediately implies that it admits an FPT algorithm, but kernels are also interesting in their own right. In particular, kernels allow us to model the performance of polynomial time pre-processing algorithms. The field of kernelization has received considerable attention, especially after the introduction of the methods for proving kernelization lower bounds [3, 7, 8, 11, 16, 20, 21]. We refer to the surveys [15, 19, 24, 26], as well as the books [6, 10, 12, 30], for a detailed treatment of the area of kernelization. In this paper, we study the kernelization complexity of the following problem.

Bipartite Permutation Vertex Deletion (BPVD) **Parameter:** k
Input: A graph G and an integer k.
Question: Does there exist a subset $S \subseteq V(G)$ of size at most k such that $G - S$ is a bipartite permutation graph?

A graph G is a *permutation graph*, if the vertices represent the elements of a permutation, and edges represent pairs of elements that are reversed by the permutation. Alternatively, a permutation graph can be defined as an intersection graph of line segments whose endpoints lie on two parallel lines $\mathcal{L}_1$ and $\mathcal{L}_2$, with one endpoint of each line segment lying on $\mathcal{L}_1$ and the other endpoint on $\mathcal{L}_2$. Due to their intriguing combinatorial properties and modelling power, the class of permutation graphs is one of the well-studied graph classes [5, 18]. As a subclass of perfect graphs, many problems that are NP-complete on general graphs can be solved efficiently on permutation graphs, such as Clique, Independent Set, Chromatic Number, Treewidth and Pathwidth. Further, there is a linear time algorithm to test whether a given graph is a permutation graph, and if so construct a permutation representing it [29]. Whether Permutation Vertex Deletion admits an FPT algorithm has been a longstanding open problem in the area. In order to make progress on this open problem, recently, Bożyk et al. [4] studied the problem of deleting vertices to a subclass of permutation graphs. The subclasses of permutation graphs include the classes of bipartite permutation graphs (characterized by Spinrad, Brandstädt & Stewart 1987 [31]) and cographs. While the fixed-parameter tractability of vertex deletion to cographs follows easily because of the finite forbidden characterization (as induced subgraphs) of cographs, no such result was known for vertex deletion to bipartite permutation graphs. Bożyk et al. [4] studied BPVD, and showed that the problem admits a factor 9-approximation algorithm as well as a FPT algorithm running in time $\mathcal{O}(9^k n^9)$. A natural follow-up question to this work, explicitly asked in [4], is whether BPVD admits a polynomial kernel. In this paper, we resolve this question in the affirmative.

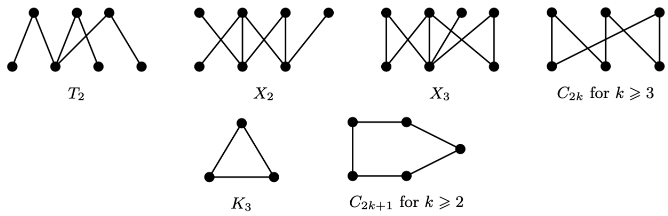

Figure 1 The set of obstructions for a bipartite permutation graph (Figure from [4]).

▶ **Theorem 1.** Bipartite Permutation Vertex Deletion *admits a polynomial kernel.*

1.1 Methods

Our kernelization heavily uses the characterization of bipartite permutation graphs in terms of their *forbidden induced subgraphs*, also called *obstructions*. Specifically, a graph H is an obstruction to the class of bipartite permutation graphs if H is not a bipartite permutation graph and $H - \{v\}$ is a bipartite permutation graph for every vertex $v \in V(H)$. A graph G is a bipartite permutation graph if and only if it does not contain any obstruction as an induced subgraph. The set of obstructions to bipartite permutation graphs have been completely characterized by Spinrad et al. [31]. It consists of T_2, X_2, X_3, K_3, as well two infinite families of graphs: even cycles of length at least 6, and odd cycles of length at least 5 (see Figure 1). We call any obstruction of size less than 45 a *small obstruction*, and call all other obstructions large obstructions. Note that every large obstruction is a hole (induced cycle) of length at least 45.

The first ingredient of our kernelization algorithm is the factor 9 polynomial time approximation algorithm for BPVD by Bożyk et al. [4]. We use this algorithm to obtain an approximate solution of size at most $9k$, or conclude that no solution of size at most k exists. We grow this approximate solution to a solution T of size $\mathcal{O}(k^{45})$, such that every set $Y \subseteq V(G)$ of size at most k is a minimal hitting set for small obstructions in G if and only if Y is a minimal hitting set for small obstructions in $G[T]$. Once we have T (also called a modulator), we know that $G - T$ is a bipartite permutation graph. Let S be a minimal (or minimum) solution of size at most k. Then, the only purpose of vertices in $S \cap (V(G) \setminus T)$ is to hit large obstructions. Next, we analyze the graph $G - T$, and reduce its size by applying various reduction rules.

For the kernelization algorithm, we look at $G-T$, and focus on *one* connected component of $G - T$. Since $G - T$ is a bipartite permutation graph, it has a "complete bipartite decomposition" [32]. For our kernelization purpose, we heavily use this known decomposition. A *biclique* or a *complete bipartite graph* is a bipartite graph where every vertex of the first part is adjacent to every vertex of the second part. We give a semi-formal definition of a complete bipartite decomposition [32]. Let $H = G - T$ and π be an ordering of $V(H)$. A sequence of vertex subsets $(Q_1, R_1, Q_2, R_2, \ldots, Q_s, R_s)$, where $Q_i, R_i \subseteq V(H)$ for every $i \in [s]$, is said to be a *complete bipartite decomposition* of H if the following holds. The vertex subsets partition $V(H)$, $H[Q_i]$ is a biclique for every $i \in [s]$, R_i is an independent set for every $i \in [s]$, and $Q_1 <_\pi R_1 <_\pi Q_2 <_\pi R_2 <_\pi \cdots <_\pi Q_s <_\pi R_s$. That is, if $X <_\pi Y$, then every vertex in X appears before every vertex in Y in π. Further, for $i, j \in [s]$, if $E(Q_i, Q_j) \neq \emptyset$, then $|i - j| \leq 1$, for $i, j \in [s]$, if $E(Q_i, R_j) \neq \emptyset$, then $i = j$, and for $i, j \in [s]$ with $i \neq j$, we have $E(R_i, R_j) = \emptyset$. Here, $E(X, Y)$ denotes the set of edges with one endpoint in X and the other in Y. Informally, $(Q_1, R_1, Q_2, R_2, \ldots, Q_s, R_s)$, is a partition of $V(H)$, where each part is either a biclique or an independent set, and each set has edges only in the neighboring parts. The complete bipartite decomposition is similar to the clique partition used by Ke et al. [23] for designing a polynomial kernel for vertex deletion to proper interval graphs.

In the first phase, we bound the maximum biclique size in $G - T$, i.e., the size of Q_i for $i \in [s]$. Our biclique-reduction procedure builds upon the clique-reduction procedure of Marx [28], which was used in the kernelizations for Chordal Vertex Deletion [1, 22] and Interval Vertex Deletion [2]. The procedure of Marx [28] as well as our procedure are based on an "irrelevant vertex rule". In particular, we find a vertex that is not necessary for a solution of size at most k, and delete it. And after this procedure we reduce the size of each biclique in $G - T$ by $k^{\mathcal{O}(1)}$. Next, using a simple marking procedure we bound the size of R_i for $i \in [s]$ as well.

In the second phase we bound the size of the connected component of $G - T$ we started with. Towards this, we first bound the number of bicliques in $Q_1, Q_2, \ldots, Q_t$ that contain a neighbor of a vertex in T (say *good bicliques*). We use small obstructions, and in particular T_2 (the subdivided claw), and K_3 (the triangle) to bound the number of good bicliques by $k^{\mathcal{O}(1)}$. This automatically divides the biclique partition into *chunks*. Mark all the good bicliques. A maximal set of unmarked bicliques between two marked bicliques form a chunk. It is clear that the number of chunks is upper bounded by $k^{\mathcal{O}(1)}$. Finally, we use structural analysis to bound the size of each chunk, which includes the design of a reduction rule that computes a minimum cut between the two good bicliques that border the chunk. In particular, we show that each chunk can be replaced by a graph of size $k^{\mathcal{O}(1)}$. We remark that the procedure also needs to handle the presence of independent sets $R_1, R_2, \ldots, R_s$, which we have completely ignored in the discussion here.

Until now we have assumed that $G-T$ is connected. Finally, again using the obstructions T_2 and K_3, we show that the number of connected components in $G-T$ is upper bounded by $k^{\mathcal{O}(1)}$. Using this bound, together with the facts that $|T| \leq k^{\mathcal{O}(1)}$, and that each connected component is of size $k^{\mathcal{O}(1)}$, we are able to deduce our polynomial kernel for BPVD.

2 Preliminaries

In this section, we define some notations and list some properties of bipartite permutation graphs.

Standard Notation. For a positive integer n, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$. For a graph G, $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively. Two vertices u, v are said to be *adjacent* if there is an edge (denoted as uv) between u and v. Given vertex subsets $X, Y \subseteq V(G)$, such that $X \cap Y = \emptyset$, $E(X, Y)$ denotes the set of edges with one endpoint in X and the other in Y. The neighbourhood of a vertex v, denoted by $N_G(v)$, is the set of vertices adjacent to v. The subscript in the notation for neighbourhood is omitted, if the graph under consideration is clear. For a set $M \subseteq V(G)$ and a vertex $u \in V(G)$, by $M(u)$ we denote $N(u) \cap M$. For a set $S \subseteq V(G)$, $G - S$ denotes the graph obtained by deleting S from G and $G[S]$ denotes the subgraph of G induced on S. A *path* $P = v_1, \ldots, v_\ell$ is a sequence of distinct vertices where every consecutive pair of vertices is adjacent. We say that P *starts* at v_1 and *ends* at v_ℓ. The vertices (or vertex set) of P, denoted by $V(P)$, is the set $\{v_1, \ldots, v_\ell\}$. The *endpoints* of P is the set $\{v_1, v_\ell\}$ and the *internal vertices* of P is the set $V(P) \setminus \{v_1, v_\ell\}$. The *length* of P is defined as $|V(P)|$. A *cycle* is a sequence $v_1, \ldots, v_\ell$ of vertices such that $v_1, \ldots, v_\ell$ is a path and $v_\ell v_1$ is an edge. A set $Q \subseteq V(G)$ of pairwise adjacent vertices in G is called a *clique*. For graph theoretic terms and definitions not stated explicitly here, we refer to [9].

2.1 Bipartite permutation graph

The characterization of bipartite permutation graphs presented below was proposed by Spinrad et al. [31]. Let G be a connected bipartite graph with vertex bipartition (A, B). A linear order $(B, <_B)$ satisfies the *adjacency property* if for each vertex $u \in A$ the set $N(u)$ consists of vertices that are consecutive in $(B, <_B)$. A linear order $(B, <_B)$ satisfies the *enclosure property* if for every pair of vertices $u, u' \in A$ such that $N(u)$ is a subset of $N(u')$, vertices in $N(u') \setminus N(u)$ occur consecutively in $(B, <_B)$. A *strong ordering* of the vertices of $A \cup B$ consists of linear orders $(A, <_A)$ and $(B, <_B)$ such that for every $(u, w'), (u', w)$ in $E(G)$, where u, u' are in A and w, w' are in B, $u <_A u'$ and $w <_B w'$ imply that $(u, w) \in E(G)$ and $(u', w') \in E(G)$. Note that whenever $(A, <_A)$ and $(B, <_B)$ form a strong ordering of $A \cup B$, then $(A, <_A)$ and $(B, <_B)$ satisfy the adjacency and enclosure properties.

▶ **Theorem 2** ([31]). *The following three statements are equivalent for a connected bipartite graph $G = (A, B, E)$:*
1. *(A, B, E) is a bipartite permutation graph.*
2. *There exists a strong ordering of $A \cup B$.*
3. *There exists a linear order $(B, <_B)$ of B satisfying adjacency and enclosure properties.*

Notation on ordering. Let G be a bipartite permutation graph with a vertex bipartition, say (A, B), of G. Fix a strong ordering, say π, of (A, B). Let π_A and π_B be the restriction of π on A and B, respectively, that is, π_A and π_B are linear orderings of the vertices of A and B. For $X \in \{A, B\}$ and a pair of vertices $x, y \in X$, we say $x <_{\pi_X} y$ if x appears before y in the ordering π_X. Similarly, for $X \in \{A, B\}$ and $Y, Y' \subseteq X$, we say $Y <_\pi Y'$ if $y <_{\pi_X} y'$ for every $y \in Y$ and $y' \in Y'$. More generally, for $Y, Y' \subseteq A \cup B$, we write $Y <_\pi Y'$ if $Y \cap A <_\pi Y' \cap A$ and $Y \cap B <_\pi Y' \cap B$. For $X \in \{A, B\}$, a set $Y \subseteq X$ and an integer q, where $1 \leq q \leq |Y|$, we write F_q^Y to denote the first q vertices of Y in the ordering π_X. Similarly, we write L_q^Y to denote the last q vertices of Y in the ordering π_X.

2.2 Complete Bipartite Decomposition

We start by defining the notion of complete bipartite decomposition.

▶ **Definition 3** (Complete Bipartite Decomposition,[32]). *Consider a bipartite permutation graph G with vertex bipartition (A, B) and a strong ordering π of (A, B). A sequence of vertex subsets $(Q_1, R_1, Q_2, R_2, \ldots, Q_s, R_s)$, where $Q_i, R_i \subseteq V(G)$ for every $i \in [s]$, is said to be a* complete bipartite decomposition *of G if the following properties hold:*

1. *$\{Q_1, R_1, Q_2, R_2, \ldots, Q_s, R_s\}$ is a partition of $V(G)$,*
2. *$G[Q_i]$ is a biclique for every $i \in [s]$,*
3. *R_i is an independent set for every $i \in [s]$,*
4. *$Q_1 <_\pi R_1 <_\pi Q_2 <_\pi R_2 <_\pi \cdots <_\pi Q_s <_\pi R_s$,*
5. *for $i, j \in [s]$, if $E(Q_i, Q_j) \neq \emptyset$, then $|i - j| \leq 1$,*
6. *for $i, j \in [s]$, if $E(Q_i, R_j) \neq \emptyset$, then $i = j$,*
7. *for $i, j \in [s]$, we have $E(R_i, R_j) = \emptyset$.*

The next lemma proves that every connected bipartite permutation graph has a complete bipartite decomposition and further, it can be computed in polynomial time.

▶ **Lemma 4** ([32]). *Every connected bipartite permutation graph has a complete bipartite decomposition. Moreover, there is a polynomial time algorithm that takes a connected bipartite permutation graph G with a fixed vertex bipartition (A, B) and a fixed strong ordering π of (A, B) as input, and returns a complete bipartite decomposition of G.*

3 Constructing a Nice Modulator

We classify the set of obstructions for bipartite permutation graphs as follows. Any obstruction of size less than 45 is known as a *small obstruction*, while other obstructions (holes) are said to be large. In this section we design a modulator, with some additional properties, of size $k^{\mathcal{O}(1)}$ to bipartite permutation graph. For this we will utilize the following known results.

▶ **Theorem 5** ([4]). *There exists a factor* 9*-approximation algorithm for* BPVD*.*

▶ **Lemma 6** ([13, Lemma 3.2]). *Let $\mathcal{F}$ be a family of sets of cardinality at most d over a universe $\mathcal{U}$ and k be a positive integer. Then there is an $\mathcal{O}(|\mathcal{F}|(k+|\mathcal{F}|))$ time algorithm that finds a non-empty set $\mathcal{F}' \subseteq \mathcal{F}$ such that*

1. *For every $Z \subseteq \mathcal{U}$ of size at most k, Z is a minimal hitting set of $\mathcal{F}$ if and only if Z is a minimal hitting set of $\mathcal{F}'$; and*
2. *$|\mathcal{F}'| \leq d!(k+1)^d$.*

We use Lemma 6 to identify a vertex subset of $V(G)$, which allows us to forget about small induced subgraphs of G, and to concentrate on long induced holes for the kernelization.

▶ **Lemma 7** (♣). [1] *Let (G,k) be an instance to* BPVD*. In polynomial time, we construct a vertex subset $T'' \subseteq V(G)$ such that*

1. *Every set $Y \subseteq V(G)$ of size at most k is a minimal hitting set of small obstructions in G if and only if it is a minimal hitting set for small obstructions in $G[T'']$, and*
2. *$|T''| \leq (45+1)!(k+1)^{45}$,*

or conclude that (G,k) is a no-instance.

Using Theorem 5, in polynomial time we construct a 9-approximate solution T', and using Lemma 7 in polynomial time we construct a vertex set T''. If $|T'| > 9k$ or $|T''| > (45+1)!(k+1)^{45}$, we conclude (G,k) is a no-instance. Otherwise we have a modulator $T = T' \cup T''$ of size $\mathcal{O}(k^{45})$, such that $G - T$ is a bipartite permutation graph, and every set $Y \subseteq V(G)$ of size at most k is a minimal hitting set of small obstructions in G if and only if it is a minimal hitting set for small obstructions in $G[T]$. Let S be a minimal (or minimum) solution of size at most k. Then, the only purpose of vertices in $S \cap (V(G) \setminus T)$ is to hit long obstructions. We call the modulator constructed above as *nice modulator*. We summarize these discussions in the next lemma.

▶ **Lemma 8** (Nice Modulator). *Let (G,k) be an instance to* BPVD*. In polynomial time, we can either construct a nice modulator $T \subseteq V(G)$ of size $\mathcal{O}(k^{45})$, or conclude that (G,k) is a no-instance.*

Furthermore, in $G - T = (A \cup B, <)$, $<$ is a strong ordering of the bipartite permutation graph that we use throughout our paper.

4 Bounding the Sizes of Bicliques and Independent Sets

In this section, we consider the modulator T of G to bipartite permutation graph obtained in the previous section, and we bound the size of each biclique and independent set in a *complete bipartite decomposition* of $G - T$.

[1] Proofs of results marked with ♣ have been omitted due to space constraints.

Throughout this section, we assume that we have fixed a bipartition (A, B) of $G - T$ and a strong ordering π of (A, B). We also assume that $G - T$ is connected. Later, we will remove this requirement. (We assume connectivity so that we can work with a complete bipartite decomposition of $G - T$.) We also fix a complete bipartite decomposition $\mathcal{D} = (Q_1, R_1, \dots, Q_s, R_s)$ of $G - T$.

4.1 Auxiliary Results

Next, we prove a few simple results that will be used later to bound the size of each biclique and independent set in the complete bipartite decomposition $\mathcal{D}$ of $G - T$.

▶ **Lemma 9.** *Let H be an induced path in G. Consider $v \in V(G) \setminus V(H)$. If v has more than 5 neighbours in $V(H)$, then $G[V(H) \cup \{v\}]$ contains a small obstruction.*

Proof. Assume that $|N(v) \cap V(H)| \geq 5$. Let H be a path from x to y for some $x, y \in V(G)$. Let $v_1, v_2, v_3, v_4, v_5 \in V(H)$ be the first 5 neighbours of v that appear as we traverse H from x to y. Note that if $v_i v_{i+1} \in E(G)$ for some $i \in [4]$, then, $\{v v_i v_{i+1}\}$ induces a triangle, which is an obstruction, and the lemma follows. So, assume that $v_i v_{i+1} \notin E(H)$ for every $i \in [4]$. This means that no two vertices from $\{v_1, v_2, v_3, v_4, v_5\}$ appear consecutively on H. For $i \in \{1, 3\}$, let u_i be the neighbour of v_i that appears between v_i and v_{i+1} as we traverse H from v_1 to v_5, and let u_5 be the neighbour of v_5 that appears between v_4 and v_5 as we traverse H from v_1 to v_5. Then, notice that $\{v, v_1, u_1, v_3, u_3, v_5, u_5\}$ induces a subdivided claw, which is an obstruction. ◀

▶ **Lemma 10 (♣).** *Let H' be a graph with a Hamiltonian cycle, and let $C = v_1 v_2 \dots, v_\ell v_1$ be a Hamiltonian cycle in H', where $\ell \geq 45$. Let $Y \subseteq V(H')$ be such that (i) $1 \leq |Y| \leq 3$, (ii) the vertices of Y appear consecutively in the cycle C (i.e., $Y = \{v_i, v_{i+1}, v_{i+2}\}$ for some $i \in [\ell - 2]$ or $Y = \{v_{\ell-1}, v_\ell, v_1\}$ or $Y = \{v_\ell, v_1, v_2\}$), (iii) $H' - Y$ is an induced path and (iv) $d_{H'}(y) \leq 5$ for every $y \in Y$. Then, H' contains an obstruction.*

Proof. Observe first that since $H' - Y$ is an induced path, all the chords in the cycle C are incident with Y. Consider $y \in Y$. Since C is a cycle, $d_C(y) = 2$. And since $d_H(y) \leq 5$, we can conclude that the cycle C has at most 3 chords that are incident with y. Note that if y is adjacent to two vertices that appear consecutively on the cycle C, i.e., if $y v_i, y v_{i+1} \in E(H')$ for some $i \in [\ell - 1]$ or $y v_\ell, y v_1 \in E(H')$, then H' contains a triangle, which is an obstruction. So, assume that there does not exist $y \in Y$ such that y is adjacent to two vertices that appear consecutively on C. Suppose that C does not contain a hole of length at least 5, then for every vertex $v_i \in C$, vertex v_{i+2} is adjacent to a vertex in Y. Intuitively every alternate vertex must have neighbour in Y so that every cycle of length at least 5 have a chord. However, $|N(Y) \cap V(C)| \leq 15$, implies that there is an induced path of length at least 5 such that it does not contain any neighbour of Y. Let P be longest induced path in C such that endpoints of P have neighbours in Y and no internal vertex of P is adjacent to any vertex of Y. Then as there is no triangle in $H'[V(C)]$, we obtain that $V(P)$ together with Y induces a hole of length at least 5, a contradiction. Hence H' contains an obstruction. ◀

4.2 Bounding the Size of a Biclique the Complete Bipartite Decomposition

In this section, we bound the size of each biclique in the complete bipartite decomposition $\mathcal{D} = (Q_1, R_1, \ldots, Q_s, R_s)$ of $G - T$. In particular, we show that if $G - T$ has a sufficiently large biclique, then in polynomial time we can find and safely delete an "irrelevant vertex" from such a biclique. We start with a marking procedure which marks a set of vertices in a given biclique.

> **Procedure** `Mark-1`**.** The procedure works in 3 steps as follows. For each $j \in [s]$, we initialise $M_j = \emptyset$, and do as follows:
> **Step 1:** For each $v \in T$, if $|(N(v) \cap Q_j \cap A) \setminus M_j| \leq k+1$, then we add $(N(v) \cap Q_j \cap A) \setminus M_j$ to M_j, otherwise we add the first $k+1$ vertices in $(N(v) \cap Q_j \cap A) \setminus M_j$ in the ordering π_A to M_j. Similarly, if $|(N(v) \cap Q_j \cap B) \setminus M_j| \leq k+1$, then we add $(N(v) \cap Q_j \cap B) \setminus M_j$ to M_j, else we add the first $k+1$ vertices in $(N(v) \cap Q_j \cap B) \setminus M_j$ in the ordering π_B to M_j.
> **Step 2:** for each $u \in F_{k+1}^{Q_j \cap A} \setminus M_j$, we add u to M_j. And for each $u \in F_{k+1}^{Q_j \cap B} \setminus M_j$, we add u to M_j.
> **Step 3:** for each $u \in L_{k+1}^{Q_j \cap A} \setminus M_j$, we add u to M_j and for each $u \in L_{k+1}^{Q_j \cap B} \setminus M_j$, we add u to M_j.

We now bound the size of the set M_j at the end of the procedure `Mark-1`.

▶ Remark 11. Observe that the Procedure `Mark-1` can be executed in polynomial time. Also note that $|M_j| \leq 2(k+1)(|T|+2)$ for every $j \in [s]$. To see this, fix $j \in [s]$. Note that for each $v \in T$, we add at most $2(k+1)$ vertices to M_j, i.e., at most $k+1$ vertices from $(N(v) \cap Q_j \cap A) \setminus M_j$ and at most $k+1$ vertices from $(N(v) \cap Q_j \cap B) \setminus M_j$. Therefore, the number of vertices we added to M_j in Step 1 is at most $2(k+1)|T|$. And in each of Steps 2 and 3, we add at most $2(k+1)$ vertices to M_j. Thus, $|M_j| \leq 2(k+1)(|T|+2)$.

▶ **Reduction Rule 12.** *If there exists a vertex $v \in Q_j \setminus M_j$ for some $j \in [s]$, then delete v.*

▶ **Lemma 13.** *Reduction Rule 12 is safe.*

Proof. Consider an application of Reduction Rule 12 in which a vertex, say $v \in Q_j \setminus M_j$ was deleted for some $j \in [s]$. We show that (G, k) is a yes-instance of BPVD if and only if $(G - v, k)$ is a yes-instance of BPVD. Observe first that if (G, k) is a yes-instance, then so is $(G - v, k)$, as $G - v$ is an induced subgraph of G. Assume now for a contradiction that $(G - v, k)$ is a yes-instance, but (G, k) is not. And let $X \subseteq V(G - v)$ be a solution of size at most k. That is $(G - v) - X$ is a bipartite permutation graph. And by our assumption that (G, k) is a no-instance, $G - X$ is not a bipartite permutation graph. Then, $G - X$ must contain an obstruction, say, H. Note that $v \in V(H)$, as otherwise, H would be an obstruction in $(G - v) - X$, which contradicts the fact that $(G - v) - X$ is a bipartite permutation graph. We first claim that H is a large obstruction. Suppose not. Note that X hits all obstructions in $G - v$. And since $G[T]$ is a subgraph of $G - v$ as $v \notin T$, X hits all obstructions in the subgraph $G[T]$ as well. In particular, X hits all small obstructions in $G[T]$. Let $Y \subseteq X$ be a minimal hitting set for all small obstructions in $G[T]$. Then, by the definitions of T and Y, we can conclude that Y hits all small obstructions in G as well. But then, as H is an obstruction in $G - X$ and $Y \subseteq X$, we can conlude that H is a small obstruction in $G - Y$, a contradiction. Thus, H is a large obstruction in $G - X$. That is, X is hole of length at least 45.

Let u and w be the neighbours of v in H, i.e., $H = uvw \dots u$. And thus $H - v$ is an induced path from w to u in G. Without loss of generality, assume that $v \in A$. Then, $u, w \in B$. We show that we can construct another hole H' in $(G - v) - X$, which will contradict the fact that $(G - v) - X$ is a bipartite permutation graph. For this, we consider different cases depending on which $Q_i \cup R_i$ or T each of the two vertices u and w belongs to. Recall that $v \in Q_j \setminus M_j$. Notice that for $x \in \{u, w\}$, if $x \notin T$, then, by the definition of a complete bipartite decomposition, $x \in Q_{j-1} \cup Q_j \cup Q_{j+1} \cup R_j$.

1. $u, w \in T$. Notice that in Step 1 of the Procedure `Mark-1`, we have added $k+1$ neighbours of u in $Q_j \cap A$ and $k+1$ neighbours of w in $Q_j \cap A$ to M_j, as otherwise, we would have added v as well to M_j. That is, we have $|M_j(u) \cap A| = k + 1$ and $|M_j(w) \cap A| = k + 1$. Since, $|X| \leq k$, we have $(M_j(u) \cap A) \setminus X \neq \emptyset$ and $(M_j(w) \cap A) \setminus X \neq \emptyset$. Let $u' \in (M_j(u) \cap A) \setminus X$ and $w' \in (M_j(w) \cap A) \setminus X$. Let $v' \in (Q_j \cap B) \setminus (V(H) \cup X)$. Let H' be the graph obtained from H by replacing the vertex v with vertices u', v', w' and edges uv, vw by edges $uu', u'v', u'w', w'w$. Notice that no vertex of H' belongs to $X \cup \{v\}$ and the graph $H' - \{u', v', w'\}$ is an induced path in G. By Lemma 9, each of the vertices u', v' and w' has at most 4 neighbours in $V(H') \setminus \{u', v', w'\}$. By Lemma 10 we conclude that H' contains an obstruction, which is also an obstruction in $(G - v) - X$, contradicts that X is a solution to $G - v$ of BPVD.
2. $u \in T, w \in Q_j \cup R_j \cup Q_{j+1}$. In Step 1 of the Procedure `Mark-1`, we have added $k + 1$ neighbours of u in $Q_j \cap A$ to M_j, as otherwise, we would have added v as well to M_j. Thus, $|M_j(u) \cap A| = k + 1$. And note that $v \notin L^{Q_j \cap A}_{k+1}$, as in Step 3 of the Procedure `Mark-1`, we have also added all the vertices in the set $L^{Q_j \cap A}_{k+1}$. Since $|X| \leq k$, $(M_j(u) \cap A) \setminus X \neq \emptyset$ and $L^{Q_j \cap A}_{k+1} \setminus X \neq \emptyset$. Let $u' \in (M_j(u) \cap A) \setminus X$ and $w' \in L^{Q_j \cap A}_{k+1} \setminus X$. Let $v' \in (Q_j \cap B) \setminus (V(H) \cup X)$. Note that by the definition of u', we have $uu' \in E(G)$. And since Q_j is a biclique, $u'v', v'w' \in E(G)$. If $w \in Q_j$, then $ww' \in E(G)$ as well. If not $w' \in R_j \cup Q_{j+1}$. But then, as $v <_{pi} w', v' <_\pi w$ and $vw, v'w' \in E(G)$, by the definition of the strong ordering, we have $ww' \in E(G)$. Let H' be the graph obtained from H by replacing the vertex v with vertices u', v', w' and edges uv, vw by edges $uu', u'v', u'w', w'w$. Notice that no vertex of H' belongs to $X \cup \{v\}$. Again, the graph $H' - \{u', v', w'\}$ is an induced path in G. And by Lemma 9, each of the vertices u', v' and w' has at most 4 neighbours in $V(H') \setminus \{u', v', w'\}$. By Lemma 10 we conclude that H' contains an obstruction, which is also an obstruction in $(G - v) - X$, contradicts that X is a solution to $G - v$ of BPVD.
3. $u \in T, w \in Q_{j-1}$. In Step 1 of the Procedure `Mark-1`, we must have marked $k+1$ neighbours of u in $Q_j \cap A$, as otherwise, we would have marked v as well. Thus, $|M_j(u) \cap A| = k + 1$. And note that $v \notin F^{Q_j \cap A}_{k+1}$, as in Step 2 of the Procedure `Mark-1`, we have also marked all the vertices in the set $F^{Q_j \cap A}_{k+1}$. Since $|X| \leq k$, $(M_j(u) \cap A) \setminus X \neq \emptyset$ and $F^{Q_j \cap A}_{k+1} \setminus X \neq \emptyset$. Let $u' \in (M_j(u) \cap A) \setminus X$, $w' \in F^{Q_j \cap A}_{k+1} \setminus X$ and $v' \in (Q_j \cap B) \setminus (V(H) \cup X)$. By the definition of u', we have $uu' \in E(G)$. Since Q_j is a biclique, $u'v', v'w' \in E(G)$. And since $w' <_\pi v, w <_\pi v'$ and $wv, w'v' \in E(G)$, by the definition of the strong ordering, we have $ww' \in E(G)$. Let H' be the graph obtained from H by replacing the vertex v with the path $u'v'w'$. Notice that no vertex of H' belongs to $X \cup \{v\}$. Again, the graph $H' - \{u', v', w'\}$ is an induced path in G. And by Lemma 9, each of the vertices u', v' and w' has at most 4 neighbours in $V(H') \setminus \{u', v', w'\}$. By Lemma 10 we conclude that H' contains an obstruction, which is also an obstruction in $(G - v) - X$, contradicts that X is a solution to $G - v$ of BPVD.

The cases where $w \in T, u \in Q_j \cup R_j \sup Q_{j+1}$ and $w \in T, u \in Q_{j-1}$ are symmetric. We have thus covered all the cases in which at least one neighbour of v in H belongs to T. Assume now that $u, w \notin T$. Then, by the definition of complete bipartite decomposition, $u, w \in Q_{j-1} \cup Q_j \cup R_j \cup Q_{j+1}$.

4. $u \in Q_{j-1} \cup Q_j \cup Q_{j+1}$ and $w \in Q_{j-1} \cup Q_j \cup Q_{j+1}$.

   Since $|X| \leq k$, $F_{k+1}^{Q_j \cap A} \setminus X \neq \emptyset$ and $L_{k+1}^{Q_j \cap A} \setminus X \neq \emptyset$. If $u \in Q_{j-1} \cup Q_j$, then let $u' \in F_{k+1}^{Q_j \cap A} \setminus X$, otherwise let $u' \in L_{k+1}^{Q_j \cap A} \setminus X$. Similarly, if $w \in Q_{j-1} \cup Q_j$, then let $w' \in F_{k+1}^{Q_j \cap A} \setminus X$, otherwise let $w' \in L_{k+1}^{Q_j \cap A} \setminus X$. Let $v' \in (Q_j \cap B) \setminus (V(H) \cup X)$. (1) If $u \in Q_{j-1}$, then since Q_{j-1} is a biclique, u must have a neighbour u'' in Q_{j-1} and as $u'' <_\pi u' <_\pi v$, i.e., u' is between u'' and v in ordering π, therefore $uu' \in E(G)$ by the definition of strong ordering. (2) If $u \in Q_j$, then since Q_j is a biclique, $uu' \in E(G)$. (3) If $u \in Q_{j+1}$, then since Q_{j+1} is a biclique, u must have a neighbour u'' in Q_{j+1} and as $v <_\pi u' <_\pi u''$, i.e., u' is between u'' and v in ordering π, therefore $uu' \in E(G)$ by the definition of strong ordering. Similar arguments follows for $w \in Q_{j-1} \cup Q_j \cup Q_{j+1}$ and implies that $ww' \in E(G)$. As Q_j is a biclique $uuv', v'w' \in E(G)$. Let H' be the graph obtained from H by replacing the vertex v with the path $u'v'w'$. Notice that no vertex of H' belongs to $X \cup \{v\}$. Again, the graph $H' - \{u', v', w'\}$ is an induced path in G. And by Lemma 9, each of the vertices u', v' and w' has at most 4 neighbours in $V(H') \setminus \{u', v', w'\}$. By Lemma 10 we conclude that H' contains an obstruction, which is also an obstruction in $(G - v) - X$, contradicts that X is a solution to $G - v$ of BPVD.

   Assume now that atleast one of u, w in R_j.

5. $u \in Q_{j-1}, w \in R_j$. Note that $v \notin F_{k+1}^{Q_j \cap A} \cup L_{k+1}^{Q_j \cap A}$, as in Steps 3,4 of the Procedure `Mark-1`, we have added all the vertices in the set $F_{k+1}^{Q_j \cap A} \cup L_{k+1}^{Q_j \cap A}$ to M_j. Since $|X| \leq k$, $F_{k+1}^{Q_j \cap A} \setminus X \neq \emptyset$ and $L_{k+1}^{Q_j \cap A} \setminus X \neq \emptyset$. Let $u' \in F_{k+1}^{Q_j \cap A} \setminus X$ and $w' \in L_{k+1}^{Q_j \cap A} \setminus X$. Let $v' \in (Q_j \cap B) \setminus (V(H) \cup X)$. By strong ordering, we have $uu' \in E(G)$. Since Q_j is a biclique, $u'v', v'w' \in E(G)$. As $v <_\pi w'$ and $v' <_\pi w$, we obtain that $vv', ww' \in E(G)$, by the properties of strong ordering. This implies that $uu', u'v', v'w', w'w \in E(G)$. Let H' be the graph obtained from H by replacing the vertex v with vertices u', v', w' and edges uv, vw by $uu', u'v', v'w', w'w$. Notice that no vertex of H' belongs to $X \cup \{v\}$ and H' is a Hamiltonian cycle. Again, the graph $H' - \{u', v', w'\}$ is an induced path in G. And by Lemma 9, each of the vertices u', v' and w' has at most 4 neighbours in $V(H') \setminus \{u', v', w'\}$. By Lemma 10 we conclude that H' contains an obstruction, which is also an obstruction in $(G - v) - X$, contradicts that X is a solution to $G - v$ of BPVD.

   The other cases where exactly one of $u, w \in R_j$ are symmetric. Assume now that both $u, w \in R_j$.

6. $u, w \in R_j$. Note that $v \notin L_{k+1}^{Q_j \cap A}$, as in Steps 4 of the Procedure `Mark-1`, we have added all the vertices in the set $L_{k+1}^{Q_j \cap A}$ to M_j. Since $|X| \leq k$, $L_{k+1}^{Q_j \cap A} \setminus X \neq \emptyset$. Let $t \in L_{k+1}^{Q_j \cap A} \setminus X$. Let $v' \in (Q_j \cap B) \setminus (V(H) \cup X)$. Since Q_j is a biclique, $vv', v'w' \in E(G)$. As $v <_\pi t$, $v' <_\pi u$ and $v' <_\pi w$, we obtain that $ut, wt \in E(G)$, by the properties of strong ordering. Let H' be the graph obtained from H by replacing the vertex v with t. and edges uv, vw by ut, tw. Notice that no vertex of H' belongs to $X \cup \{v\}$. Again, the graph $H' - \{t\}$ is an induced path in G. And by Lemma 9, t has at most 4 neighbours in $V(H') \setminus \{t\}$. Thus, $|N(\{t\}) \cap (V(H') \setminus \{t\}| \leq 4$. By Lemma 10 we conclude that H' contains an obstruction, which is also an obstruction in $(G - v) - X$, contradicts that X is a solution to $G - v$ of BPVD. ◀

4.3 Bounding the Size of an Independent Set in the Complete Bipartite Decomposition

In this section we bound the number of vertices in each independent set R_i for each $i \in [s]$ in the complete bipartite decomposition $\mathcal{D}$ of $G - T$. First, we describe construction of a set M_j with respect to an independent set R_j, $j \in [s]$ in the complete bipartite decomposition $\mathcal{D}$ of $G - T$.

Procedure `Mark-2`**.** The procedure works in 4 steps as follows. For each $j \in [s]$, we initialise $M_j = \emptyset$, and do as follows:
Step 1: For each $v \in T$, if $|(N(v) \cap R_j \cap A) \setminus M_j| \leq k+1$, then we add $(N(v) \cap R_j \cap A) \setminus M_j$ to M_j, and otherwise we add the first $k+1$ vertices in $(N(v) \cap R_j \cap A) \setminus M_j$ in the ordering π_A to M_j. Similarly, if $|(N(v) \cap R_j \cap B) \setminus M_j| \leq k+1$, then add $(N(v) \cap R_j \cap B) \setminus M_j$ to M_j, and else we add the first $k+1$ vertices in $(N(v) \cap R_j \cap B) \setminus M_j$ in the ordering π_B to M_j.
Step 2: For each pair $x, y \in T$, if $|(N(x) \cap N(y) \cap R_j \cap A) \setminus M_j| \leq k+1$, then we add $(N(x) \cap N(y) \cap R_j \cap A) \setminus M_j$ to M_j, else we add the first $k+1$ vertices in $(N(x) \cap N(y) \cap R_j \cap A) \setminus M_j$ in the sequence π to M_j. Similarly, for each pair $x, y \in T$, if $|(N(x) \cap N(y) \cap R_j \cap B) \setminus M_j| \leq k+1$, then we add $(N(x) \cap N(y) \cap R_j \cap B) \setminus M_j$ to M_j, else we add first $k+1$ vertices in $(N(x) \cap N(y) \cap R_j \cap B) \setminus M_j$ in the sequence π to M_j.
Step 3: for each $u \in F_{k+1}^{R_j \cap A} \setminus M_j$, we add u to M_j. And for each $u \in F_{k+1}^{R_j \cap B} \setminus M_j$, we add u to M_j.
Step 4: for each $u \in L_{k+1}^{R_j \cap A} \setminus M_j$, we add u to M_j and for each $u \in L_{k+1}^{R_j \cap B} \setminus M_j$, we add u to M_j.

We now bound the size of the set M_j at the end of the procedure `Mark-2`.

▶ Remark 14. Observe that the Procedure `Mark-2` can be executed in polynomial time. Observe also that $|M_j| \leq (k+1)(|T| + |T|^2 + 1)$ for every $j \in [s]$. To see this, fix $j \in [s]$. Note that for each $v \in T$, we added at most $2(k+1)$ neighbours to v to M_j, i.e., at most $2(k+1)$ vertices from $(N(v) \cap R_j) \setminus M_j$. Therefore the number of vertices we added to M_j in Step 1 is at most $2(k+1)|T|$. And in Step 2, for each pair $x, y \in T$, we added at most $2(k+1)$ common neighbours of x and y to M_j, and therefore the number of vertices we added to M_j in Step 2 is at most $2(k+1)|T|^2$. In each of Steps 3 and 4, we added at most $2(k+1)$ vertices to M_j. Thus, $|M_j| \leq 2(k+1)(|T| + |T|^2 + 2)$.

Using the set M_j constructed by Procedure `Mark-2`, we get the following reduction rule.

▶ **Reduction Rule 15.** *If there exists $v \in R_j \setminus M_j$ for some $j \in [s]$, then delete v.*

▶ **Lemma 16.** *Reduction Rule 15 is safe.*

Proof. Consider an application of Reduction Rule 15 in which a vertex, say $v \in R_j \setminus M_j$ was deleted for some $j \in [s]$. We show that (G, k) is a yes-instance of BPVD if and only if $(G - v, k)$ is a yes-instance of BPVD. Observe first that if (G, k) is a yes-instance, then so is $(G - v, k)$, as $G - v$ is an induced subgraph of G. Assume now for a contradiction that $(G - v, k)$ is a yes-instance, but (G, k) is not. And let $X \subseteq V(G - v)$ be a solution of size at most k. That is $(G - v) - X$ is a bipartite permutation graph. And by our assumption that (G, k) is a no-instance, $G - X$ is not a bipartite permutation graph. Then, $G - X$

must contain an obstruction, say, H. Note that $v \in V(H)$, as otherwise, H would be an obstruction in $(G - v) - X$, which contradicts the fact that $(G - v) - X$ is a bipartite permutation graph.

We first claim that H is a large obstruction. Suppose not. Note that X hits all obstructions in $G - v$. And since $G[T]$ is a subgraph of $G - v$ as $v \notin T$, X hits all obstructions in the subgraph $G[T]$ as well. In particular, X hits all small obstructions in $G[T]$. Let $Y \subseteq X$ be a minimal hitting set for all small obstructions in $G[T]$. Then, by the definitions of T and Y, we can conclude that Y hits all small obstructions in G as well. But then, as H is an obstruction in $G - X$ and $Y \subseteq X$, we can conclude that H is a small obstruction in $G - Y$, a contradiction. Thus, H is a large obstruction in $G - X$. That is, H is hole of length at least 45.

Let u and w be the neighbours of v in H, i.e., $H = uvw \dots u$. And thus $H - v$ is an induced path from w to u. Without loss of generality, assume that $v \in A$. Then, $u, w \in B$. We show that we can construct another hole H' in $(G - v) - X$, which will contradict the fact that $(G - v) - X$ is a bipartite permutation graph. For this, we consider different cases depending on which $Q_i \cup R_i$ or T each of the two vertices u and w belongs to. Recall that $v \in R_j \setminus M_j$. Notice that for $x \in \{u, w\}$, if $x \notin T$, then, by the definition of a complete bipartite decomposition, $x \in Q_j$.

1. $u, w \in T$. Notice that as $v \in (N(u) \cap N(w) \cap R_j \cap A) \setminus M_j$, by Step 2 of the Procedure `Mark-2`, we must have marked $k + 1$ common neighbours of u, w in $R_j \cap A$, i.e., we have added $k + 1$ vertices in $(N(u) \cap N(w) \cap R_j \cap A)$ to M_j as otherwise, we would have added v to M_j as well. That is, we have $|M_j \cap N(u) \cap N(w) \cap A| \geq k + 1$. Since, $|X| \leq k$, we have $(M_j \cap N(u) \cap N(w) \cap A) \setminus X \neq \emptyset$. Also notice that $N(u) \cap N(w) \cap V(H) = \{v\}$, as H is a hole. Let $v' \in M_j \cap N(u) \cap N(w) \cap A) \setminus X$ and H' be the graph obtained from H by replacing the vertex v by v' and by replacing edges uv, vw by $uv', v'w$. Notice that no vertex of H' belongs to $X \cup \{v\}$ and the graph $H' - v$ is an induced path in G. And H' is a cycle of length at least 45in G. By Lemma 9, v' have at most 4 neighbours in $H' - v'$. By Lemma 10 we conclude that H' contains an obstruction, which is also an obstruction in $(G - v) - X$, contradicts that X is a solution to $G - v$ of BPVD.
2. $u \in Q_j, w \in T$. (analogous arguments follows for the case $u \in T, w \in Q_j$) In Step 1 of the Procedure `Mark-2`, we have added $k + 1$ neighbours of u in $R_j \cap A$ to M_j which are before v in sequence π, as otherwise, we would have added v as well to M_j. Thus, $|N(u) \cap M_j \cap A| = k + 1$. Let $v' \in N(w) \cap R_j \cap A \setminus X$. As $v' <_\pi v$, we have $v'u \in E(G)$, by the definition of the strong ordering, as Q_j is a non-trivial biclique and hence u must have a neighbour u' in $Q_j \cap A$ and hence all the vertices between u' to v in π are neighbours of u, which implies $v' \in N(u)$. Let H' be the graph obtained from H by replacing the vertex v with vertex v' and edge uv, vw by edges $uv', v'w$. And by Lemma 9, each of the vertices u', v' and w' has at most 4 neighbours in $V(H') \setminus \{u', v', w'\}$. By Lemma 10 we conclude that H' contains an obstruction, which is also an obstruction in $(G - v) - X$, contradicts that X is a solution to $G - v$ of BPVD.
3. $u, w \in Q_j$. Note that $v \notin F_{k+1}^{R_j \cap A}$, as in Step 3 of the Procedure `Mark-2`, we have added all the vertices in the set $F_{k+1}^{R_j \cap A}$ to M_j. Since $|X| \leq k$, $F_{k+1}^{R_j \cap A} \setminus X \neq \emptyset$. Let $v' \in F_{k+1}^{R_j \cap A} \setminus X$. As $v' <_\pi v$, we have $v'u, v'w \in E(G)$, by the definition of the strong ordering, as Q_j is a non-trivial biclique and hence u, w must have a neighbour u' in $Q_j \cap A$ and hence all the vertices between u' to v in π are neighbours of u, w, which implies $v' \in N(u) \cap N(w)$. Let H' be the graph obtained from H by replacing the vertex v with vertex v' and edge uv, vw by edges $uv', v'w$. And by Lemma 9, each of the vertices u', v' and w' has at most 4 neighbours in $V(H') \setminus \{u', v', w'\}$. By Lemma 10 we conclude that H' contains an obstruction, which is also an obstruction in $(G - v) - X$, contradicts that X is a solution to $G - v$ of BPVD. ◀

▶ **Observation 17.** *After an exhaustive application of Reduction Rule 12, note that for every* $j \in [s]$, $M_j \setminus Q_j = \emptyset$. *Thus, by Remark 11,* $|Q_j| = |M_j| = \mathcal{O}(k \cdot |T|)$.

▶ **Observation 18.** *After an exhaustive application of Reduction Rule 15, note that for every* $j \in [s]$, $M_j \setminus R_j = \emptyset$. *Thus, by Remark 14,* $|R_j| = |M_j| = \mathcal{O}(k \cdot |T|^2)$.

Observation 17 and 18 together imply the following result.

▶ **Lemma 19.** *Given an instance* (G, k) *of* BPVD *and a nice modulator* $T \subseteq V(G)$ *of size* $k^{\mathcal{O}(1)}$, *in polynomial time, we can construct an equivalent instance* (G', k) *such that* G' *is an induced subgraph of* G, $T \subseteq V(G')$, *and for each connected component of* $G' - T$ *with a complete bipartite decomposition* $(Q_1, R_1, \ldots, Q_s, R_s)$, *we have* $|Q_j \cup R_j| = \mathcal{O}(k \cdot |T|^2)$.

5 Bounding the Size of a Connected Component

In this section we bound the size of each connected component in $G - T$. Recall that in previous sections we bounded the size of each Q_i and R_i, $i \in [s]$, in nice decomposition of $G - T$. Our aim in this section is to bound the number of Q_i and R_i in each connected component of $G - T$. Let $\mathcal{C}$ be a connected component in $G - T$. Without loss of generality let $\mathcal{C} = \bigcup_i (Q_i \cup R_i)$. For a pair Q_i, R_i of biclique and independent set, the set $Q_i \cup R_i$ is called a *block*.

▶ **Reduction Rule 20.** *Let* v *be a vertex in* T *such that* v *is contained in at least* $k + 1$ *disjoint triangles* (v, a_i, b_i, v) *intersecting exactly at* $\{v\}$, *where* $a_i, b_i \in V(G) \setminus T$, *then delete* v *from* G, *and reduce* k *by* 1. *The resultant instance is* $(G - v, k - 1)$.

The correctness of above reduction rule is easy to see as every solution to (G, k) of BPVD must contain v. From now onwards we assume that Reduction Rule 20 is not applicable.

▶ **Reduction Rule 21.** *Let* v *be a vertex in* T. *If* v *has more than* $6(k + 1)$ *neighbours* a_i*'s in different* $Q_i \cup R_i$ *such that there exists* $b_i \in N(a_i) \cap Q_i \setminus N(v)$, *then delete* v *from* G, *and reduce* k *by* 1. *The resultant instance is* $(G - v, k - 1)$.

▶ **Lemma 22.** *Reduction Rule 21 is correct.*

Proof. Notice that (v, a_i, b_i) is an induced P_3. By pigeon hole principle there are at least $3(k+1)$ non-consecutive blocks $Q_i \cup R_i$ which contains a pair (a_i, b_i) such that (v, a_i, b_i) is an induced P_3. Let $\mathcal{P}$ be the set of such induced P_3s. That is, $\mathcal{P}$ is a set of distinct induced P_3s, (v, a_i, b_i) intersecting exactly at $\{v\}$ and for every pair of P_3s, (v, a_i, b_i) and (v, a_j, b_j), where $a_i, b_i \in Q_i \cup R_i$ and $a_j, b_j \in Q_j \cup R_j$, the blocks $Q_i \cup R_i$ and $Q_j \cup R_j$ are not consecutive. Notice that we can construct a set of $k + 1$ induced subdivided claws intersecting exactly at v using $\mathcal{P}$, which implies that any solution to (G, k) of BPVD must contain v. ◀

From now onwards we assume that Reduction Rules 20 and 21 are not applicable.

▶ **Lemma 23 (♣).** *Let* $\mathcal{C}$ *be a connected component in* $G - T$. *Then there are at most* $7|T|(k+1)$ *many disjoint blocks* $(Q_i \cup R_i)$ *in nice decomposition of* $\mathcal{C}$ *such that* $N(T) \cap (Q_i \cup R_i) \neq \emptyset$.

If $\mathcal{C}$ has $3500|T|k(k + 1)$ disjoint blocks, then by the pigeon hole principle and Lemma 23, there are at least $500k$ consecutive blocks in $\mathcal{C}$ that do not contain any vertex from $N(T)$. Let $Q_1 \cup R_1, \ldots, Q_{500k} \cup R_{500k}$ be the set of $500k$ such consecutive blocks in $\mathcal{C}$ that are disjoint from $N(T)$. Let $j = 500k/2$. Consider $\mathcal{D}_L = \{Q_i \cup R_i | i \in [j - 2k, j - 3]\} \setminus R_{j-3}$ and $\mathcal{D}_R = \{Q_i \cup R_i | i \in [j + 3, j + 2k]\}$. Let $F = \{R_{j-3}\} \cup \{Q_i \cup R_i | i \in [j - 2, j + 2]\}$

and $Z = \{Q_i | i \in [j-2k, j+2k]\}$. Observe that, for a vertex $v \in \mathcal{D}_L \cup \mathcal{D}_R$ and a vertex $u \in T$, $\text{dist}_G(u,v) \geq 240k$. This observation will be used in in proving further results. Let $Q = Q_{j-3}$ and $Q' = Q_{j+3}$. Let Y be a $Q_i, Q_{i'}$ cut in $G - T$, where $i \in [j-2k, j-3]$ and $i' \in [j+3, j+2k]$, where Y must contain vertices from only block $Q_a \cup R_a$, $a \in [i+1, i'-1]$. Let τ be the size of minimum $Q_i, Q_{i'}$ cut in $G - T$ over all pairs i, i', $i \in [j-2k, j-3]$ and $i' \in [j+3, j+2k]$.

▶ **Reduction Rule 24.** *Let F be as defined above. Delete all the vertices of F from G. Introduce three new bicliques $S_1 = K_{k^2,k^2}, S_2 = K_{\lceil \tau/2 \rceil, \lfloor \tau/2 \rfloor}, S_3 = K_{k^2,k^2}$. Also add edges such that $G[V(Q) \cup S_1]$ and $G[S_1 \cup S_2]$, $G[S_2 \cup S_3]$ and $G[V(Q') \cup S_3]$ are complete bipartite graphs. The bicliques appear in the order Q, S_1, S_2, S_3, Q'.*

Let G' be the reduced graph after an application of Reduction Rule 24. Let $S = S_1 \cup S_2 \cup S_3$. Notice that $G' - T$ is a bipartite permutation graph by construction.

▶ **Observation 25 (♣).** *There are no small obstructions containing any vertices from $F \cup \mathcal{D}_L \cup \mathcal{D}_R$ in G. There are no small obstructions containing any vertices from $S \cup \mathcal{D}_L \cup \mathcal{D}_R$ in G'.*

▶ **Observation 26.** *Any hole H in G which contains a vertex from $F \cup \mathcal{D}_L \cup \mathcal{D}_R$, intersects all bicliques in $F \cup \mathcal{D}_L \cup \mathcal{D}_R$. And such H is of length at least $500k$.*

Proof. Since there are no *large* holes in $G - T$, $V(H) \cap T \neq \emptyset$. Without loss of generality, suppose that H intersects a block $Q_i \cup R_i$ but does not intersect some $Q_{i+1} \in Z$. Then any biclique $Q_{i'}$ where $i' < i$ contains at least two vertices from the hole H. Let a_1 and a_2 be two such vertices such that they have an induced path between them in H. Let $H = (s, v_1, v_2, ...a_1, ...a_2, ..., s)$. Notice that a_1 and a_2 can not belong to different partitions of Q_{i-21} since H is a hole. But Q_{i-21} has some vertex v in its other partition. But then we get a cycle $C = (s, ..., a_1, v, a_2, ..., s)$. But v can have at most 5 neighbors on the induced path of the hole $(a_1, ..., s, ..., a_2)$, otherwise there is a small obstruction containing v which is completely contained in $G - T$ which is not possible. Since cycle C has at length at least 40, we can construct a new hole H_1 such that $V(H_1) \subseteq V(C)$ which is completely contained in $G - T$, which is a contradiction. Notice that any such hole must have one vertex from each of the $500k$ consecutive bicliques. Hence the hole has length more than $500k$. ◀

The following claim can be argued similarly.

▶ **Observation 27.** *Any hole H in G' which contains a vertex from $S \cup \mathcal{D}_L \cup \mathcal{D}_{\mathcal{R}}$, intersects all the bicliques in $S \cup \mathcal{D}_L \cup \mathcal{D}_R$.*

▶ **Lemma 28 (♣).** *Reduction Rule 24 is safe.*

With the above reduction rule we obtain the following result.

▶ **Lemma 29.** *Given an instance (G, k) of* BPVD *and a nice modulator $T \subseteq V(G)$ of size $k^{\mathcal{O}(1)}$, in polynomial time, we can construct an equivalent instance (G', k) such that, $T \subseteq V(G')$, T is a nice modulator for G' and for each connected component $\mathcal{C}$ of $G' - T$ with a complete bipartite decomposition $(Q_1, R_1, \ldots, Q_s, R_s)$, the number of blocks $(Q_i \cup R_i)$s in the connected component $\mathcal{C}$ is at most $3500|T|k^2 = \mathcal{O}(k^2 \cdot |T|)$.*

6 Bounding the Number of Connected Components

Until now we have assumed that $G - T$ is connected. Further, in Section 5, we showed that the size of any connected component is upper bounded by $k^{\mathcal{O}(1)}$. In this section we show that the number of connected components in $G - T$ is also upper bounded by $k^{\mathcal{O}(1)}$. This together with the fact that $|T| \le k^{\mathcal{O}(1)}$, result in a polynomial kernel for BPVD.

A connected component that has no neighbor in T is a bipartite permutation graph. Hence, we can safely remove it from our instance.

▶ **Reduction Rule 30.** *If there is a connected component $\mathcal{C}$ in $G-T$ such that $N(T) \cap V(\mathcal{C}) = \emptyset$, then reduce (G, k) to $(G - V(\mathcal{C}), k)$.*

From now onwards, we assume that the Reduction Rules 20, 21 and 30 are not applicable. We partition the set of all the connected components in $G - T$ into two sets $\mathbb{C}_{\ge 2}$ and $\mathbb{C}_{=1}$, where $\mathbb{C}_{\ge 2}$ contains all the connected components of size at least 2 whereas, $\mathbb{C}_{=1}$ contains all the connected components of size exactly 1. First, we bound the size of $\mathbb{C}_{\ge 2}$.

▶ **Lemma 31.** $|\mathbb{C}_{\ge 2}| \le 7|T|(k + 1)$.

Proof. Consider any vertex $v \in T$ such that v has a neighbor, say, a, in a connected component, say, $\mathcal{C}_i$, where $\mathcal{C}_i \in \mathbb{C}_{\ge 2}$. Note that for vertex a_i, there exists a neighbor $b_i \in \mathcal{C}_i$ since $\mathcal{C}_i$ has size at least 2.

Case 1: (The vertex b_i is adjacent to v.) Therefore, we have a triangle (v, a_i, b_i, v). If v has more than $k + 1$ such different pairs of (a_i, b_i) such that b_i is adjacent to v, then there are $k + 1$ triangles of the form (v, a_i, b_i, v) having a common vertex v. It implies that any solution of size k must contain v. By non-applicability of Reduction Rule 20 such case cannot occur. Hence, for any vertex $v \in T$, v has neighbors (a_i's) in at most $k + 1$ different components $\mathcal{C}_i \in \mathbb{C}_{\ge 2}$ such that there is a vertex $b_i \in \mathcal{C}_i \cap N(v)$.

Case 2: (The vertex b_i is not adjacent to v.) Therefore, (v, a_i, b_i) is an induced P_3. Let v has more than $6(k + 1)$ neighbors (a_i's) in different $\mathcal{C}_i$ such that there exists $b_i \in \mathcal{C}_i \setminus N(v)$. Therefore, there exists some $Q_i \cup R_i$ in component $\mathcal{C}_i$ such that $a_i \in Q_i \cup R_i$ and $b_i \in N(a_i) \cap Q_i \setminus N(v)$. Since vertex v has more that $6(k + 1)$ such neighbors a_i, Reduction Rule 21 would be applicable. By non-applicability of Reduction Rule 21 such case cannot occur. Hence, for any vertex $v \in T$, v has neighbors (a_i's) in at most $6(k+1)$ different components $\mathcal{C}_i$ such that there is a vertex $b_i \in N(a_i) \cap Q_i \setminus N(v)$.

Thus, every vertex $v \in T$ has neighbors at most in $(k + 1) + 6(k + 1)$, that is, $7(k + 1)$ different components $\mathcal{C}_i$'s. Hence, $|\mathbb{C}_{\ge 2}| \le 7|T|(k + 1)$. ◀

Next, we proceed to bound the size of the set $\mathbb{C}_{=1}$. Towards that we will utilize the next marking scheme.

Procedure `Mark-3`. We initialise $M = \emptyset$ and for each $\{x, y\} \subseteq T$, we initialise $M(x, y) = \emptyset$, and do as follows: For each $\{x, y\} \subseteq T$, if $|M(x, y)| \le k + 1$ and if there exists $u \in \mathbb{C}_{=1}$ such that $u \in (N(x) \cap N(y)) \setminus M$, then we add u to $M(x, y)$ and M, i.e., we set $M(x, y) \leftarrow M(x, y) \cup \{u\}$ and $M \leftarrow M \cup \{u\}$.

▶ Remark 32. Observe first that $M = \bigcup_{\{x,y\} \subseteq T} M(x, y)$. And in the procedure `Mark-3`, corresponding to each $\{x, y\} \subseteq T$, we add at most $k+1$ vertices to $M(x, y)$. Thus, $|M(x, y)| \le k + 1$, and therefore, $|M| \le (k + 1)\binom{|T|}{2}$, as there are $\binom{|T|}{2}$ many distinct sets $\{x, y\} \subseteq T$.

▶ **Reduction Rule 33.** *If there exists $v \in \mathbb{C}_{=1} \setminus M$, then delete v.*

▶ **Lemma 34** (♣). *Reduction Rule 33 is safe.*

Observe that by Remark 32 and by applying the Reduction Rule 33 repeatedly, we can reduce the graph such that in the reduced instance, $|\mathbb{C}_{=1}| \leq (k+1)\binom{|T|}{2}$. This reduction and Lemma 31 implies the following result:

▶ **Lemma 35.** *Given an instance (G, k) and a nice modulator $T \subseteq V(G)$ of size $k^{\mathcal{O}(1)}$, in polynomial time, we can construct an equivalent instance (G', k) such that the number of connected component in $G' - T$ is $\mathcal{O}(k \cdot |T|^2)$.*

7 Kernel size analysis

Now we are ready to prove the main result of our paper, that is, Theorem 1. Before proceeding with the proof, let us state all the bounds that contributes to the kernel size.

Size of nice modulator T: $\mathcal{O}(k^{45})$
Number of connected components in $G - T$: $\mathcal{O}(k \cdot |T|^2)$.
Number of blocks in any connected component in $G - T$: $\mathcal{O}(k^2 \cdot |T|)$
Size of any block $(Q_i \cup R_i)$ in $G - T$: $\mathcal{O}(k \cdot |T|^2)$.

Proof of Theorem 1. Let (G, k) be an instance to the BPVD problem. First we show that if G is not connected we can reduce it to the connected case. If there is a connected component $\mathcal{C}$ that is a bipartite permutation graph, we delete it. Clearly, (G, k) is a yes instance if and only if $(G \setminus \mathcal{C}, k)$ is a yes instance. We repeat this process until every connected component of G is not a bipartite permutation graph. At this stage if the number of connected components is at least $k + 1$, then we conclude that G can not be made into a bipartite permutation graph by deleting at most k vertices. Thus, we assume that G has at most k connected components. Now we show how to obtain a kernel for the case when G is connected, and for the disconnected case, we just run this algorithm on each connected component. This only increases the kernel size by a factor of k. From now onwards we assume that G is connected.

From Lemma 8, in polynomial time, we can obtain a nice modulator $T \subseteq V(G)$ of size $\mathcal{O}(k^{45})$ or concludes that (G, k) is a no-instance.

Note that, $G - T$ is a bipartite permutation graph. Next, we take the complete bipartite decomposition of each component in $G - T$. Now by Theorem 35, in polynomial time we return a graph G such that $G - T$ has $\mathcal{O}(k \cdot |T|^2)$ components.

Next, we show how to obtain a kernel for one connected component in $G - T$ and we just run this algorithm on each connected component. This only increases the kernel size by a factor of $\mathcal{O}(k \cdot |T|^2)$. From now onwards we assume that G' is a connected component in $G - T$. By Lemma 29, in polynomial time we can reduce the graph G' such that G' has at most $\mathcal{O}(k^2 \cdot |T|)$ blocks $Q_i \cup R_i$. Next, we bound the size of each block $Q_i \cup R_i$ in G'. By Lemma 19, in polynomial time we can reduce the graph G' such that for each block $Q_i \cup R_i$, $|Q_j \cup R_j| = \mathcal{O}(k \cdot |T|^2)$. Therefore the total number of vertices in any connected component G is at most $\mathcal{O}(k \cdot |T|^2) \cdot \mathcal{O}(k^2 \cdot |T|)$, that is, $\mathcal{O}(k^3 \cdot |T|^3)$.

As the graph $G - T$ has at most $\mathcal{O}(k \cdot |T|^2)$ number of components, the total size of the graph $G - T$ is at most $\mathcal{O}(k \cdot |T|^2) \cdot \mathcal{O}(k^3 \cdot |T|^3)$, that is, $\mathcal{O}(k^4 \cdot |T|^5)$. It follows that $|V(G)| = \mathcal{O}(k^4 \cdot |T|^5) + |T|$, that is, $\mathcal{O}(k^4 \cdot |T|^5)$. Recall that $|T| = \mathcal{O}(k^{45})$. Therefore, the size of the obtained kernel is $\mathcal{O}(k^4 \cdot |T|^5)$, that is, $\mathcal{O}(k^{229})$. ◀

8 Conclusion

In this paper we studied Bipartite Permutation Vertex Deletion from the perspective of kernelization complexity, and designed a polynomial kernel of size $\mathcal{O}(k^{229})$. This answers an open question posed by Bożyk et al. [4]. We remark that the size of kernel can be brought closer to $\mathcal{O}(k^{100})$ by doing more careful case analysis. However, getting a kernel of size $\mathcal{O}(k^{20})$ would require significantly new ideas and we leave that as an open problem. Indeed, showing whether Permutation Vertex Deletion is FPT remains a challenging open problem.

References

1 Akanksha Agrawal, Daniel Lokshtanov, Pranabendu Misra, Saket Saurabh, and Meirav Zehavi. Feedback vertex set inspired kernel for chordal vertex deletion. In *Proceedings of the 28th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1383–1398, 2017.

2 Akanksha Agrawal, Pranabendu Misra, Saket Saurabh, and Meirav Zehavi. Interval vertex deletion admits a polynomial kernel. In Timothy M. Chan, editor, *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019*, pages 1711–1730. SIAM, 2019.

3 Hans L. Bodlaender, Rodney G. Downey, Michael R. Fellows, and Danny Hermelin. On problems without polynomial kernels. *Journal of Computer and System Sciences*, 75(8):423–434, 2009.

4 Lukasz Bozyk, Jan Derbisz, Tomasz Krawczyk, Jana Novotná, and Karolina Okrasa. Vertex deletion into bipartite permutation graphs. In Yixin Cao and Marcin Pilipczuk, editors, *15th International Symposium on Parameterized and Exact Computation, IPEC 2020, December 14-18, 2020, Hong Kong, China (Virtual Conference)*, volume 180 of *LIPIcs*, pages 5:1–5:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

5 Andreas Brandstädt, Van Bang Le, and Jeremy P Spinrad. *Graph classes: a survey*. SIAM, 1999.

6 Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer-Verlag, 2015.

7 Holger Dell and Dániel Marx. Kernelization of packing problems. In *Proceedings of the 23rd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 68–81, 2012.

8 Holger Dell and Dieter van Melkebeek. Satisfiability allows no nontrivial sparsification unless the polynomial-time hierarchy collapses. *Journal of the ACM*, 61(4):23:1–23:27, 2014.

9 Reinhard Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.

10 Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013.

11 Andrew Drucker. New limits to classical and quantum instance compression. *SIAM Journal on Computing*, 44(5):1443–1479, 2015.

12 Jörg Flum and Martin Grohe. *Parameterized Complexity Theory*. Springer-Verlag, Berlin, 2006.

13 F. V. Fomin, S. Saurabh, and Y. Villanger. A Polynomial Kernel for Proper Interval Vertex Deletion. *SIAM Journal on Discrete Mathematics*, 27(4):1964–1976, 2013.

14 Fedor V. Fomin, Daniel Lokshtanov, Neeldhara Misra, and Saket Saurabh. Planar f-deletion: Approximation, kernelization and optimal FPT algorithms. In *53rd Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 470–479, 2012.

15 Fedor V. Fomin and Saket Saurabh. Kernelization methods for fixed-parameter tractability. In *Tractability*, pages 260–282. Cambridge Univ. Press, Cambridge, 2014.

16 Lance Fortnow and Rahul Santhanam. Infeasibility of instance compression and succinct PCPs for NP. *Journal of Computer and System Sciences*, 77(1):91–106, 2011.
17 Toshihiro Fujito. A unified approximation algorithm for node-deletion problems. *Discrete Applied Mathematics*, 86:213–231, 1998.
18 Martin Charles Golumbic. *Algorithmic Graph Theory and Perfect Graphs*. Academic Press, 1980.
19 Jiong Guo and Rolf Niedermeier. Invitation to data reduction and problem kernelization. *SIGACT News*, 38(1):31–45, 2007.
20 Danny Hermelin, Stefan Kratsch, Karolina Soltys, Magnus Wahlström, and Xi Wu. A completeness theory for polynomial (turing) kernelization. *Algorithmica*, 71(3):702–730, 2015.
21 Danny Hermelin and Xi Wu. Weak compositions and their applications to polynomial lower bounds for kernelization. In *Proceedings of the 23rd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 104–113, 2012.
22 Bart M. P. Jansen and Marcin Pilipczuk. Approximation and kernelization for chordal vertex deletion. In *Proceedings of the 28th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1399–1418, 2017.
23 Yuping Ke, Yixin Cao, Xiating Ouyang, Wenjun Li, and Jianxin Wang. Unit interval vertex deletion: Fewer vertices are relevant. *J. Comput. Syst. Sci.*, 95:109–121, 2018.
24 Stefan Kratsch. Recent developments in kernelization: A survey. *Bulletin of the EATCS*, 113, 2014.
25 John M. Lewis and Mihalis Yannakakis. The node-deletion problem for hereditary properties is NP-complete. *Journal of Computer and System Sciences*, 20(2):219–230, 1980.
26 Daniel Lokshtanov, Neeldhara Misra, and Saket Saurabh. Kernelization - preprocessing with a guarantee. In *The Multivariate Algorithmic Revolution and Beyond - Essays Dedicated to Michael R. Fellows on the Occasion of His 60th Birthday*, pages 129–161, 2012.
27 Carsten Lund and Mihalis Yannakakis. On the hardness of approximating minimization problems. *Journal of the ACM*, 41:960–981, 1994.
28 Dániel Marx. Chordal deletion is fixed-parameter tractable. *Algorithmica*, 57(4):747–768, 2010.
29 Ross M. McConnell and Jeremy P. Spinrad. Modular decomposition and transitive orientation. *Discret. Math.*, 201(1-3):189–241, 1999.
30 Rolf Niedermeier. *Invitation to fixed-parameter algorithms*, volume 31. Oxford University Press, 2006.
31 Jeremy P. Spinrad, Andreas Brandstädt, and Lorna Stewart. Bipartite permutation graphs. *Discret. Appl. Math.*, 18(3):279–292, 1987.
32 Ryuhei Uehara and Gabriel Valiente. Linear structure of bipartite permutation graphs and the longest path problem. *Inf. Process. Lett.*, 103(2):71–77, 2007.
33 Mihalis Yannakakis. Node- and edge-deletion np-complete problems. In *Proceedings of the 10th Annual ACM Symposium on Theory of Computing (STOC)*, pages 253–264, 1978.