Enclosing Depth and Other Depth Measures

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Abstract

We study families of depth measures defined by natural sets of axioms. We show that any such depth measure is a constant factor approximation of Tukey depth. We further investigate the dimensions of depth regions, showing that the *Cascade conjecture*, introduced by Kalai for Tverberg depth, holds for all depth measures which satisfy our most restrictive set of axioms, which includes Tukey depth. Along the way, we introduce and study a new depth measure called *enclosing depth*, which we believe to be of independent interest, and show its relation to a constant-fraction Radon theorem on certain two-colored point sets.

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1 Introduction

Medians are an important tool in the statistical analysis and visualization of data. Due to the fact that medians only depend on the order of the data points, and not their exact positions, they are very robust against outliers. However, in many applications, data sets are multidimensional, and there is no clear order of the data set. For this reason, various generalizations of medians to higher dimensions have been introduced and studied, see e.g. [1, 17, 21] for surveys. Many of these generalized medians rely on a notion of *depth* of a query point within a data set, a median then being a query point with the highest depth among all possible query points. Several such depth measures have been introduced over time, most famously Tukey depth [28] (also called halfspace depth), simplicial depth [16], or convex hull peeling depth (see, e.g., [1]). In particular, just like the median, all of these depth measures only depend on the relative positions of the involved points. More formally, let $S^{\mathbb{R}^d}$ denote the family of all finite sets of points in $\mathbb{R}^d$. A depth measure is a function $\varrho : (S^{\mathbb{R}^d}, \mathbb{R}^d) \to \mathbb{R}_{\geq 0}$ which assigns to each pair (S, q) consisting of a finite set of data points S and a query point q a value, which describes how deep the query point q lies within the data set S. A depth measure ϱ is called *combinatorial* if it depends only on the order type of $S \cup \{q\}$, that is, if it only depends on the orientations of the simplices spanned by the points, but not on their actual positions. In this paper, we consider general classes of combinatorial depth measures, defined by a small set of axioms, and prove relations between them and concrete depth measures, such as *Tukey depth* (TD) and *Tverberg depth* (TvD). Let us first briefly discuss these two depth measures.

▶ **Definition 1.** *Let S be a finite point set in $\mathbb{R}^d$ and let q be a query point. Then the Tukey depth of q with respect to S, denoted by $\boldsymbol{TD}(S, q)$, is the minimum number of points of S in any closed half-space containing q.*

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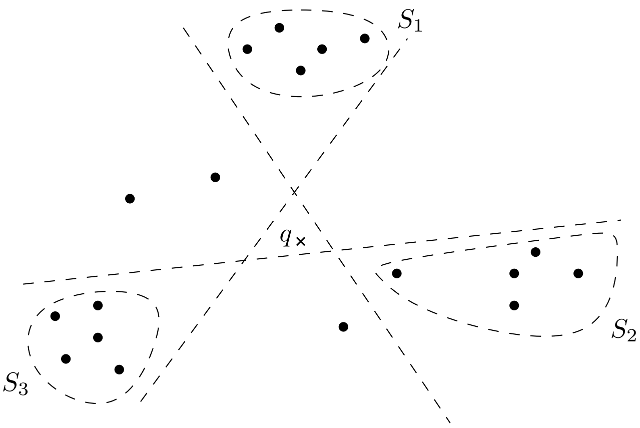

Figure 1 The point q has enclosing depth 5.

Tukey depth, also known as *halfspace depth*, was independently introduced by Joseph L. Hodges in 1955 [11] and by John W. Tukey in 1975 [28] and has received significant attention since, both from a combinatrial as well as from an algorithmic perspective, see e.g. Chapter 58 in [27] and the references therein. Notably, the *centerpoint theorem* states that for any point set $S \subset \mathbb{R}^d$, there exists a point $q \in \mathbb{R}^d$ for which $\mathrm{TD}(S, q) \geq \frac{|S|}{d+1}$ [22].

In order to define Tverberg depth, we need a preliminary definition: given a point set S in $\mathbb{R}^d$, an *r-partition* of S is a partition of S into r pairwise disjoint subsets $S_1, \ldots, S_r \subset S$ with $\bigcap_{i=1}^r \mathrm{conv}(S_i) \neq \emptyset$, where $\mathrm{conv}(S_i)$ denotes the convex hull of S_i. We call $\bigcap_{i=1}^r \mathrm{conv}(S_i)$ the *intersection* of the r-partition.

▶ **Definition 2.** *Let S be a finite point set in $\mathbb{R}^d$ and let q be a query point. Then the Tverberg depth of q with respect to S, denoted by $\boldsymbol{TvD}(S, q)$, is the maximum r such that there is an r-partition of S whose intersection contains q.*

Tverberg depth is named after Helge Tverberg who proved in 1966 that any set of $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ allows an r-partition [29]. In particular, this implies that there is a point q with $\mathrm{TvD}(S, q) \geq \frac{|S|}{d+1}$. Just as for Tukey depth, there is an extensive body of work on Tverbergs theorem, see the survey [4] and the references therein.

In $\mathbb{R}^1$, both Tukey and Tverberg depth give a very natural depth measure: it counts the number of points of S to the left and to the right of q and then returns the minimum of the two numbers. We call this measure the *standard depth* in $\mathbb{R}^1$. In particular, for all of them there is always a point $q \in \mathbb{R}^1$ for which we have $\varrho(S, q) \geq \frac{|S|}{2}$, that is, a median.

Another depth measure that is important in this paper is called enclosing depth. For an illustration of this depth measure, see Figure 1 We say that a point set S of size $(d+1)k$ in $\mathbb{R}^d$ *k-encloses* a point q if S can be partitioned into $d+1$ pairwise disjoint subsets $S_1, \ldots, S_{d+1}$, each of size k, in such a way that for every transversal $p_1 \in S_1, \ldots, p_{d+1} \in S_{d+1}$, the point q is in the convex hull of $p_1, \ldots, p_{d+1}$. Intuitively, the points of S are centered around the vertices of a simplex with q in its interior.

▶ **Definition 3.** *Let S be a finite point set in $\mathbb{R}^d$ and let q be a query point. Then the enclosing depth of q with respect to S, denoted by $\boldsymbol{ED}(S, q)$, is the maximum k such that there exists a subset of S which k-encloses q.*

It is straightforward to see that enclosing depth also gives the standard depth in $\mathbb{R}^1$. The centerpoint theorem [22] and Tverberg's theorem [29] show that both for Tukey as well as Tverberg depth, there are deep points in any dimension. The question whether a depth measure enforces deep points is a central question in the study of depth measures. We will show that this also holds for enclosing depth. In fact, we will show that enclosing depth can be bounded from below by a constant fraction of Tukey depth. We will further show that all depth measures considered in this paper can be bounded from below by enclosing depth. From this we get one of the main results of this paper: all depth measures that satisfy the axioms given later are a constant factor approximation of Tukey depth.

Another area of study in depth measures are *depth regions*, also called depth contours. For some depth measure ϱ and $\alpha \in \mathbb{R}$, we define the *α-region* of a point set $S \subset \mathbb{R}^d$ as the set of all points in $\mathbb{R}^d$ that have depth at least α with respect to S. We denote the α-region of S by $D^S_\varrho(\alpha) := \{q \in \mathbb{R}^d \mid \varrho(S,q) \geq \alpha\}$. Note that for $\alpha < \beta$ we have $D^S_\varrho(\alpha) \supset D^S_\varrho(\beta)$, that is, the depth regions are nested. The structure of depth regions has been studied for several depth measures, see e.g. [20, 32] In particular, depth regions in $\mathbb{R}^2$ have been proposed as a tool for data visualization [28]. From a combinatorial point of view, Gil Kalai introduced the following conjecture [13]

▶ **Conjecture 4** (Cascade Conjecture). *Let S be a point set of size n in $\mathbb{R}^d$. For each $i \in \{1, \ldots, n\}$, denote by d_i the dimension of $D^S_{TvD}(i)$, where we set* $\dim(\emptyset) = -1$. *Then*

$$\sum_{i=1}^{n} d_i \geq 0.$$

The conjecture is known to be true when S is in so-called *strongly* general position [23], for general position in some dimensions [24, 25, 26] (see also [4] for more information), and without any assumption of general position for $d \leq 2$ in an unpublished M. Sc thesis in Hebrew by Akiva Kadari (see [15]).

While Kalai's conjecture is specifically about Tverberg depth, the sum of dimensions of depth regions can be computed for any depth measure, and thus the conjecture can be generalized to other depth measures. In fact, in a talk Kalai conjectured that the Cascade conjecture is true for Tukey depth, mentioning on his slides that "this should be doable" [14]. In this work, we will prove the conjecture to be true for a family of depth measures that includes Tukey depth.

Structure of the paper

We start the technical part by introducing a first set of axioms in Section 2, defining what we call *super-additive* depth measures. For these depth measures, we show that they lie between Tukey and Tverberg depth. In Section 3 we then prove the cascade conjecture for additive depth measures whose depth regions are convex. We then give a second set of axioms in Section 4, defining *central* depth measures, and show how to bound them from below by enclosing depth. Finally, in Section 5, we give a lower bound for enclosing depth in terms of Tukey depth. In order to prove this bound, we notice a close relationship of enclosing depth with a version of Radon's theorem on certain two-colored point sets.

2 A first set of axioms

The first set of depth measures that we consider are *super-additive* depth measures[1]. A combinatorial depth measure $\varrho : (S^{\mathbb{R}^d}, \mathbb{R}^d) \to \mathbb{R}_{\geq 0}$ is called super-additive if it satisfies the following conditions:

[1] We name both our families of depth measures after one of the conditions they satisfy. The reason for this is that the condition they are named after is the condition which separates this family from the other one.

(i) for all $S \in S^{\mathbb{R}^d}$ and $q, p \in \mathbb{R}^d$ we have $|\varrho(S,q) - \varrho(S \cup \{p\}, q)| \leq 1$ (sensitivity),
(ii) for all $S \in S^{\mathbb{R}^d}$ and $q \in \mathbb{R}^d$ we have $\varrho(S,q) = 0$ for $q \notin \mathrm{conv}(S)$ (locality),
(iii) for all $S \in S^{\mathbb{R}^d}$ and $q \in \mathbb{R}^d$ we have $\varrho(S,q) \geq 1$ for $q \in \mathrm{conv}(S)$ (non-triviality),
(iv) for any disjoint subsets $S_1, S_2 \subseteq S$ and $q \in \mathbb{R}^d$ we have $\varrho(S,q) \geq \varrho(S_1,q) + \varrho(S_2,q)$ (super-additivity).

It is not hard to show that a one-dimensional depth measure which satisfies these conditions has to be the standard depth measure (in fact, the arguments are generalized to higher dimensions in the following two observations) and that no three conditions suffice for this. Further, it can be shown that both Tukey depth and Tverberg depth are super-additive.

We first note that the first two axioms suffice to give an upper bound:

▶ **Observation 5.** *For every depth measure ϱ satisfying (i) sensitivity and (ii) locality and for all $S \in S^{\mathbb{R}^d}$ and $q \in \mathbb{R}^d$ we have $\varrho(S,q) \leq TD(S,q)$.*

Proof. By the definition of Tukey depth, $\mathrm{TD}(S,q) = k$ implies that we can remove a subset S' of k points from S so that q is not in the convex hull of $S \setminus S'$. In particular, $\varrho(S \setminus S', q) = 0$ by locality. By sensitivity we further have $\varrho(S \setminus S', q) \geq \varrho(S,q) - k$, which implies the claim. ◀

Further, the last two axioms can be used to give a lower bound:

▶ **Observation 6.** *For every depth measure ϱ satisfying (iii) non-triviality and (iv) super-additivity and for all $S \in S^{\mathbb{R}^d}$ and $q \in \mathbb{R}^d$ we have $\varrho(S,q) \geq TvD(S,q)$.*

Proof. Let $\mathrm{TvD}(S,q) = k$ and consider a k-partition $S_1, \ldots, S_k$ with q in its intersection. By non-triviality we have $\varrho(S_i, q) \geq 1$ for each S_i. Using super-additivity and induction we conclude that $\varrho(\bigcup_{i=1}^k S_i, q) \geq \sum_{i=1}^k \varrho(S_i, k) \geq k$. ◀

Finally, it is not too hard to show that $\mathrm{TvD}(S,q) \geq \frac{1}{d}\mathrm{TD}(S,q)$, see e.g. [10] for an argument. Combining these observations, we thus get the following.

▶ **Corollary 7.** *Let ϱ be a super-additive depth measure. Then for every point set S and query point q in $\mathbb{R}^d$ we have*

$$TD(S,q) \geq \varrho(S,q) \geq TvD(S,q) \geq \frac{1}{d} TD(S,q).$$

Let us note here that it could be that the factor $\frac{1}{d}$ in the last inequality could be improved. Indeed, in the plane, we have that $\mathrm{TvD} = \min\{\mathrm{TD}, \lceil \frac{|S|}{3} \rceil\}$ [23]. This fails already in dimension 3 [3]. It would be interesting to see how much the factor $\frac{1}{d}$ can be improved.

From Corollary 7 it follows that for any super-additive depth measure and any point set there is always a point of depth at least $\frac{|S|}{d+1}$, for example any Tverberg point. On the other hand, there are depth measures which give the standard depth in $\mathbb{R}^1$ which are not super-additive, for example convex hull peeling depth or enclosing depth.

▶ **Observation 8.** *Enclosing depth satisfies conditions (i)–(iii) and (v), but not the super-additivity condition (iv).*

Proof. It follows straight from the definition that enclosing depth satisfies the conditions (i)–(iii) and (v). To see that the super-additivity condition is not satisfied, consider the example in Figure 2. The point q has enclosing depth 1 with respect to both the set of blue points and the set of red points. However, it can be seen that the enclosing depth of q with respect to both the red and the blue points is still 1. ◀

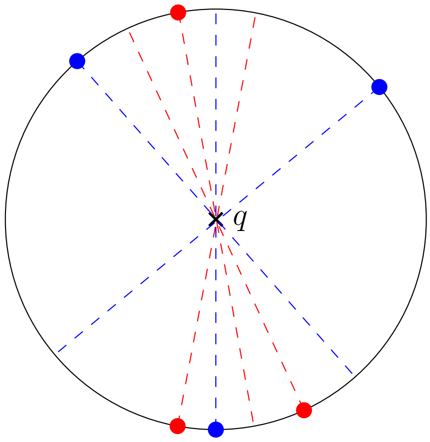

Figure 2 Enclosing depth does not satisfy the super-additivity condition: the point q has enclosing depth 1 with respect to both the blue and the red points, but its enclosing depth with respect to the union of the two sets is still 1.

3 The Cascade Conjecture

In this section we prove the cascade conjecture for super-additive depth measures whose depth regions are convex. In fact, we will prove the cascade conjecture for the case of *weighted point sets*. A weighted point set is a point set S together with a weight function $w : S \to \mathbb{R}_{\geq 0}$ which assigns a weight $w(p)$ to each $p \in S$. We say that a weighted point set S' is a strict subset of S, denoted by $S' \subset S$, if the underlying point set of S' is a strict subset of the underlying point set of S, and $w'(p) \leq w(p)$ for every $p \in S'$, where w' is the weight function on S'. In particular, if $S' \subset S$, there is a point which is in S but not in S'. For two weighted point sets A and B with weight functions w_A and w_B, respectively, the weight function on their union $A \cup B$ is defined as the sum of the respective weight functions. That is, we have $w(p) = w_A(p)$ for $p \in A \setminus B$, $w(p) = w_B(p)$ for $p \in B \setminus A$ and $w(p) = w_A(p) + w_B(p)$ for $p \in A \cap B$. Further, for a set S of points we define the weight of S as $w(S) := \sum_{p \in S} w(p)$. Similarly, by a partition of a weighted point set S into parts A and B we mean two weight functions w_A and w_B, such that $w(p) = w_A(p) + w_B(p)$ for $p \in S$, and by a partition into strict subsets A and B, we mean that both weighted point sets A and B must be strict subsets of S, that is, there are points p_A, p_B in S for which $w_A(p_A) = 0$ and $w_B(p_B) = 0$. The axioms for super-additive depth measures extend to weighted point sets in the following way:

(i) for all $S \in S^{\mathbb{R}^d}$ and $q, p \in \mathbb{R}^d$ we have $|\varrho(S, q) - \varrho(S \cup \{p\}, q)| \leq w(p)$ (sensitivity),
(ii) for all $S \in S^{\mathbb{R}^d}$ and $q \in \mathbb{R}^d$ we have $\varrho(S, q) = 0$ for $q \notin \text{conv}(S)$ (locality),
(iii) for all $S \in S^{\mathbb{R}^d}$ and $q \in \mathbb{R}^d$ we have $\varrho(S, q) \geq \min\{w(p) : p \in S\}$ for $q \in \text{conv}(S)$ (non-triviality),
(iv) for any disjoint subsets $S_1, S_2 \subseteq S$ and $q \in \mathbb{R}^d$ we have $\varrho(S, q) \geq \varrho(S_1, q) + \varrho(S_2, q)$ (super-additivity).

Clearly, each point set can be considered as a weighted point set by assigning weight 1 to each point. On the other hand, by placing several points at the same location, normalizing and using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, each depth measure defined on point sets can be extended to weighted point sets. Further, we can again define depth regions $D^S_\varrho(\alpha) := \{q \in \mathbb{R}^d \mid \varrho(S, q) \geq \alpha\}$. We will also use a special depth region, called the *median region*, denoted by $M_\varrho(S)$, which is the deepest non-empty depth region. More formally, let α_0 be the supremum value for which $D_\varrho^S(\alpha_0) \neq \emptyset$. Then $M_\varrho(S) := D_\varrho^S(\alpha_0)$. In the setting of weighted point sets, the cascade condition translates to

$$\int_0^{w(S)} d_\alpha \mathrm{d}\alpha \geq 0.$$

Note that the cascade conjecture for a depth measure on weighted point sets implies the cascade conjecture for that depth measure on unweighted point sets. If for a depth measure ϱ the above integral is non-negative for any weighted point set S, we say that ϱ is *cascading*.

In the following, we will show that super-additive depth measures whose depth regions are convex are cascading in two steps. First we will show that if we partition a weighted point set into two parts whose median regions intersect and the cascade condition holds for both parts, then the cascade condition holds for the whole set. In a second step, we prove that we can always partition a point set in such a way, further enforcing that none of the parts contains all points, that is, each part is a strict subset. The claim then follows by induction.

► **Lemma 9.** *Let ϱ be a super-additive depth measure whose depth regions are convex and let S_1 and S_2 be two weighted point sets in $\mathbb{R}^d$ whose median regions intersect. Assume that the cascade condition holds for S_1 and S_2. Then the cascade condition holds for $S_1 \cup S_2$.*

Before we prove this, let us describe a way to compute $\int_0^{w(S)} d_\alpha \mathrm{d}\alpha$. Consider some depth region $D_\varrho^S(\alpha)$ of dimension k. Being convex, this depth region lies in some k-dimensional affine subspace $H \subset \mathbb{R}^d$. Considering all depth regions, they lie in a sequence of nested affine subspaces, also known as a *flag*. Assuming that the origin lies in the median region, we can find a basis $F = \{f_1, \ldots, f_d\}$ of $\mathbb{R}^d$ such that each relevant affine subspace is spanned by a subset of the basis vectors. In fact, there are many choices of bases. Further, we can assign to each basis vector f_i a *survival time* α_i defined by the following property: for each $\alpha \in \mathbb{R}$, the affine subspace in which $D_\varrho^S(\alpha)$ lies is spanned by the subset $\{f_i \in F \mid \alpha_i \geq \alpha\}$. As above, we let α_0 be the supremum value for which $D_\varrho^S(\alpha_0) \neq \emptyset$, that is, we view α_0 as the survival time of the origin. Using this formulation, we note that

$$\int_0^{w(S)} d_\alpha d\alpha = \sum_{i=0}^{d} \alpha_i - w(S),$$

see Figure 3 for an illustration.

Proof of Lemma 9. We may assume without loss of generality that the origin is in both median regions. Further, we can choose a basis $F = \{f_1, \ldots, f_d\}$ of $\mathbb{R}^d$ such that all relevant affine subspaces both of S_1 and S_2, and thus also of $S_1 \cup S_2$, are spanned by subsets of F. Let α_i, β_i and γ_i denote the survival times of f_i for S_1, S_2 and $S_1 \cup S_2$, respectively. It follows from the super-additivity condition that $\gamma_i \geq \alpha_i + \beta_i$. Thus we get

$$\begin{aligned}\sum_{i=0}^{d} \gamma_i - w(S_1 \cup S_2) &\geq \sum_{i=0}^{d} (\alpha_i + \beta_i) - (w(S_1) + w(S_2)) \\ &\geq \sum_{i=0}^{d} \alpha_i - w(S_1) + \sum_{i=0}^{d} \beta_i - w(S_2) \geq 0. \qquad (1)\end{aligned}$$

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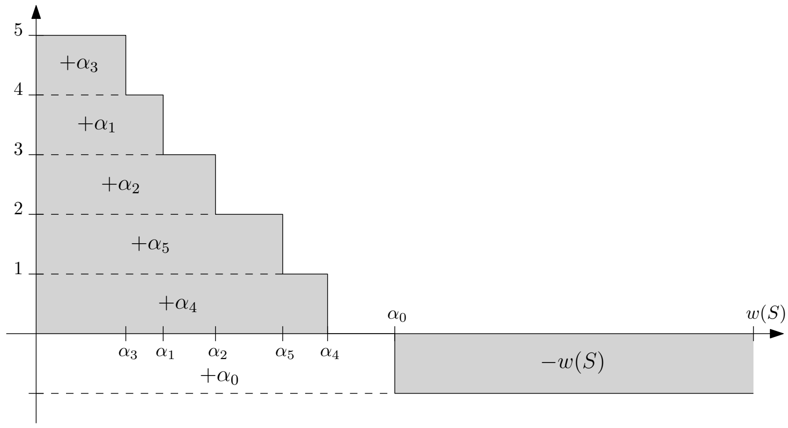

Figure 3 $\int_0^{w(S)} d_\alpha d\alpha = \sum_{i=0}^{d} \alpha_i - w(S)$.

▶ **Lemma 10.** *Let ϱ be a super-additive depth measure whose depth regions are convex and let S be a weighted point set in $\mathbb{R}^d$ with $|S| \geq d+2$. Then there exists a partition of S into strict subsets S_1 and S_2 whose median regions intersect.*

Proof. Assume without loss of generality that $w(p) = 1$ for every $p \in S$ (otherwise just multiply the weights of p in S_1 and S_2 with $w(p)$ after finding the partition). Consider the barycentric subdivision B of the boundary $\partial\Delta$ of the simplex with vertices S. There is a natural identification of the vertices of B with strict subsets of S (see Figure 4). Linearly extending this assignment to $\partial\Delta$ defines a map which assigns to each point b on $\partial\Delta$ a strict weighted subset $S(b)$ of S. Further, under the natural antipodality on $\partial\Delta$, we get complements of the weighted subsets, that is, $S(-b) = S(b)^C$.

We claim that for some point b on $\partial\Delta$ we have that the median regions of $S(b)$ and $S(-b)$ intersect. If this is true, our claim follows by setting $S_1 = S(b)$ and $S_2 = S(-b)$. Using Proposition 1 from [31], for each b we may assume that the median region of $S(b)$ is a single point $m(b)$ in $\mathbb{R}^d$ ant that the map m which sends b to $m(b)$ is continuous. We thus want to find a point b for which $m(b) = m(-b)$. Further, $\partial\Delta$ is homeomorphic to the sphere $S^{|S|-2}$, and the antipodality on $\partial\Delta$ corresponds to the standard antipodality on the sphere. As $|S| \geq d+2$, the existence of a point b for which $m(b) = m(-b)$ thus follows from the Borsuk-Ulam theorem. ◀

While we have only shown that there is a partition, Bourgin-Yang-type theorems [6, 30] tell us, that the space of possible partitions has to be large. In particular, it has dimension at least $|S| - d - 2$. Depending on the application, this might be used to enforce other conditions on the partitions.

▶ **Theorem 11.** *Let ϱ be a super-additive depth measure whose depth regions are convex. Then ϱ is cascading.*

Proof. Let S be a weighted point set in $\mathbb{R}^d$ and assume without loss of generality that its affine hull is $\mathbb{R}^d$ (otherwise, we can just consider S to be a weighted point set in some lower dimensional space). We want to show that the cascade condition holds for S. We prove this by induction on $|S|$. If $|S| \leq d+1$, then S must be the vertices of a simplex, and in this case it is not hard to check that the cascade condition holds. So, assume now that $|S| \geq d+2$. By Lemma 10, we can partition S into S_1 and S_2 whose median regions intersect. Note that $|S_1|, |S_2| < |S|$, so by the induction hypothesis the cascade condition holds for both S_1 and S_2. Thus, by Lemma 9, the cascade condition also holds for S. ◀

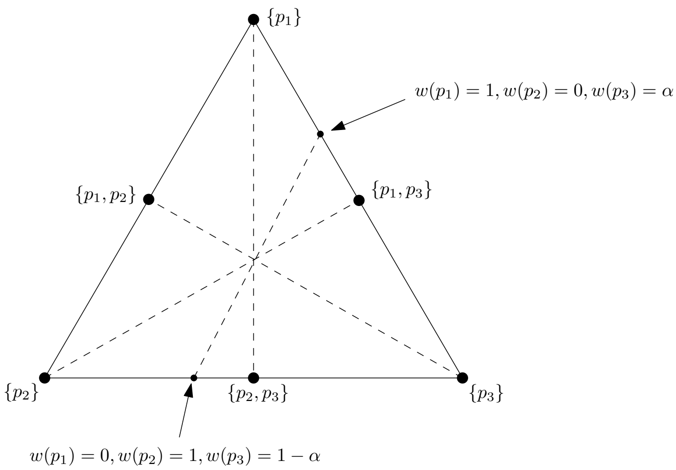

Figure 4 Vertices of the barycentric subdivision correspond to strict subsets.

As noted above, an example of a super-additive depth measure with convex depth regions is Tukey depth. Thus, we get the following.

▶ **Corollary 12.** *Tukey depth is cascading.*

On the other hand, while Tverberg depth is super-additive, its depth regions are in general not convex; in fact, they are not even connected. A weak version of Kalai's cascade conjecture claims that the cascade condition holds for the convex hull of Tverberg depth regions. These depth regions are convex by definition, but the resulting depth measure is in general not super-additive anymore. So while our approach proves the cascade conjecture for an entire family of depth measures, solving Kalai's cascade conjecture even in its weak form likely requires additional ideas. As every super-additive depth measure is bounded from below by Tverberg depth, solving the strong version of Kalai's cascade conjecture would imply that all super-additive depth measures are cascading. Further, it can be seen that any cascading depth measure must enforce deep points. More precisely, if ϱ is a cascading depth measure and S is a point set in $\mathbb{R}^d$, then there must be a point $q \in \mathbb{R}^d$ for which $\varrho(S, q) \geq \frac{|S|}{d+1}$. Indeed, if there was no such point, we would have $d_{|S|/(d+1)} = -1$, and even if $d_i = d$ for all $i < \frac{|S|}{d+1}$, the sum $\sum_{i=1}^{|S|} d_i$ would still be negative. The existence of deep points is the main feature of the next family of depth measures that we study.

4 A second set of axioms

The second family of depth measures we consider are *central* depth measures. A combinatorial depth measure $\varrho : (S^{\mathbb{R}^d}, \mathbb{R}^d) \to \mathbb{R}_{\geq 0}$ is called central if it satisfies the following conditions:

(i) for all $S \in S^{\mathbb{R}^d}$ and $q, p \in \mathbb{R}^d$ we have $|\varrho(S, q) - \varrho(S \cup \{p\}, q)| \leq 1$ (sensitivity),

(ii) for all $S \in S^{\mathbb{R}^d}$ and $q \in \mathbb{R}^d$ we have $\varrho(S, q) = 0$ for $q \notin \mathrm{conv}(S)$ (locality),

(iii') for every $S \in S^{\mathbb{R}^d}$ there is a $q \in \mathbb{R}^d$ for which $\varrho(S, q) \geq \frac{1}{d+1}|S|$ (centrality).

(iv') for all $S \in S^{\mathbb{R}^d}$ and $q, p \in \mathbb{R}^d$ we have $\varrho(S \cup \{p\}, q) \geq \varrho(S, q)$ (monotonicity),

Note that conditions (i) and (ii) are the same as for super-additive depth measures, so by Observation 5 we have $\varrho(S, q) \leq \mathrm{TD}(S, q)$ for every central depth measure. Further, the centrality condition (iii') is stronger than the non-triviality condition (iii) for super-additive depth measures. On the other hand, the super-additivity condition (iv) is stronger than the monotonicity condition (iv'), so at first glance, the families of super-additive depth measures and central depth measures are not comparable. However, we have seen before that any super-additive depth measure indeed satisfies the centrality condition, so central depth measures are a superset of super-additive depth measures. It is actually a strict superset, as for example the depth measure whose depth regions are defined as the convex hulls of Tverberg depth regions is central but not super-additive.

While central depth measures enforce deep points by definition, they might still differ a lot locally. In the following, we will show that we can bound by how much they differ locally, showing that every central depth measure is a constant factor approximation of Tukey depth.

▶ **Theorem 13.** *Let ϱ be a central depth measure in $\mathbb{R}^d$. Then there exists a constant $c = c(d)$, which depends only on the dimension d, such that*

$$TD(S, q) \geq \varrho(S, q) \geq ED(S, q) - (d + 1) \geq c \cdot TD(S, q) - (d + 1).$$

Here the first inequality is just Observation 5. As for the second inequality, we would like to argue that if S k-encloses q then $\varrho(S, q) = k$. By centrality, there must indeed be a point q' with $\varrho(S, q') = k$ (note that $|S| = k(d+1)$ by definition of k-enclosing), but this point can lie anywhere in the centerpoint region of S and not every point in the centerpoint region is k-enclosed by S. However, by adding $d + 1$ points very close to q, we can ensure that q is the only possible centerpoint in the new point set, and the second inequality then follows from sensitivity and monotonicity after removing these points again.

This argument can be generalized even to a relaxation of central depth measures: We say that a combinatorial depth measure as α-central if it satisfies conditions (i), (ii) and (iv'), and the following weak version of condition (iii'): for every $S \in S^{\mathbb{R}^d}$ there is a $q \in \mathbb{R}^d$ for which $\varrho(S, q) \geq \alpha|S|$ (α-centrality)

▶ **Lemma 14.** *Let $\alpha > \frac{1}{d+2}$, and let ϱ be an α-central depth measure. Then there exists a constant $c_1 = c_1(d)$ such that*

$$\varrho(S, q) \geq c_1 \cdot ED(S, q) - (d + 1).$$

Proof. Let $\mathrm{ED}(S, q) = k$ and let S' be a witness subset. Recall that by monotonicity, we have $\varrho(S, q) \geq \varrho(S', q)$. Further, note that $\mathrm{TD}(S', q) = k$ and $\mathrm{TD}(S', q') \leq k$ for all $q' \in \mathbb{R}^d$. Let $\alpha' := (d + 1)\alpha$ and let $m := \lfloor \frac{1-\alpha'}{\alpha'} k + 1 \rfloor$. Add $(d + 1)m$ points very close to q such that the new point set P $(k + m)$-encloses q. The new point set P has $(d+1)(k+m)$ many points, and we have

$$\alpha|P| = \alpha'(k + m) > \alpha'(k + \frac{1 - \alpha'}{\alpha'} k) = \alpha' k + (1 - \alpha')k = k.$$

In particular, the only points q' for which $\varrho(P, q') \geq \alpha|P|$ is possible are by construction very close to q. As they were in the same cell as q before adding the new points, we can assume without loss of generality that we have $\varrho(P, q) \geq \alpha|P|$. By sensitivity we now have

$$\begin{aligned}
&\varrho(S', q) \geq \varrho(P, q) - (d + 1)m \\
&\geq \alpha'(k + m) - (d + 1)m \\
&\geq \alpha' k - (d + 1 - \alpha')m \\
&\geq \alpha' k - (d + 1 - \alpha')(\frac{1 - \alpha'}{\alpha'} + 1) \\
&= \alpha' k - \frac{(d + 1 - \alpha')(1 - \alpha')}{\alpha'} k - (d + 1) + \alpha
\end{aligned}$$

$$\geq (\alpha'^2 - (d+1) + \alpha' + (d+1)\alpha' - \alpha'^2)\frac{k}{\alpha'} - (d+1)$$
$$= \frac{(d+2)\alpha' - (d+1)}{\alpha'}k - (d+1). \quad (2)$$

Plugging in $\alpha' := (d+1)\alpha$ we get

$$\varrho(S,q) \geq \frac{(d+2)(d+1)\alpha - (d+1)}{(d+1)\alpha}k - (d+1) = (d+2-\frac{1}{\alpha})k - (d+1).$$

As $(d+2-\frac{1}{\alpha}) > 0$ for $\alpha > \frac{1}{d+2}$, the claim follows. ◀

The most involved part of Theorem 13 is the last inequality, which we will prove in the next section.

5 A lower bound for enclosing depth

In this section, we will prove a lower bound on the enclosing depth in terms of Tukey depth:

▶ **Theorem 15** ($E(d)$). *There is a constant $c_1 = c_1(d)$ such that for all $S \in S^{\mathbb{R}^d}$ and $q \in \mathbb{R}^d$ we have $ED \leq c_1 \cdot TD(S,q)$.*

We will denote this statement in dimension d by $E(d)$. Note that $E(1)$ is true and $c_1(1) = 1$. The general result could be proved using the semi-algebraic same type lemma due to Fox, Pach and Suk [9], combined with the first selection lemma (see e.g. [19]). Here we will give a different proof for two reasons: first, the bounds on c_1 that our proof gives are better than the bounds we would get from the proof using the semi-algebraic same type lemma. Second, our proof shows an intimate relation of enclosing depth to a positive fraction Radon theorem on certain bichromatic point sets.

Let $P = R \cup B$ be a bichromatic point set with color classes R (red) and B (blue). We say that B *surrounds* R if for every halfspace h we have $|B \cap h| \geq |R \cap h|$. Note that this in particular implies $|B| \geq |R|$. The positive fraction Radon theorem is now the following:

▶ **Theorem 16** ($R(d)$). *Let $P = R \cup B$ be a bichromatic point set where B surrounds R. Then there is a constant $c_2 = c_2(d)$ such that there are integers a and b and pairwise disjoint subsets $R_1, \ldots, R_a \subseteq R$ and $B_1, \ldots, B_b \subseteq B$ with*
1. *$a + b = d + 2$,*
2. *$|R_i| \geq c_2 \cdot |R|$ for all $1 \leq i \leq a$,*
3. *$|B_i| \geq c_2 \cdot |R|$ for all $1 \leq i \leq b$,*
4. *for every transversal $r_1 \in R_1, \ldots, r_a \in R_a, b_1 \in B_1, \ldots, b_b \in B_b$, we have*

$$conv(r_1, \ldots, r_a) \cap conv(b_1, \ldots, b_b) \neq \emptyset.$$

In other words, the Radon partition respects the color classes. We will denote the above statement in dimension d by $R(d)$.

▶ **Lemma 17.** *$R(1)$ can be satisfied choosing $a = 1$, $b = 2$ and $c_2(1) = \frac{1}{3}$.*

Proof. Consider two points x_1 and x_2 such that there are exactly $\frac{|R|}{3}$ blue points to the left of x_1 and to the right of x_2, respectively. Define B_1 as the set of blue points left of x_1 and B_2 as the set of blue points right x_2. We then have $|B_1| = |B_2| = \frac{1}{3}|R|$. Further, as B surrounds R, we have at most $\frac{|R|}{3}$ red points to the left of x_1, and also to the right of x_2. In particular, there are at least $\frac{|R|}{3}$ red points between x_1 and x_2. Let now R_1 be any subset of $\frac{|R|}{3}$ red points between x_1 and x_2. It follows from the construction that $\mathrm{conv}(R_1) \cap \mathrm{conv}(B_1, B_2) \neq \emptyset$. ◀

In the following, we will prove that $R(d-1) \Rightarrow E(d)$ and that $E(d-1) \Rightarrow R(d)$. By induction, these two claims then imply the above theorems.

▶ **Lemma 18.** *$R(d-1) \Rightarrow E(d)$.*

Proof. Assume that $\mathrm{TD}(S, q) = k$ and let h be a witnessing hyperplane which contains q but no points of S. Without loss of generality, assume that q is the origin and that h is the hyperplane through the equator on $S^{d-1} \subseteq \mathbb{R}^d$, with exactly k points below. Color the points below h red and the points above h blue. Now, for every point $p \in S$, consider the line through p and q and let p' be the intersection of that line with the tangent hyperplane to the north pole of S^{d-1}. Color p' the same color as p. This gives a bichromatic point set $S' = R \cup B$ in $\mathbb{R}^{d-1}$. Further, in S', we have that B surrounds R: Assume there is a hyperplane ℓ (in $\mathbb{R}^{d-1}$) with r red points and b blue points on its positive side, where $r > b$. In $\mathbb{R}^d$, this lifts to a hyperplane containing q with $k - r$ red points and b blue points on its positive side (note that there are exactly k red points). However, $k - r + b < k$, whenever $r > b$, thus we would have $\mathrm{TD}(s, q) < k$, which is a contradiction.

As we now have a point set in $\mathbb{R}^{d-1}$, in which B surrounds R, we can apply $R(d-1)$ to find families of $d + 2$ subsets of S', each of size $c_2 \cdot k$, some red and some blue, such that in each transversal the color classes form a Radon partition. We claim that the corresponding subsets of S $c_2 \cdot k$-enclose q. Pick some transversal (which we call the original red and blue points) and consider the corresponding subset in S'. Let z be a point in the intersection of the convex hulls of the two color classes, and let g be the line through z and q. As z is in the convex hull of the blue points, there is a point z^+ on g which is in the convex hull of the original blue points, and thus above h. Similarly, there is a point z^- on g which is in the convex hull of the original red points, and thus below h. As q is in the convex hull of z^+ and z^-, it is thus in the convex hull of the original blue and red points. ◀

In particular, this proof shows that $c_1(d) = c_2(d-1)$.

For the proof of the second implication, we need to recall a few results, starting with the *Same Type Lemma* by Bárány and Valtr [5].

▶ **Theorem 19** (Theorem 2 in [5]). *For every two natural numbers d and m there is a constant $c_3(d, m) > 0$ with the following property: Given point sets $X_1, \ldots, X_m \subseteq \mathbb{R}^d$ such that $X_1 \cup \ldots \cup X_m$ is in general position, there are subsets $Y_i \subseteq X_i$ with $|Y_i| \geq c_3 \cdot |X_i|$ such that all transversals of the Y_i have the same order type.*

From the proof in [5], we get $c_3(d, m) = 2^{-m^{O(d)}}$. This bound has been improved in [9] to $c_3(d, m) = 2^{-O(d^3 m \log m)}$.

The second result that we will need is the *Center Transversal Theorem*, proved independently by Dol'nikov [8] as well as Zivaljević and Vrećica [31]. We will only need the version for two colors, so we state it in this restricted version:

▶ **Theorem 20** (Center Transversal for two colors). *Let μ_1 and μ_2 be two finite Borel measures on $\mathbb{R}^d$. Then there exists a line ℓ such that for every closed halfspace H which contains ℓ and every $i \in \{1, 2\}$ we have $\mu_i(H) \geq \frac{\mu_i(\mathbb{R}^d)}{d}$.*

Such a line ℓ is called a *center transversal*. By a standard argument (replacing points with balls of small radius, see e.g. [18]), the same result also holds for two point sets P_1, P_2 in general position, where $\mu_i(H)$ is replaced by $|P_i \cap H|$. As we will need similar ideas later, we will briefly sketch a proof of the above Theorem. Consider some $(d-1)$-dimensional linear subspace F, i.e., a hyperplane through the origin, and project both measures to it.

For each projected measure, consider the centerpoint region (i.e., the region of Tukey depth $\geq \frac{\mu_i(\mathbb{R}^d)}{(d-1)+1}$). This is a non-empty, convex set, so it has a unique center of mass, which we will denote by $c_i(F)$. Rotating the subspace F in continuous fashion, these centers of mass also move continuously, so the $c_i(F)$ are two continuous assignments of points to the set of all $(d-1)$-dimensional linear subspaces. The result then follows from the following Lemma, again proved independently by Dol'nikov [8] as well as Zivaljević and Vrećica [31]:

▶ **Lemma 21.** *Let g_1 and g_2 be two continuous assignments of points to the set of all $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$. Then there exists such a subspace F in which $g_1(F) = g_2(F)$.*

Note that in order to apply this Lemma, we had to choose in a continuous way a centerpoint. If the two measures can be separated by a hyperplane, we can do something similar with the center transversal:

▶ **Lemma 22.** *Let μ_1 and μ_2 be two finite Borel measures on $\mathbb{R}^d$, which can be separated by a hyperplane. Then there is a unique canonical choice of a center transversal.*

Proof. Let $x_1, \ldots, x_d$ be the basis vectors of $\mathbb{R}^d$ and assume without loss of generality that the hyperplane $H : x_d = 0$ separates the two measures μ_1, μ_2. For any $d-1$-dimensional linear subspace F, consider the projection $\pi_F : \mathbb{R}^d \to F$. Note that if F is orthogonal to H, then $\pi_F(H)$ separates $\pi_F(\mu_1)$ and $\pi_F(\mu_1)$, so there is no center transversal parallel to H. It thus suffices to consider only (oriented) subspaces which point upwards (in the sense that the x_d-component in their normal vector is > 0). The space of these subspaces is homeomorphic to the upper hemisphere S^+ of S^{d-1}. Let now C be the set of all such subspaces in which we have $g_1(F) = g_2(F)$. We claim that C is a convex set in S^+. Consider two subspaces F_1 and F_2 with $g_1(F_1) = g_2(F_1)$ and $g_1(F_2) = g_2(F_2)$. The shortest path between F_1 and F_2 corresponds to a rotation around a $(d-2)$-dimensional axis. Rotate from F_1 to F_2 with constant speed and consider a point in the support of a measure. The projection of this point moves along a line in the projection. In fact, all points in move along parallel lines with direction $\overrightarrow{d}$, and the points in the support of μ_1 move in the opposite direction of the points in the support of μ_2. Further, for any points $p_1$in the support of μ_1 and $p_2$in the support of μ_2, their projections move towards one another, until they are on a common hyperplane with normal vector $\overrightarrow{d}$, and the away from one another. The same arguments hold for the centerpoint regions of the projections and their centers of mass, which shows that if $g_1(F_1) = g_2(F_1)$ and $g_1(F_2) = g_2(F_2)$ then $g_1(F) = g_2(F)$ for every subspace F along the rotation. Thus, the set C is indeed convex, and we can choose the unique solution corresponding to the center of mass of C. ◀

Again, the same statement holds for point sets in general position. With these tools at hand, we are now ready to prove the second part of the induction.

▶ **Lemma 23.** $E(d-1) \Rightarrow R(d)$.

Proof. Let ℓ be a line through the origin. Sweep a hyperplane orthogonal to ℓ from one side to the other (without loss of generality from left to right). Let h_1 be a sweep hyperplane with exactly $\frac{|R|}{3}$ blue points to the left, and let A_1 be the set of these blue points. Similarly, let A_2 be a set of exactly $\frac{|R|}{3}$ blue points to the right of a sweep hyperplane h_2. Let c be the unique center transversal of A_1 and A_2 given by Lemma 22 and let g be the $(d-1)$-dimensional linear subspace which is orthogonal to c. Note that it follows from the proof of Lemma 22 that g cannot be orthogonal to the sweep hyperplanes. We denote the projection of c to g as c_A. Note that c_A is a centerpoint of the projections of A_1 and of A_2 to g. Now, consider the set M of all red points between h_1 and h_2 and note that as the blue points surround the red points we have $|M| \geq \frac{|R|}{3}$. Project M to g and denote by c_M the center of mass of the centerpoint region of the projected point set. We claim that there exists a choice of a line ℓ, such that $c_M = c_A$. Indeed, as g is not orthogonal to a sweep hyperplane, there is a unique shortest rotation which rotates g to a hyperplane orthogonal to ℓ, thus the space of all g's is homeomorphic to the space of all $(d-1)$-dimensional linear subspaces. Further, c_A and c_M are continuous assignments of points, thus the above claim follows from Lemma 21.

So assume now that $c_M = c_A$. In particular, c is a center transversal for A_1, A_2 and M. Project A_1 to g. The projection of c is a centerpoint of the projection of A_1 in g and g has dimension $d-1$, thus by the statement $E(d-1)$ there are three subsets $A_{1,1}, \ldots A_{1,d}$ of A_1, each of size $c_1 \cdot |A_1|$ whose projections enclose the projection of c. The analogous arguments gives subsets $A_{2,1}, \ldots, A_{2,d}$ of A_2 and $M_1, \ldots, M_d$ of M. Consider now these $3d$ subsets. By Theorem 19 there are subsets $A'_{1,1}, \ldots, M'_d$, each of size linear in the size of the original subset, such that each transversal of the subsets has the same order type. Consider such a transversal. By construction, the d points of A_1 contain in their convex hull a point on c which is to the left of h_1. Similarly, the d points of A_2 contain in their convex hull a point on c to the right of h_2. Finally, the d points of M contain in their convex hull a point on c between h_1 and h_2. Thus, the convex hulls of the blue points (from A_1 and A_2) and the red points (from M) intersect. In particular, there is a subset of $d+2$ red and blue points, which form a Radon partition. By choosing the subsets from which these points were selected, we now get the subsets required for $R(d)$. ◀

This proof show that $c_2(d) = \frac{c_3(d,d+2)}{3d} c_1(d-1)$. Using the bound on c_3 from [9] and $c_1(d) = c_2(d-1)$, we thus get $c_2(d) = \Omega(\frac{c_2(d-2)}{3d \cdot 2^{d^4 \log d}}) = \ldots = \Omega(\frac{1}{3^{d/2} d!! \cdot 2^{d^5 \log d}})$, and as $c_1(d) = c_2(d-1)$ we get the same asymptotics for c_1.

Combining this with the results from Section 4, we get that any central depth measure is an approximation of Tukey depth. In fact, by Lemma 14 this even holds for many α-central depth measures.

▶ **Corollary 24.** *Let ϱ be an α-central depth measure on $\mathbb{R}^d$ where $\alpha > \frac{1}{d+2}$. Then there exists a constant $c = c(d)$ such that for every point set S and query point q in $\mathbb{R}^d$ we have*

$$TD(S,q) \geq \varrho(S,q) \geq c \cdot TD(S,q).$$

6 Conclusion

We have introduced two families of depth measures, called super-additive depth measures and central depth measures, where the first is a strict subset of the second. We have shown that all these depth measures are a constant-factor approximation of Tukey depth.

It is known that Tukey depth is coNP-hard to compute when both $|S|$ and d is part of the input [12], and it is even hard to approximate [2] (see also [7]). Our result is thus an indication that central depth measures are hard to compute. However, this does not follow directly, as our constant has a doubly exponential dependence on d. It is an interesting open problem whether the approximation factor can be improved.

Further, we have introduced a new depth measure called enclosing depth, which is neither super-additive nor central, but still is a constant-factor approximation of Tukey depth. As it turns out, this depth measure is intimately related to a constant fraction Radon theorem on bi-colored point sets. Finally, we have shown that any super-additive depth measure whose depth regions are convex is cascading.

This last result is motivated by Kalai's cascade conjecture, which, in the terminology of this paper, states that Tverberg depth is cascading. While this conjecture remains open, we hope that our results might be useful for an eventual proof.

There is a depth measure which has attracted a lot of research, which does not fit into our framework: simplicial depth (SD). The reason for this is that while the depth studied in this paper are linear in the size of the point set, simplicial depth has values of size $O(|S|^{d+1})$. However, after the right normalization, simplicial depth can be reformulated to satisfy all conditions except super-additivity and centrality. It would be interesting to see whether there is some function g depending on point sets and query points such that the depth measure $\frac{\mathrm{SD}(S,q)}{g(S,q)}$ is super-additive. Such a function, if it exists, could potentially be used to improve bounds for the first selection lemma (see e.g. [19]).

References

1. Greg Aloupis. Geometric measures of data depth. In *Data Depth: Robust Multivariate Analysis, Computational Geometry and Applications*, pages 147–158, 2003. `doi:10.1090/dimacs/072/10`.
2. Edoardo Amaldi and Viggo Kann. The complexity and approximability of finding maximum feasible subsystems of linear relations. *Theoretical Computer Science*, 147(1):181–210, 1995.
3. David Avis. The m-core properly contains the m-divisible points in space. *Pattern recognition letters*, 14(9):703–705, 1993.
4. Imre Bárány and Pablo Soberón. Tverberg's theorem is 50 years old: a survey. *Bulletin of the American Mathematical Society*, 55(4):459–492, 2018.
5. Imre Bárány and Pavel Valtr. A positive fraction Erdös-Szekeres theorem. *Discrete & Computational Geometry*, 19(3):335–342, 1998.
6. DG Bourgin. On some separation and mapping theorems. *Commentarii Mathematici Helvetici*, 29(1):199–214, 1955.
7. Dan Chen, Pat Morin, and Uli Wagner. Absolute approximation of Tukey depth: Theory and experiments. *Computational Geometry*, 46(5):566–573, 2013. Geometry and Optimization.
8. VL Dol'nikov. Transversals of families of sets in in $\mathbb{R}^n$ and a connection between the Helly and Borsuk theorems. *Russian Academy of Sciences. Sbornik Mathematics*, 79(1):93, 1994.
9. Jacob Fox, János Pach, and Andrew Suk. A polynomial regularity lemma for semialgebraic hypergraphs and its applications in geometry and property testing. *SIAM Journal on Computing*, 45(6):2199–2223, 2016.
10. Sariel Har-Peled and Timothy Zhou. Improved approximation algorithms for tverberg partitions. *arXiv preprint*, 2020. `arXiv:2007.08717`.
11. Joseph L Hodges. A bivariate sign test. *The Annals of Mathematical Statistics*, 26(3):523–527, 1955.
12. D.S. Johnson and F.P. Preparata. The densest hemisphere problem. *Theoretical Computer Science*, 6(1):93–107, 1978.
13. GIL KALAI. Combinatorics with a geometric flavor. *Visions in Mathematics: GAFA 2000 Special Volume, Part II*, page 742, 2011.
14. Gil Kalai. Problems in geometric and topological combinatorics. Lecture at FU Berlin, 2011.
15. Gil Kalai. Problems for imre bárány's birthday. *Discrete Geometry and Convexity in Honour of Imre Bárány*, page 59, 2017.
16. R. Y. Liu. On a notion of data depth based on random simplices. *The Annals of Statistics*, 18(1):405–414, 1990.
17. Regina Y. Liu, Jesse M. Parelius, and Kesar Singh. Multivariate analysis by data depth: descriptive statistics, graphics and inference. *Ann. Statist.*, 27(3):783–858, June 1999.
18. Jiří Matoušek. *Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry.* Springer Publishing Company, Incorporated, 2007.
19. Jivr'i Matouvsek. *Lectures on discrete geometry*, volume 212 of *Graduate texts in mathematics*. Springer, 2002.

20 Kim Miller, Suneeta Ramaswami, Peter Rousseeuw, J. Antoni Sellarès, Diane Souvaine, Ileana Streinu, and Anja Struyf. Efficient computation of location depth contours by methods of computational geometry. *Statistics and Computing*, 13(2):153–162, 2003.

21 Karl Mosler. *Depth Statistics*, pages 17–34. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.

22 Richard Rado. A theorem on general measure. *Journal of the London Mathematical Society*, 21:291–300, 1947.

23 John R Reay. Several generalizations of tverberg's theorem. *Israel Journal of Mathematics*, 34(3):238–244, 1979.

24 Jean-Pierre Roudneff. Partitions of points into simplices withk-dimensional intersection. part i: The conic tverberg's theorem. *European Journal of Combinatorics*, 22(5):733–743, 2001.

25 Jean-Pierre Roudneff. Partitions of points into simplices withk-dimensional intersection. part ii: Proof of reay's conjecture in dimensions 4 and 5. *European Journal of Combinatorics*, 22(5):745–765, 2001.

26 Jean-Pierre Roudneff. New cases of reay's conjecture on partitions of points into simplices with k-dimensional intersection. *European Journal of Combinatorics*, 30(8):1919–1943, 2009.

27 Csaba D Toth, Joseph O'Rourke, and Jacob E Goodman. *Handbook of discrete and computational geometry*. Chapman and Hall/CRC, 2017.

28 John W. Tukey. Mathematics and the picturing of data. In *Proc. International Congress of Mathematicians*, pages 523–531, 1975.

29 Helge Tverberg. A generalization of Radon's theorem. *Journal of the London Mathematical Society*, 1(1):123–128, 1966.

30 Chung-Tao Yang. On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson, I. *Annals of Mathematics*, pages 262–282, 1954.

31 Rade T. Zivaljević and Siniša T Vrećica. An extension of the ham sandwich theorem. *Bulletin of the London Mathematical Society*, 22(2):183–186, 1990.

32 Yijun Zuo and Robert Serfling. Structural properties and convergence results for contours of sample statistical depth functions. *Ann. Statist.*, 28(2):483–499, April 2000.