On Geometric Priority Set Cover Problems

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Abstract

We study the priority set cover problem for simple geometric set systems in the plane. For pseudo-halfspaces in the plane we obtain a PTAS via local search by showing that the corresponding set system admits a planar support. We show that the problem is APX-hard even for unit disks in the plane and argue that in this case the standard local search algorithm can output a solution that is arbitrarily bad compared to the optimal solution. We then present an LP-relative constant factor approximation algorithm (which also works in the weighted setting) for unit disks via quasi-uniform sampling. As a consequence we obtain a constant factor approximation for the capacitated set cover problem with unit disks. For arbitrary size disks, we show that the problem is at least as hard as the vertex cover problem in general graphs even when the disks have nearly equal sizes. We also present a few simple results for unit squares and orthants in the plane.

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1 Introduction

The priority set cover problem is defined as follows. Given a ground set X and a set $\mathcal{S}$ of subsets of X, the where each element $x \in X$ has an associated priority $\pi(x)$ and each set $S \in \mathcal{S}$ has an associated priority $\pi(S)$, the goal is to pick the smallest cardinality subset $\mathcal{S}' \subseteq \mathcal{S}$ s.t. for each $x \in X$, there is some $S \in \mathcal{S}'$ contain x s.t. $\pi(S) \geq \pi(x)$.

We study the priority set cover problem for simple geometric regions in the plane. It is a natural generalization of the set cover problem, and is interesting in its own right. It is also related to the capacitated covering problem via the work of Chakarabarty et al. [11]. Another motivation for studying such problems is that it leads to a better understanding of the limitations of the current techniques and forces us to extend them.

By treating the priority as an additional dimension, these problems can be seen as special cases of three dimensional set cover problems. However, these problems turn out to be significantly harder than the corresponding problems without priority. For example, while the set cover problem with disks in the plane admits a PTAS, the same problem with priorities surprisingly turns out to be APX-hard even for unit disks. This is one of the few problems known that is APX-hard for unit disks. One standard technique that yields a constant factor

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approximation for many geometric covering problems is quasi-uniform sampling [28, 13]. However this fails for the priority set cover problem for unit disks since the *shallow cell complexity* (defined in Section 2) of the corresponding set system can be quadratic. Another common technique used to obtain approximation algorithms is *local search*. The analysis of local search requires showing the existence of a *support*, or the existence of a *local search graph* that come from a hereditary family with sublinear size separators (see [26] and the references therein). However, even for the priority set cover problem with unit disks, such graphs do not exist. In fact we show that the standard local search algorithm may produce solutions that are arbitrarily bad compared to an optimal solution.

We develop techniques to obtain the first $O(1)$-factor approximation algorithms for the priority set cover problem with unit disks. The algorithm relies on tools we develop to study the priority set cover problem defined by points and *pseudo-halfspaces* in the plane. For the latter problem we show that the shallow cell complexity of the corresponding set system is linear. For the set cover problem without priorities (or equivalently the priority set cover problem with uniform priorities), the problem is known to be solvable in polynomial time [21]. However, we show that the introduction of priorities renders the problem NP-hard.

The proof is non-trivial and uses a novel approach that might be useful in other settings. We also obtain a PTAS for the priority set cover problem for pseudo-halfspaces via local search. For this, we prove that the corresponding set system admits a planar support. Again due to priorities, the proof is much more subtle than for pseudo-halfspaces without priorities.

An identical proof also yields an $O(1)$-approximation for the priority set cover problem with unit squares. In this case, we do not know if the problem is APX-hard, and this remains an intriguing open question.

As a consequence of our results for the priority problem, we immediately obtain $O(1)$-approximation algorithms for the capacitated covering problems when combined with the results of Bansal and Pruhs [6], and Chakrabarty et al. [11]. We start with the necessary definitions and results in Section 2, and describe related work in Section 3. In Section 4 we present our results for pseudo-halfspaces. We present our results for disks in Section 5. We conclude with open problems in Section 6.

2 Preliminaries

Let P be a set of points and let $\mathcal{R}$ be a set of regions in the plane and let $\pi : P \cup \mathcal{R} \to \mathbb{R}$ be a function that assigns a priority to each point and each region. We say that a region R *covers* a point p if R contains p and $\pi(R) \geq \pi(p)$. We use the notation $p \prec R$ for "R covers p". For any region R, we denote by $R(P)$, the set of points in P covered by R. We denote the set system $(P, \{R(P) : R \in \mathcal{R}\})$ by $(P, \mathcal{R}, \pi)$ and call it the "set system defined by P and $\mathcal{R}$". For any point p, we denote by $p(\mathcal{R})$ the set of regions in $\mathcal{R}$ covering p and we denote the set system $(\mathcal{R}, \{p(\mathcal{R} : p \in P\})$ by $(\mathcal{R}, P, \pi)$ and call it the "dual set system defined by P and $\mathcal{R}$".

The *Priority Set Cover problem* defined by P, $\mathcal{R}$ and π is the set cover problem on the set system $(P, \mathcal{R}, \pi)$. In other words, the goal is to find the smallest subset $\mathcal{R}' \subseteq \mathcal{R}$ s.t. each point in P is covered by at least one of the regions in $\mathcal{R}'$. In the *weighted* variant of this problem, we have a weight w_R with each region R and the goal is to minimize the total weight of the regions in $\mathcal{R}'$ instead of its cardinality. We also consider the *Capacitated Set Cover* problem studied by Chakrabarty et al. [11]. In this problem, we are given a set system $(X, \mathcal{S})$, where X is a set of elements, $\mathcal{S}$ is a collection of subsets of X with a weight function $w : \mathcal{S} \to \mathbb{R}_+$, and a capacity function $c : \mathcal{S} \to \mathbb{R}_+$. Each element $x \in X$ has a demand $d(x) > 0$. The objective is to select the smallest weight sub-collection $\mathcal{S}' \subseteq \mathcal{S}$ such that the total capacity of the sets in $\mathcal{S}'$ containing any element x is at least $d(x)$. A special case of this problem is the *Set Multicover problem* where the capacity of each set in $\mathcal{S}'$ is 1.

A finite collection of unbounded x-monotone curves is called a *family of pseudolines* if every pair of curves intersect in at most one point and at this point the curves cross [3]. Pseudoline arrangements have a rich history and a rich combinatorial structure. See the book [20] for further results on pseudolines. A *family of pseudo-halfspaces* is a collection of closed unbounded regions in the plane whose boundaries form a family of pseudolines.

Given a set system $(X, \mathcal{S})$, a *support* is a graph $G = (X, E)$ s.t. any set $S \in \mathcal{S}$ induces a connected subgraph of G. A *planar support* is a support that is planar.

A *plane graph* is a drawing of a planar graph in the plane where the vertices are drawn as points and edges are drawn as interior disjoint simple Jordan curves connecting the points corresponding to the incident vertices. A *triangulation* is a plane graph in which all faces have three vertices.

Given a set system $(X, \mathcal{S})$, let $x(\mathcal{S}) = \{S \in \mathcal{S} : x \in S\}$. The set system $(X, \mathcal{S})$ has *shallow cell complexity* [13] function $f(\cdot, \cdot)$ if for any subset $\mathcal{S}' \subseteq \mathcal{S}$, and any $k \in \mathbb{N}$, the number of sets of size at most k in $\{x(\mathcal{S}') : x \in X\}$ is at most $f(|\mathcal{S}'|, k)$. If $f(|\mathcal{S}'|, k) \leq \phi(|\mathcal{S}'|) \cdot k^c$ for some constant c where $\phi(\cdot)$ is a linear function of its argument, we say that the shallow cell complexity of the set system is "linear". Similarly, if $\phi(\cdot)$ is a quadratic function of its argument, we say that the shallow cell complexity is "quadratic".

3 Related Work

Packing and covering problems are central topics in computational geometry literature, and studied intensively over several decades. Broadly, there are three main algorithmic techniques: LP-rounding, local search and separator based methods.

The technique of Bronimann and Goodrich [9] reduces any covering problem to an ϵ-net question so that if the set system admits an ϵ-net of size $\frac{1}{\epsilon} \cdot f(\frac{1}{\epsilon})$, then we obtain an LP-relative approximation algorithm with approximation factor $f(\text{OPT})$ where OPT is the size of the optimal solution. Since set systems of finite VC- dimension admit ϵ-nets of size $O(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$, this implies an $O(\log \text{OPT})$ approximation algorithm for covering problems involving such set systems. Similarly, for set systems with low shallow cell complexity, we obtain algorithms with correspondingly small approximation factors (see [28, 13]). In particular if the shallow cell complexity is linear, we obtain constant factor approximation algorithms. Varadarajan [28] showed via the *quasi-uniform* sampling technique how these results can be made to work in the weighted setting. His technique was optimized by Chan et al. [13] who also introduced the notion of *shallow cell complexity* generalizing the notion of *union complexity* from geometric set systems to abstract set systems. Some of these algorithms have also been extended to work in the multicover setting (see [15], [6]). One limitation of the approach in [9] is that even for simple set systems with linear shallow cell complexity, the lower bound on the size of the ϵ-net may involve a large constant factor which then translates into a lower bound on the approximation ratio of the corresponding rounding algorithm. For simple set systems, such as that for points and halfspaces in the plane, Har-Peled and Lee [21] gave a polynomial time dynamic programming algorithm. Bringmann et al. [8] show a tight $O(n^{\sqrt{k}})$ exact algorithm to check if there is a set cover of size at most k, while for dimensions larger than 3, they show that under ETH, it is not possible to improve upon brute force enumeration.

The local search framework, where one starts with any feasible solution and tries to improve the solution by only making constant size *swaps* (i.e., adding/removing a constant number of elements from the solution), yields a PTAS for several packing and covering problems (see e.g. [25, 14, 5, 19, 26, 7]). This framework also has major limitations: it is not as broadly applicable as the LP-rounding technique (in particular it works only for the unweighted setting so far), often hard to analyse, and the PTASes they yield have a running time like $n^{O(1/\epsilon^2)}$ with large constants in the exponent, making them irrelevant for practical applications.

The third type of algorithms consists of separator based methods where some kind of *separator* is used to split the problem instance into smaller problems which can be solved independently and combined to obtain an approximate solution. Hochbaum and Maass [22] used this idea to obtain PTASes for several packing and covering problems. More recently, Adamaszek and Wiese [1, 2] have used this kind of idea for obtaining a QPTAS for independent set problems. Mustafa et al. [24] extend the idea of Adamaszek and Wiese to obtain a QPTAS for weighted set cover problem with pseudodisks in the plane and halfspaces in $\mathbb{R}^3$. For unit disks, there have also been attempts to obtain better approximation algorithms that run fast. See [17] for an 18-approximation algorithm that runs in $O(mn)$ for m points and n unit disks.

The priority set cover problem was introduced by Chakrabarty et al. [11] as an approach to solve the capacitated set cover problem. In particular, they showed that an LP-relaxation for the capacitated covering problem with *knapsack cover inequalities*, introduced by Carr et al. [10] in the context of approximation algorithms has an $O(1)$-approximation algorithm if there is an LP-relative $O(1)$-approximation for the set multicovering problem, and an $O(1)$ LP-relative approximation for the priority set cover problem.

4 Pseudo-halfspaces

In this section we study the priority set cover problem for pseudo-halfspaces in the plane. For the set cover problem without priorities (or equivalently the priority set cover problem with uniform priorities), the problem is known to be solvable in polynomial time [21]. However, in the full version of the paper we show that the introduction of priorities renders the problem NP-hard.

Let $H = \{h_1, \cdots, h_n\}$ be a set of pseudo-halfspaces and let P be a set of points in $\mathbb{R}^2$. We denote the boundary of h_i by ℓ_i. We assume that curves $\ell_1, \cdots, \ell_n$ lie in general position i.e., no more that two of them intersect at any point in the plane. Each pseudo-halfspace and each point also has an associated priority. We assume without loss of generality that the priorities of all the pseudo-halfspaces are distinct. For any point p, let $H(p)$ denote the subset of pseudo-halfspaces covering p. We define *depth*(p) as $|H(p)|$.

▶ **Lemma 1.** *Let t be a positive integer. Let P' be any subset of the points in P s.t. for any point $p \in P'$, depth$(p) \leq t$ and for any two distinct points $p, q \in P'$, $H(p) \neq H(q)$. Then, $|P'|$ is $O(nt^2)$.*

Note that the above lemma implies that the shallow cell complexity of the set system $(P, \{H(p) : p \in P\})$ is linear.

For any point p, we can assume without loss of generality that p is contained in a bounded cell in the arrangement of the boundaries of the pseudo-halfspaces in $H(p)$. This can be guaranteed by adding three *dummy* pseudo-halfspaces of priority larger than all the points in P so that P is contained in a bounded cell defined by them. The dummy halfspaces increase the depth of each point by 3 but this does affect the upper bound claimed above.

In order to prove Lemma 1, we define a new set Q of points as follows. For every triple of pseudo-halfspaces h_i, h_j, h_k s.t. $\pi(h_i) < \pi(h_j), \pi(h_k)$ and h_i contains $\ell_j \cap \ell_k$, we define a point $q = q(i, j, k)$ located at $\ell_j \cap \ell_k$ with priority $\pi(h_i)$. Note that Q may contain several points with the same location but with different priorities. However no two points in Q have the same location and priority.

We map each $p \in P'$ to a point in Q as follows. We consider the arrangement of the boundaries of all the pseudo-halfspaces whose priority is at least that of p. As mentioned above, we can assume that p lies in a bounded cell C of this arrangement. Let h_i be the pseudo-halfspace with the lowest priority in $H(p)$. Note that the cell C must have at least three vertices since this is a pseudo-line arrangement. Thus, it must have a vertex defined by the boundries of two pseudo-halfspaces h_j and h_k other than h_i. Note that $\pi(h_i) < \pi(h_j), \pi(h_k)$ and h_i contains $\ell_j \cap \ell_k$. We map p to $q(i,j,k)$.

Since $\ell_j \cap \ell_k$ is adjacent to at most four cells in the arrangement each of which can contain at most one point of P', at most four points in P' are mapped to $q(i,j,k)$. Note also that if p is mapped to $q(i,j,k)$ then the depth of p and $q(i,j,k)$ differ by at most 2 depending on whether h_j and h_k contain p. Thus, in order to prove the upper bound in Lemma 1, it suffices to prove the upper bound on the number of points in Q of depth at most t.

▷ Claim 2. Let t be a positive integer. The number of points in Q of depth at most t is $O(nt^2)$.

Proof. First note that any point $q = q(i,j,k) \in Q$ has depth at least three since it is contained in the three pseudo-halfspaces h_i, h_j and h_k. We first prove that the number of points $q(i,j,k)$ of depth 3 is $O(n)$. We then use the Clarkson-Shor technique [16] to prove the lemma.

If a point $q(i,j,k) \in Q$ has depth 3 then note that h_i is the pseudo-halfspace of the highest priority below $\min\{\pi(h_j), \pi(h_k)\}$ containing $\ell_j \cap \ell_k$. This means that if we imagine inserting the pseudo-halfspaces into an initially empty arrangement in the decreasing order of their priorities then $\ell_j \cap \ell_k$ is a vertex on the boundary of the arrangement until h_i is inserted at which point it is no longer a vertex on the boundary of the arrangement. Since inserting any pseudo-halfspace can create at most two new vertices on the boundary of the arrangement, the total number of vertices that appear on the boundary of the arrangement throughout the process is $O(n)$ and since only one point of depth 3 in Q is located at any such point, the number of points in Q of depth 3 is also $O(n)$.

We now bound the number N_t of points in Q of depth $\leq t$ as follows. Imagine picking a sample of the pseudo-halfspaces where each pseudo-halfspace is picked independently with probability $\rho = 1/t$. Let Q' be the subset of points in Q that are still present and have depth 3 in the sample. The probability that a point $q(i,j,k)$ is still present in the sample and has depth 3 in the sample is $\rho^3(1-\rho)^{t-3}$ since this happens iff h_i, h_j and h_k are in the sample but none of remaining $t-3$ pseudo-halfspaces covering $q(i,j,k)$ are. Thus, $\mathbb{E}(Q') \geq N_t \cdot \rho^3(1-\rho)^{t-3}$. On the other hand since the expected number of pseudo-halfspaces in the sample is ρn, we also have $\mathbb{E}(Q') = O(\rho n)$. Thus, $N_t = O\left(\frac{n}{\rho^2(1-\rho)^{t-3}}\right) = O(nt^2)$. ◁

Lemma 1 now follows. The following theorem is an immediate consequence of Lemma 1 and the results in [13].

▶ **Theorem 3.** *There is a polynomial time $O(1)$ LP-relative*[2] *approximation for the weighted priority set cover problem defined by a set of points and a set of pseudo-halfspaces in the plane.*

▶ **Theorem 4.** *The capacitated set cover problem defined by points and pseudo-halfspaces in the plane has a polynomial time $O(1)$-approximation algorithm.*

[2] with respect to the standard LP relaxation for set cover

Proof. Since the shallow-cell complexity of the set system defined by points and pseudo-halfspaces (without priorities) is linear, the result of Bansal and Pruhs [6] implies an $O(1)$ LP-relative approximation for the multicover problem of this set system. Now, Lemma 1 implies that the set system defined by pseudo-halfspaces and points with priorities is linear, and therefore implies an $O(1)$ LP-relative approximation for the priority problem via the result of Chan et al. [13]. Now, by the result of Chakrabarty et al. [11], the result follows. ◀

Next, we show that the set system (P, H, π) has a planar support. Our result only requires the boundaries of the pseudo-halfspaces and not the direction of the pseudo-halfspace. Therefore, we consider the following problem: the input is a set P of points and a set L of pseudolines with priorities. We construct a plane graph G with vertex set P such that the subgraphs induced on the points covered by $h^+(\ell)$ and $h^-(\ell)$ are connected. Here, $h^+(\ell)$ and $h^-(\ell)$ are two pseudo-halfspaces with priority $\pi(\ell)$ and boundary ℓ. Such a graph G is called a support on P with respect to L.

The proof is constructive: we process the points in increasing order of priority, and maintain a support graph on the processed points with respect to L. We additionaly maintain that this support graph is a triangulation on the processed points and has the property that each edge in the graph is a simple curve that crosses any $\ell \in L$ at most once. In order to construct the graph, we use the following well known result called the Levi extension lemma.

▶ **Lemma 5** (Levi extension Lemma [20]). *Given a pseudoline arrangement L and two points p and q not lying on the same pseudoline in L, there exists a simple curve ℓ through p and q such that $L \cup \ell$ is a pseudoline arrangement.*

Using the Lemma above, we now construct a planar support.

▶ **Theorem 6.** *Let P be a set of n points, L a family of pseudolines in $\mathbb{R}^2$, and $\pi : P \cup L \to \mathbb{R}$ be priorities. Then, there exists a graph T which is a support on P w.r.t. L such that each edge in T crosses any pseudoline $\ell \in L$ at most once. Further, for any $n \geq 3$, T is a triangulation.*

Proof. Let $p_1, \ldots, p_n$ be an ordering of the points P in increasing order of priority. We process the points in this order, and for each $i = 1, \ldots, n$, we maintain a graph T_i that is a support graph on the points $p_1, \ldots, p_i$ with respect to L and such that: any edge of T_i crosses the boundary of any pseudoline in L at most once. We call such a graph with its embedding a *nice graph*.

For $i = 1$, the graph T_1 consisting of p_1 and no edges clearly satisfies both the conditions, and is a nice graph. For $i = 2$, by Lemma 5, there is a curve γ_{12} between p_1 and p_2 such that $L_2 = \gamma_{12} \cup L$ is a pseudoline arrangement. The edge e_{12} is defined as $\gamma_{12}[p_1, p_2]$, i.e., the segment on γ_{12} between p_1 and p_2. It is clear that T_2 is a nice graph.

For $i = 3$, we use Lemma 5 with the pseudoline arrangement L_2 to construct a pseudoline γ_{13} through points p_1 and p_3, such that $L_2 \cup \gamma_{13}$ is a pseudoline arrangment. Invoking Lemma 5 again with $L_2 \cup \gamma_{13}$, we obtain a pseudoline γ_{23} between p_2 and p_3, such that $L_3 = L_2 \cup \gamma_{13} \cup \gamma_{23}$ is a pseudoline arrangement. The new pseudolines added define edges $e_{13} = \gamma_{13}[p_1, p_3]$ and $e_{23} = \gamma_{23}[p_2, p_3]$. Set $T_3 = T_2 \cup e_{13} \cup e_{23}$. It is easy to see that T_3 is a nice graph.

Let T_{i-1} be the graph constructed for the first $i - 1$ points, and let L_{i-1} be the union of L and the additional pseudolines added in the first $i - 1$ iterations. We construct T_i as follows: Let p_i lie in a triangle Δ defined by points p_a, p_b, and p_c - note that p_i may lie in the external face. Suppose that p_i lies in an interior face of T_{i-1}. Using Lemma 5 with the arrangement L_{i-1} lets us construct a pseudoline γ_{ia} through p_i and p_a, such that $L_{i-1} \cup \gamma_{ia}$ is a pseudoline arrangment. This gives us the edge $e_{ia} = \gamma_{ia}[p_i, p_a]$. Note that the interior of e_{ia} lies in Δ. Otherwise, if e_{ia} crosses Δ, it crosses one of the pseudolines defining the boundaries of Δ, which it can not cross again without violating the assumption that $L_{i-1} \cup \gamma_{ia}$ is a pseudoline arrangement. If p_i lies in an external face of T_{i-1}, we first pick a point o that lies in an internal face of T_{i-1}, does lie on any pseudolines in L_{i-1} and is not one of the points in P. Let C be a unit circle centered at o. We temporarily apply an inversion[3] with respect to C to turn the external face in which p_i lies into an internal face. We then proceed as before and apply the inversion again to undo the first inversion.

By a similar argument, we construct edges e_{ib} and e_{ic}, and set $L_i = L_{i-1} \cup \gamma_{ia} \cup \gamma_{ib} \cup \gamma_{ic}$, and $T_i = T_{i-1} \cup e_{ia} \cup e_{ib} \cup e_{ic}$. Since p_i lies on γ_{ia}, γ_{ib} and γ_{ic} it follows that the interiors of e_{ia}, e_{ib} and e_{ic} are pairwise internally disjoint. By construction, the edges e_{ia}, e_{ib} and e_{ic} lie in the same face of T_{i-1}, and therefore do not intersect the interiors of edges in T_{i-1}.

To show that T_i is a nice graph, we still need to show that T_i is a support on P_i with respect to L. Consider a pseudo-halfspace $h = h(\ell)$ defined by a pseudoline $\ell \in L$. First, we will show that $h' = h \setminus \cup_{e \in T_i} interior(e)$ is path connected. If not, let p and q be two points in h' that cannot be connected by a continous path in h'. Let H_p be the set of points reachable from p i.e., H_p is the set of points in h to which there is path from p in h', and let H_q be the set of points reachable from q. Since h is connected and we have removed only the interiors of the edges that are pairwise non-intersecting, there are no vertices on the boundary of H_p and H_q. Since each edge is a simple curve, it implies that there is an edge e that separates H_p and H_q, i.e., the boundary of e crosses ℓ twice. This leads to a contradiction.

Now, let u and v be any two points in P_i that are contained in h. We show that there is a path between u and v in the subgraph of T_i induced by $h \cap P_i$. Since u and v do not lie in the interior of any of the edges in T_i, $u, v \in h'$. By the previous argument, there is a simple curve σ in h' joining u and v. Observe that adjacent points of P_i on σ lie in the same face of T_i, and therefore are adjacent in T_i, since T_i is a triangulation. This implies that there is a path in the subgraph of T_i induced by $h \cap P_i$ joining u and v.

Let p_k be the point of highest priority that is contained h. Then, by the fact that T_k is a support on P_k with respect to L and h covers exactly the points in $P_k \cap h$, we conclude that the subgraph of T_k induced by the points covered by h is connected. Since T_n contains T_k as subgraph, the subgraph of T_n induced by the points in $P = P_n$ covered by h is also connected. The theorem follows. ◀

An immediate consequence of the above theorem is the following.

▶ **Corollary 7.** *The set system (P, H, π) admits a planar support.*

▶ **Corollary 8.** *The set system (H, P, π) admits a planar support if the union of the pseudo-halfspaces in H do not cover the entire plane.*

Proof. If there is a point o in the plane that is not covered by any of the pseudo-halfspaces in H then using the duality between points and pseudolines [4], we can map points to pseudo-halfspaces and pseudo-halfspaces to points while maintaining incidences. Then, Corollary 7 implies that the set system (H, P, π) admits a planar support. ◀

[3] `https://en.wikipedia.org/wiki/Inversive_geometry`

The next theorem follows directly from the above result and the results in [26] and [27] which show that the existence of a suitable planar support implies a PTAS.

▶ **Theorem 9.** *The Set Cover, Hitting Set, Point Packing, and Region Packing problems with priorities defined by a set of points and pseudo-halfspaces in the plane admit a PTAS. The set multi-cover problem defined by a set of points and pseudo-halfspaces with priorities admits a $(2+\epsilon)$-approximation algorithm for any $\epsilon > 0$.*

The definitions of the problems mentioned in the theorem above without the priorities can be found in [26] and [27] and naturally extend to the version with priorities.

5 Disks

Chan et al. [12] showed that the set cover problem with horizontal and vertical strips in the plane is APX-hard. The input is a set P of n points in the plane and a set $\mathcal{S}$ of vertical or horizontal strips of the form $V(a,b) = \{(x,y) \in \mathbb{R}^2 : a \le x \le b\}$ or $H(a,b) = \{(x,y) \in \mathbb{R}^2 : a \le y \le b\}$. We show below that the set system defined by axis aligned strips and points in the plane can be implemented using disks and points with priorities. This implies that the set cover problem defined by points and axis aligned strips in the plane can be reduced to the priority set cover problem defined by a set of points and unit disks in the plane in polynomial time.

▶ **Theorem 10.** *Given a set $\mathcal{S}$ of horizontal and vertical strips and a set of points P in the plane, we can map each strip $S \in \mathcal{S}$ to a unit radius disk S' and each point $p \in P$ to another point p' with appropriate priorities so that S' covers p' iff S contains p.*

Proof. Let n be the number of points in P. We can assume without loss of generality that the points in P lie on an $n \times n$ grid G and have cartesian coordinates (i,j) where $i,j \in [n]$. For convenience, we will assume that P consists of all points in the grid since if the theorem holds for such a P then it certainly holds for any subset of it. We refer to the point with cartesian coordinates (i,j) in G as p_{ij}.

Let D be a unit radius disk centered at the origin o. We first define $n+2$ points $z_0, z_1, \cdots, z_{n+1}$ on the boundary of D as follows. The point z_i has polar coordinates $(1,\theta_i)$ i.e., cartesian coordinates $(\cos\theta_i, \sin\theta_i)$ where $\theta_i = i \cdot \frac{\pi}{4(n+1)}$. Let $\hat{u}$ be a unit vector along the positive x-axis. We map the point p_{ij} on the grid to the point $q_{ij} = z_i + j \cdot \frac{\epsilon}{n}\hat{u}$ where ϵ is a sufficiently small constant. We assign priority $n-j$ to the point q_{ij}. Note that the points on j^{th} column of the grid are mapped to the points $q_{1j}, \cdots, q_{nj}$ that lie on the boundary of a unit radius disk whose center is at $o + j \cdot \frac{\epsilon}{n}\hat{u}$. We will call this disk C_j. The points on the i^{th} row of the grid G are mapped to points on a horizontal segment of length ϵ whose left end-point is z_i. We denote the set of points $\{q_{ij} : i,j \in [n]\}$ by Q.

Consider any vertical strip $S = V(a,b)$ which contains the points in columns $a, a+1, \cdots, b$ of G. We map this strip to the disk S' which is identical to C_b but has priority $n-a$. It can be verified that a point $q_{ij} \in Q$ is covered by S' iff the corresponding point p_{ij} is covered by S. Now consider any horizontal strip $S = H(a,b)$ which contains the points in rows $a, a+1, \cdots, b$ of G. We map this strip to a disk S' defined as follows. Let u be the mid-point on the arc on ∂D joining z_{a-1} and z_a. Similarly, let v be the mid-point on the arc on ∂D joining z_b and z_{b+1}. S' is the unique disk of unit radius whose center lies outside D and whose boundary intersects the boundary of D at the points u and v. For sufficiently small ϵ, S' contains exactly the subset of points in Q that correspond to the points in P contained in S. We assign a priority of n to S' so that it covers all the points it contains. The theorem follows. ◀

The following is an immediate consequence of Theorem 10 and the results in [12].

▶ **Corollary 11.** *The priority set cover problem defined by a set of points P and a set of unit radius disks $\mathcal{D}$ in the plane is APX-hard.*

▶ **Corollary 12.** *The shallow cell complexity of the set system defined by unit radius disks and points with priority is quadratic.*

Proof. This follows from Theorem 10 and the fact that the shallow cell complexity of the set system defined by axis aligned strips and points in the plane is quadratic. To see the latter, consider n disjoint horizontal strips $H_1, \cdots, H_n$ and n disjoint vertical strips $V_1, \cdots, V_n$. Then for every pair of indices $i, j \in [n]$, H_i and V_j intersect at a point in the plane that is not contained in any other strip. ◀

▶ Remark. Since the shallow cell complexity is quadratic, the quasi-uniform sampling technique [28, 13] cannot be directly applied to obtain a constant factor approximation for the priority set cover problem defined by points and unit disks in the plane.

The next lemma shows that the standard local search algorithm does not work for the priority set cover problem defined by unit disks and points in the plane. For minimization problems, the standard local search algorithm is the following. It has a fixed parameter k. The algorithm starts with any feasible solution and tries to decrease the size of the solution by removing at most k elements from the current solution and adding fewer elements to the solution without violating feasibility. When such improvements are not possible it returns the current solution.

▶ **Lemma 13.** *For any positive interger k, there exist instances of the priority set cover problem defined by unit disks and points in the plane such that the standard local search algorithm with parameter k does not yield a solution with a bounded approximation ratio.*

Proof. We will construct an instance of the set cover problem defined by points and axis aligned strips in the plane for which the standard local search algorithm with swap size k does not yield a solution with a bounded approximation ratio. This along with Theorem 10 implies the statement in the theorem.

Let $H_1, \cdots, H_m$ be m disjoint horizontal strips and let $V_1, \cdots, V_n$ be n disjoint vertical strips where $m \gg n > k$. For any $i, j \in [n]$, let p_{ij} be a point in $H_i \cap V_j$. Consider the set cover problem defined the all the horizontal and vertical strips and the points $\{p_{ij} : i, j \in [n]\}$. Then, the horizontal strips form a locally optimal solution i.e., the solution cannot be improved by swapping out at most k strips from this solution and swapping in fewer strips. This is because if any horizontal strip is dropped, we would need to add all the vertical strips, of which there are more than k, in order to obtain a feasible solution. Since $m \gg n$ this solution is arbitrarily large compared to the optimal solution formed by the vertical strips. ◀

We now show that we can construct an arbitrarily set of disks such that every pair intersects at a depth 2. This implies that the shallow-cell complexity of this set system is is quadratic.

▶ **Theorem 14.** *For any positive integer n, there exist a set of n disks $\mathcal{D} = \{D_1, \cdots, D_n\}$ whose radii are nearly equal (i.e., the ratio of any two of the radii can be made arbitrarily close to 1) and a set of points P with $\binom{n}{2}$ points s.t. for any pair of disks $i, j \in [n]$ s.t. $i < j$, there exists a point $p_{ij} \in P$ which is covered by only the disks D_i and D_j among the disks in $\mathcal{D}$.*

Figure 1 Constructing disk D_{k+1} from disk D_k. The radius of D_{k+1} only slightly bigger than the radius of D_k, the difference being arbitrarily small. The centers of the two disks are also arbitrarly close to each other.

Proof. We show the existence of a family of disks $\mathcal{D}$ so that for any $j \in [n]$, if we consider the arrangement of the disks in $\mathcal{D}_j := \{D_1, \cdots, D_j\}$, then i) the boundary of every disk in $\mathcal{D}_j$ contributes at least one arc to the boundary of the union of the disks in $\mathcal{D}_j$ and ii) the boundary of D_j intersects the boundary of every other disk non-tangentially on the boundary of the union of the disks in $\mathcal{D}_j$

We show the existence of such disks by induction on the number of disks. The base case is $n = 1$ and is trivially true. Suppose that we have shown this for $n = k$ for some $k \geq 1$. We now show that we can add a disk D_{k+1} to the existing collection so that the above properties i) and ii) hold for $n = k + 1$. By the inductive hypothesis, D_k contributes an arc to the boundary of the union of the disks in $\mathcal{D}_k$. Let p be the mid-point of such an arc. Without loss of generality assume that the radius of D_k is 1. We construct the disk D_{k+1} in two steps. See Figure 1.

First we tentatively set D_{k+1} to be a disk of radius $1 + \delta_{k+1}$ for some $\delta_{k+1} > 0$ so that D_{k+1} contains D_k and the boundaries of D_k and D_{k+1} intersect tangentially at p. We set δ_{k+1} to be sufficiently small so that none of the vertices in the arrangement of disks in $\mathcal{D}_k$ lie in the region $D_{k+1} \setminus D_k$. At this point D_{k+1} almost satisfies the required properties. Since δ_{k+1} is very small D_{k+1} is almost the same as D_k and is obtained by "growing" D_k slightly. This means that for each $j < k$, D_j still contributes an arc to the boundary of the union of disks in $\mathcal{D}_k$ and the boundary of D_{k+1} intersects the boundary D_j on the boundary of the union of the disks in $\mathcal{D}_{k+1}$. D_{k+1} however intersects D_k tangentially at p which also means that the boundary of D_k does not contribute any arc to the boundary of the union of the disks in $\mathcal{D}_{k+1}$. To fix these problems, we move the center of D_{k+1} by a distance $\epsilon_{k+1} > 0$ in the direction $c - p$ where c is the center of D_k. We choose ϵ_{k+1} to be sufficiently small so that during the movement, the boundary of D_k does not touch any of the vertices in the arrangement of the disks in $\mathcal{D}_k$. It can be checked that after this movement the set of disks $\mathcal{D}_{k+1}$ satisfies the two properties. This concludes the inductive proof. By making ϵ_{k+1} and δ_{k+1} appropriately small for every k, we can ensure that the disks have nearly equal radii.

We assign the priority $n - i$ to the disk D_i. For any $i < j$, we define the point p_{ij} to be the point located where the boundaries of D_i and D_j intersect on the boundary of the union of the disks in $\mathcal{D}_j$ and having priority $n - j$. Note that the point p_{ij} is covered by both D_i and D_j. It is however not contained in any of other disks in $\{D_1, \cdots, D_j\}$ and it is not covered by any of the disks in $\{D_{j+1}, \cdots, D_n\}$ since those disks have a lower priority than that of p_{ij}. ◀

▶ Remark. It can be shown that the values of ϵ_{k+1} and δ_{k+1} in the above proof can be chosen so that they can be encoded using poly(n) bits. The formal proof of this statement will appear in the extended version of the paper.

▶ **Corollary 15.** *The priority set cover problem defined by a set of points and nearly equal size disks in the plane does not admit a strongly polynomial time approximation algorithm with approximation factor smaller than* 1.36 *unless* $P = NP$*. Under the unique games conjecture, this implies that the priority set cover problem does not admit a strongly polynomial time algorithm with approximation factor smaller than* 2*.*

Proof. We give an approximation preserving reduction from vertex cover to priority set cover problem defined by a set of points and a set of nearly equal sized disks in the plane. The corollary then follows from the results known for the vertex cover problem [18, 23]. Given a graph G with n vertices $v_1, \cdots, v_n$ and m edges, we use Theorem 14 to obtain a set $\mathcal{D}$ of n disks and a set P with $\binom{n}{2}$ points. The disk D_i corresponds to the vertex v_i of G. For each edge $\{v_i, v_j\}$ in G, we retain the point $p_{ij} \in P$. We remove all other points. Let Q be the set of points retained. Then, the priority set cover problem defined by the points in Q and the disks in $\mathcal{D}$ is equivalent to the vertex cover problem in G. ◀

We now show that there exists a polynomial time constant factor approximation algorithm for the weighted priority set cover problem defined by a set of points P and a set of unit disks $\mathcal{D}$ in the plane. We first define an LP-relaxation for this problem:

$$\text{minimize} \sum_{D \in \mathcal{D}} w_D x_D \quad \text{s.t. for each } p \in P : \sum_{D \in \mathcal{D} \,:\, p \prec D} x_D \geq 1.$$

We will show that there is a polynomial time algorithm that outputs a solution of size at most a constant times the value OPT_{LP} of an optimal solution to the above LP.

▶ **Theorem 16.** *There is a polynomial time LP-relative* $O(1)$*-approximation algorithm for the weighted priority set cover problem defined by a set of points* P *and a set of unit radius disks* $\mathcal{D}$ *in the plane.*

Proof. We first prove the theorem for disks containing a common point which without loss of generality is assumed to be the origin o. Let x^* be an optimal solution to the LP-relaxation. Consider any one of the four quadrants formed by the axes and consider the priority set cover problem restricted to that quadrant. Note that x^* is also a feasible solution to this problem. Since the boundaries of any two disks containing o intersect at most once in the quadrant, they behave like pseudo-halfspaces with respect to the quadrant. By Lemma 1, the shallow cell complexity of the corresponding set system is linear and therefore quasi-uniform sampling [13] yields an LP-relative $O(1)$-approximation for this problem. Since there are four quadrants, by taking the union of the solutions for each of the quadrants, we obtain an LP-relative $O(1)$ approximation for the priority set cover problem where all disks contain a common point o.

Now, we consider the case of a general set of unit disks in the plane. We partition the given set of disks into a constant number of families s.t. each family consists of disjoint groups of disks so that disks in each group intersect at a common point but disks from different groups do not intersect.

An $O(1)$-approximation for the priority set cover problem defined by such a family of disks follows from the fact that disks in different groups don't interact and for each group we have an $O(1)$-approximation.

The families of the required type are obtained as follows. We place a uniform grid over the plane having cells of size $\sqrt{2} \times \sqrt{2}$ i.e., each cell is a square with diameter 2. Since there are only a finite number of disks, we can also choose the grid in such a way that the center of each disk lies in the interior of some cell. We associate each unit disk with the cell containing its center. Note that the disks associated with a cell contain the center of the cell. Next we color the cells with a constant number of colors so that two cells whose centers have distance less than 4 get distinct colors. The disks associated with cells of a single color then is a family of the required type. Each distinct color defines a family. Let $\mathcal{F}_1, \cdots \mathcal{F}_k$ be the families obtained. As argued above, there is an $O(1)$-approximation algorithm for the priority set cover problem defined by any particular family. However, a point can belong to disks of several different families. So, we need a way to assign each point to a particular family. To do this, we consider an optimal solution x^* to the priority cover problem defined by all disks and points. Since, for any point p, we have that $\sum_{D\in\mathcal{D}\,:\,p\prec D} x_D \geq 1$, there is some family $\mathcal{F}_i$ s.t. $\sum_{D\in\mathcal{F}_i\,:\,p\prec D} x_D \geq 1/k$. We assign the point p to one such family.

Let P_i be the subset of points assigned to family $\mathcal{F}_i$. Then note that $\hat{x}_D = k \cdot x^*_D$, $D \in \mathcal{F}_i$ is a feasible solution to the LP-relaxation for the priority set cover problem defined by the points in P_i and the disks in $\mathcal{F}_i$. Let S_i be a solution to this problem using an LP-relative $O(1)$-approximation. This means that S_i has weight at most $O(1) \cdot \sum_{D\in\mathcal{F}_i} w_D \hat{x}_D$. Note that $S = \bigcup_{i=1}^k S_i$ is a solution to the weighted priority set cover problem defined by all disks and points and has weight at most

$$O(1) \cdot \sum_{i=1}^{k} \sum_{D\in\mathcal{F}_i} w_D \hat{x}_D \leq O(1) \cdot \sum_{i=1}^{k} \sum_{D\in\mathcal{F}_i} k \cdot w_D x^*_D = O(1) \cdot k \cdot \sum_{D} w_D x^*_D = O(1) \cdot \sum_{D} w_D x^*_D$$

since k is a constant. S is therefore an LP-relative $O(1)$-approximation to the weighted priority set cover problem defined by all disks and points. ◀

▶ **Theorem 17.** *There is a polynomial time LP-relative $O(1)$-approximation algorithm for the weighted priority set cover problem defined by a set of points P and a set of unit squares $\mathcal{S}$ in the plane.*

Proof. The proof is identical to the proof of Theorem 16, except that we have unit squares instead of unit disks. ◀

▶ **Theorem 18.** *The capacitated set cover problem with unit squares, or unit disks admits an $O(1)$-approximation.*

Proof. The result of Chakrabarty et al. [11] shows that there exists an $O(1)$-approximation for the capacitated set cover problem whenever we have an $O(1)$-LP-relative approximation for the multicover problem, and an $O(1)$-LP-relative approximation for the priority cover problem. The result of Bansal and Pruhs [6] implies an $O(1)$-LP-relative approximation for the multicover problem, and Theorem 16 implies an $O(1)$-LP-relative approximation for the priority problem. The result for unit disks follows. For unit sqaures, the result similarly follows from that of Bansal and Pruhs [6] and Theorem 17. ◀

6 Conclusion

We studied the priority set cover problem for several simple geometric set systems in the plane. For pseudo-halfspaces in the plane we were able to obtain a PTAS but for unit disks in the plane the problem is APX-hard and we obtained a constant factor approximation. Obtaining a relatively small approximation factor is an interesting open question. For unit squares in the plane we also obtain a constant factor approximation but it is not clear if the problem is APX-hard. In fact even for orthants in the plane (possibly containing orthants of opposite types), it is not clear if there is a PTAS. In particular, we do not know if the standard local search yields a PTAS. There are instances showing that the corresponding set system does not admit a planar support. Another interesting open problem is to obtain a constant factor approximation algorithms for disks or square of arbitrary size in the plane.

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