On the Kernel and Related Problems in Interval Digraphs

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Abstract

Given a digraph G, a set $X \subseteq V(G)$ is said to be an *absorbing set* (resp. *dominating set*) if every vertex in the graph is either in X or is an in-neighbour (resp. out-neighbour) of a vertex in X. A set $S \subseteq V(G)$ is said to be an *independent set* if no two vertices in S are adjacent in G. A kernel (resp. solution) of G is an independent and absorbing (resp. dominating) set in G. The problem of deciding if there is a kernel (or solution) in an input digraph is known to be NP-complete. Similarly, the problems of computing a minimum cardinality kernel, absorbing set (or dominating set) and the problems of computing a maximum cardinality kernel, independent set are all known to be NP-hard for general digraphs. We explore the algorithmic complexity of these problems in the well known class of *interval digraphs.* A digraph G is an interval digraph if a pair of intervals (S_u, T_u) can be assigned to each vertex u of G such that $(u, v) \in E(G)$ if and only if $S_u \cap T_v \neq \emptyset$. Many different subclasses of interval digraphs have been defined and studied in the literature by restricting the kinds of pairs of intervals that can be assigned to the vertices. We observe that several of these classes, like interval catch digraphs, interval nest digraphs, adjusted interval digraphs and chronological interval digraphs, are subclasses of the more general class of *reflexive interval digraphs* – which arise when we require that the two intervals assigned to a vertex have to intersect. We see as our main contribution the identification of the class of reflexive interval digraphs as an important class of digraphs. We show that all the problems mentioned above are efficiently solvable, in most of the cases even linear-time solvable, in the class of reflexive interval digraphs, but are APX-hard on even the very restricted class of interval digraphs called *point-point digraphs*, where the two intervals assigned to each vertex are required to be degenerate, i.e. they consist of a single point each. The results we obtain improve and generalize several existing algorithms and structural results for reflexive interval digraphs. We also obtain some new results for undirected graphs along the way: (a) We get an $O(n(n+m))$ time algorithm for computing a minimum cardinality (undirected) independent dominating set in cocomparability graphs, which slightly improves the existing $O(n^3)$ time algorithm for the same problem by Kratsch and Stewart; and (b) We show that the Red Blue Dominating Set problem, which is NP-complete even for planar bipartite graphs, is linear-time solvable on *interval bigraphs*, which is a class of bipartite (undirected) graphs closely related to interval digraphs.

2012 ACM Subject Classification Mathematics of computing → Graph algorithms

Keywords and phrases Interval digraphs, kernel, absorbing set, dominating set, independent set, algorithms, approximation hardness

Digital Object Identifier 10.4230/LIPIcs.ISAAC.2021.17

Related Version *Full Version*: `https://arxiv.org/abs/2107.08278` [11]

Funding *Pavol Hell*: Support from an NSERC Discovery Grant is gratefully acknowledged. Part of this work was also supported by the Smt Rukmini Gopalakrishnachar Chair Professorship at IISc (November-December, 2019).

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32nd International Symposium on Algorithms and Computation (ISAAC 2021).
Editors: Hee-Kap Ahn and Kunihiko Sadakane; Article No. 17; pp. 17:1–17:17
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1 Introduction

Let $H = (V, E)$ be an undirected graph. A set $S \subseteq V(H)$ is said to be an *independent set* in H if for any two vertices $u, v \in S$, $uv \notin E(H)$. A set $S \subseteq V(H)$ is said to be a *dominating set* in H if for any $v \in V(H) \setminus S$, there exists $u \in S$ such that $uv \in E(H)$. A set $S \subseteq V(H)$ is said to be an *independent dominating set* in H if S is dominating as well as independent. Note that any maximal independent set in H is an independent dominating set in H, and therefore every undirected graph contains an independent dominating set, which implies that the problem of deciding whether an input undirected graph contains an independent dominating set is trivial. On the other hand, finding an independent dominating set of *maximum cardinality* is NP-complete for general graphs, since independent dominating sets of maximum cardinality are exactly the independent sets of maximum cardinality in the graph. The problem of finding a minimum cardinality independent dominating set is also NP-complete for general graphs [12] and also in many special graph classes (refer [18] for a survey). We study the directed analogues of these problems, which are also well-studied in the literature.

Let $G = (V, E)$ be a directed graph. A set $S \subseteq V(G)$ is said to be an *independent set* in G, if for any two vertices $u, v \in S$, $(u, v), (v, u) \notin E(G)$. A set $S \subseteq V(G)$ is said to be an *absorbing (resp. dominating) set* in G, if for any $v \in V(G) \setminus S$, there exists $u \in S$ such that $(v, u) \in E(G)$ (resp. $(u, v) \in E(G)$). As any set of vertices that consists of a single vertex is independent and the whole set $V(G)$ is absorbing as well as dominating, the interesting computational problems that arise here are that of finding a maximum independent set, called INDEPENDENT-SET, and that of finding a minimum absorbing (resp. dominating) set in G, called ABSORBING-SET (resp. DOMINATING-SET). A set $S \subseteq V(G)$ is said to be an *independent dominating (resp. absorbing) set* if S is both independent and dominating (resp. absorbing). Note that unlike undirected graphs, the problem of finding a maximum cardinality independent dominating (resp. absorbing) set is different from the problem of finding a maximum cardinality independent set for directed graphs.

Given a digraph G, a collection $\{(S_u, T_u)\}_{u \in V(G)}$ of pairs of intervals is said to be an *interval representation* of G if $(u, v) \in E(G)$ if and only if $S_u \cap T_v \neq \emptyset$. A digraph G that has an interval representation is called an *interval digraph* [6]. We consider a loop to be present on a vertex u of an interval digraph if and only if $S_u \cap T_u \neq \emptyset$. An interval digraph is a *reflexive interval digraph* if there is a loop on every vertex. Let G be a digraph. If there exists an interval representation of G such that $T_u \subseteq S_u$ for each vertex $u \in V(G)$ then G is called an *interval nest digraph* [26]. If G has an interval representation in which intervals S_u and T_u for each vertex $u \in V(G)$ are required to have a common left end-point, the interval digraphs that arise are called *adjusted interval digraphs* [9]. Note that the class of reflexive interval digraphs is a superclass of both interval nest digraphs and adjusted interval digraphs. Another class of interval digraphs, called *interval-point digraphs* arises when the interval T_u for each vertex u is required to be degenerate (it is a point) [6]. Note that interval-point digraphs may not be reflexive. We call a digraph G a *point-point digraph* if there is an interval representation of G in which both S_u and T_u are degenerate intervals for each vertex u. Clearly, point-point digraphs form a subclass of interval-point digraphs and they are also not necessarily reflexive.

In this paper, we show that the reflexivity of an interval digraph has a huge impact on the algorithmic complexity of several problems related to domination and independent sets in digraphs. In particular, we show that all the problems we study are efficiently solvable on reflexive interval digraphs, but are NP-complete and/or APX-hard even on point-point digraphs. Along the way we obtain new characterizations of both these graph classes, which reveal some of the properties of these digraphs.

An undirected graph is a *comparability* graph if its edges can be oriented in such a way that it becomes a partial order. The complements of comparability graphs are called *cocomparability graphs*.

Our results. We provide a vertex-ordering characterization for reflexive interval digraphs and two simple characterizations for point-point digraphs including a forbidden structure characterization. Our characterization of point-point digraphs directly yields a linear time recognition algorithm for that class of digraphs (note that Müller's [22] recognition algorithm for interval digraphs directly gives a polynomial-time recognition algorithm for reflexive interval digraphs). From our vertex-ordering characterization of reflexive interval digraphs, it follows that the underlying undirected graphs of every reflexive interval digraph is a cocomparability graph. Also a natural question that arises here is whether the underlying graphs of reflexive interval digraphs is the same as the class of cocomparability graphs. We show that this is not the case by demonstrating that the underlying graphs of reflexive interval digraphs cannot contain an induced $K_{3,3}$. This can be used to strengthen a result of Prisner [26] about interval nest digraphs: our results imply that the underlying undirected graphs of interval nest digraphs and their reversals are $K_{3,3}$-free weakly triangulated cocomparability graphs. Also, as the INDEPENDENT SET problem is linear time solvable on cocomparability graphs [19], the problem is also linear time solvable on reflexive interval digraphs. This improves and generalizes the $O(nm)$-time algorithm for the same problem on interval nest digraphs. In contrast, we prove that the INDEPENDENT SET problem is APX-hard for point-point digraphs.

Domination in digraphs is a topic that has been explored less when compared to its undirected counterpart. Even though bounds on the minimum dominating sets in digraphs have been obtained by several authors (see the book [13] for a survey), not much is known about the computational complexity of finding a minimum cardinality absorbing set (or dominating set) in directed graphs. Even for tournaments, the best known algorithm for DOMINATING-SET does not run in polynomial-time [20, 27]. In [20], the authors give an $n^{O(\log n)}$ time algorithm for the DOMINATING-SET problem in tournaments and they also note that SAT can be solved in $2^{O(\sqrt{v})}n^K$ time (where v is the number of variables, n is the length of the formula and K is a constant) if and only if the DOMINATING-SET in a tournament can be solved in polynomial time. Thus, determining the algorithmic complexity of the DOMINATING-SET problem even in special classes of digraphs seems to be much more challenging than the algorithmic question of finding a minimum cardinality dominating set in undirected graphs.

For a bipartite graph having two specified partite sets A and B, a set $S \subseteq B$ such that $\bigcup_{u \in B} N(u) = A$ is called an *A-dominating set*. Note that the graph does not contain an A-dominating set if and only if there are isolated vertices in A. The problem of finding an A-dominating set of minimum cardinality in a bipartite graph with partite sets A and B is more well-known as the RED-BLUE DOMINATING SET problem, which was introduced for the first time in the context of the European railroad network [30] and plays an important role in the theory of fixed parameter tractable algorithms [7]. This problem is equivalent to the well known SET COVER and HITTING SET problems [12] and therefore, it is NP-complete for general bipartite graphs. The problem remains NP-complete even for planar bipartite graphs [1]. The class of *interval bigraphs* are closely related to the class of interval digraphs. These are undirected bipartite graphs with partite sets A and B such that there exists a collection of intervals $\{S_u\}_{u \in V(G)}$ such that $uv \in E(G)$ if and only if $u \in A$, $v \in B$, and $S_u \cap S_v \neq \emptyset$.

Our results. We observe that the problem of solving Absorbing-Set on a reflexive interval digraph G can be reduced to the problem of solving Red-Blue Dominating Set on an interval bigraph whose interval representation can be constructed from an interval representation of G in linear time. Further, we show that Red-Blue Dominating Set is linear time solvable on interval bigraphs (given an interval representation). Thus the problem Absorbing-Set (resp. Dominating-Set[1]) is linear-time solvable on reflexive interval digraphs, given an interval representation of the digraph as input. If no interval representation is given, Müller's algorithm [22] can be used to construct one in polynomial time, and therefore these problems are polynomial time solvable on reflexive interval digraphs even when no interval representation of the input graph is known. In contrast, we prove that the Absorbing-Set (resp. Dominating-Set) problem remains APX-hard even for point-point digraphs.

An independent absorbing set in a directed graph is more well-known as a *kernel* of the graph, a term introduced by Von Neumann and Morgenstern [21] in the context of game theory. They showed that for digraphs associated with certain combinatorial games, the existence of a kernel implies the existence of a winning strategy. Most of the work related to domination in digraphs has been mainly focused on kernels. We follow the terminology in [26] and call an independent dominating set in a directed graph a *solution* of the graph. It is easy to see that a kernel in a directed graph G is a solution in the directed graph obtained by reversing every arc of G and vice versa. Note that unlike in the case of undirected graphs, a kernel need not always exist in a directed graph. Therefore, besides the computational problems of finding a minimum or maximum sized kernel, called Min-Kernel and Max-Kernel respectively, the comparatively easier problem of determining whether a given directed graph has a kernel in the first place, called Kernel, is itself a non-trivial one. In fact, the Kernel problem was shown to be NP-complete in general digraphs by Chvátal [4]. Later, Fraenkel [10] proved that the Kernel problem remains NP-complete even for planar digraphs of degree at most 3 having in- and out-degrees at most 2. It can be easily seen that the Min-Kernel and Max-Kernel problems are NP-complete for those classes of graphs for which the Kernel problem is NP-complete. A digraph is said to be *kernel-perfect* if every induced subgraph of it has a kernel. Several sufficient conditions for digraphs to be kernel-perfect has been explored [28, 8, 21]. The Kernel problem is trivially solvable in polynomial-time on any kernel-perfect family of digraphs. But the algorithmic complexity status of the problem of computing a kernel in a kernel-perfect digraph also seems to be unknown [24]. Prisner [26] proved that interval nest digraphs and their reversals are kernel-perfect, and a kernel can be found in these graphs in time $O(n^2)$ if a representation of the graph is given. Note that the Min-Kernel problem can be shown to be NP-complete even in some kernel-perfect families of digraphs that have polynomial-time computable kernels (see Remark 13).

Our results. We show that reflexive interval digraphs are kernel-perfect and hence the Kernel problem is trivial on this class of digraphs. We construct a linear-time algorithm that computes a kernel in a reflexive interval digraph, given an interval representation of

[1] Note that the reversal of a reflexive interval digraphs is also a reflexive interval digraph. It is easy to see that the reversal of any digraph can be constructed in linear time. Moreover, an interval representation of the reversal of a reflexive interval digraph G can be obtained in linear time given an interval representation of G (if $\{(S_u, T_u)\}_{u \in V(G)}$ is the interval representation of G, then $\{(T_u, S_u)\}_{u \in V(G)}$ is an interval representation of the reversal of G). Therefore, all our results about the Kernel, Min-Kernel, Max-Kernel, and Absorbing-Set problems can be adapted to obtain similar results about the Solution, Min-Solution, Max-Solution, and Dominating-Set problems.

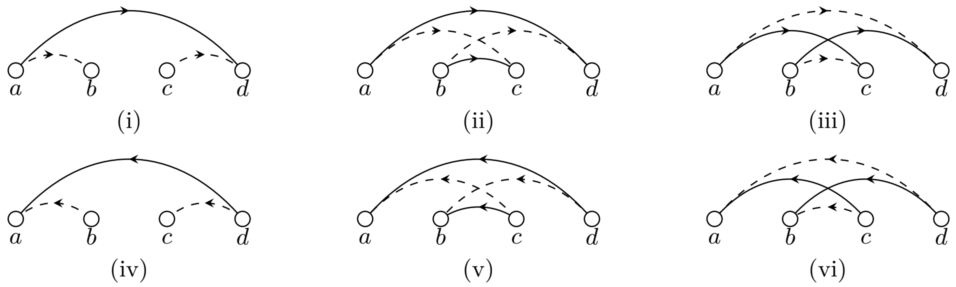

Figure 1 Forbidden structures for reflexive interval digraphs (possibly $b = c$ in (i), (ii), (iv) and (v)). A dashed arc from u to v indicates the absence of the edge (u, v) in the graph.

digraph as an input. This improves and generalizes Prisner's similar results about interval nest digraphs mentioned above. Moreover, we give an $O((n + m)n)$ time algorithm for the Min-Kernel and Max-Kernel problems for a superclass of reflexive interval digraphs. As a consequence, we obtain an improvement over the $O(n^3)$ time algorithm for finding a minimum independent dominating set in cocomparability graphs that was given by Kratsch and Stewart [17]. Our algorithms for Min-Kernel and Max-Kernel problems have a better running time of $O(n^2)$ for adjusted interval digraphs. On the other hand, we show that even the problem Kernel is NP-complete for point-point digraphs. Moreover, the Min-Kernel and Max-Kernel problems are APX-hard on point-point digraphs.

1.1 Notation

For a closed interval $I = [x, y]$ of the real line (here $x, y \in \mathbb{R}$ and $x \leq y$), we denote by $l(I)$ the left end-point x of I and by $r(I)$ the right end-point y of I. We use the following observation throughout the paper: if I and J are two intervals, then $I \cap J = \emptyset \Leftrightarrow (r(I) < l(J)) \vee (r(J) < l(I))$.

Let $G = (V, E)$ be a directed graph. For $u, v \in V(G)$, we say that u is an *in-neighbour* (resp. *out-neighbour*) of v if $(u, v) \in E(G)$ (resp. $(v, u) \in E(G)$). For a vertex v in G, we denote by $N_G^+(v)$ and $N_G^-(v)$ the set of out-neighbours and the set of in-neighbours of the vertex v in G respectively. When the graph G under consideration is clear from the context, we abbreviate $N_G^+(v)$ and $N_G^-(v)$ to just $N^+(v)$ and $N^-(v)$ respectively.

For $i, j \in \mathbb{N}$ such that $i \leq j$, let $[i, j]$ denote the set $\{i, i + 1, \ldots, j\}$. Let G be a digraph with vertex set $[1, n]$. Then for $i, j \in [1, n]$, we define $N^+_{>j}(i) = N^+(i) \cap [j + 1, n]$, $N^-_{>j}(i) = N^-(i) \cap [j + 1, n]$, $N^+_{<j}(i) = N^+(i) \cap [1, j - 1]$, and $N^-_{<j}(i) = N^-(i) \cap [1, j - 1]$. We shorten $N^+_{>i}(i)$ and $N^-_{>i}(i)$ to $N^+_>(i)$ and $N^-_>(i)$ respectively. Let $N_>(i) = N^+_>(i) \cup N^-_>(i)$ and $\overline{N_>(i)} = [i + 1, n] \setminus N_>(i)$.

2 Ordering characterization

We first show that a digraph is a reflexive interval digraph if and only if there is a linear ordering of its vertex set such that none of the structures shown in Figure 1 are present.

▶ **Theorem 1** ($\star$).[2] *A digraph G is a reflexive interval digraph if and only if $V(G)$ has an ordering $<$ in which for any $a, b, c, d \in V(G)$ such that $a < b < c < d$, none of the structures in Figure 1 occur (b and c can be the same vertex in (i), (ii), (iv), (v) of Figure 1).*

Now we define the following.

▶ **Definition 2** (DUF-ordering). *A* directed umbrella-free ordering *(or in short a* DUF-ordering*) of a digraph G is an ordering $<$ of $V(G)$ satisfying the following properties for any three distinct vertices $i < j < k$:*
1. *if $(i, k) \in E(G)$, then either $(i, j) \in E(G)$ or $(j, k) \in E(G)$, and*
2. *if $(k, i) \in E(G)$, then either $(k, j) \in E(G)$ or $(j, i) \in E(G)$.*

▶ **Definition 3** (DUF-digraph). *A digraph G is a* directed umbrella-free digraph *(or in short a* DUF-digraph*) if it has a DUF-ordering.*

Then the following corollary is an immediate consequence of Theorem 1.

▶ **Corollary 4.** *Every reflexive interval digraph is a DUF-digraph.*

Let G be an undirected graph. We define the *symmetric digraph of G* to be the digraph obtained by replacing each edge of G by symmetric arcs.

The following is a characterization of cocomparability graphs due to Kratsch and Stewart [17].

▶ **Theorem 5** ([17]). *An undirected graph G is a cocomparability graph if and only if there is an ordering $<$ of $V(G)$ such that for any three vertices $i < j < k$, if $ik \in E(G)$, then either $ij \in E(G)$ or $jk \in E(G)$.*

Let G be a DUF-digraph with a DUF-ordering $<$. Let H be the underlying undirected graph of G. Clearly, $<$ is an ordering of $V(H)$ that satisfies the property given in Theorem 5, implying that H is a cocomparability graph. Thus we have the following corollary.

▶ **Corollary 6.** *The underlying undirected graph of every DUF-digraph is a cocomparablity graph.*

Note that there exist digraphs which are not DUF-digraphs but their underlying undirected graphs are cocomparability (for example, a directed triangle with edges $(a, b), (b, c)$ and (c, a)). But we can observe that the class of underlying undirected graphs of DUF-digraphs is precisely the class of cocomparability graphs, since the symmetric digraphs of cocomparability graphs are all DUF-digraphs (for any cocomparability graph H, a vertex ordering of H that satisfies the property given in Theorem 5 is also a DUF-ordering of the symmetric digraph of H). In contrast, the class of underlying undirected graphs of reflexive interval digraphs forms a strict subclass of cocomparability graphs. We prove this by showing that no directed graph that has $K_{3,3}$ as its underlying undirected graph can be a reflexive interval digraph ($K_{3,3}$ can easily be seen to be a cocomparability graph). This would also imply by Corollary 4 that the class of reflexive interval digraphs forms a strict subclass of DUF-digraphs.

▶ **Theorem 7** ($\star$). *The underlying undirected graph of a reflexive interval digraph cannot contain $K_{3,3}$ as an induced subgraph.*

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[2] The proofs of the statements marked with a ($\star$) are omitted due to space constraints. Refer [11] for the omitted proofs.

Prisner [26] proved that the underlying undirected graphs of interval nest digraphs are weakly triangulated graphs. By Corollaries 4, 6 and Theorem 7, we can conclude that the underlying undirected graphs of reflexive interval digraphs are $K_{3,3}$-free cocomparability graphs. This strengthens the result of Prisner, since now we have that the underlying undirected graphs of interval nest digraphs are $K_{3,3}$-free weakly triangulated cocomparability graphs.

3 Algorithms for reflexive interval digraphs

In this section, we present polynoimal-time algorithms for the Kernel, Min-Kernel, Max-Kernel, Absorbing-Set, and Independent-Set problems on reflexive interval digraphs.

3.1 Kernel

We use the following result of Prisner that is implied by Theorem 4.2 of [26].

▶ **Theorem 8** ([26]). *Let $\mathcal{C}$ be a class of digraphs that is closed under taking induced subgraphs. If in every graph $G \in \mathcal{C}$, there exists a vertex z such that for every $y \in N^-(z)$, $N^+(z) \setminus N^-(z) \subseteq N^+(y)$, then the class $\mathcal{C}$ is kernel-perfect.*

▶ **Lemma 9** ($\star$). *Let G be a reflexive interval digraph G with interval representation $\{(S_u, T_u)\}_{u \in V(G)}$. Let z be the vertex such that $r(S_z) = \min\{r(S_v) : v \in V(G)\}$. Then for every $y \in N^-(z)$, $N^+(z) \setminus N^-(z) \subseteq N^+(y)$.*

Since reflexive interval digraphs are closed under taking induced subgraphs, by Theorem 8 and Lemma 9, we have the following.

▶ **Theorem 10.** *Reflexive interval digraphs are kernel-perfect.*

It follows from the above theorem that the decision problem Kernel is trivial on reflexive interval digraphs. As explained below, we can also compute a kernel in a reflexive interval digraph efficiently, if an interval representation of the digraph is known.

Let G be a reflexive interval digraph with an interval representation $\{(S_u, T_u)\}_{u \in V(G)}$. Let $G_0 = G$ and z_0 be the vertex in G such that $r(S_{z_0}) = \min\{r(S_v) : v \in V(G)\}$. For $i \geq 1$, recursively define G_i to be the induced subdigraph of G with $V(G_i) = V(G_{i-1}) \setminus (\{z_{i-1}\} \cup N^-(z_{i-1}))$ and if $V(G_i) \neq \emptyset$, define z_i to be the vertex such that $r(S_{z_i}) = \min\{r(S_v) : v \in V(G_i)\}$. Let t be smallest integer such that $V(G_{t+1}) = \emptyset$. Note that this implies that $V(G_t) = \{z_t\} \cup N^-_{G_t}(z_t)$. Clearly $t \leq n$ and $r(S_{z_0}) < r(S_{z_1}) < \cdots < r(S_{z_t})$. By Lemma 9, we have that for each $i \in \{1, 2, \ldots, t\}$, z_i has the following property: for any $y \in N^-_{G_i}(z_i)$ we have $N^+_{G_i}(z_i) \setminus N^-_{G_i}(z_i) \subseteq N^+_{G_i}(y)$.

We now recursively define a set $K_i \subseteq V(G_i)$ as follows: Define $K_t = \{z_t\}$. For each $i \in \{t-1, t-2, \ldots, 0\}$,

$$K_i = \begin{cases} \{z_i\} \cup K_{i+1} & \text{if } (z_i, z_j) \notin E(G), \text{ where } j = \min\{l : z_l \in K_{i+1}\} \\ K_{i+1} & \text{otherwise.} \end{cases}$$

▶ **Lemma 11** ($\star$). *For each $i \in \{1, 2, \ldots, t\}$, K_i is a kernel of G_i.*

By the above lemma, we have that K_0 is a kernel of G. We can now construct an algorithm that computes a kernel in a reflexive interval digraph G, given an interval representation of it. We assume that the interval representation of G is given in the form of a list of left and

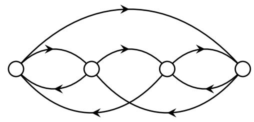

Figure 2 Example of a DUF-digraph that has no kernel.

right endpoints of intervals corresponding to the vertices. We can process this list from left to right in a single pass to compute the list of vertices $z_0, z_1, \ldots, z_t$ in $O(n+m)$ time. We then process this new list from right to left in a single pass to generate a set K as follows: initialize $K = \{z_t\}$ and for each $i \in \{t-1, t-2, \ldots, 0\}$, add z_i to K if it is not an in-neighbor of the last vertex that was added to K. Clearly, the set K can be generated in $O(n+m)$ time. It is easy to see that $K = K_0$ and therefore by Lemma 11, K is a kernel of G. Thus, we have the following theorem.

▶ **Theorem 12.** *A kernel of a reflexive interval digraph can be computed in linear-time, given an interval representation of the digraph as input.*

The linear-time algorithm described above is an improvement and generalization of a result of Prisner [26], who showed that interval nest digraphs are kernel-perfect, and a kernel can be found in these graphs in time $O(n^2)$ if an interval representation of the graph is given.

Now it is interesting to note that even for some kernel perfect digraphs with a polynomial-time computable kernel, the problems Min-Kernel and Max-Kernel turn out to be NP-complete. The following remark provides an example of such a class of digraphs.

▶ Remark 13. Let $\mathcal{C}$ be the class of symmetric digraphs of undirected graphs. Note that the class $\mathcal{C}$ is kernel-perfect, as for any $G \in \mathcal{C}$ the kernels of the digraph G are exactly the independent dominating sets of its underlying undirected graph. Note that any maximal independent set of an undirected graph is also an independent dominating set of it. Therefore, as a maximal independent set of any undirected graph can be found in linear-time, the problem Kernel is linear-time solvable for the class $\mathcal{C}$. On the other hand, note that the problems Min-Kernel and Max-Kernel for the class $\mathcal{C}$ is equivalent to the problems of finding a minimum cardinality independent dominating set and a maximum cardinality independent set for the class of undirected graphs, respectively. Since the latter problems are NP-complete for the class of undirected graphs, we have that the problems Min-Kernel and Max-Kernel are NP-complete in $\mathcal{C}$.

Note that unlike the class of reflexive interval digraphs, the class of DUF-digraphs are not kernel-perfect. Figure 2 provides an example for a DUF-digraph that has no kernel. Since that graph is a semi-complete digraph (i.e. each pair of vertices is adjacent), and every vertex has an out-neighbor which is not its in-neighbor, it cannot have a kernel. The ordering of the vertices of the graph that is shown in the figure can easily be verified to be a DUF-ordering. In contrast to Remark 13, even though DUF-digraphs may not have kernels, we show in the next section that the problems Kernel, Min-Kernel, and Max-Kernel can be solved in polynomial time in the class of DUF-digraphs. In fact we give a polynomial time algorithm that, given a DUF-digraph G with a DUF-ordering as input, either finds a minimum (or maximum) sized kernel in G or correctly concludes that G does not have a kernel.

3.2 Minimum sized kernel

Let G be a DUF-digraph with vertex set $[1, n]$. We assume without loss of generality that $<= (1, 2, \ldots, n)$ is a DUF-ordering of G.

For any vertex $i \in \{1, 2, \ldots, n\}$, let $P_i = \{j : j \in \overline{N_>(i)}$ such that $[i+1, j-1] \subseteq N^-(i) \cup N^-(j)\}$ and let $G[i, n]$ denote the subgraph induced in G by the set $[i, n]$. Note that we consider $[i+1, j-1] = \emptyset$, if $j = i+1$. For a collection of sets $\mathcal{S}$, we denote by $\mathrm{Min}(\mathcal{S})$ an arbitrarily chosen set in $\mathcal{S}$ of the smallest cardinality. For each $i \in \{1, 2, \ldots, n\}$, we define a set $K(i)$ as follows. Here, when we write $K(i) = \infty$, we mean that the set $K(i)$ is undefined.

$$K(i) = \begin{cases} \{i\}, & \text{if } N_>^-(i) = \{i+1, \ldots, n\} \\ \{i\} \cup \mathrm{Min}\{K(j) \neq \infty : j \in P_i\}, & \text{if } P_i \neq \emptyset \text{ and } \exists j \in P_i \text{ such that } K(j) \neq \infty \\ \infty, & \text{otherwise} \end{cases}$$

Note that it follows from the above definition that $K(n) = \{n\}$. For each $i \in \{1, 2, \ldots, n\}$, let $OPT(i)$ denote a minimum sized kernel of $G[i, n]$ that also contains i. If $G[i, n]$ has no kernel that contains i, then we say that $OPT(i) = \infty$. We then have the following lemma.

▶ **Lemma 14** ($\star$). *The following hold.*
1. *If $K(i) \neq \infty$, then $K(i)$ is a kernel of $G[i, n]$ that contains i, and*
2. *if $OPT(i) \neq \infty$, then $K(i) \neq \infty$ and $|K(i)| = |OPT(i)|$.*

Suppose that G has a kernel. Now let OPT denote a minimum sized kernel in G. Let $\mathcal{K} = \{K(j) \neq \infty : [1, j-1] \subseteq N^-(j)\}$. Note that we consider $[1, j-1] = \emptyset$ if $j = 1$. By Lemma 14(1), it follows that every member of $\mathcal{K}$ is a kernel of G. So if G does not have a kernel, then $\mathcal{K} = \emptyset$. The following lemma shows that the converse is also true.

▶ **Lemma 15** ($\star$). *If G has a kernel, then $\mathcal{K} \neq \emptyset$ and $|OPT| = \big|\mathrm{Min}(\mathcal{K})\big|$.*

We thus have the following theorem.

▶ **Theorem 16.** *The DUF-digraph G has a kernel if and only if $K(j) \neq \infty$ for some j such that $[1, j-1] \subseteq N^-(j)$. Further, if G has a kernel, then the set $\{K(j) \neq \infty : [1, j-1] \subseteq N^-(j)\}$ contains a kernel of G of minimum possible size.*

▶ **Theorem 17** ($\star$). *The* Min-Kernel *problem can be solved for DUF-digraphs in $O((n+m)n)$ time if the DUF-ordering is known. Consequently, for a reflexive interval digraph, the* Min-Kernel *problem can be solved in $O((n+m)n)$ time if the interval representation is given as input.*

▶ **Corollary 18** ($\star$). *An independent dominating set of minimum possible size can be found in $O((n+m)n)$ time in cocomparability graphs.*

The above corollary is an improvement over the results of Kratsch and Stewart [17], who proved that an independent dominating set of minimum possible size can be computed in $O(n^3)$ time in cocomparability graphs.

We now show that a minimum sized kernel of an adjusted interval digraph, whose interval representation is known, can be computed more efficiently than in the case of DUF-digraphs.

▶ **Corollary 19** ($\star$). *The* Min-Kernel *problem can be solved in adjusted interval digraphs in $O(n^2)$ time, given an interval representation of the digraph.*

▶ Remark 20. Note that the Max-Kernel problem can also be solved in $O((n+m)n)$ time for the class of DUF-digraphs, by a minor modification of our algorithm that solves Min-Kernel problem (replace $\mathrm{Min}\{K(j) \neq \infty : j \in P_i\}$ in the recursive definition of $K(i)$ by $\mathrm{Max}\{K(j) \neq \infty : j \in P_i\}$ and follow the same procedure. Then we have that if a kernel exists, then a maximum sized kernel is given by $\mathrm{Max}(\mathcal{K})$). Further, the recursive definition can also be easily adapted to the weighted versions of the problems Min-Kernel and Max-Kernel to obtain $O((n+m)n)$ time algorithms for those problems too.

3.3 Minimum absorbing set

Given any digraph G, the splitting bigraph B_G is defined as follows: $V(B_G)$ is partitioned into two sets $V' = \{u' : u \in V(G)\}$ and $V'' = \{u''\colon u \in V(G)\}$, and $E(B_G) = \{u'v''\colon (u,v) \in E(G)\}$. Müller [22] observed that G is an interval digraph if and only if B_G is an interval bigraph (since if $\{(S_u, T_u)\}_{u\in V(G)}$ is an interval representation of a digraph G, then $\{\{S_u\}_{u'\in V'}, \{T_u\}_{u''\in V''}\}$ is an interval bigraph representation of the bipartite graph B_G).

Recall that for a bipartite graph having two specified partite sets A and B, a set $S \subseteq B$ such that $\bigcup_{u\in B} N(u) = A$ is called an A-dominating set (or a red-blue dominating set). If G is a reflexive interval digraph, then every V'-dominating set of B_G corresponds to an absorbing set of G and vice versa. To be precise, if $S \subseteq V''$ is a V'-dominating set of B_G, then $\{u \in V(G)\colon u'' \in S\}$ is an absorbing set of G and if $S \subseteq V(G)$ is an absorbing set of G, then $\{u'' \in V''\colon u \in S\}$ is a V'-dominating set of B_G (note that this is not true for general interval digraphs). Thus finding a minimum cardinality absorbing set in G is equivalent to finding a minimum cardinality V'-dominating set in the bipartite graph B_G. We show in this section that the problem of computing a minimum cardinality A-dominating set is linear time solvable for interval bigraphs. This implies that the Absorbing-Set problem can be solved in linear time on reflexive interval digraphs.

Consider an interval bigraph H with partite sets A and B. Let $\{I_u\}_{u\in V(H)}$ be an interval representation for H; i.e. $uv \in E(H)$ if and only if $u \in A$, $v \in B$ and $I_u \cap I_v \neq \emptyset$. Let $|A| = t$. We assume without loss of generality that $A = \{1, 2, \ldots, t\}$, where $r(I_i) < r(I_j) \Leftrightarrow i < j$. We also assume that there are no isolated vertices in A, as otherwise H does not have any A-dominating set. For each $i \in \{1, 2, \ldots, t\}$, we compute a minimum cardinality subset $DS(i)$ of B that dominates $\{i, i+1, \ldots, t\}$, i.e. $\{i, i+1, \ldots, t\} \subseteq \bigcup_{u\in DS(i)} N(u)$. Then $DS(1)$ will be a minimum cardinality A-dominating set of H. We first define some parameters that will be used to define $DS(i)$.

Let $i \in \{1, 2, \ldots, t\}$. We define $\rho(i) = \max_{u\in N(i)} r(I_u)$ and let $R(i)$ be a vertex in $N(i)$ such that $r(I_{R(i)}) = \rho(i)$. Since A does not contain any isolated vertices, $\rho(i)$ and $R(i)$ exist for each $i \in \{1, 2, \ldots, t\}$. Let $\lambda(i) = \min\{j\colon \rho(i) < l(I_j)\}$. Note that $\lambda(i)$ may not exist.

▶ **Lemma 21** (⋆). *Let $i \in \{1, 2, \ldots, t\}$. If $\lambda(i)$ exists, then $R(i)$ dominates every vertex in $\{i, i+1, \ldots, \lambda(i)-1\}$ and otherwise, $R(i)$ dominates every vertex in $\{i, i+1, \ldots, t\}$.*

We now explain how to compute $DS(i)$ for each $i \in \{1, 2, \ldots, t\}$. We recursively define $DS(i)$ as follows:

$$DS(i) = \begin{cases} \{R(i)\} \cup DS(\lambda(i)) & \text{if } \lambda(i) \text{ exists} \\ \{R(i)\} & \text{otherwise.} \end{cases}$$

▶ **Lemma 22** (⋆). *For each $i \in \{1, 2, \ldots, t\}$, the set $DS(i)$ as defined above is a minimum cardinality subset of B that dominates $\{i, i+1, \ldots, t\}$.*

It is not difficult to verify that given an interval representation of the interval bigraph H with partite sets A and B, the parameters $R(i)$ and $\lambda(i)$ can be computed for each $i \in A$ in $O(n+m)$ time. Also, given a reflexive interval digraph G, the interval bigraph B_G can be constructed in linear time. Thus we have the following theorem.

▶ **Theorem 23.** *The* Red-Blue Dominating Set *problem can be solved in interval bigraphs in linear time, given an interval representation of the bigraph as input. Consequently, the* Absorbing-Set *problem can be solved in linear time in reflexive interval digraphs, given an interval representation of the input digraph.*

Note that even if an interval representation of an interval bigraph is not known, it can be computed in polynomial time using Müller's algorithm [22]. Thus given just the adjacency list of the graph as input, the Red-Blue Dominating Set problem is polynomial-time solvable on interval bigraphs and the Absorbing-Set problem is polynomial-time solvable on reflexive interval digraphs.

3.4 Maximum independent set

We have the following theorem due to McConnell and Spinrad [19].

▶ **Theorem 24** ([19]). *An independent set of maximum possible size can be computed for cocomparability graphs in $O(n+m)$ time.*

Let G be a DUF-digraph. Let H be the underlying undirected graph of G. Then by Corollary 6, we have that H is a cocomparability graph. Note that the independent sets of G and H are exactly the same. Therefore any algorithm that finds a maximum cardinality independent set in cocomparability graphs can be used to solve the Independent-Set problem in DUF-digraphs. Thus by the above theorem, we have the following corollary.

▶ **Corollary 25.** *The* Independent-Set *problem can be solved for DUF-digraphs in $O(n+m)$ time. Consequently, the* Independent-Set *problem can be solved for reflexive interval digraphs in $O(n+m)$ time.*

The above corollary generalizes and improves the $O(mn)$ time algorithm due to Prisner's [26] observation that underlying undirected graph of interval nest digraphs are weakly triangulated and the fact that maximum cardinality independent set problem can be solved for weakly triangulated graphs in $O(mn)$ time [14]. Note that the weighted Independent-Set problem can also be solved in DUF-digraphs in $O(n+m)$ time, as the problem of finding a maximum weighted independent set in a cocomparability graph can be solved in linear time [16].

4 Hardness results for point-point digraphs

4.1 Characterizations for point-point digraphs

In this section we give a characterization for point-point digraphs which will be further useful for proving our NP-completeness results for this class. Let $G=(V,E)$ be a digraph. We say that a,b,c,d is an *anti-directed walk* of length 3 if $a,b,c,d \in V(G)$, $(a,b),(c,b),(c,d) \in E(G)$ and $(a,d) \notin E(G)$ (the vertices a,b,c,d need not be pairwise distinct, but it follows from the definition that $a \neq c$ and $b \neq d$). Recall that $B_G=(X,Y,E)$ is a splitting bigraph of G, where $X=\{x_u : u \in V(G)\}$ and $Y=\{y_u : u \in V(G)\}$ and $x_u y_v \in E(G_B)$ if and only if $(u,v) \in E(G)$. We then have the following theorem.

▶ **Theorem 26.** *Let G be a digraph. Then the following conditions are equivalent:*
1. *G is a point-point digraph.*
2. *G does not contain any anti-directed walk of length 3.*
3. *The splitting bigraph of G is a disjoint union of complete bipartite graphs.*

Proof. ($\mathbf{1} \Rightarrow \mathbf{2}$): Let G be a point-point digraph with a point-point representation $\{(S_u,T_u)\}_{u \in V(G)}$. Suppose that there exist vertices a,b,c,d in G such that $(a,b),(c,b),(c,d) \in E(G)$. By the definition of point-point representation, we then have $S_a=T_b=S_c=T_d$. This implies that $(a,d) \in E(G)$. Therefore we can conclude that G does not contain any anti-directed walk of length 3.

(**2** ⇒ **3**): Suppose that G does not contain any anti-directed walk of length 3. Let $B_G = (X, Y, E)$ be the splitting bigraph of G. Let $x_u y_v$ be any edge in B_G, where $u, v \in V(G)$. Clearly, by the definition of B_G, $(u, v) \in E(G)$. We claim that the graph induced in B_G by the vertices $N(x_u) \cup N(y_v)$ is a complete bipartite graph. Suppose not. Then it should be the case that there exist two vertices $x_a \in N(y_v)$ and $y_b \in N(x_u)$ such that $x_a y_b \notin E(B_G)$, where $a, b \in V(G)$. By the definition of B_G, we then have that $(a, v), (u, v), (u, b) \in E(G)$ and $(a, b) \notin E(G)$. So a, v, u, b is an anti-directed walk of length 3 in G, which is a contradiction to 2. This proves that for every $p \in X$ and $q \in Y$ such that $pq \in E(B_G)$, the set $N(p) \cup N(q)$ induces a complete bipartite subgraph in B_G. Therefore, each connected component of B_G is a complete bipartite graph. (This can be seen as follows: Suppose that there is a connected component C of B_G that is not complete bipartite. Choose $p \in X \cap C$ and $q \in Y \cap C$ such that $pq \notin E(B_G)$ and the distance between p and q in B_G is as small as possible. Let t be the distance between p and q in B_G. Clearly, t is odd and $t \geq 3$. Consider a shortest path $p = z_0, z_1, z_2, \ldots, z_t = q$ from p to q in B_G. By our choice of p and q, we have that $z_1 z_{t-1} \in E(B_G)$. But then $p \in N(z_1)$, $q \in N(z_{t-1})$ and $pq \notin E(B_G)$, contradicting our observation that $N(z_1) \cup N(z_{t-1})$ induces a complete bipartite graph in B_G.)
(**3** ⇒ **1**): Suppose that G is a digraph such that the splitting bigraph B_G is a disjoint union of complete bipartite graphs, say $H_1, H_2, \ldots, H_k$. Now we can obtain a point-point representation $\{(S_u, T_u)\}_{u \in V(G)}$ of the digraph G as follows: For each $i \in \{1, 2, \ldots, k\}$, define $S_u = i$ if $x_u \in V(H_i)$ and $T_v = i$ if $y_v \in V(H_i)$. Note that $(u, v) \in E(G)$ if and only if $x_u y_v \in E(B_G)$ if and only if $x_u, y_v \in V(H_i)$ for some $i \in \{1, 2, \ldots, k\}$. Therefore we can conclude that $(u, v) \in E(G)$ if and only if $S_u = T_v = i$ for some $i \in \{1, 2, \ldots, k\}$. Thus the digraph G is a point-point digraph. ◀

▶ **Corollary 27.** *Point-point digraphs can be recognized in linear time.*

4.2 Subdivision of an irreflexive digraph

For an undirected graph G, the *k-subdivision* of G, where $k \geq 1$, is defined as the graph H having vertex set $V(H) = V(G) \cup \bigcup_{ij \in E(G)} \{u_{ij}^1, u_{ij}^2, \ldots, u_{ij}^k\}$, obtained from G by replacing each edge $ij \in E(G)$ by a path $i, u_{ij}^1, u_{ij}^2, \ldots, u_{ij}^k, j$. The 0-subdivision of an undirected graph G is defined to be G itself.

The following theorem is adapted from Theorem 5 of Chlebík and Chlebíková [3].

▶ **Theorem 28** ([3]). *Let G be an undirected graph having m edges.*
1. *The problem of computing a maximum cardinality independent set is APX-complete when restricted to $2k$-subdivisions of 3-regular graphs for any fixed integer $k \geq 0$.*
2. *The problem of finding a minimum cardinality dominating set (resp. independent dominating set) is APX-complete when restricted to $3k$-subdivisions of graphs having degree at most 3 for any fixed integer $k \geq 0$.*

Note that the independent sets, dominating sets and independent dominating sets of an undirected graph G are exactly the independent sets, dominating sets (which are also the absorbing sets), and solutions (which are also the kernels) of the symmetric digraph of G. Clearly the symmetric digraph of G is irreflexive. Since the Max-Kernel problem is equivalent to the Independent-Set problem in symmetric digraphs, we then have the following corollary of Theorem 28.

▶ **Corollary 29** ($\star$). *The problems* Independent-Set*,* Absorbing-Set*,* Min-Kernel *and* Max-Kernel *are APX-complete on irreflexive symmetric digraphs of in- and out-degree at most 3.*

Note that for $k \geq 1$, the symmetric digraph of the $2k$-subdivision or $3k$-subdivision of an undirected graph contains an anti-directed walk of length 3 (unless the graph contains no edges), and therefore by Theorem 26, is not a point-point digraph. Thus we cannot directly deduce the APX-hardness of the problems under consideration for point-point digraphs from Theorem 28.

We define the subdivision of an irreflexive digraph, so that the techniques of Chlebík and Chlebíková can be adapted for proving hardness results on point-point digraphs.

▶ **Definition 30.** *Let G be an irreflexive digraph (i.e. G contains no loops). For $k \geq 1$, define the* k-subdivision *of G to be the digraph H having vertex set $V(H) = V(G) \cup \bigcup_{(i,j)\in E(G)} \{u_{ij}^1, u_{ij}^2, \ldots, u_{ij}^k\}$, obtained from G by replacing each edge $(i,j) \in E(G)$ by a directed path $i, u_{ij}^1, u_{ij}^2, \ldots, u_{ij}^k, j$.*

Note that the k-subdivision of any irreflexive digraph is also an irreflexive digraph. We then have the following lemma.

▶ **Lemma 31.** *For any $k \geq 1$, the k-subdivision of any irreflexive digraph is a point-point digraph.*

Proof. Let $k \geq 1$ and let G be any irreflexive digraph. By Theorem 26, it is enough to show that the k-subdivision H of G does not contain any anti-directed walk of length 3. Note that by the definition of k-subdivision, all the vertices in $V(H) \setminus V(G)$ have both in-degree and out-degree exactly equal to one. Further, for every vertex v in H such that $v \in V(G)$, we have that $N^+(v), N^-(v) \subseteq V(H) \setminus V(G)$. Suppose for the sake of contradiction that u, v, w, x is an anti-directed walk of length 3 in H. Recall that we then have $(u,v), (w,v), (w,x) \in E(H)$, $u \neq w$ and $v \neq x$. By the above observations, we can then conclude that $v \in V(G)$ and further that $u, w \in V(H) \setminus V(G)$. Then since $(w,x) \in E(H)$ and $v \neq x$, we have that w has out-degree at least 2, which contradicts our earlier observation that every vertex in $V(H) \setminus V(G)$ has out-degree exactly one. This proves the lemma. ◀

Now we have the following theorem.

▶ **Theorem 32** ($\star$)**.** *The problem* Independent-Set *is APX-hard for point-point digraphs.*

4.3 Kernels

As a first step towards determining the complexity of the problems Kernel, Min-Kernel and Max-Kernel for point-point digraphs, we show the following.

▶ **Lemma 33** ($\star$)**.** *Let G be an irreflexive digraph and let $k \geq 1$. Then G has a kernel if and only if the $2k$-subdivision of G has a kernel. Moreover, G has a kernel of size q if and only if the $2k$-subdivision of G has a kernel of size $q + km$. Further, given a kernel of size $q + km$ of the $2k$-subdivision of G, we can construct a kernel of size q of G in polynomial time.*

The above lemma can be used to show the existence of a polynomial-time reduction from the Kernel problem in general digraphs to the Kernel problem in point-point digraphs. Thus we have the following theorem.

▶ **Theorem 34** ($\star$)**.** *The problem* Kernel *is NP-complete for point-point digraphs.*

Note that Kernel is known to be NP-complete even on planar digraphs having degree at most 3 and in- and out-degrees at most 2 [10]. The above reduction transforms the input digraph in such a way that every newly introduced vertex has in- and out-degree exactly

1 and the in- and out-degrees of the original vertices remain the same. Moreover, if the input digraph is planar, the digraph produced by the reduction is also planar. Thus we can conclude that the problem Kernel remains NP-complete even for planar point-point digraphs having degree at most 3 and in- and out-degrees at most 2.

An *L-reduction* as defined below is an approximation-preserving reduction for optimization problems.

▶ **Definition 35** ([23]). *Let A and B be two optimization problems with cost functions c_A and c_B respectively. Let f be a polynomially computable function that maps the instances of problem A to the instances of problem B. Then f is said to be an L-reduction form A to B if there exist a polynomially computable function g and constants $\alpha, \beta \in (0, \infty)$ such that the following conditions hold:*

1. *If y' is a solution to $f(x)$ then $g(y')$ is a solution to x, where x is an instance of the problem A.*
2. *$OPT_B(f(x)) \leq \alpha OPT_A(x)$, where $OPT_B(f(x))$ and $OPT_A(x)$ denote the optimum value of respective instances for the problems B and A respectively.*
3. *$|OPT_A(x) - c_A(g(y'))| \leq \beta |OPT_B(f(x) - c_B(y')|$.*

In order to prove that a problem P is APX-hard, it is enough to show that the problem P has an L-reduction from an APX-hard problem. Using Lemma 33, one can construct an L-reduction from the Min-Kernel and Max-Kernel problems for irreflexive symmetric digraphs having in- and out-degree at most 3 to the Min-Kernel and Max-Kernel problems for $2k$-subdivisions of irreflexive symmetric digraphs having in- and out-degree at most 3. Thus, by Corollary 29, we have the following theorem.

▶ **Theorem 36** ($\star$). *For $k \geq 1$, the problems* Min-Kernel *and* Max-Kernel *are APX-hard for $2k$-subdivisions of irreflexive symmetric digraphs having in- and out-degree at most 3. Consequently, the problems* Min-Kernel *and* Max-Kernel *are APX-hard for point-point digraphs.*

4.4 Minimum Absorbing set

In order to show the approximation hardness of the Absorbing-set problem for point-point digraphs, first we prove the following lemma.

▶ **Lemma 37** ($\star$). *Let G be an irreflexive digraph and let $k \geq 1$. Then G has an absorbing set of size at most q if and only if the $2k$-subdivision of G has an absorbing set of size at most $q + km$. Further, given an absorbing set of size at most $q + km$ in the $2k$-subdivision of G, we can construct in polynomial time an absorbing set of size at most q in G.*

The above lemma can be used to construct an L-reduction from the Absorbing-Set problem for irreflexive symmetric digraphs having in- and out-degree at most 3 to the Absorbing-Set problem for $2k$-subdivisions of irreflexive symmetric digraphs having in- and out-degree at most 3. By Corollary 29, we then have the following theorem.

▶ **Theorem 38** ($\star$). *For $k \geq 1$, the problem* Absorbing-set *is APX-hard for $2k$-subdivisions of irreflexive symmetric digraphs having in- and out-degree at most 3. Consequently, the problem* Absorbing-set *is APX-hard for point-point digraphs.*

5 Conclusion

The class of interval nest digraphs coincides with the class of totally bounded bitolerance digraphs which was introduced by Bogart and Trenk [2]. Thus totally bounded bitolerance digraphs are a subclass of reflexive interval digraphs, and all the results that we obtain for reflexive interval digraphs hold also for this class of digraphs. Figure 3 shows the inclusion relations between the classes of digraphs that were studied in this paper ($\star$). After work on this paper had been completed, we have been made aware of a recent manuscript of Jaffke, Kwon and Telle [15], in which unified polynomial time algorithms have been obtained for the problems considered in this paper for some classes reflexive intersection digraphs including reflexive interval digraphs. Since their algorithms are more general in nature, their time complexities are much larger than the ones for the algorithms presented in this paper.

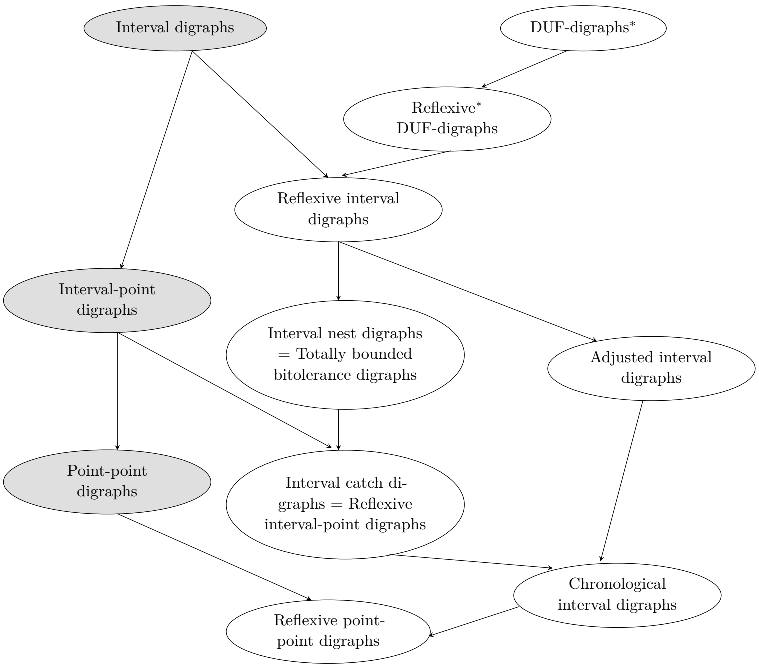

Figure 3 Inclusion relations between graph classes. In the diagram, there is an arrow from $\mathcal{A}$ to $\mathcal{B}$ if and only if the class $\mathcal{B}$ is contained in the class $\mathcal{A}$. Moreover, each inclusion is strict. The problems studied are efficiently solvable in the classes shown in white, while they are NP-hard and/or APX-hard in the classes shown in gray (* the complexity of the ABSORBING-SET problem on DUF-digraphs and reflexive DUF-digraphs remain open).

Müller [22] showed the close connection between interval digraphs and interval bigraphs and used this to construct the only known polynomial time recognition algorithm for both these classes. Since this algorithm takes $O(nm^6(n+m)\log n)$ time, the problem of finding a forbidden structure characterization for either of these classes or a faster recognition algorithm are long standing open questions in this field. But many of the subclasses of interval digraphs, like adjusted interval digraphs [29], chronological interval digraphs [5], interval catch digraphs [25], and interval point digraphs [26] have simpler and much more efficient recognition algorithms. It is quite possible that simpler and efficient algorithms for recognition exist also for reflexive interval digraphs. As for the case of interval nest digraphs, no polynomial time recognition algorithm is known. The complexities of the recognition problem and ABSORBING-SET problem for DUF-digraphs also remain as open problems.

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