From Local to Global Determinacy in Concurrent Graph Games

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Abstract

In general, finite concurrent two-player reachability games are only determined in a weak sense: the supremum probability to win can be approached via stochastic strategies, but cannot be realized.

We introduce a class of concurrent games that are determined in a much stronger sense, and in a way, it is the largest class with this property. To this end, we introduce the notion of *local interaction* at a state of a graph game: it is a *game form* whose outcomes (i.e. a table whose entries) are the next states, which depend on the concurrent actions of the players. By definition, a game form is *determined* iff it always yields games that are determined via deterministic strategies when used as a local interaction in a Nature-free, one-shot reachability game. We show that if all the local interactions of a graph game with Borel objective are determined game forms, the game itself is determined: if Nature does not play, one player has a winning strategy; if Nature plays, both players have deterministic strategies that maximize the probability to win. This constitutes a clear-cut separation: either a game form behaves poorly already when used alone with basic objectives, or it behaves well even when used together with other well-behaved game forms and complex objectives.

Existing results for positional and finite-memory determinacy in turn-based games are extended this way to concurrent games with determined local interactions (CG-DLI).

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1 Introduction

We consider games that involve two players and that are played on infinite (unless otherwise stated) graphs. On such games, we consider several flavors of determinacy properties. Specifically, the existence of a winning deterministic strategy for either of the players, of optimal deterministic strategies for both players, of almost-sure stochastic strategies for either of the players, and of ϵ-optimal stochastic strategies for both of the players. Generic determinacy results have been established on many classes of games. We illustrate these notions on turn-based and concurrent games with either a deterministic or stochastic Nature on Figures 1, 2, 3 and 4 (see [21] for the introduction of the most general setting, the stochastic concurrent games). In all cases the game starts in q_0 and the goal for Player A is to see y at some point while Player B wins if y never occurs.

Consider the turn-based game in Figure 1. There, Player A chooses either the self-loop or the edge to q_1; a symbol x, called a color, is seen in either case; then the game proceeds to state q_0 or q_1. In q_1 Player B chooses either the y-labeled self-loop or the x-labeled edge to q_0. This generates an infinite sequence over $\{x, y\}$. Player B has a winning strategy, which consists in never using the self-loop in q_1: however Player A may play, the generated sequence

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is x^ω. Thus, the game is said to be determined, in a very strong sense, and many sorts of objectives enjoy similar properties on such turn-based games. More generally, Martin [14, 15] proved that turn-based games with Borel objective are deterministically determined.

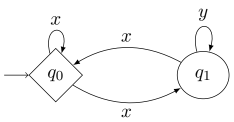

Figure 1 A turn-based game w/o Nature with diamond-shaped nodes for Player A, circle-shaped nodes for Player B and color labels on edges.

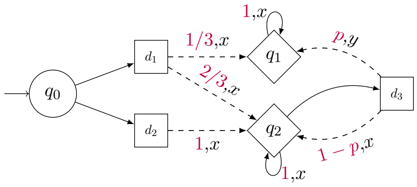

Figure 2 A turn-based game with Nature with probabilities displayed in purple on Nature-to-player edges, and colors in black on the same edges for convenience.

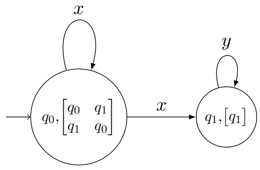

Figure 3 A concurrent game w/o Nature, with two actions for each player: Player A chooses a row, Player B chooses a column.

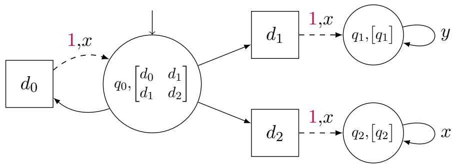

Figure 4 A concurrent game with Nature (albeit deterministic) with probabilities displayed in purple on Nature-to-player edges.

Now consider the turn-based game with (stochastic) Nature in Figure 2. In q_0 Player B moves to Nature state d_1 or d_2. In d_1 Nature goes to q_1 and q_2 with probability $\frac{1}{3}$ and $\frac{2}{3}$, respectively; in d_2 with probability 1 to q_2. In q_1 there is only a self-loop, which is a shorthand for an edge towards a Nature state that goes back to q_1 with probability 1. In q_2 Player A stays in q_2 or moves to d_3. In d_3 Nature goes to q_1 and q_2 with probabilities $p, 1-p \in]0,1[$. The edge (q_2, q_1) is labeled with y, and the other edges between the q_i are labeled with x. From q_0, Player B has a strategy that minimizes the probability to see y (to $\frac{2}{3}$), namely to play towards d_1; and Player A maximizes this probability (to the same value $\frac{2}{3}$) by playing d_3 when in q_2. Note that from q_2, this Player A's strategy wins almost surely but not surely. These optimal strategies of the Players are deterministic. The game is said to be determined, in a sense that is rather strong but weaker than above without Nature, and several objectives (though fewer than above) enjoy similar properties on turn-based games with Nature. More generally, it was proved [4, 23] that turn-based parity games played on finite graphs with stochastic Nature have deterministic optimal strategies.

Consider the game in Figure 3. The table depicted within state q_0 records the concurrent interaction between the two players at q_0: Player A chooses a row of the table while Player B independently chooses a column of the table; depending on the two choices, the game proceeds either to state q_0 again (first row first column, or second row second column) or to state q_1. In the two cases x is seen. In q_1 the interaction is trivial, i.e. each player has only one option, and y is seen. It is easy to see that Player A has no deterministic winning strategy, but a stochastic strategy that wins almost surely: in q_0, she picks each row with probability one half.

In the game from [8, 5] in Figure 4, Player A has no stochastic strategy that wins almost surely, but for all $\epsilon \in]0,1]$, she has a stochastic strategy that wins with a probability at least $1-\epsilon$: in q_0, she chooses the second row with probability ϵ. More generally, Martin [16] proved that such a weak determinacy holds in games with Borel objective if the local interactions involve finitely many rows and columns.

The above examples and existing results suggest that what prevents the existence of optimal strategies is more the structure of the local interaction rather than the presence of a stochastic Nature. This article substantiates this impression.

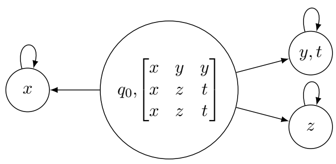

Figure 5 A concurrent game reachability game that is determined for every set of states as target set for either of the players.

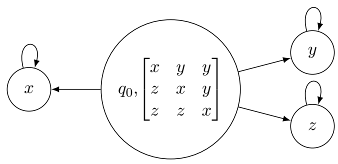

Figure 6 A concurrent reachability game that is not determined if a player tries to reach the set of states $T = \{x\}$.

Our contribution. A game form is a table whose entries are called outcomes, see e.g. Figures 5, 6, 7. By definition, it is determined if replacing each outcome with 1 or 0 yields a table with a row full of 1 (Player A wins) or a column full of 0 (Player B wins). It is easy to show that it is determined iff every "one-shot" reachability game using it as local interaction is deterministically determined. E.g. consider the one-shot reachability arena in Figure 5, involving a determined game form. Setting any subset of $\{x, y, z, t\}$ as target for either of the players yields a deterministically determined game. However, the game form in Figure 6 is non-determined, e.g. by setting $x := 1$ and $y, z := 0$. Equivalenty, setting the target of Player A to $\{x\}$ yields a game with no winning strategies. Thus, the determinacy of a game form amounts to its good behavior when used individually as local interaction in very simple games. We will show that individually well-behaved game forms are collectively well-behaved. More specifically, we extend various determinacy results from turn-based [14, 3, 4, 23, 11] to concurrent games with determined local interactions (CG-DLI). Fix a set K of colors. Each edge of our games is labeled with some color, and the winning objective is expressed as a subset of K^ω. We prove the following:

1. In all CG-DLI with Borel (parity) objective, one player has a (positional) winning strategy.
2. In all CG-DLI with Borel (parity) objective and stochastic Nature, both players have (positional) optimal strategies.
3. Let $\mathcal{M}$ be a memory skeleton (DFA on K, explained later). The following are equivalent.
   - W and $\mathsf{K}^\omega \setminus W$ are $\mathcal{M}$-monotone and $\mathcal{M}$-selective (notions recalled later).
   - All CG-DLI with finitely many states and actions, and objective W has a finite-memory winning strategy implemented via $\mathcal{M}$.

Moreover in the three statements above, the winning/optimal strategies can be chosen both deterministic and dependent only on the history of observed colors, rather than visited states.

Conversely, let G be any non-determined game form. As hinted at above, one can show that for all Borel objectives $\emptyset \subsetneq W \subsetneq \mathsf{K}^\omega$, there is a Nature-free game with one single non-trivial state whose local interaction is G, and no deterministic optimal (or winning) strategy, even one that would depend on the history of visited states. A similar result holds for finite-memory strategies. Hence, these results provide a clear-cut separation: determined game forms are well-behaved basic bricks that collectively build well-behaved-only concurrent games, while non-determined game forms are ill-behaved already when used alone.

A large part of the proofs of the above extensions is factored out by our following theorem: a CG-DLI is (finite-memory, positionnaly, "plainly") determined (via winning deterministic or optimal stochastic strategies) if and only if its sequential version is. The sequential version of the game is obtained by letting one player (whichever, but keep the convention) act first at each state of the game, and the opponent act second. Although most of the extensions are straightforward applications of this theorem, the finite-memory case is different: the result in [3] requires the objective to satisfy specific properties, and it is rather long to prove that these properties satisfy the assumptions of the theorem.

Outline. Section 2 contains notations; Section 3 recalls the notion of game form; Section 4 presents the game-theoretic formalism; Section 5 defines the sequentialization and parallelization, and proves related preservation results; Section 6 presents determinacy extensions.

Additional details and complete proofs are available in the arXiv version of this paper [1].

2 Preliminaries

Consider a non-empty set D. We denote by $D^\uparrow := D^* \cup D^\omega$ the set of finite or infinite sequences in D. For a sequence $\pi = \pi_0\pi_1 \ldots \pi_n \in D^*$, we denote by $\mathsf{lt}(\pi)$ the last element of the sequence: $\mathsf{lt}(\pi) = \pi_n$.

For a function $\mathsf{f} : E \to F$ and $F' \subseteq F$, the notation $\mathsf{f}^{-1}[F']$ refers to the preimage $\{e \in E \mid \mathsf{f}(e) \in F'\}$ of F' by the function f. Furthermore, a function $\mathsf{f} : E \to F$ can be lifted into a function $\bar{\mathsf{f}} : E^\uparrow \to F^\uparrow$ defined by: $\bar{\mathsf{f}}(\varepsilon) = \varepsilon$, $\bar{\mathsf{f}}(e) = \mathsf{f}(e)$ for all $e \in E$, and $\bar{\mathsf{f}}(\pi \cdot \pi') = \bar{\mathsf{f}}(\pi) \cdot \bar{\mathsf{f}}(\pi')$ for all $\pi \in E^*$ and $\pi' \in E^\uparrow$. For a set $E' \subseteq E$, we define the projection function $\phi_{E,E'} : E^\uparrow \to E'^\uparrow$ such that $\phi_{E,E'}(e) = e$ if $e \in E'$, $\phi_{E,E'}(e) = \varepsilon$ otherwise and $\phi_{E,E'}(\pi \cdot \pi') = \phi_{E,E'}(\pi) \cdot \phi_{E,E'}(\pi')$ for all $\pi \in E^*$ and $\pi' \in E^\uparrow$. For a set Q and a function $\mathsf{f} : Q \times Q \to T$, we denote by $\mathsf{tr}_\mathsf{f} : Q^+ \to T^* \times Q$ the function that associates to a sequence $\pi \in Q^+$, its trace $\mathsf{tr}_\mathsf{f}(\pi) = (\bar{\mathsf{f}}(\pi), \mathsf{lt}(\pi))$. For instance, $\mathsf{tr}_\mathsf{f}(a \cdot b \cdot c) = (f(a,b) \cdot f(b,c), c)$.

Let us now recall the definition of cylinder sets. For a non-empty set Q, for all $\pi \in Q^*$, the cylinder set $\mathsf{Cyl}(\pi)$ generated by π is the set $\mathsf{Cyl}(\pi) = \{\pi \cdot \rho \in Q^\omega \mid \rho \in Q^\omega\}$. We denote by Cyl_Q the set of all cylinder sets on Q^ω. The open sets of Q^ω are the sets equal to an arbitrary union of cylinder sets. The set of Borel sets on Q^ω, denoted $\mathsf{Borel}(Q)$, is then equal to the smallest set containing all open sets that is closed under complementation and countable union. Recall that, considering two probability measures $\nu, \nu' : \mathsf{Borel}(Q) \to [0,1]$, if for all $C \in \mathsf{Cyl}_Q$, we have $\nu(C) = \nu'(C)$, then $\nu = \nu$.

3 Game Forms and Win/Lose Games

Informally, game forms (used in [10, 12]) are games without objectives, see Definition 1 and examples in Figure 7. They are similar to what is sometimes called arena, but are presented in normal form, i.e. by ignoring their possible underlying graph or tree structure.

▶ **Definition 1** (Game form and win/lose game). *A* game form *is a tuple* $\mathcal{F} = \langle \mathsf{S_A}, \mathsf{S_B}, \mathsf{O}, \varrho \rangle$ *where* $\mathsf{S_A}$ *(resp.* $\mathsf{S_B}$*) is the non-empty set of strategies available to Player* A *(resp.* B*),* O *is a non-empty set of possible outcomes, and* $\varrho : \mathsf{S_A} \times \mathsf{S_B} \to \mathsf{O}$ *is a function that associates an outcome to each pair of strategies. A* win/lose game *is a pair* $\mathcal{G} = \langle \mathcal{F}, \mathcal{V} \rangle$ *where* $\mathcal{F}$ *is a game form and* $\mathcal{V} \subseteq \mathsf{O}$ *is the set of winning outcomes for Player* A *whereas* $\mathsf{O} \setminus \mathcal{V}$ *is the set of winning outcomes for Player* B*.*

As in Definition 1, a player wins if she obtains an outcome that makes her win, hence winning for Player A means reaching an outcome in $\mathcal{V}$, whereas winning for Player B means reaching an outcome in $\mathsf{O} \setminus \mathcal{V}$. So, one player wins if and only if the other player loses, hence the terminology. In the context of win/lose games, we can define the notion of winning strategy, that is, a strategy for a player that ensures winning regardless of his opponent's strategy. The definition of determinacy follows.

$$I_1 = \begin{bmatrix} x & y \\ y & x \end{bmatrix} \; I_2 = \begin{bmatrix} x & x \\ y & z \end{bmatrix} \; I_3 = \begin{bmatrix} x & x & z \\ x & y & y \\ z & y & z \end{bmatrix} \; I_4 = \begin{bmatrix} x & y & z \\ y & x & z \\ z & z & z \end{bmatrix} \; I_5 = \begin{bmatrix} x & x & z \\ x & z & y \\ z & y & y \end{bmatrix}$$

Figure 7 Five game forms: I_1 and I_5 are not determined, whereas I_2, I_3, and I_4 are.

▶ **Definition 2** (Winning Strategies and Determinacy). *Consider a game form $\mathcal{F} = \langle \mathsf{S_A}, \mathsf{S_B}, \mathsf{O}, \varrho \rangle$ and a subset of outcomes $\mathcal{V} \subseteq \mathsf{O}$. In the win/lose game $\mathcal{G} = \langle \mathcal{F}, \mathcal{V} \rangle$, a* winning strategy *$\mathsf{s_A} \in \mathsf{S_A}$ (resp. $\mathsf{s_B} \in \mathsf{S_B}$) for Player A (resp. B) is a strategy such that, for all $\mathsf{s_B} \in \mathsf{S_B}$ (resp. $\mathsf{s_A} \in \mathsf{S_A}$), we have $\varrho(\mathsf{s_A}, \mathsf{s_B}) \in \mathcal{V}$ (resp. $\mathsf{O} \setminus \mathcal{V}$). We write $\mathcal{W}_\mathsf{A}(\mathcal{F}, \mathcal{V})$ (resp. $\mathcal{W}_\mathsf{B}(\mathcal{F}, \mathsf{O} \setminus \mathcal{V})$) the set of winning strategies for Player A (resp. Player B) with objective $\mathcal{V}$ (resp. $\mathsf{O} \setminus \mathcal{V}$). The win/lose game $\mathcal{G}$ is* determined *if either of the players has a winning strategy: $\mathcal{W}_\mathsf{A}(\mathcal{F}, \mathcal{V}) \cup \mathcal{W}_\mathsf{B}(\mathcal{F}, \mathsf{O} \setminus \mathcal{V}) \neq \emptyset$. The game form $\mathcal{F}$ is said to be* determined *if, for all $\mathcal{V} \subseteq \mathsf{O}$, the win/lose game $\mathcal{G} = \langle \mathcal{F}, \mathcal{V} \rangle$ is determined. We denote by $\mathsf{Det_{GF}}$ the set of determined game forms.*

▶ **Example 3.** Consider the game forms represented in Figure 7. We argue that I_2, I_3, and I_4 are determined, while I_1 and I_5 are not. Consider any subset $\mathcal{V}$ of the outcomes and, in I_2, I_3, and I_4, replace each occurence of outcome in $\mathcal{V}$ with $\mathsf{w_A}$ (indicating winning outcomes for Player A) and the others with $\mathsf{w_B}$ (indicating winning outcomes for Player B). There is always a row of $\mathsf{w_A}$ or a column of $\mathsf{w_B}$, so these game forms are determined. However, rewriting x with $\mathsf{w_A}$ and y with $\mathsf{w_B}$ in I_1 yields the well-known matching-penny game, which clearly has no winning strategies. Similarly, rewriting z with $\mathsf{w_A}$ and x, y with $\mathsf{w_B}$ in I_5 leads to no row full of $\mathsf{w_A}$ and no column full of $\mathsf{w_B}$.

As we shall see, determined game forms are exactly the game forms that share enough similarities with "turn-based interactions", so that our determinacy transfer may hold. Hence, we may ask whether the determined game forms are nothing but turn-based interactions in disguise. Of course, the answer depends on what we mean by "in disguise". For a natural notion of being similar to a turn-based interaction, the answer is negative. Thus, determined game forms are more than turn-based interactions.

In addition to the toy examples in Figure 7, let us exemplify that determined game forms arise naturally in computer science. A parity game ([7, 17, 22]) is defined on a priority arena, i.e. a graph where each vertex is controlled by one player and every edge is labeled with a natural number less than a fixed bound. The outcome of an infinite run in such an arena is the maximum of all the numbers that occur infinitely often during the run. If the priorities are seen not as concrete numbers but as abstract outcomes, the priority arena can be seen as a game form. By a slight generalization of [7, 17, 22] described, e.g., in [19, Corollary 3.8], it is moreover a determined game form. So, as we shall see, choosing the next state following a local interaction given by a parity game will be a *well-behaved* interaction.

Finally, consider the complexity of deciding if a given game form is determined (the corresponding decision problem is denoted DetGF). It is straightforwardly in coNP since proving that a game form is not determined amounts to exhibiting a $\{\mathsf{w_A}, \mathsf{w_B}\}$-valuation for which there is neither a row full of $\mathsf{w_A}$s nor a column full of $\mathsf{w_B}$s, which can be checked in polynomial time. In fact, in [2] (where determinacy is refered to as tightness), the authors mentioned that DetGF could be solved in quasi-polynomial time via a reduction to the dualization of monotone CNF formulae (denoted $\mathsf{MonotoneDual}$), which can be solved in quasi-polynomial time [9]. Note that it is an open problem if $\mathsf{MonotoneDual}$ is in P or is coNP-complete [6]. In fact, we can show that DetGF is equivalent (modulo polytime reduction) to $\mathsf{MonotoneDual}$ thus showing that answering if DetGF is in P or coNP-complete directly answers the same question for $\mathsf{MonotoneDual}$.

4 Concurrent Graph Games and Strategies

Colored stochastic win/lose concurrent graph games. Informally, a stochastic concurrent game is played on a graph as follows: from a given state, both players simultaneously choose an action, and the next state is set according to a probability distribution that depends on the two actions. We want to consider the ways the two players interact at each state (which we call the local interactions of the game) as game forms. To facilitate this, we decouple the concurrent interaction of the players from the stochastic choice of Nature; we therefore add intermediate states belonging to Nature, and ensure that they do not impact winning conditions by assigning colors to ordered pairs of player states, thus hiding the Nature states that are visited. To sum up, the outcome of an interaction of the players is a Nature state from which the next (relevant) state of the game is chosen via a probability distribution.

▶ **Definition 4** (Stochastic concurrent games). *A colored* stochastic concurrent graph arena $\mathcal{C}$ *is a tuple* $\langle A, B, Q, q_0, \mathsf{D}, \delta, \mathsf{dist}, \mathsf{K}, \mathsf{col}\rangle$ *where* A *(resp.* B*) is the non-empty set of actions available to Player* A *(resp.* B*),* Q *is the (non-empty) set of states,* $q_0 \in Q$ *is the initial state,* D *is the set of Nature states,* $\delta : Q \times A \times B \to \mathsf{D}$ *is the transition function,* $\mathsf{dist} : \mathsf{D} \to \mathsf{Dist}(Q)$ *is the distribution function,* K *is a non-empty set of colors, and* $\mathsf{col} : Q \times Q \to \mathsf{K}$ *is a coloring function. The composition of the transition and distribution functions* $\mathsf{dist} \circ \delta : Q \times A \times B \to \mathsf{Dist}(Q)$ *will be denoted* Δ*. A* win/lose concurrent graph game *is a pair* $\langle \mathcal{C}, W\rangle$ *where* $W \in \mathsf{Borel}(\mathsf{K})$ *is the set of winning sequences of colors (for Player* A*).*

In the following, the arena $\mathcal{C}$ will always refer to the tuple $\langle A, B, Q, q_0, \mathsf{D}, \delta, \mathsf{dist}, \mathsf{K}, \mathsf{col}\rangle$ unless otherwise stated. In section 6, we will be able to apply some of our results only to finite arenas, i.e. when A, B and $Q \cup \mathsf{D}$ are finite.

Strategies and their values. We consider two kinds of strategies: those that only depend on the sequence of colors seen (and the current state) and that output a specific action – called chromatic strategies [13] – and those that may depend on the sequence of states seen and that output a distribution over the available actions – called state strategies.

▶ **Definition 5** (State and chromatic strategies). *Let* $\mathcal{C}$ *be an arena.*

- *A state strategy, for Player* A *is a function* $\mathsf{s_A} : Q^+ \to \mathsf{Dist}(A)$ *and the set of all such strategies in arena* $\mathcal{C}$ *for that player is denoted* $\mathsf{StaSt}^{\mathsf{A}}_{\mathcal{C}}$*.*
- *A chromatic strategy for Player* A *is a function* $\mathsf{s_A} : \mathsf{K}^* \times Q \to A$ *and the set of all such strategies in arena* $\mathcal{C}$ *for that player is denoted* $\mathsf{ColSt}^{\mathsf{A}}_{\mathcal{C}}$*. From a chromatic strategy* $\mathsf{s_A} \in \mathsf{ColSt}^{\mathsf{A}}_{\mathcal{C}}$*, we can extract the state strategy* $\widetilde{\mathsf{s_A}} : Q^+ \to \mathsf{Dist}(A)$ *defined by* $\widetilde{\mathsf{s_A}} = \mathsf{s_A} \circ \mathsf{tr}_{\mathsf{col}}$*.*

The definitions are likewise for Player B*. Two state strategies* $\mathsf{s_A}$ *and* $\mathsf{s_B}$ *for Players* A *and* B *then induce a probability of occurrence of finite paths and, following, of cylinder sets. This, in turn, induces a probability measure* $\mathbb{P}^{\mathcal{C}}_{\mathsf{s_A},\mathsf{s_B}}$ *over all Borel sets.*

In a game $\langle \mathcal{C}, W\rangle$, Player A tries to maximize the probability to be in the set W whereas Player B tries to minimize it. We will show that the concurrent games we consider are determined and that chromatic strategies are sufficient to play optimally. However, since the games considered are stochastic, for a strategy to be optimal, it has to achieve the optimal value against all strategies – i.e. state strategies – of the antagonist player. For convenience in the proofs, we give below this assymetric definition of values, where one player plays with chromatic strategies while the other is allowed to use state strategies. This is without restriction as we will be able to prove that the color values of the two players coincide.

▶ **Definition 6** (Value of strategies and color value of the game). *Let $\mathcal{C}$ be an arena. The corresponding winning set for Player* A *to a Borel set* $W \subseteq \mathsf{K}^\omega$ *is equal to* $U_W = \overline{\mathsf{col}}^{-1}[W] \subseteq Q^\omega$*, which is also Borel.*[1] *Let* $\mathsf{s_A} \in \mathsf{ColSt}^{\mathsf{A}}_{\mathcal{C}}$ *be a chromatic strategy for Player* A*. The* value *of strategy* $\mathsf{s_A}$ *is equal to* $\chi^{\mathcal{C}}_{\mathsf{s_A}}[W] := \inf_{\mathsf{s_B} \in \mathsf{StaSt}^{\mathsf{B}}_{\mathcal{C}}} \mathbb{P}^{\mathcal{C}}_{\widetilde{\mathsf{s_A},\mathsf{s_B}}}[U_W]$*. The* color value $\chi^{\mathcal{C}}_{\mathsf{A}}$ *of the game for Player* A *is:* $\chi^{\mathcal{C}}_{\mathsf{A}}[W] := \sup_{\mathsf{s_A} \in \mathsf{ColSt}^{\mathsf{A}}_{\mathcal{C}}} \chi^{\mathcal{C}}_{\mathsf{s_A}}[W]$*. The definitions are likewise for Player* B*, by reversing the supremum and infimum.*

A win/lose stochastic concurrent graph game $\langle \mathcal{C}, W\rangle$ *is* limit-determined *if* $\chi^{\mathcal{C}}_{\mathsf{A}}[W] = \chi^{\mathcal{C}}_{\mathsf{B}}[W]$*. If in addition there are strategies* $\mathsf{s_A} \in \mathsf{ColSt}^{\mathsf{A}}_{\mathcal{C}}$ *and* $\mathsf{s_B} \in \mathsf{ColSt}^{\mathsf{B}}_{\mathcal{C}}$ *such that* $\chi^{\mathcal{C}}_{\mathsf{s_A}}[W] = \chi^{\mathcal{C}}_{\mathsf{A}}[W]$ *and* $\chi^{\mathcal{C}}_{\mathsf{s_B}}[W] = \chi^{\mathcal{C}}_{\mathsf{B}}[W]$*, we say that the game is* determined*. In this case, such strategies are called* optimal *strategies.*

Let us look at what the *local determinacy* of a concurrent game refers to, which will yield the definition of locally determined stochastic concurrent games.

▶ **Definition 7** (Local interactions). *The* local interaction *in a stochastic concurrent graph arena* $\mathcal{C}$ *at state* $q \in Q$ *is the game form* $\mathcal{F}_q = \langle A, B, \mathsf{D}, \delta(q,\cdot,\cdot)\rangle$ *where the strategies available for Player* A *(resp.* B*) are the actions in* A *(resp.* B*) and the outcomes are the Nature states. For a set of game forms* $\mathcal{I}$*, we say that a concurrent arena* $\mathcal{C} = \langle A, B, Q, q_0, \mathsf{D}, \delta, \mathsf{dist}, \mathsf{K}, \mathsf{col}\rangle$ *is* built on $\mathcal{I}$ *if, for all* $q \in Q$*, we have* $\mathcal{F}_q \in \mathcal{I}$ *(up to a renaming of the outcomes). A stochastic concurrent graph arena/game is* locally determined *if it is built on* $\mathsf{Det_{GF}}$.

Turn-based games. Usually, turn-based games and concurrent games are described in two different formalisms. Indeed, in a turn-based game, a player plays only in the states that she controls, whereas in a concurrent game, in each state both players play an action and subsequently the next (Nature) state is reached. However, turn-based games can be seen as a special case of concurrent games, where at each state, the next (Nature) state is chosen regardless of one of the player's action. We choose the second option .

Section 5 will translate locally determined concurrent games into turn-based games, then transfer existing determinacy results on turn-based games back into extension results for the more general locally determined concurrent games.

Chromatic strategy implementations. We recall the notion of memory skeleton (see, for instance, [3]) and we see how it can implement the chromatic strategies. For a set of colors K and a set of states Q, a memory skeleton on K is a triple $\mathcal{M} = \langle M, m_{\text{init}}, \mu\rangle$, where M is a non-empty set called the memory, $m_{\text{init}} \in M$ is the initial state of the memory and $\mu : M \times \mathsf{K} \to M$ is the update function. An action map with memory M is a function $\lambda : M \times Q \to T$ for a non-empty set T. Note that T is a set of possible decisions that can be made. Here, T will be instantiated with the set of actions of either of the players. In fact, a memory skeleton and an action map implement a chromatic strategy.

▶ **Definition 8** (Implementation of strategies). *Consider a concurrent colored arena* $\mathcal{C}$*, a player* $p \in \{\mathsf{A}, \mathsf{B}\}$ *and the corresponding set of actions* $T \in \{A, B\}$*. A memory skeleton* $\mathcal{M} = \langle M, m_{\text{init}}, \mu\rangle$ *on* K *and an action map* $\lambda : M \times Q \to T$ *implement the chromatic strategy* $\mathsf{s} : \mathsf{K}^* \times Q \to T$ *that is defined by* $\mathsf{s}(\rho, q) = \lambda(\overline{\mu}(m_{\text{init}}, \rho), q) \in T$ *for all* $(\rho, q) \in \mathsf{K}^* \times Q$.

A strategy s *is* finite memory *if there exists a memory skeleton* $\mathcal{M} = \langle M, m_{\text{init}}, \mu\rangle$*, with* M *finite, and an action map* λ *implementing* s*. If* M *is reduced to a singleton,* s *is* positional*, aka memoryless. The amount of memory used to implement the strategy* s *is* $|M|$.

[1] As the preimage of a Borel set by the continuous function $\overline{\mathsf{col}}$.

Note that any chromatic strategy $\mathsf{s} : \mathsf{K}^* \times Q \to T$ can be implemented with a (possibly infinite) memory skeleton and an action map: consider the memory skeleton $\mathcal{M} = \langle \mathsf{K}^*, \epsilon, \mu \rangle$ where $\mu : \mathsf{K}^* \times \mathsf{K} \to \mathsf{K}^*$ is defined by $\mu(\rho, k) = \rho \cdot k$ for all $\rho \in \mathsf{K}^*$ and $k \in \mathsf{K}$.

▶ **Definition 9** (Finite-memory determinacy). *A game is said to be* finite-memory *(resp.* positionally*) determined if it is determined and optimal strategies can be found among finite-memory (resp. positional) strategies.*

5 Sequentialization of Games

In this section, we explain how we sequentialize a concurrent graph game. We then show "correctness" of this sequentialization in a sense that we will make precise.

Sequential version of a concurrent graph game. The sequential version of an arbitrary colored stochastic concurrent graph arena consists of a turn-based graph arena where Player A plays first and then Player B responds.

▶ **Definition 10** (Sequentialization of a concurrent arena and game). *Consider a concurrent arena $\mathcal{C} = \langle A, B, Q, q_0, \mathsf{D}, \delta, \mathsf{dist}, \mathsf{K}, \mathsf{col} \rangle$ and an objective $W \in \mathsf{Borel}(\mathsf{K})$.*

- *The* sequential version *of $\mathcal{C}$ is the turn-based arena $\mathsf{Seq}(\mathcal{C}) = \langle A, B, V, q_0, \mathsf{D}_\mathsf{A} \uplus \mathsf{D}_\mathsf{B}, \delta_\mathcal{C}, \mathsf{dist}_\mathcal{C}, \mathsf{K}_\mathcal{C}, \mathsf{col}_\mathcal{C} \rangle$ where $V = V_\mathsf{A} \uplus V_\mathsf{B}$ with $V_\mathsf{A} = Q$ and $V_\mathsf{B} = Q \times A$, $\mathsf{D}_\mathsf{A} = V_\mathsf{B}$ and $\mathsf{D}_\mathsf{B} = \mathsf{D}$. Furthermore, for all $q \in V_\mathsf{A}$, $a \in A$ and $b \in B$, we have $\delta_\mathcal{C}(q, a, b) = (q, a) \in V_\mathsf{B} = \mathsf{D}_\mathsf{A}$ and $\mathsf{dist}_\mathcal{C}((q, a))[(q, a)] = 1$. In addition, for all $d \in \mathsf{D}$, we have $\mathsf{dist}_\mathcal{C}(d) = \mathsf{dist}(d)$ and for all $a' \in A$, $b \in B$, and $(q, a) \in V_\mathsf{B}$ we have $\delta_\mathcal{C}((q, a), a', b) = \delta(q, a, b) \in \mathsf{D} = \mathsf{D}_\mathsf{B}$. Finally, we have $\mathsf{K}_\mathcal{C} = \mathsf{K} \cup \{k_\mathcal{C}\}$ for some fresh color $k_\mathcal{C} \notin \mathsf{K}$ and $\mathsf{col}_\mathcal{C}(q, (q, a)) = k_\mathcal{C}$ if $q \in V_\mathsf{A}$ and $(q, a) \in V_\mathsf{B}$ and $\mathsf{col}_\mathcal{C}((q, a), q') = \mathsf{col}(q, q')$ if $(q, a) \in V_\mathsf{B}$ and $q' \in V_\mathsf{A}$. The function $\mathsf{col}_\mathcal{C}$ is defined arbitrarily on other pairs of states.*
- *The* sequential version *of the concurrent game $\langle \mathcal{C}, W \rangle$ is the turn-based game $\langle \mathsf{Seq}(\mathcal{C}), \mathsf{Seq}(W) \rangle$, where $\mathsf{Seq}(W) = (\phi_{\mathsf{K}_\mathcal{C}, \mathsf{K}})^{-1}[W]$ is the preimage of the winning set W by the projection function $\phi_{\mathsf{K}_\mathcal{C}, \mathsf{K}} : \mathsf{K}_\mathcal{C}^\uparrow \to \mathsf{K}^\uparrow$.*

In the above definition, one can notice that the states in V_A belong to Player A whereas states in V_B belong to Player B.

▶ **Example 11.** Sequentialization of an arena is a rather simple operation that we illustrate in Figure 8. Note that the initial concurrent arena (from Figure 4) has deterministic Nature (all probabilities that appear equal 1), and the sequential version also does. From q_0, Player A selects either the first row (top choice in the figure) or the second row (bottom choice in the figure), then Player B selects one of the options, i.e. one of the next states offered in the subset – this corresponds to choosing a column in the game form. The fresh color $k_\mathcal{C}$ appear after the choice of Player A, and the original colors appear after the choice of Player B.

One can notice here that in the original concurrent game and its sequential version, the value of the game for the players are different: in the turn-based game, from q_0, Player B has a strategy to ensure never seeing the color y (which induces a value of 0 for Player B) whereas it is not the case in the original game. As we will see along that paper, this is due to the fact that the local interaction at q_0 is not determined.

We make several remarks: paths in a concurrent arena and in its sequential version relate via a projection and, if W is Borel, so is $\mathsf{Seq}(W)$ as the continuous preimage of a Borel set.

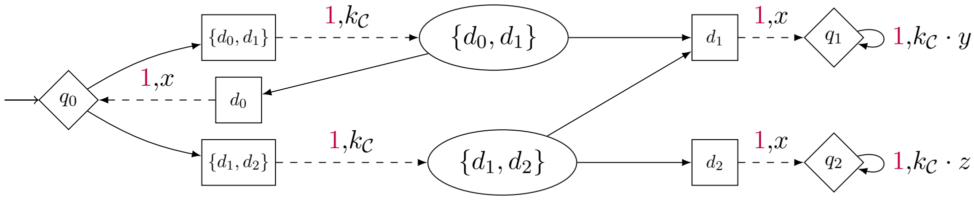

Figure 8 Sequentialization of the concurrent arena from Figure 4. Diamond-shaped nodes belong to Player A, ellipse-shaped ones belong to Player B and the rectangle-shaped are Nature states. On the edges, probabilities appear in purple and colors in black. The pairs in $Q \times A$ are represented as the corresponding set of states $\delta(q, a, B) \subseteq \mathcal{P}(\mathsf{D})$.

Main theorem. We now state the main result, and discuss it in the rest of Section 5.

▶ **Theorem 12.** *Consider a concurrent game $\langle \mathcal{C}, W \rangle$ and assume that it is locally determined. Then, it is (resp. finite-memory, resp. positionnaly) determined if and only if its sequential version $\langle \mathsf{Seq}(\mathcal{C}), \mathsf{Seq}(W) \rangle$ is (resp. finite-memory, resp. positionnaly) determined.*

We assume for the rest of the section that $\langle \mathcal{C}, W \rangle$ with $\mathcal{C} = \langle A, B, Q, q_0, \mathsf{D}, \delta, \mathsf{dist}, \mathsf{K}, \mathsf{col} \rangle$ is a concurrent graph game, and $\langle \mathsf{Seq}(\mathcal{C}), \mathsf{Seq}(W) \rangle$ with $\mathsf{Seq}(\mathcal{C}) = \langle A, B, V, q_0, \mathsf{D}_\mathsf{A} \uplus \mathsf{D}_\mathsf{B}, \delta_\mathcal{C}, \mathsf{dist}_\mathcal{C}, \mathsf{K}_\mathcal{C}, \mathsf{col}_\mathcal{C} \rangle$, is its sequential version.

The idea of the proof is to show that the same values are achieved by both players in the two games while preserving the memory which is used (memory skeletons have to be slightly adapted to take care of removing the new color $k_\mathcal{C}$ used in $\mathsf{Seq}(\mathcal{C})$). To do so:

- given a chromatic strategy s (for either of the players) in the concurrent game $\mathcal{C}$, build a sequentialized version $\mathsf{Seq}(\mathsf{s})$ (which is also a chromatic strategy) in $\mathsf{Seq}(\mathcal{C})$ that is at least as good in $\mathsf{Seq}(\mathcal{C})$ as s is in $\mathcal{C}$. For both players, this is not difficult to achieve since histories in $\mathsf{Seq}(\mathcal{C})$ give at least as much information for taking a decision as in $\mathcal{C}$; it is even the case that Player B has more information in $\mathsf{Seq}(\mathcal{C})$ since she plays second but she doesn't use it.
- given a chromatic strategy σ (for either of the players) in the sequential game $\mathsf{Seq}(\mathcal{C})$, build a parallelized version $\mathsf{Par}(\sigma)$ (which is also a chromatic strategy) in $\mathcal{C}$ that is at least as good in $\mathcal{C}$ as σ is in $\mathsf{Seq}(\mathcal{C})$. This is not difficult to prove (while preserving the same memory) for Player A. Indeed, she has the same information in both games on histories when she has to take a decision (removing $k_\mathcal{C}$'s). The case of Player B is very different since she plays second, hence has more information in the sequentialized version than in the original concurrent game. The next paragraph is dedicated to this case, highlighting the role of the local determinacy hypothesis.

Overall, we obtain that the values of the concurrent game $\langle \mathcal{C}, W \rangle$ and its sequential version $\langle \mathsf{Seq}(\mathcal{C}), \mathsf{Seq}(W) \rangle$ are equal for both players, which implies Theorem 12. Formally:

▶ **Theorem 13.** *If the game $\langle \mathcal{C}, W \rangle$ is locally determined, we have the following:*

- $\chi_\mathsf{A}^\mathcal{C}[W] = \chi_\mathsf{A}^{\mathsf{Seq}(\mathcal{C})}[\mathsf{Seq}(W)]$ *and* $\chi_\mathsf{B}^\mathcal{C}[W] = \chi_\mathsf{B}^{\mathsf{Seq}(\mathcal{C})}[\mathsf{Seq}(W)]$;
- *the game $\langle \mathcal{C}, W \rangle$ is limit-determined iff its sequential version $\langle \mathsf{Seq}(\mathcal{C}), \mathsf{Seq}(W) \rangle$ is;*
- *if a chromatic strategy s is optimal in $\langle \mathcal{C}, W \rangle$, so is $\mathsf{Seq}(s)$ is in $\langle \mathsf{Seq}(\mathcal{C}), \mathsf{Seq}(W) \rangle$;*
- *if a chromatic strategy σ is optimal in $\langle \mathsf{Seq}(\mathcal{C}), \mathsf{Seq}(W) \rangle$, so is $\mathsf{Par}(\sigma)$ is in $\langle \mathcal{C}, W \rangle$.*

Parallelization of the strategy of Player B. We now give arguments for the difficult case, the preservation of the value for Player B from $\mathsf{Seq}(\mathcal{C})$ back to $\mathcal{C}$.

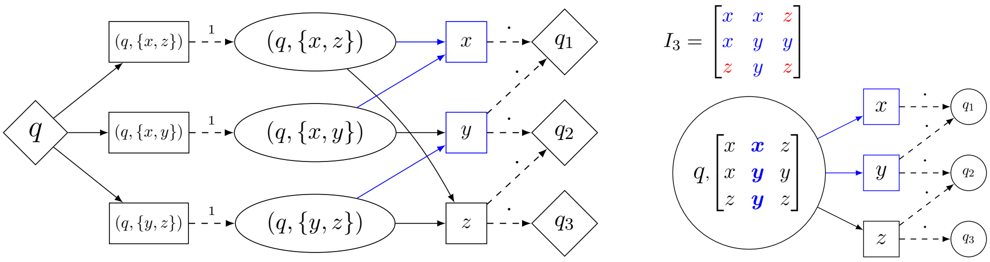

Figure 9 On the left-hand side, we have a portion of a turn-based graph arena, with all the states reachable in at most two steps from the state q. This turn-based arena corresponds to the sequentialization of the portion of the concurrent arena on the right-hand side with the local interaction in state q being I_3 from Figure 7. Out of q, Player A has three choices (corresponding to the three rows), hence the three outgoing edges; leading to three Nature states from which a specific state belonging to Player B is reached with probability 1. From each of these three states, Player B has two choices, leading to two out of the three x, y and z Nature states. A strategy for Player B is represented in blue arrows in the turn-based arena, with the Nature states reachable with that strategy represented in blue. It is done similarly in the local interaction I_3, with the state that is not reachable, i.e. z, in red. Finally, in the concurrent arena, the blue states are the Nature states reachable if Player B opts for the second column, which is the winning strategy for Player B in the win/lose game obtained from the game form I_3 if she has $\{x, y\}$ as winning set.

Consider a strategy σ for Player B in $\mathsf{Seq}(\mathcal{C})$. Such a strategy takes a finite sequence of colors in $\mathsf{K}_\mathcal{C}$ and the current vertex $(q, a) \in Q \times A$ to make the next decision, where a is the last action played by Player A. We assume that σ is implemented by a memory skeleton $\mathcal{M} = \langle M, m_{\text{init}}, \mu\rangle$ on $\mathsf{K}_\mathcal{C}$ and an action map $\lambda : M \times V_\mathsf{B} \to B$. The parallelization $\mathsf{Par}(\sigma)$ will be implemented by the memory skeleton $\mathsf{Par}(\mathcal{M})$ and the action map $\mathsf{Par}(\lambda)$, where $\mathsf{Par}(\mathcal{M}) = \langle M, m_{\text{init}}, \mathsf{Par}(\mu)\rangle$ only adds occurrences of $k_\mathcal{C}$: $\mathsf{Par}(\mu)(m, k) = \mu(\mu(m, k_\mathcal{C}), k)$. The parallelization $\mathsf{Par}(\lambda)\colon M \times Q \to B$ of the action map $\lambda\colon M \times V_\mathsf{B} \to B$ is more difficult to define since λ has more information than is supposed to have $\mathsf{Par}(\lambda)$.

Since our goal is to ensure that the value of the game does not worsen, we want the new strategy to ensure that the Nature states reachable in $\mathcal{C}$ with the parallel version of the action map are also reachable in $\mathsf{Seq}(\mathcal{C})$ with the original action maps: that way, every path that can be generated with some probability in the concurrent game could also be generated (up to projection) with the same probability in the turn-based game.

We fix a memory state $m \in M$ and a state $q \in V_\mathsf{A} = Q$ in $\mathsf{Seq}(\mathcal{C})$. We define $\mathsf{Rch}^\sigma_{m,q} = \{\delta(q, a, \lambda(\mu(m, k_\mathcal{C}), (q, a))) \mid a \in A\} \subseteq \mathsf{D}$ the set of Nature states that can be reached in two steps when applying strategy σ from memory state m and state q in $\mathsf{Seq}(\mathcal{C})$, taking into account all possible choices of Player A. The crux of the construction relies Lemma 14.

▶ **Lemma 14.** *If the local interaction $\mathcal{F}_q$ is determined, $\mathcal{W}_\mathsf{B}(\mathcal{F}_q, \mathsf{Rch}^\sigma_{m,q}) \neq \emptyset$.*

Proof. Consider an action $a \in A$. There exists $b \in B$ such that $\delta(q, a, b) \in \mathsf{Rch}^\sigma_{m,q}$. Since this is true for all $a \in A$, it implies that Player A has no strategy to avoid the set $\mathsf{Rch}^\sigma_{m,q}$ in the game form $\mathcal{F}_q$, i.e. she has no winning strategy in the win/lose game $\langle \mathcal{F}_q, Q \setminus \mathsf{Rch}^\sigma_{m,q}\rangle$. In other words, $\mathcal{W}_\mathsf{A}(\mathcal{F}_q, Q \setminus \mathsf{Rch}^\sigma_{m,q}) = \emptyset$, which implies $\mathcal{W}_\mathsf{B}(\mathcal{F}_q, \mathsf{Rch}^\sigma_{m,q}) \neq \emptyset$ by determinacy of $\mathcal{F}_q$: Player B has a winning strategy in this game. ◀

The parallelization of the action map λ for Player B follows: any winning action in $\mathcal{W}_\mathsf{B}(\mathcal{F}_q, \mathsf{Rch}^\sigma_{m,q})$ will ensure that the strategy σ is correctly mimicked by $\mathsf{Par}(\sigma)$.

▶ **Example 15.** We illustrate the definition on the example on Figure 9. We keep the notations used before. We consider Player B strategy depicted in the left-hand side of Figure 9 from the state q and an arbitrary state of the memory m omitted on the figure. For each possible choice of Player A (which corresponds to the rows of the local interaction I_3), Player B reacts with his strategy and either x or y is reached (in blue on the left figure). Specifically, writing m' for $\mu(m, k_{\mathcal{C}})$, we have $\delta(q, a_1, \lambda(m', (q, a_1))) = x$, $\delta(q, a_2, \lambda(m', (q, a_2))) = x$ and $\delta(q, a_3, \lambda(m', (q, a_3))) = y$, where a_i represents the action for player A for the i-th row (similarly, b_i represents the action for Player B for the i-th column). Then, we must define the action for Player B to play in the concurrent game at state q, that is $\mathsf{Par}(\lambda)(m, q) \in B$, so that only the states x and y can be reached. To choose $\mathsf{Par}(\lambda)(m, q)$ we consider the local interaction $\mathcal{F}_q = I_3$. We know that for each action of Player A, there is one for Player B to reach the set $\{x, y\}$ (it is given by the strategy depicted in the turn-based arena). It follows that Player A has no winning strategy in the win/lose game I_3 with $\{x, y\}$ as winning set for Player B. Since the local interaction I_3 is determined, Player B has a winning strategy that ensures reaching a state in $\{x, y\}$. By opting for this strategy, which corresponds to choosing the second column in the local interaction, it follows that the states reachable in the concurrent arena from q are $\{x, y\}$ (depicted in blue). Hence, we set $\mathsf{Par}(\lambda)(m, q) = b_2$.

6 Applications

6.1 Games with deterministic Nature (i.e. without Nature)

We consider the special case of games $\mathcal{C}$ with a deterministic Nature (i.e. all probabilities are equal to 1: for all Nature states $d \in \mathsf{D}$, there is a state $q \in Q$ such that $\mathsf{dist}(d)(q) = 1$), as, on turn-based games, this setting enjoys more determinacy results than the stochastic one. In such a setting, it is relevant to consider infinite paths compatible with a chromatic strategy, not only their probabilities. A winning strategy is a strategy whose set of compatible paths is included in the winning set – U_W for Player A and $Q^\omega \setminus U_W$ for Player B.

A deterministic concurrent game $\langle \mathcal{C}, W \rangle$ is (resp. positionally, resp. finite-memory) *exactly-determined* if either of the player has a (resp. positional, resp. finite-memory) winning strategy. In the literature this notion is sometimes called “sure winning”, while winning with probability 1 is called “almost-sure winning”. However, in deterministic concurrent games with chromatic strategies (recall that they are deterministic strategies), we have an equivalence between the two notions. This immediately gives us:

▶ **Corollary 16.** *A deterministic CG-DLI is (resp. positionally, resp. finite-memory) exactly-determined if and only if it is (resp. positionally, resp. finite-memory) determined.*

In the following, the determinacy of a deterministic game will refer to exact-determinacy. We consider the transfer of determinacy results from turn-based games to CG-DLI.

Borel determinacy. We apply Theorem 12 and Corollary 16 to prove the Borel determinacy of CG-DLI. By rephrasing the famous result of Borel determinacy in our formalism, we have that a deterministic turn-based graph arena $\mathcal{C}$ is determined for all Borel winning set $W \subseteq \mathsf{Borel}(\mathsf{K})$. Note that this theorem is not directly given by the results proved by Martin in [14, 15, 16]. To obtain this theorem, we additionally need to apply a result from [20] since a strategy depends on color history instead of state history. We use this result to prove the determinacy of CG-DLI.

▶ **Theorem 17.** *For all Borel winning set W, for all locally determined deterministic concurrent graph arena $\mathcal{C}$, the concurrent game $\langle \mathcal{C}, W \rangle$ is determined. Conversely, for all non-trivial Borel winning set $\emptyset \subsetneq W \subsetneq \mathsf{K}^\omega$, for all non-determined game form $\mathcal{F}$, there exists a deterministic concurrent arena $\mathcal{C}$ with only $\mathcal{F}$ as a local interaction that is not determined such that the game $\langle \mathcal{C}, W \rangle$ is not determined.*

Finite-memory determinacy. The next application only applies to finite arenas. In [3], the authors proved an equivalence between the shape of a winning set and the existence of winning strategies that can be implemented with a given memory skeleton[2] $\mathcal{M}$. They defined the properties of $\mathcal{M}$-selectivity and $\mathcal{M}$-monotony and proved that for $\mathcal{M}$ a memory skeleton and $W \subseteq \mathsf{K}^\omega$, we have that W and $\mathsf{K}^\omega \setminus W$ are $\mathcal{M}$-monotone and $\mathcal{M}$-selective is equivalent to every finite deterministic turn-based game with W as winning set is determined with winning strategies for both players that can be found among strategies implemented with memory skeleton $\mathcal{M}$.

Let $\langle \mathcal{C}, W \rangle$ be a deterministic concurrent game, $\langle \mathsf{Seq}(\mathcal{C}), \mathsf{Seq}(W) \rangle$ be its sequential version, and $\mathcal{M}$ a memory skeleton on K. In fact, W is $\mathcal{M}$-monotone and $\mathcal{M}$-selective if and only if $\mathsf{Seq}(W)$ is $\mathsf{Seq}(\mathcal{M})$-monotone and $\mathsf{Seq}(\mathcal{M})$-selective. The proof of this fact, longer than the other applications, requires establishing algebraic properties of the projection function $\phi_{\mathsf{K}_\mathcal{C},\mathsf{K}} : \mathsf{K}_\mathcal{C}^\uparrow \to \mathsf{K}^\uparrow$. In turn, finite deterministic CG-DLI ensure the following theorem:

▶ **Theorem 18.** *Let $\mathcal{M}$ be a memory skeleton and $W \subseteq \mathsf{K}^\omega$. The following two assertions are equivalent:*

1. *every finite deterministic locally determined concurrent game $\langle \mathcal{C}, W \rangle$ with finite action sets is determined with winning strategies for both players that can be found among strategies implemented with memory skeleton $\mathcal{M}$;*
2. *W and $\mathsf{K}^\omega \setminus W$ are $\mathcal{M}$-monotone and $\mathcal{M}$-selective.*

As for Borel determinacy, local determinacy is somehow a necessary condition as a one-shot reachability game, with a non-determined initial local interaction, may not be determined (see Figures 5, 6). In fact, this theorem can written as a more involved equivalence.

6.2 Stochastic Games (i.e. with Nature)

There are fewer determinacy results on stochastic games, especially with deterministic strategies. Let us translate some of them into locally determined concurrent games. We consider parity objectives and the more general case of tail-objectives (a.k.a. prefix-independent).

Parity Objectives. As already mentioned in Section 3, parity objectives are defined as follows. For a set of colors $\mathsf{K} = [\![m, n]\!]$ for some $m, n \in \mathbb{N}$, a parity objective on K is the winning set $W = \{\rho \in \mathsf{K}^\omega \mid \max(\mathsf{n}_\infty(\rho)) \text{ is even } \}$ where $\mathsf{n}_\infty(\rho)$ is the set of colors seen infinitely often in ρ. A result from [4, 23] gives us that any finite turn-based parity game is positionally determinedThis result can be directly transferred to locally determined concurrent games thanks to Theorem 12. Note that, as in the two previous cases, the local determinacy assumption is somewhat necessary.

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[2] In fact, they looked at the existence of Nash equilibria with antagonistic preference relations instead of winning sets. However, a winning set $W \subseteq \mathsf{K}^\omega$ can be directly translated into an equivalent preference relation $\prec_W \subseteq \mathsf{K}^\omega \times \mathsf{K}^\omega$ by $\rho \prec_W \rho' \Leftrightarrow \rho \notin W \wedge \rho' \in W$. In the following we will refer to the preference relation $\prec_W$ when mentionning the winning set W.

▶ **Theorem 19.** *Consider a (stochastic) locally determined finite concurrent graph arena $\mathcal{C}$ with color set $\mathsf{K} = [\![m, n]\!]$ for some $m, n \in \mathbb{N}$. For all parity objective $W \in \mathsf{Borel}(\mathsf{K})$ on K, the concurrent game $\langle \mathcal{C}, W \rangle$ is positionally determined.*

Tail Objectives. We consider more general objectives than the parity objectives. In particular, positional determinacy does not hold in the general case for these objectives (consider, for instance, the Muller objectives). A tail objective is a winning set that is closed by adding and removing finite prefixes, that is, for a set of colors K, a winning set $W \in \mathsf{Borel}(\mathsf{K})$ is a tail-objective if, for all $\rho \in \mathsf{K}^\omega$ and $\pi \in \mathsf{K}^*$, we have $\rho \in W \Leftrightarrow \pi \cdot \rho \in W$. In particular, a parity objective is a tail objective. In fact, we have that every finite turn-based game that is limit-determined with value 0 or 1 is determined.

This result can be directly transferred to locally determined concurrent games. As usual, the local determinacy is a somewhat necessary condition.

▶ **Theorem 20.** *Consider a (stochastic) locally determined finite concurrent graph arena $\mathcal{C}$ with finite action sets. Then, for all Borel winning set $W \subseteq \mathsf{Borel}(\mathsf{K})$ that is a tail objective, on K, if $\chi_\mathsf{A}^\mathcal{C}[W] = 1$ or $\chi_\mathsf{B}^\mathcal{C}[W] = 0$, then the game is determined.*

7 Future Work

Several applications from Section 6 can be generalized in the setting of non-antagonistic preferences instead of the win/lose setting that was used in this article. Apart from formatting bureaucracy, most of these generalizations are automatic corollaries of the combination of this paper's results and [18]. The latter is a general transfer result, from determinacy results in the win/lose setting, into existence of Nash equilibria in the setting of non-antagonistic preferences. We intend to state and detail these generalizations in the journal version of this paper.

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