On the Expressive Equivalence of TPTL in the Pointwise and Continuous Semantics

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Abstract

We consider a first-order logic with linear constraints interpreted in a pointwise and continuous manner over timed words. We show that the two interpretations of this logic coincide in terms of expressiveness, via an effective transformation of sentences from one logic to the other. As a consequence it follows that the pointwise and continuous semantics of the logic TPTL with the since operator also coincide. Along the way we exhibit a useful normal form for sentences in these logics.

2012 ACM Subject Classification Theory of computation → Modal and temporal logics

Keywords and phrases Real-Time Logics, First-Order Logics

Digital Object Identifier 10.4230/LIPIcs.FSTTCS.2021.45

1 Introduction

Several real-time logics proposed in the literature have been interpreted over timed behaviours in two natural ways which have come to be known as the "pointwise" and "continuous" interpretations. In the pointwise semantics, formulas may be asserted only at points where an action occurs (the so-called "action points"), while in the continuous semantics formulas may be asserted at arbitrary time points. To illustrate these semantics, consider the popular timed temporal logic Metric Temporal Logic (MTL) [10, 1, 12], which extends the U operator of classical LTL with an interval index, to allow formulas of the form $\theta U_I \eta$ which says that, with respect to the current time point, there is a future time point where η is satisfied and which lies at a distance that falls within the interval I, and at all time points in between θ is satisfied. Consider a timed word σ depicted in Fig. 1 below, in which the first action is an a at time 2, followed subsequently by only b's. Then the MTL formula $\Diamond(\Diamond_{[1,1]}a)$ is satisfied in σ in the continuous semantics, but not in the pointwise semantics since there is no action point at time 1.

The Timed Temporal Logic (TPTL) of Alur and Henzinger [2, 3] is a well-known timed temporal logic for specifying real-time behaviors. The logic is interpreted over timed words and extends classical LTL with the "freeze" quantifier $x.\theta$ which binds x to the value of the current time point, along with the ability to constrain these time points using linear constraints of the form $x \sim y + c$. For example the formula $x.(\Diamond y.(a \wedge y = x + 2))$ says that with respect to the current time point, an action a occurs exactly two time units later. Then the TPTL formula $\Diamond x.\Diamond y.(a \wedge y = x + 1)$ is satisfied in σ in the continuous semantics, but not in the pointwise semantics, since there is no action point at time 1. It is not difficult to

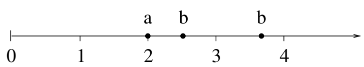


Figure 1 Timed word σ.

41st IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2021).
Editors: Mikołaj Bojańczyk and Chandra Chekuri; Article No. 45; pp. 45:1–45:21
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

see that for a typical timed temporal logic the continuous semantics is at least as expressive as the pointwise one, since one can ask for a time point to be an action point by asserting $\bigvee_{a\in\Sigma} a$ at each quantified time point.

There have been several results in the literature which show that for the logic MTL and its variants the continuous semantics is in fact strictly more expressive than the pointwise one. In particular, the logics MTL over infinite words [4, 5] and finite words [13]; MTL_S (MTL with the "since" operator S) and MTL_{S_I} (MTL with the S_I operator), over both infinite and finite words [7]; are all strictly more expressive in the continuous semantics than their pointwise counterparts. In addition, Ho et al [9] show the strict inclusion of MTL in a first-order logic $FO(<, +1)$ over finite words in the pointwise semantics, in contrast to their equivalence in the continuous semantics [11].

In this paper we show, somewhat surprisingly, that the logic TPTL_S (TPTL with the "since" operator) has the same expressive power in both the pointwise and continuous semantics. We do this by considering a natural first-order logic $\mathrm{FO}(<, +\mathbb{Q})$ interpreted over timed words, which is similar in flavour to TPTL_S. The logic allows atomic predicates of the form $a(x)$ which says that an a-event occurs at time point x, and constraints of the form $x < y + c$. The interpretation of the quantifier $\exists x$ depends on the pointwise or continuous semantics: in the pointwise it is interpreted as "there exists an action point x", while in the continuous semantics it is interpreted as "there exists a time point x." The main technical result in this paper is that the expressiveness of $\mathrm{FO}(<, +\mathbb{Q})$ in the pointwise and continuous interpretations coincide.

The main proof idea is to show that we can go from an arbitrary sentence in the continuous version of $\mathrm{FO}(<, +\mathbb{Q})$ to an equivalent sentence in $\mathrm{FO}(<, +\mathbb{Q})$ which uses only "active" quantifiers, where each $\exists x$ is qualified by an assertion that x is an action point. A sentence in which all quantifiers are active, is clearly equivalent to a pointwise formula.

The technique in this paper builds on the work reported in a preprint [6] by giving a more transparent argument for a key step of the proof via Thm. 2. In the next few sections of this paper we focus on the result for $\mathrm{FO}(<, +\mathbb{Q})$, and turn to its application to TPTL in Sec. 7.

2 Preliminaries

We begin with some preliminary definitions. Let $\mathbb{R}_{\geq 0}$ denote the set of non-negative real numbers, $\mathbb{Q}$ the set of rational numbers, and $\mathbb{N}$ the set of non-negative integers. We use the standard notation to represent intervals, which are convex subsets of $\mathbb{R}$. For example $[1, \infty)$ denotes the set $\{t \in \mathbb{R} \mid 1 \leq t\}$.

For an alphabet A we denote by A^ω the set of infinite words over A. Let Σ be a finite alphabet of actions, which we fix for the rest of this paper. An (infinite) timed word σ over Σ is an element of $(\Sigma \times \mathbb{R}_{\geq 0})^\omega$ of the form $(a_0, t_0)(a_1, t_1) \cdots$, satisfying the conditions that: for each $i \in \mathbb{N}$, $t_i < t_{i+1}$ (*monotonicity*), and for each $t \in \mathbb{R}_{\geq 0}$ there exists an $i \in \mathbb{N}$ such that $t < t_i$ (*progressiveness*). For convenience, we will also assume in this paper that $t_0 = 0$, so that the timed word begins with an action at time 0. We will sometimes represent the timed word σ above as a pair (α, τ), where $\alpha = a_0 a_1 \cdots$ and $\tau = t_0 t_1 \cdots$. Thus $\alpha(i)$ and $\tau(i)$ denote the the action and the time stamp respectively, in σ at position i. We write $T\Sigma^\omega$ to denote the set of all timed words over Σ.

We now introduce the linear constraints we use in this paper, and some notation for manipulating them. We assume a supply of variables $Var = \{x, y, \ldots\}$ which we will use in constraints as well as later in our logics. We use restricted linear constraints of the form $x \sim y + c$ or $x \sim c$, where x and y are variables in Var, $\sim$ is one of the relations $\{<, \leq, =, \geq, >\}$, and c is in $\mathbb{Q}$. We call these constraints *simple constraints*. In general, by (an unqualified) "constraint" we will mean a conjunction of simple constraints.

(a)	(b)	(c)	(d)
$0 \le x \le 1$ $x+1 \le y \le x+1.2$ $0 \le y$	$0 \le x \le 1$ $y-1.2 \le x \le y-1$ $0 \le y$	$0 \le y-1$ $y-1.2 \le 1$ $y-1.2 \le y-1$ $0 \le y$	$1 \le y \le 2.2$

Figure 2 Illustrating steps of the Fourier-Motzkin elimination method.

An *assignment* for variables is a map $\mathbb{I} : \mathit{Var} \to \mathbb{R}_{\geq 0}$. For $t \in \mathbb{R}_{\geq 0}$ and $x \in \mathit{Var}$ we will use $\mathbb{I}[t/x]$ to represent the assignment which sends x to t, and agrees with $\mathbb{I}$ on all other variables. When we are interested in a finite set of variables $\{x_1, \dots, x_n\}$ we will write $[t_1/x_1, \dots, t_n/x_n]$ to represent an assignment that maps each x_i to t_i. For an assignment $\mathbb{I}$ and a constraint δ, we write $\mathbb{I} \models \delta$ to mean that the constraint δ is satisfied in the assignment $\mathbb{I}$, and defined in the expected way.

As a final piece of notation, we will make use of the well-known Fourier-Motzkin method for eliminating variables from constraints. Given a conjunction π of simple constraints, some of which contain a variable x, the technique gives us a conjunction π' of simple constriants not containing x, such that the formula $\exists x \pi$ is logically equivalent to π' (assuming a standard first-order logic interpreted over rationals or reals). When we are interested in the domain of $\mathbb{R}_{\geq 0}$ (like in this paper), we assume that π implicitly contains the constraint $z \geq 0$ for each variable z in π. As an illustration of the method, consider the conjunction π of the constraints in Fig. 2(a). To eliminate x from π, we first rewrite the constraints involving x as lower and upper bounds on x, as shown in the first two constraints in Fig. 2(b), and carry forward the constraints not involving x (like the third one). Next we relate each lower bound of x to each upper bound of x, as shown in the first three constraints of Fig. 2(c), while carrying forward the constraints not involving x. Finally, we simplify the constraints by dropping looser bounds and removing redundant constraints like $0 \leq 1$, to obtain the constraint π' in Fig. 2(d). The constraint π' can be seen to be logically equivalent to $\exists x \pi$.

We will use the notation $\mathrm{FME}_x(\pi)$ to refer to the constraint π'. We refer the reader to [14] for the details of this technique.

3 The $\mathrm{FO}(<, +\mathbb{Q})$ logic

We now define our first order logic with simple constraints $\mathrm{FO}(<, +\mathbb{Q})$, which is interpreted over timed words over the alphabet Σ. The formulas of $\mathrm{FO}(<, +\mathbb{Q})$ are given by:

$$\varphi ::= a(x) \mid g \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x \varphi,$$

where $a \in \Sigma$, $x \in \mathit{Var}$, and g is a simple constraint.

We first define the *continuous* semantics for $\mathrm{FO}(<, +\mathbb{Q})$. Let φ be a formula in $\mathrm{FO}(<, +\mathbb{Q})$. Let $\sigma = (\alpha, \tau)$ be a timed word over Σ, and let $\mathbb{I}$ be an assignment for variables. Then the satisfaction relation $\sigma, \mathbb{I} \models_c \varphi$ (read "σ satisfies φ with the assignment $\mathbb{I}$ in the continuous semantics") is inductively defined as:

$$\begin{array}{lll}
\sigma, \mathbb{I} \models_c a(x) & \text{iff} & \exists i : \tau(i) = \mathbb{I}(x) \text{ and } \alpha(i) = a \\
\sigma, \mathbb{I} \models_c g & \text{iff} & \mathbb{I} \models g \\
\sigma, \mathbb{I} \models_c \neg\nu & \text{iff} & \sigma, \mathbb{I} \not\models_c \nu \\
\sigma, \mathbb{I} \models_c \nu \vee \psi & \text{iff} & \sigma, \mathbb{I} \models_c \nu \text{ or } \sigma, \mathbb{I} \models_c \psi \\
\sigma, \mathbb{I} \models_c \exists x \nu & \text{iff} & \exists t \in \mathbb{R}_{\geq 0} \text{ such that } \sigma, \mathbb{I}[t/x] \models_c \nu.
\end{array}$$

The derived connectives $\wedge$, $\supset$ (implies), $\forall$, etc are defined in the standard way. A variable x is said to occur *free* in a formula φ if there is an occurrence of x that is not within the scope of any $\exists x$ quantifier in φ. A *sentence* is a formula in which there are no free occurrences of variables. The satisfaction of a sentence in a timed word is independent of an assignment for variables. The timed language defined by an $\mathrm{FO}(<, +\mathbb{Q})$ sentence φ in the continuous semantics is given by $L^c(\varphi) = \{\sigma \in T\Sigma^\omega \mid \sigma \models_c \varphi\}$.

We can similarly define the *pointwise* semantics of the logic $\mathrm{FO}(<, +\mathbb{Q})$, where the quantification is over action points in the timed word. The satisfaction relation $\sigma, \mathbb{I} \models_{pw} \varphi$ is defined as above, except for the $\exists$ clause which is interpreted as follows:

$$\sigma, \mathbb{I} \models_{pw} \exists x \nu \quad \text{iff} \quad \exists t \in \mathbb{R}_{\geq 0} \text{ such that } t = \tau(i) \text{ for some } i \in \mathbb{N}, \text{ and } \sigma, \mathbb{I}[t/x] \models_{pw} \nu.$$

The timed language defined by a sentence φ in the pointwise semantics is given by $L^{pw}(\varphi) = \{\sigma \in T\Sigma^\omega \mid \sigma \models_{pw} \varphi\}$.

The formulas of the logic $\mathrm{FO}(<, +\mathbb{Q})$ can be seen to be essentially that of a first-order logic over the signature $(0, \{+c\}_{c\in\mathbb{Q}}, <, \{a\}_{a\in\Sigma})$, where each $+c$ is a function that adds the rational c to its argument, and each $a \in \Sigma$ is a unary predicate. The logic is interpreted over timed words in the expected way, with the domain being $\mathbb{R}_{\geq 0}$ in the continuous interpretation, and the set of action points in the pointwise interpretation. In the sequel we will write $\mathrm{FO}^c(<, +\mathbb{Q})$ to denote the logic with the continuous interpretation, and similarly $\mathrm{FO}^{pw}(<, +\mathbb{Q})$ to denote the pointwise interpretation.

4 A normal form for FO sentences

In this section we exhibit a normal form for $\mathrm{FO}^c(<, +\mathbb{Q})$ sentences which will be useful in our proofs. We begin with a normal form for formulas of the form $\exists x \varphi$. An $\mathrm{FO}^c(<, +\mathbb{Q})$ formula is said to be in $\exists$-*normal form* if it is of the form $\exists x(a(x) \wedge \pi(x) \wedge \nu)$, where $a \in \Sigma$, $\pi(x)$ is a conjunction of simple constraints each containing x, and ν is a conjunction of formulas of the form ψ or $\neg\psi$, where each ψ is again in $\exists$-normal form. In addition, we allow any of the components $a(x)$ and ν to be absent. We say a formula is in *negated* $\exists$-normal form if it is the negation of a formula in $\exists$-normal form. Fig. 3 depicts a sentence which is a boolean combination of $\exists$-normal form sentences.

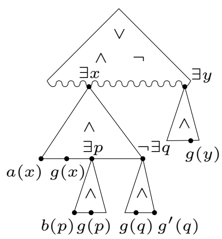


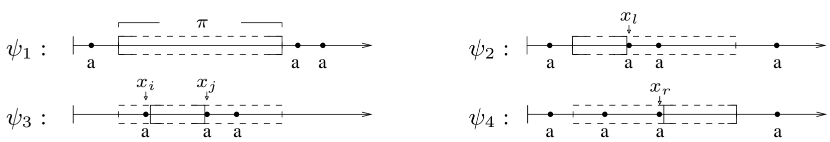
Figure 3 Boolean combination of sentences in $\exists$-normal form.

▶ **Theorem 1.** *Any* $\mathrm{FO}^c(<, +\mathbb{Q})$ *sentence can be equivalently expressed as a boolean combination of sentences in* $\exists$*-normal form.*

Proof. Let φ be an $\mathrm{FO}^c(<, +\mathbb{Q})$ sentence. Since φ is a sentence it must be a boolean combination of sentences of the form $\exists x \varphi'$. We transform φ into an equivalent sentence which is a boolean combination of sentences in $\exists$-normal form, by repeatedly transforming the formula tree of φ as follows:

1. In every subtree rooted at a $\exists$-node, in the formula tree of φ, push every $\neg$ operator downwards over $\vee$, $\wedge$, and all the way through g nodes, till it reaches a $\exists$-node or an a (action) node. After this step, the subtree below every $\exists$ node contains only conjunctions and disjunctions of a, $\neg a$, $\exists$, $\neg\exists$, and g nodes.
2. For convenience, in the next couple of steps, we will consider $\neg\exists$ as a single composite node in the formula tree. Pull all the $\vee$'s upwards in the resulting formula tree for φ, using the following identities: $\nu_1 \wedge (\nu_2 \vee \nu_3) \equiv (\nu_1 \wedge \nu_2) \vee (\nu_1 \wedge \nu_3)$, $\exists x(\nu_1 \vee \nu_2) \equiv (\exists x\nu_1) \vee (\exists x\nu_2)$ and $\neg\exists x(\nu_1 \vee \nu_2) \equiv (\neg\exists x\nu_1) \wedge (\neg\exists x\nu_2)$. It is not difficult to see that using these identities we obtain a formula φ' in which each $\exists$-node or $\neg\exists$-node contains only conjunctions of a, $\neg a$, $\exists$, $\neg\exists$, and g nodes.
3. In this step we pull up from a subtree rooted at an $\exists x$ node, all nodes which are independent of x, namely nodes of the form $b(y)$, $\neg b(y)$ (with $y \neq x$), and g where g does not contain x. This is done by recursively applying following equivalences starting from the lower most $\exists x$ or $\neg\exists x$ nodes: $\exists x(b(y) \wedge \nu) \equiv b(y) \wedge \exists x(\nu)$ and $\neg\exists x(b(y) \wedge \nu) \equiv \neg b(y) \vee \neg\exists x(\nu)$. We can use similar equivalences for $\neg b(y)$ and g to pull them up the tree. Finally, we move all the newly generated $\vee$'s up the tree using Step 2.
   After this step, the subtrees rooted at each $\exists x$ node is a conjunction of $a(x)$, $\neg a(x)$, $\exists$, $\neg\exists$ and $g(x)$ nodes.
4. We now update the formula tree with the following equivalences: $a(x) \wedge b(x) \equiv \perp$ and $a(x) \wedge \neg b(x) \equiv a(x)$, where $a, b \in \Sigma$ with $a \neq b$. After this step, the only action-related nodes in a subtree rooted at a $\exists x$ node are a single action node $a(x)$ or a conjunction of negation of actions of the form $\bigwedge_{a\in X} \neg a(x)$ for some $X \subseteq \Sigma$.
5. We can now replace formulas of the form $\bigwedge_{a\in A} \neg a(x)$ by a disjunction of formulas which contain at most one action, as described below. We then pull up the newly generated $\vee$ nodes up the tree using Step 2. After this step, the subtree rooted at every $\exists x$ node contains only conjunctions of $a(x)$, $\exists$, $\neg\exists$ and $g(x)$ nodes. We can collect the $g(x)$ nodes together to get a single conjunction of constraints $\pi(x)$. Thus finally each subtree rooted at $\exists$ node is in $\exists$-normal form.

To see how we can replace formulas of the form $\bigwedge_{a\in A} \neg a(x)$ by a disjunction of formulas in $\exists$-normal form, consider a formula ψ of the form $\exists x(\bigwedge_{a\in A} \neg a(x) \wedge \pi(x) \wedge \nu)$. Let $A(x)$ be shorthand for the formula $\bigvee_{a\in A} a(x)$. Then, $\psi = \exists x(\neg A(x) \wedge \pi(x) \wedge \nu)$. We claim that $\varphi \equiv \psi_1 \vee \psi_2 \vee \psi_3 \vee \psi_4$ with each ψ_i defined as follows. The figure below illustrates these cases. We view the constraint $\pi(x)$ as an interval determined by the values assigned to the variables other than x.



If an A-action does not occur anywhere in the interval $\pi(x)$, then φ is satisfied if ν is satisfied for any x in $\pi(x)$:

$$\psi_1 = \quad \neg\exists x(A(x) \wedge \pi(x)) \wedge \exists x(\pi(x) \ \wedge \ \nu)$$

If there are one or more actions $A(x)$ in $\pi(x)$ then φ is satisfied iff ν is satisfied before the first occurrence of $A(x)$, or between any two consecutive occurrences of $A(x)$, or after the last occurrence of $A(x)$, in $\pi(x)$. These three cases are formulated as follows:

$$\begin{aligned}
\psi_2 = {} & \exists x_l(A(x_l) \wedge \pi(x_l) \wedge \neg\exists x'(A(x') \wedge \pi(x') \wedge x' < x_l) \wedge \exists x(\pi(x) \wedge x < x_l \ \wedge \ \nu)) \\
\psi_3 = {} & \exists x_i(A(x_i) \wedge \pi(x_i) \wedge \exists x_j(A(x_j) \wedge \pi(x_j) \wedge \neg\exists x'(A(x') \wedge \pi(x') \wedge x_i < x' < x_j) \\
& \qquad \wedge \exists x(\pi(x) \wedge x_i < x \wedge x < x_j \ \wedge \ \nu))) \\
\psi_4 = {} & \exists x_r(A(x_r) \wedge \pi(x_r) \wedge \neg\exists x'(A(x') \wedge \pi(x') \wedge x_r < x') \wedge \exists x(\pi(x) \wedge x_r < x \ \wedge \ \nu))
\end{aligned}$$

This completes the proof of the normal form transformation. ◀

5 Equivalence of FO^c and FO^{pw} semantics

In this section our aim is to show that the logics $\mathrm{FO}^{pw}(<, +\mathbb{Q})$ and $\mathrm{FO}^c(<, +\mathbb{Q})$ are expressively equivalent. It is easy to translate an $\mathrm{FO}^{pw}(<, +\mathbb{Q})$ sentence φ to an equivalent $\mathrm{FO}^c(<, +\mathbb{Q})$ sentence by simply replacing every $\exists x\varphi'$ subformula, by $\exists x(\bigvee_{a\in\Sigma} a(x) \wedge \varphi'')$, where φ'' is obtained by similarly replacing $\exists$-subformulas in φ'.

In the converse direction, let us call an $\mathrm{FO}^c(<, +\mathbb{Q})$ formula φ *actively quantified* (or simply *active*) if every $\exists$-subformula is of the form $\exists x(a(x) \wedge \varphi')$ for some action $a \in \Sigma$ and formula φ'. Then, an active $\mathrm{FO}^c(<, +\mathbb{Q})$ formula clearly defines the same language of timed words, regardless of the semantics being pointwise or continuous. Hence, our aim in the rest of this section is to show how we can go from an arbitrary formula in $\mathrm{FO}^c(<, +\mathbb{Q})$ to an equivalent active formula.

5.1 Proof Idea

A formula in the continuous semantics has the obvious advantage of being able to associate any value in $\mathbb{R}_{\geq 0}$ to its variables, whereas an actively quantified variable can refer only to the action points in a timed word. Consider the sentence below where x is passively quantified:

$$\exists x(0 \leq x \wedge x \leq 1 \wedge \exists y(a(y) \wedge x + 1 \leq y \wedge y \leq x + 1.2)). \tag{1}$$

In the continuous semantics this is essentially asking for an a action sometime in the interval $[1, 2, 2]$. However, if we interpret this sentence in the pointwise semantics we get a strictly stronger requirement of there being an action point in the interval $[0, 1]$ from which we have an a-action at a distance of 1 to 1.2. In our approach we transform the given sentence to an equivalent active sentence (all in the continuous semantics), which we can do as follows. The given sentence is equivalent to the sentence (2) below by simple logical manipulation. Then we apply Fourier-Motzkin elimination in the $\exists x$ part of (2) to get the sentence (3), which is now an equivalent active formula.

$$\exists y(a(y) \wedge \exists x(0 \leq x \wedge x \leq 1 \wedge x + 1 \leq y \wedge y \leq x + 1.2)) \tag{2}$$

$$\exists y(a(y) \wedge 1 \leq y \wedge y \leq 2.2) \tag{3}$$

As another example, consider the language of all timed words over a and b, where for every b in the interval [1,2], there is an a in [0,1] exactly one time unit earlier. This can be written easily in FO^c as:

$$\neg\exists x((\neg a(x) \wedge 0 \leq x \wedge x \leq 1 \wedge \exists y(b(y) \wedge y = x + 1)). \tag{4}$$

But if we interpret this sentence in the pointwise semantics it does not describe the same property. The given sentence is not in $\exists$-normal form and the normalization yields a disjunction of four formulas $\psi_1, \psi_2, \psi_3, \psi_4$, where x is the only variable which is passively

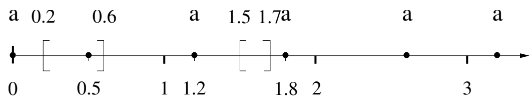


Figure 4 Timed word σ satisfying formula (5).

quantified. If we can eliminate x from ψ_1, ψ_2, ψ_3, and ψ_4 without introducing any new passively quantified variables, the disjunction of these actively quantified formulas recognizes the required language. The subformula involving x in each ψ_i looks like $\exists x(\pi(x) \wedge \exists y(b(y) \wedge y = x+1))$. This can be equivalently written as $\exists y(b(y) \wedge \exists x(\pi(x) \wedge y = x+1))$. We can now use Fourier-Motzkin elimination to eliminate x from $\pi(x) \wedge y = x+1$ to obtain a constraint $\pi'(y)$ on y. The above formula can now be expressed equivalently as $\exists y(b(y) \wedge \pi'(y))$, thereby eliminating the passively quantified variable x.

As a final example, consider the following modified version of formula (2):

$$\exists x(0 \leq x \wedge x \leq 1 \wedge \neg \exists y(a(y) \wedge x+1 \leq y \wedge y \leq x+1.2)). \tag{5}$$

For the sake of simplicity, let us consider a scenario where a is the only action in Σ. The above formula is true iff there is a point x in $[0,1]$ such that there is no action point a in the interval $[x+1, x+1.2]$. To eliminate the passively quantified variable x from this formula, we will consider the interval $\pi_1 = w_1 < x+1 < w_2 \wedge w_3 < x+1.2 < w_4$ and find an assignment to the variables w_1–w_4 such that for every point x which lies in the intersection of the intervals $[0,1]$ and π_1, there is no action point a in the interval $[x+1, x+1.2]$. Consider as an example the timed word σ shown in Fig. 4. This timed word satisfies the formula (5) with the valuation $x = 0.5$ since there is no action point a in the interval $[1.5, 1.7]$. Now consider the following assignment, $w_1 = 1.2$ is the first action point in σ before 1.5 and $w_2 = 1.8$ is the first action point in σ after 1.5. Similarly, $w_3 = 1.2$ is the first action point in σ before 1.7 and $w_4 = 1.8$ is the first action point in σ after 1.7. With this assignment, we get the interval $\pi_1 = 1.2 < x+1 < 1.8 \wedge 1.2 < x+1.2 < 1.8$ which is equivalent to $0.2 < x < 0.6$. It is easy to see that for any x in the intersection of the intervals $[0,1]$ and π_1 there is no action point a in the interval $[x+1, x+1.2]$. Hence, the timed word σ will also satisfy the equivalent formula:

$$\begin{aligned}
&\exists w_1 \exists w_2 \exists w_3 \exists w_4 (a(w_1) \wedge a(w_2) \wedge a(w_3) \wedge a(w_4) \\
&\qquad \wedge \forall x((0 \leq x \leq 1 \wedge \pi_1) \supset \neg \exists y(a(y) \wedge x+1 \leq y \wedge y \leq x+1.2))) && (6)\\
\equiv\ &\exists w_1 \exists w_2 \exists w_3 \exists w_4 (a(w_1) \wedge a(w_2) \wedge a(w_3) \wedge a(w_4) \\
&\qquad \wedge \neg \exists x(0 \leq x \leq 1 \wedge \pi_1 \wedge \exists y(a(y) \wedge x+1 \leq y \wedge y \leq x+1.2))) && (7)\\
\equiv\ &\exists w_1 \exists w_2 \exists w_3 \exists w_4 (a(w_1) \wedge a(w_2) \wedge a(w_3) \wedge a(w_4) \\
&\qquad \wedge \neg \exists y(a(y) \wedge \exists x(0 \leq x \leq 1 \wedge \pi_1 \wedge x+1 \leq y \wedge y \leq x+1.2))). && (8)
\end{aligned}$$

We get (7) from (6) using the equivalence $\forall x \varphi \equiv \neg \exists x \neg \varphi$. Finally, we eliminate x from the innermost part of formula (8) using Fourier-Motzkin elimination to get a formula which is completely actively quantified. We will prove in the later part of this section that it is always possible to identify the interval π_1 using the syntax of $\mathrm{FO}^c(<, +\mathbb{Q})$.

5.2 Equivalence Proof

We begin with some definitions. The *quantifier depth* of an FO formula is the maximum nesting depth of quantifiers in the formula. Given a formula $\varphi(x)$ (where x is free in φ) and a timed word σ, we will call an assignment $\mathbb{I}$ an *x-restricted assignment* for φ w.r.t. the timed word σ iff for every atomic subformula $y \sim x + c$ of φ, $\mathbb{I}(x) + c$ is not an action point of σ, and for every atomic subformula $y \sim x - c$ of φ, $\mathbb{I}(x) - c$ is not an action point of σ. Finally, consider a formula φ of the form $\exists x(\pi(x) \wedge \psi)$, where π is a conjunction of simple constraints and ψ is a formula in $\exists$-normal form. We say that a timed word σ *strongly satisfies* φ if there exists an x-restricted assignment $\mathbb{I}$ for ψ w.r.t. σ such that $\sigma, \mathbb{I} \models \pi(x) \wedge \psi$.

Also, observe that for any formula in $\exists$-normal form, we can replace all the equality atomic formulas, i.e. atomic formulas of the form $x = y + c$, with the equivalent formula $x \leq y + c \wedge x \geq y + c$. Hence, we will first remove all the equalities in our formula using this replacement. Furthermore, for simplicity, we will assume that the set of actions Σ is a singleton set i.e. $\Sigma = \{a\}$. This idea can be generalised to a finite set of actions Σ. Now we have the following theorem:

▶ **Lemma 2.** *Consider a formula of the form $\varphi = \exists x(\pi(x) \wedge \psi)$ where π is a conjunction of simple constraints and ψ is an actively quantified formula in $\exists$-normal form or negated $\exists$-normal form. Then, we can construct a formula θ which is a disjunction of formulas of the form:*

$$\exists w_1 \exists w_2 \cdots \exists w_n (\bigwedge_{i=1}^{i=n} a(w_i) \wedge \exists x(\pi(x) \wedge \pi_1(x, w_1, \ldots, w_n)) \wedge \forall x((\pi(x) \wedge \pi_1(x, w_1, \ldots, w_n)) \supset \psi)),$$

such that for any timed word σ which strongly satisfies φ, we also have $\sigma \models \theta$.

We will prove this theorem in the next section. In the rest of this section we see how to use it to prove the equivalence of the pointwise and continuous semantics of FO($<, +\mathbb{Q}$).

▶ **Theorem 3.** *Given any formula φ in $\exists$-normal form of the form $\exists x(\pi(x) \wedge \psi)$ where ψ is actively quantified, we can construct an equivalent formula ν which is a disjunction of formulas that are either actively quantified formulas in $\exists$-normal form or conjunctions of simple constraints that do not contain x. In other words, we can eliminate the passive variable x from φ.*

Proof. We prove this by induction on the quantifier depth of the formula ψ.

Base case (quantifier depth 0): In this case our formula φ is of the form $\varphi = \exists x \pi(x)$. We can use Fourier Motzkin elimination here to eliminate the variable x and get a formula which is a conjunction of simple constraints.

Inductive case: Now assume that we have proved the theorem for quantifier depth up to n, and consider the case when ψ has quantifier depth $n + 1$. We consider three different cases for the form of ψ.

Case 1a (Single positive conjunct): $\psi = \exists y(a(y) \wedge \delta \wedge \psi')$. In this case, we have

$$\begin{aligned}\varphi &= \exists x(\pi(x) \wedge \exists y(a(y) \wedge \delta \wedge \psi')) \\ &\equiv \exists y(a(y) \wedge \exists x(\pi(x) \wedge \delta \wedge \psi')).\end{aligned}$$

By the induction hypothesis the formula $\exists x((\pi(x) \wedge \delta) \wedge \psi')$ can be expressed as an equivalent formula $\nu = \nu_1 \vee \cdots \vee \nu_k$ with each ν_i actively quantified in $\exists$-normal form or negated $\exists$-normal form. Hence, we get

$$\begin{aligned}\varphi &\equiv \exists y(a(y) \wedge \nu) \\ &\equiv \exists y(a(y) \wedge (\nu_1 \vee \cdots \vee \nu_k)) \\ &\equiv \exists y(a(y) \wedge \nu_1) \vee \cdots \vee \exists y(a(y) \wedge \nu_k),\end{aligned}$$

which completes the proof for this case.

Case 1b (Single negative conjunct): $\psi = \neg\exists y(a(y) \wedge \delta \wedge \psi')$. In this case, we construct ν as follows.

We apply Lemma 2 on the formula φ to get a disjunct θ such that for any timed word σ if σ strongly satisfies φ, we have that $\sigma \models \theta$.

Each formula in the disjunct θ is of the form:

$$\begin{aligned}\theta_i = \exists w_1 \cdots \exists w_n (& \bigwedge_{i=1}^{n} a(w_i) \\ & \wedge \exists x(\pi(x) \wedge \pi_i(x, w_1, \ldots, w_n)) \\ & \wedge \forall x((\pi(x) \wedge \pi_i(x, w_1, \ldots, w_n)) \supset \psi)).\end{aligned}$$

In the conjunct $\exists x(\pi(x) \wedge \pi_i(x, w_1, \ldots, w_n))$ we can eliminate the passive variable x using Fourier-Motzkin elimination. The third conjunct is $\forall x((\pi(x) \wedge \pi_i(x, w_1, \ldots, w_n)) \supset \psi)$. Substituting $\psi = \neg\exists y(a(y) \wedge \delta \wedge \psi')$, we get

$$\begin{aligned} & \forall x((\pi \wedge \pi_i) \supset \neg\exists y(a(y) \wedge \delta \wedge \psi')) \\ \equiv & \neg\exists x(\pi \wedge \pi_i \wedge \exists y(a(y) \wedge \delta \wedge \psi')) \\ \equiv & \neg\exists y(a(y) \wedge \exists x(\pi \wedge \pi_i \wedge \delta \wedge \psi')).\end{aligned}$$

Now rewrite the interval $\pi \wedge \pi_i \wedge \delta$ as π' and apply the induction hypothesis on the formula $\exists x(\pi' \wedge \psi')$ to replace it with an equivalent disjunct $\nu' = \nu'_1 \vee \nu'_2 \vee \cdots \vee \nu'_k$ where each disjunct is actively quantified. Hence, after these manipulations, θ is a disjunction of actively quantified formulas in $\exists$-normal form and we have that for any timed word σ, if σ strongly satisfies φ, we have $\sigma \models \theta$.

Now we have to take care of the corner cases where $\sigma \models \varphi$ but σ does not strongly satisfy φ. For this, we do the following:

For each atomic formula of the form $x \sim v + c$, where v is some variable other than x, appearing in the formula ψ, define a formula

$$\mu = \exists w(a(w) \wedge \pi(w + c) \wedge \psi[(w + c)/x]).$$

Define $\mathcal{D}_1$ to be the disjunction of all such μ's. Similarly, for each atomic formula of the form $x + c \sim v$ appearing in the formula ψ, define a formula

$$\mu = \exists x(a(w) \wedge \pi(w - c) \wedge \psi[(w - c)/x]).$$

Define $\mathcal{D}_2$ to be the disjunction of all such μ's. Finally, define

$$\nu = \mathcal{D}_1 \vee \mathcal{D}_2 \vee \theta.$$

Observe that all the formulas in the disjuncts $\mathcal{D}_1$ and $\mathcal{D}_2$ are actively quantified formulas in $\exists$-normal form. Hence, ν is a disjunction of actively quantified formula in $\exists$ normal form.

Now, we need to show that for any timed word σ, we have $\sigma \models \varphi \iff \sigma \models \nu$. Assume that $\sigma \models \varphi$, i.e. there is an assignment $x = t_1$ such that $\sigma, [t_1/x] \models \pi \wedge \psi$. Then, this can happen in three ways:

1. For some atomic formula $x \sim v + c$ occurring in ψ, $t_1 - c$ is an action point of σ. In this case, $\sigma \models \mathcal{D}_1$.
2. For some atomic formula $x + c \sim v$ occurring in ψ, $t_1 + c$ is an action point of σ. In this case, $\sigma \models \mathcal{D}_2$.
3. $x = t_1$ is an x-restricted assignment for φ w.r.t σ. In this case, from Lemma 2, we get that $\sigma \models \theta$.

The other direction is straightforward.

Case 2 (Multiple Conjuncts): Now we consider the case where ψ has more than one conjunct. For simplicity, let ψ be the conjunct $\psi_1 \wedge \psi_2$. Our original formula is thus $\varphi = \exists x(\pi(x) \wedge \psi_1 \wedge \psi_2)$. We apply Lemma 2 to the formulas $\varphi_1 = \exists x(\pi(x) \wedge \psi_1)$ and $\varphi_2 = \exists x(\pi(x) \wedge \psi_2)$ to get two formulas

$$\theta'_1 = \bigvee_{j=1}^{k} \theta_{1j} \text{ and } \theta'_2 = \bigvee_{j=1}^{m} \theta'_{2j},$$

where each θ'_{ij} is of the form:

$$\begin{aligned}\theta'_{ij} = \exists w_1 \cdots \exists w_n (&\bigwedge_{i=1}^{p} a(w_i) \\ &\wedge \exists x(\pi(x) \wedge \pi_{ij}(x, w_1, \ldots, w_n)) \\ &\wedge \forall x((\pi(x) \wedge \pi_{ij}(x, w_1, \ldots, w_n)) \supset \psi_i).\end{aligned}$$

Here note that p might be different for each (i, j) . Now for each $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, m\}$ we construct a formula θ_{ij} below:

$$\begin{aligned}\exists w_1 \cdots \exists w_l \exists w'_1 \cdots \exists w'_n (&\bigwedge_{i=1}^{l} a(w_i) \bigwedge_{i=1}^{n} a(w'_i) \\ &\wedge \exists x(\pi(x) \wedge \pi_{1i}(x, w_1, \ldots, w_l) \wedge \pi_{2j}(x, w'_1, \ldots, w'_n)) \\ &\wedge \forall x((\pi(x) \wedge \pi_{1i}(x, w_1, \ldots, w_l) \wedge \pi_{2j}(x, w'_1, \ldots, w'_n)) \supset (\psi_1 \wedge \psi_2)),\end{aligned}$$

and define $\theta = \bigvee_{i=1}^{k} \bigvee_{j=1}^{m} \theta_{ij}$.

We will now show that if σ strongly satisfies φ, then $\sigma \models \theta$. Suppose σ strongly satisfies φ, then σ strongly satisfies both φ_1 and φ_2. Now by the single conjunct case, we know that there exists i_0 and j_0 such that $\sigma \models \theta'_{1i_0}$ and $\sigma \models \theta'_{2j_0}$. We will now show that $\sigma \models \theta_{i_0 j_0}$. Since $\sigma \models \theta_{1i_0}$, we have an assignment $\mathcal{W}$ such that

$$\begin{aligned}\sigma, \mathcal{W} \models (&\bigwedge_{i=1}^{n} a(w_i) \\ &\wedge \exists x(\pi(x) \wedge \pi_{1i_0}(x, w_1, \ldots, w_n)) \\ &\wedge \forall x((\pi(x) \wedge \pi_{1i_0}(x, w_1, \ldots, w_n)) \supset \psi_1).\end{aligned}$$

Similarly, since $\sigma \models \theta_{2j_0}$, we have an assignment $\mathcal{W}'$ such that

$$\begin{aligned}\sigma, \mathcal{W}' \models (&\bigwedge_{i=1}^{n} a(w_i) \\ &\wedge \exists x(\pi(x) \wedge \pi_{2j_0}(x, w_1, \ldots, w_n)) \\ &\wedge \forall x((\pi(x) \wedge \pi_{2j_0}(x, w_1, \ldots, w_n)) \supset \psi_2).\end{aligned}$$

Now, it is easy to see that

$$\begin{aligned}\sigma, \mathcal{W}, \mathcal{W}' \models (&\bigwedge_{i=1}^{l} a(w_i) \bigwedge_{i=1}^{n} a(w'_i) \\ &\wedge \exists x(\pi(x) \wedge \pi_{1i_0}(x, w_1, \ldots, w_l) \wedge \pi_{2j_0}(x, w'_1, \ldots, w'_n)) \\ &\wedge \forall x((\pi(x) \wedge \pi_{1i_0}(x, w_1, \ldots, w_l) \wedge \pi_{2j_0}(x, w'_1, \ldots, w'_n)) \supset (\psi_1 \wedge \psi_2)).\end{aligned}$$

Hence, $\sigma \models \theta_{i_0 j_0}$ and $\sigma \models \theta$.

Recall that we still have to eliminate x from the $\theta'_{ij}s$ which are of the form:

$$\exists w_1 \cdots \exists w_l \exists w'_1 \cdots \exists w'_n (\bigwedge_{i=1}^{l} a(w_i) \bigwedge_{i=1}^{n} a(w'_i)$$

$$\wedge \exists x(\pi(x) \wedge \pi_{1i}(x, w_1, \dots, w_l) \wedge \pi_{2j}(x, w'_1, \dots, w'_n)) \tag{9}$$

$$\wedge \forall x((\pi(x) \wedge \pi_{1i}(x, w_1, \dots, w_l) \wedge \pi_{2j}(x, w'_1, \dots, w'_n)) \supset (\psi_1 \wedge \psi_2)). \tag{10}$$

We can eliminate x from (9) using Fourier-Motzkin elimination. As for (10), we do the following manipulations (we drop the free variables w_i's to remove some clutter):

$$\begin{aligned} &\forall x((\pi \wedge \pi_{1i} \wedge \pi_{2j}) \supset (\psi_1 \wedge \psi_2)) \\ \equiv &\forall x((\pi \wedge \pi_{1i} \wedge \pi_{2j}) \supset \psi_1) \wedge \forall x((\pi \wedge \pi_{1i} \wedge \pi_{2j}) \supset \psi_2) \\ \equiv &\neg \exists x(\pi \wedge \pi_{1i} \wedge \pi_{2j} \wedge \neg \psi_1) \wedge \neg \exists x(\pi \wedge \pi_{1i} \wedge \pi_{2j} \wedge \neg \psi_2). \end{aligned} \tag{11}$$

Now (11) has two formulas containing the passive variable x, but both of them can be handled by the single conjunct case (*Case 1* above). Hence, we have successfully eliminated x for the multiple conjunct case.

To handle the cases where $\sigma \models \varphi$ but σ does not strongly satisfy φ, we can do a construction similar to the one done for *Case 1b*.

This completes the proof of the theorem. ◀

▶ **Theorem 4.** *Every* FO^c *sentence can be transformed to an equivalent active sentence.*

Proof. Any FO^c sentence φ can be written as a boolean combination of sentences in $\exists$-normal form. Since translation is intact across the boolean operations, it is sufficient if we can eliminate all the passive variables from formulas in $\exists$-normal form. We will now show that given a formula φ in $\exists$-normal form which has passive variables, we can construct an equivalent formula which is a disjunction of actively quantified formulas in $\exists$-normal form or negated $\exists$-normal form. Observe that if we can show this, the theorem follows immediately.

We will show this by induction on the number of passive variables n of the formula φ. We assume all quantified variables of φ are distinct, by renaming them if necessary.

Base Case: Suppose φ has 1 passive variable. Let T be the formula tree of φ, and let N be the corresponding node in T. The subtree roooted at N is thus of the form $\exists x(\pi(x) \wedge \psi)$ where ψ is actively quantified. Now we can use Theorem 3 to replace the subtree at N with a disjunction of actively quantified formulas in $\exists$-normal form. We then pull up the disjuncts to the top of the tree to get a disjunction of actively quantified formulas in $\exists$-normal form.

Inductive Case: Now suppose we have shown our hypothesis for formulas with up to n passive variables. Let φ be a formula in $\exists$-normal form with $n + 1$ passive variables. Let T be the formula tree of φ. We call a subtree of T rooted at node N a *maximal passive subtree* if N is a passive node and has no passive nodes as ancestors.

Now, if T has more than one maximal passive subtree, then the corresponding formula for each subtree is a formula in $\exists$-normal form with at most n passive variables. By our induction hypothesis, we can replace any such subtree with a disjunction of actively quantified formulas. We then pull out the disjunction to the top of T to get a finite disjunction of formulas in $\exists$-normal form, each of which has at most n passive variables. We can then apply the induction hypothesis on each of these to replace them with equivalent actively quantified formulas.

If T has exactly one maximal passive subtree, the formula corresponding to this subtree is of the form $\exists x(\pi(x) \wedge \psi)$ where $\psi = \psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_m$ where each ψ_i is in $\exists$-normal form or negated $\exists$-normal form. Observe that since $\exists x(\pi(x) \wedge \psi)$ has $n+1$ passive variables, each ψ_i can have atmost n passive variables. If ψ_i is in negated $\exists$-normal form, it is of the form $\neg\mu_i$ where μ_i is in $\exists$-normal form and has atmost n passive variables. We can now apply the induction hypothesis on μ_i and replace it with a disjunction of acitvely quantified formulas, i.e.

$$\mu_i = \nu_{i1} \vee \nu_{i2} \vee \cdots \vee \nu_{ik}.$$

Therefore,

$$\psi_i = \neg\mu_i = \neg\nu_{i1} \wedge \neg\nu_{i2} \wedge \cdots \wedge \neg\nu_{ik}.$$

We first replace all such ψ_i's. Now the remaining ψ_i's are in $\exists$-normal form and each have at most n passive variables. We apply the induction hypothesis on each of these and replace them with disjunctions of actively quantified formulas in $\exists$-normal form. Hence, we get

$$\exists x(\pi(x) \wedge \psi) = \exists x(\pi(x) \wedge \bigwedge_{i=1}^{k} \bigvee_{j=1}^{l_i} \nu_{ij}).$$

Let $S := \{1, 2, \dots, l_1\} \times \{1, 2, \dots, l_2\} \times \cdots \times \{1, 2, \dots, l_k\}$. Then

$$\exists x(\pi(x) \wedge \psi) = \bigvee_{(a_1,\dots,a_p)\in S} \exists x(\pi(x) \wedge \bigwedge_{i=1}^{p} \nu_{ia_i}). \tag{12}$$

Now each disjunct in (12) is in $\exists$-normal form with exactly one passive variable. We apply our induction hypothesis here and replace each of them with a disjunction of actively quantified formulas in $\exists$-normal form. We then pull up the disjuncts to the top of the tree T to get a disjunction of actively quantified formulas in $\exists$-normal form.

Thus we have shown that for any formula in $\exists$-normal form, we have an equivalent actively quantified formula, and we are done. ◀

To summarise:

▶ **Theorem 5.** *The logics* FO^c *and* FO^{pw} *are expressively equivalent. Moreover there is an effective procedure to translate a sentence in one logic to an equivalent one in the other.* ◀

6 Proof of Lemma 2

Here we will give a proof of Lemma 2 in a simplified setting. A more detailed exposition is given in the Appendix. For now, we will consider a simplified setting where our timed words are two way infinite timed words i.e. the timeline is not $[0, \infty)$ but instead $(-\infty, \infty)$ and for any timed word σ and a point t_0 there are action points before and after the point t_0. Recall the definition of x-restricted assignment and strong satisfaction from Section 5. Now in this setting, we have the following lemma which says that given a timed word σ, an interval $\delta(x, y)$ which is an interval for y determined by x (for example: $x + 1 \le y \le x + 2$), and a value $x = t_1$ for x, we can construct an interval $\pi_1(x)$ (using some other variables) such that for any $x = t'_1$ in the interval $\pi_1(x)$, the set of action points $a(y)$ in the intervals $\delta(t_1, y)$ and $\delta(t'_1, y)$ are exactly the same, i.e. for any point in the interval π_1 the set of action points in the interval δ are preserved.

▶ **Lemma 6** (Preservation of action points). *Consider a formula $\varphi = a(y) \wedge \delta(x, y)$ where x, y are free variables, a is an action and δ is a conjunction of simple constraints. Then, we can construct an interval $\pi_1(x, w_1, \ldots, w_n)$ where the w_i's are newly introduced free variables, such that given any timed word σ and a x-restricted assignment $\mathbb{I}$ for φ w.r.t. σ (let $\mathbb{I}(x) = t_1$), there exists an assignment $\mathcal{W} = [b_1/w_1, \ldots, b_n/w_n]$ such that the b_i's are action points of σ and, for any t_1' which satisfies $[t_1'/x], \mathcal{W} \models \pi_1(x)$, and for any t_0, we have*

$$\sigma, [t_1/x, t_0/y] \models \varphi \iff \sigma, [t_1'/x, t_0/y] \models \varphi.$$

Furthermore, $[t_1/x], \mathcal{W} \models \pi_1(x, w_1, \ldots, w_n)$.

Proof. We can think of δ as an interval for y that is determined by the value of x. This lemma says that for any t_1' in the interval π_1, the set of action points $a(y)$ in the interval $\delta([t_1'/x])$ is the same as that of the interval $\delta([t_1/x])$.

We construct the interval π_1 using the right and left boundaries of δ. The right boundary of δ will be of the form $y \sim x \pm c$ where $\sim \in \{<, \le\}$ and the left boundary will be of the form $y \sim x \pm c$ where $\sim \in \{>, \ge\}$. W.l.o.g, take the left boundary to be $y \ge x - c_1$ and the right boundary to be $y \le x + c_2$. Define four new variables w_1, w_2, w_3 and w_4 and define π_1 as

$$\pi_1 := w_1 < x - c_1 < w_2 \wedge w3 < x + c_2 < w_4.$$

Now take any timed word σ and a x-restricted assignment $\mathbb{I}$ for φ w.r.t. σ, with $\mathbb{I}(x) = t_1$. The assignment $\mathcal{W}$ is defined as follows:

1. Let b_1 be the first action point of σ that precedes the point $t_1 - c_1$. Assign $w_1 := b_1$
2. Let b_2 be the first action point of σ that succeeds the point $t_1 - c_1$. Assign $w_2 := b_2$
3. Let b_3 be the first action point of σ that precedes the point $t_1 + c_2$. Assign $w_3 := b_3$
4. Let b_4 be the first action point of σ that succeeds the point $t_1 + c_2$. Assign $w_4 := b_4$

With this assignment, it is easy to see that for any $x = t_1'$ in the interval π_1 i.e. $[x/t_1'], \mathcal{W} \models \pi_1$, the set of action points in the intervals $\delta([x/t_1])$ and $\delta([x/t_1'])$ is the same. Also, $[x/t_1], \mathcal{W} \models \pi_1(x, w_1, \ldots, w_n)$. And hence, the lemma follows. ◀

With this lemma we will prove the version of Lemma 2 in our simplified setting, which we state below:

▶ **Lemma 7.** *Consider a formula of the form $\varphi = \exists x(\pi(x) \wedge \psi)$ where π is a conjunction of simple constraints and ψ is an actively quantified formula in $\exists$-normal form or negated $\exists$-normal form. Then, we can construct an equivalent formula μ which is of the form:*

$$\exists w_1 \cdots \exists w_n (\bigwedge_{i=1}^{n} a(w_i) \wedge \exists x(\pi(x) \wedge \pi_1(x, w_1, \ldots, w_n)) \wedge \forall x((\pi(x) \wedge \pi_1(x, w_1, \ldots, w_n)) \supset \psi))$$

such that for any timed word σ which strongly satisfies φ, we also have $\sigma \models \mu$

Proof. We will prove the lemma by inducting on the quantifier depth of the formula ψ.

Base Case (quantifier depth = 1): In this case, $\psi = \exists y(a(y) \wedge \delta)$ or $\psi = \neg\exists y(a(y) \wedge \delta)$. We apply the preservation of action points lemma on the formula $a(y) \wedge \delta$ to get the interval $\pi_1(w_1, w_2, \ldots, w_n)$ for which action points in the interval δ are preserved. We define μ as follows:

$$\exists w_1, w_2, \ldots, w_n (\bigwedge_{i=1}^{n} a(w_i) \wedge \exists x(\pi(x) \wedge \pi_1(x, w_1, \ldots, w_n)) \wedge \forall x((\pi(x) \wedge \pi_1(x, w_1, \ldots, w_n)) \supset \psi)).$$

Now pick any timed word σ such that σ strongly satisfies φ. The preservation of action points lemma will give us a valuation $\mathcal{W}$ for the variables $w_1, \ldots, w_n$. It is easy to see that $\sigma \models \mu$ with the valuation $\mathcal{W}$

Inductive Case (quantifier depth = n): In this case, $\psi = \exists y(a(y) \wedge \delta \wedge \nu)$ or $\psi = \neg\exists y(a(y) \wedge \delta \wedge \nu)$ where ν is of quantifier depth $n-1$. To construct θ, we first look at the formula $\varphi' = \exists x(\pi(x) \wedge \psi')$ where $\psi' = \nu$ if $\psi = \exists y(a(y) \wedge \delta \wedge \nu)$ and $\psi' = \neg\nu$ otherwise. Applying induction hypothesis to this formula, we get the formula

$$\mu' = \exists w'_1, w'_2, \ldots, w'_m(\bigwedge_{i=1}^{m} a(w'_i) \wedge \exists x(\pi(x) \wedge \pi'_1(x, w_1, \ldots, w_m)) \\ \wedge \forall x((\pi(x) \wedge \pi'_1(x, w_1, \ldots, w_m)) \supset \psi')).$$

We apply the preservation of action points lemma on the formula $a(y) \wedge \delta$ to get the interval $\pi_1(w_1, w_2, \ldots, w_n)$ for which action points in the interval δ are preserved. We construct μ using π_1 and μ' as follows:

$$\mu = \exists w_1, \ldots, w_n \exists w'_1, w'_2, \ldots, w'_m(\bigwedge_{j=1}^{n} a(w_j) \bigwedge_{i=1}^{m} a(w'_i) \\ \wedge \exists x(\pi(x) \wedge \pi_1(x, w_1, \ldots, w_n) \wedge \pi'_1(x, w'_1, \ldots, w'_m)) \\ \wedge \forall x((\pi(x) \wedge \pi_1(x, w_1, \ldots, w_n) \wedge \pi'_1(x, w'_1, \ldots, w'_m)) \supset \psi)).$$

Pick any timed word σ such that σ strongly satisfies φ i.e. there is a x-restricted assigment $\mathbb{I}(x) = t_1$ such that $\sigma, [t_1/x] \models \pi(x) \wedge \psi$. We need to show that for any $x = t'_1$ in the interval $\pi \wedge \pi_1 \wedge \pi'_1$ we have ψ. Now from the preservation of action points lemma, we know that for any $x = t'_1$ in the interval π_1, the action points in the interval δ are preserved. Also, by the induction, the interval π'_1 ensures that for any $x = t'_1$ in the interval π'_1, $\sigma, [t'_1/x] \models \psi'$. Hence, for any $x = t'_1$ in the intersection of these intervals, i.e. the interval $\pi \wedge \pi_1 \wedge \pi'_1$ we have $\sigma, [t'_1/x] \models \psi$. Hence, we get that $\sigma \models \mu$. ◀

7 Equivalence of FO and TPTL$_S$

We can now argue that the logic TPTL$_S$ is expressively equivalent in its continuous and pointwise semantics. We recall that the formulas of TPTL$_S$ [3, 5] over the alphabet Σ, are defined as follows: $\theta ::= a \mid g \mid \neg\theta \mid \theta \vee \theta \mid \theta U \theta \mid \theta S \theta \mid x.\theta$ where $a \in \Sigma$, x is a variable in *Var*, and g is a simple constraint.

In the pointwise semantics, for a TPTL$_S$ formula θ, timed word $\sigma = (\alpha, \tau)$ over Σ, $i \in \mathbb{N}$, and an assignment for variables $\mathbb{I}$, we define the satisfaction relation $\sigma, i, \mathbb{I} \models_{pw} \theta$, as:

$\sigma, i, \mathbb{I} \models_{pw} a$	iff	$\alpha(i) = a$
$\sigma, i, \mathbb{I} \models_{pw} g$	iff	$\mathbb{I} \models g$
$\sigma, i, \mathbb{I} \models_{pw} \theta U \eta$	iff	$\exists k : i < k$ s.t. $\sigma, k, \mathbb{I} \models_{pw} \eta$ and $\forall j : i < j < k : \sigma, j, \mathbb{I} \models_{pw} \theta$
$\sigma, i, \mathbb{I} \models_{pw} \theta S \eta$	iff	$\exists k : 0 \leq k < i$ s.t. $\sigma, k, \mathbb{I} \models_{pw} \eta$ and $\forall j : k < j < i, \sigma, j, \mathbb{I} \models_{pw} \theta$
$\sigma, i, \mathbb{I} \models_{pw} x.\theta$	iff	$\sigma, i, \mathbb{I}[\tau(i)/x] \models_{pw} \theta$,

with boolean operators handled in the expected way. For a "closed" TPTL$_S$ formula θ, in which every occurrence of x is within the scope of a freeze quantifier "$x.$", we set the language defined by it to be $L^{pw}(\theta) = \{\sigma \in T\Sigma^\omega \mid \sigma, 0 \models_{pw} \theta\}$.

In the continuous semantics, for a TPTL$_S$ formula θ, a timed word $\sigma = (\alpha, \tau)$ over Σ, $t \in \mathbb{R}_{\geq 0}$, and an assignment $\mathbb{I}$, the satisfaction relation $\sigma, t, \mathbb{I} \models_c \theta$ is defined similarly, except that:

$$\begin{array}{lll}
\sigma, t, \mathbb{I} \models_c a & \text{iff} & \exists i : \alpha(i) = a \text{ and } \tau(i) = t \\
\sigma, t, \mathbb{I} \models_c \theta U \eta & \text{iff} & \exists t' : t < t' \text{ s.t. } \sigma, t', \mathbb{I} \models_c \eta \text{ and } \forall t'' : t < t'' < t', \sigma, t'', \mathbb{I} \models_c \theta \\
\sigma, t, \mathbb{I} \models_c \theta S \eta & \text{iff} & \exists t' : 0 \leq t' < t \text{ s.t. } \sigma, t', \mathbb{I} \models_c \eta \text{ and } \forall t'' : t' < t'' < t, \sigma, t'', \mathbb{I} \models_c \theta \\
\sigma, t, \mathbb{I} \models_c x.\theta & \text{iff} & \sigma, t, \mathbb{I}[t/x] \models_c \theta.
\end{array}$$

We use the standard syntactic abbreviations of $\Diamond, \diamondminus, \Box$ and $\boxminus$ defined in a reflexive manner: $\Diamond\theta = \theta \vee (\top U \theta)$, $\diamondminus\theta = \theta \vee (\top S \theta)$, $\Box\theta = \neg\Diamond\neg\theta$, and $\boxminus\theta = \neg\diamondminus\neg\theta$. We note that in TPTL_S it is possible to express the U and S operators using $\Diamond$ and $\diamondminus$ operators, in both the continuous and pointwise semantics. For instance, $\theta U \eta \ \equiv x.\Diamond y.(\eta \wedge x < y \wedge \boxminus z.(x < z \wedge z < y \Rightarrow \theta))$. Hence we concentrate only on these operators in the translations below.

▶ **Theorem 8.** *The logics* TPTL_S^c *and* TPTL_S^{pw} *are expressively equivalent.* ◀

Proof. We first show that we can translate a formula in TPTL_S to an equivalent one in $\mathrm{FO}(<, +\mathbb{Q})$, and vice-versa. For a closed formula θ in TPTL_S we show how to give a formula *tptl-fo*(θ) in $\mathrm{FO}(<, +\mathbb{Q})$, which has a single free variable z, such that for any timed word σ, $\sigma, t \models_c \theta$ if and only if $\sigma, [t/z] \models_c$ *tptl-fo*(θ) (and similarly in pointwise semantics). The translation *tptl-fo* is defined inductively on the structure of θ as follows:

$$\begin{array}{lll}
\mathit{tptl\text{-}fo}(a) & = & a(z) \\
\mathit{tptl\text{-}fo}(g) & = & g \\
\mathit{tptl\text{-}fo}(\Diamond\theta') & = & \exists x(x \geq z \wedge \mathit{tptl\text{-}fo}(\theta')[x/z]) \\
\mathit{tptl\text{-}fo}(\diamondminus\theta') & = & \exists x(x \leq z \wedge \mathit{tptl\text{-}fo}(\theta')[x/z]) \\
\mathit{tptl\text{-}fo}(x.\theta') & = & (\mathit{tptl\text{-}fo}(\theta'))[z/x]
\end{array}$$

with boolean operators handled in the expected manner. We can now translate a closed formula θ in TPTL_S to an $\mathrm{FO}(<, +\mathbb{Q})$ sentence $\varphi = \exists z(z = 0 \wedge \mathit{tptl\text{-}fo}(\theta))$, with $L^c(\theta) = L^c(\varphi)$.

In the other direction, we translate an $\mathrm{FO}(<, +\mathbb{Q})$ sentence φ, to an equivalent closed TPTL_S formula *fo-tptl*(φ) as follows. We first transform φ into its normal form as given in Thm 1. The translation *fo-tptl* is defined inductively in a similar manner to *tptl-fo* above, with $\exists$-subformulas being translated via the rule:

$$\mathit{fo\text{-}tptl}(\exists x(a(x) \wedge \pi(x) \wedge \nu)) \quad = \quad \Diamond x.(a \wedge \pi(x) \wedge \mathit{fo\text{-}tptl}(\nu)) \vee \diamondminus x.(a \wedge \pi(x) \wedge \mathit{fo\text{-}tptl}(\nu)).$$

It is easy to see that $L^c(\varphi) = L^c(\mathit{fo\text{-}tptl}(\varphi))$.

We can now prove the non-trivial direction of the theorem. Consider a closed formula θ of TPTL_S^c. We go over to an equivalent $\mathrm{FO}(<, +\mathbb{Q})$ formula $\varphi = \mathit{tptl\text{-}fo}(\theta)$, obtain an equivalent pointwise $\mathrm{FO}(<, +\mathbb{Q})$ formula φ' using Thm. 5, and finally obtain an equivalent pointwise TPTL_S formula θ', with $L^c(\theta) = L^{pw}(\theta')$. ◀

8 Conclusion

In this paper we have shown the expressive equivalence of, and in fact given an effective translation between, the pointwise and continuous versions of two natural logics $\mathrm{FO}(<, +\mathbb{Q})$ and TPTL_S over timed words. Some interesting directions include addressing a similar question for TPTL (i.e. without the since operator). One may be able to use an argument like [8] to say that TPTL is as expressive as TPTL_S in the pointwise setting. Another interesting question is about first-order logic with Presburger constraints in general. Here it appears that we can express strong properties in the continuous interpretation which seem difficult to express in the pointwise setting.

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A Proof of Lemma 2

▶ **Lemma 9** (Preservation of action points). *Consider a formula $\varphi = a(y) \wedge \delta(x, y)$ where x, y are free variables, a is an action and δ is a conjunction of simple linear constraints. Then, we can construct a finite number of intervals $\pi_1(x, w_1, w_2, \ldots, w_n), \ldots, \pi_m(x, w_1, w_2, \ldots, w_k)$ where the w_i's are newly introduced free variables, such that given any timed word σ and an x-restricted assignment, $x = t_1$ for φ w.r.t σ, there exists an $i \in \{1, \ldots, m\}$ and an assignment $\mathcal{W} = \{w_1 = b_1, w_2 = b_2, \ldots, w_n = b_n\}$ with the b_i's being action points of the timed word σ, such that for any t_1' which satisifes $x = t_1', \mathcal{W} \models \pi_i(x)$, and for any valuation $y = t_0$ for y, we have*

$$\sigma, x = t_1, y = t_0 \models \varphi \iff \sigma, x = t_1', y = t_0 \models \varphi.$$

Furthermore, $x = t_1, \mathcal{W} \models \pi_i$

Proof. Observe that the constraints in δ are defining an interval for y. Based on the boundaries of this interval, we can have four different cases:

1. Both the left end and right end of δ are of the form $y \sim x + c$
2. the left end of δ is of the form $y \sim x - c$ and the right end of δ is of the form $y \sim x + c$
3. Both the left and right end of δ are of the form $y \sim x - c$
4. the left end of δ is of the form $y \sim x + c$ and the right end is of the form $y \sim x - c$

We now provide the constructions for each of the cases.

Case 1: Let the left end of δ be $y \sim x + c_1$ and the right end be $y \sim x + c_2$. In this case, we have only 1 interval π_1. We introduce 4 new variables w_1, w_2, w_3, w_4 and construct $\pi_1 = w_1 < x + c_1 < w_2 \wedge w_3 < x + c_2 < w_4$

Case 2: Let the left end of δ be $y \sim x - c_1$ and the right end be $y \sim x + c_2$. In this case, we have two intervals:

$$\pi_1 = w_1 < x - c_1 < w_2 \wedge w_3 < x + c_2 < w4$$
$$\pi_2 = 0 < x < c_1 \wedge w_3 < x + c_2 < w4$$

Case 3: Let the left end of δ be $y \sim x - c_1$ and the right end be $y \sim x - c_2$. We have 4 intervals:

$$\pi_1 = w_1 < x - c_1 < w_2 \wedge w_3 < x - c_2 < w_4$$
$$\pi_2 = 0 < x < c_1 \wedge w_3 < x - c_2 < w_4$$
$$\pi_3 = w_1 < x - c_1 < w_2 \wedge 0 < x < c_2$$
$$\pi_4 = 0 < x < c_1 \wedge 0 < x < c_2$$

Case 4: Let the left end of δ be $y \sim x + c_1$ and the right end be $y \sim x - c_2$. We have 2 intervals:

$$\pi_1 = w_1 < x + c_1 < w_2 \wedge w_3 < x - c_2 < w_4$$
$$\pi_2 = w_1 < x + c_1 < w_2 \wedge 0 < x < c_2$$

We will prove the theorem for case 2. The proof is similar for the others. Pick any timed word σ and an x-restricted valuation $x = t_1$ for φ w.r.t σ. W.l.o.g, let the left end of δ be $y \geq x - c_1$ and the right end be $y \leq x + c_2$. We can have two cases:

Case 1 : $t_1 - c_1 > 0$. For this case, pick the interval $\pi_1 = w_1 < x - c_1 < w_2 \wedge w_3 < x + c_2 < w4$. we define the assignment $\mathcal{W}$ as follows:

- Set w_1 to be the first action point of σ that is less than $t_1 - c_1$
- Set w_2 to be the first action point of σ that is greater than $t_1 - c_1$
- Set w_3 to be the first action point of σ that is less than $t_1 + c_2$
- Set w_4 to be the first action point of σ that is greater than $t_1 + c_2$

Now we need to show that for any assignment $y = t_0$ for y and any t_1' such that $x = t_1', \mathcal{W} \models \pi_1(x)$, we have $\sigma, x = t_1, y = t_0 \models \varphi \iff \sigma, x = t_1', y = t_0 \models \varphi$. Pick any assignment $y = t_0$ for y and any t_1' such that $x = t_1', \mathcal{W} \models \pi_1(x)$. Assume $\sigma, x = t_1, y = t_0 \models \varphi$. Suppose that $\sigma, x = t_1', y = t_0 \not\models \varphi$, then,

$$\begin{aligned} &\sigma, x = t_1', y = t_0 \not\models \delta \\ \implies &\sigma, x = t_1', y = t_0 \not\models y \geq x - c_1 \text{ OR } \sigma, x = t_1', y = t_0 \not\models y \leq x + c_2 \end{aligned}$$

WLOG suppose $\sigma, x = t_1', y = t_0 \not\models y \geq x - c_1$ or in other words, $\sigma, x = t_1', y = t_0 \models y < x - c_1$. Now observe that $t_1', \mathcal{W} \models \pi_1(x)$ and hence, $t_1' - c_1 < w_2$. Furthermore,

$$\begin{aligned} &\sigma, x = t_1, y = t_0 \models \varphi \implies \sigma, x = t_1, y = t_0 \models \delta \\ \implies &\sigma, x = t_1, y = t_0 \models y \geq x - c_1 \implies t_0 \geq t_1 - c_1 \end{aligned}$$

Hence, we get the inequality, $t_1 - c_1 \leq t_0 < t'_1 - c_1 < w_2$. This is saying that t_0 is an action point of σ which lies in between $t_1 - c_1$ and w_2. This directly contradicts our assignment of w_2. Hence, our assumption is wrong. Therefore, $\sigma, x = t'_1, y = t_0 \models y \geq x - c_1$. An exactly similar proof will show the other direction.

Case 2: $t_1 - c_1 < 0$. For this case, pick the interval $\pi_2 = 0 < x < c_1 \wedge w_3 < x + c_2 < w_4$. We use the same assignment $\mathcal{W}$ as in case 1, but restricted to the variables w_3 and w_4. One can argue similar to above to show that the lemma holds in this case also.

Hence, we get our Lemma. ◀

▶ **Theorem 10.** *Consider a formula of the form $\varphi = \exists x(\pi(x) \wedge \psi)$ where π is a conjunction of simple constraints and ψ is an actively quantified formula in $\exists$-normal form or the negation of an actively quantified formula in $\exists$-normal form. Then, we can construct a formula θ which is a disjunction of formulas of the form:*

$$\theta_i = \exists w_1 \exists w_2 \cdots \exists w_n (\bigwedge_{i=1}^{n} a(w_i) \wedge \exists x(\pi(x) \wedge \pi_i(x, w_1, \ldots, w_n)) \\ \wedge \forall x((\pi(x) \wedge \pi_i(x, w_1, \ldots, w_n)) \supset \psi))$$

such that for any timed word σ which strongly satisfies φ we also have $\sigma \models \theta$.

Proof. Assume that σ strongly satisfies φ with an x-restricted assignment $x = t_1$ for φ w.r.t. σ. Observe that if $\sigma \models \theta$, then there is an i_0 such that $\sigma \models \theta_{i_0}$. We will not only show that $\sigma \models \theta_{i_0}$ for some i_0, but we will also further show that $\sigma \models \theta_{i_0}$ with an assignment $\mathcal{W}$ for the w_i's such that $x = t_1, \mathcal{W} \models \pi_{i_0}(x, w_1, \ldots, w_n)$ and the choice of the assignment $\mathcal{W}$ depends only on σ and the assignment $x = t_1$ for x.

We prove this by inducting on the quantifier depth of the formula ψ.

Base Case: Quantifier depth = 1
In this case, we have $\varphi = \exists x(\pi(x) \wedge \psi)$ where

$$\psi = \exists y(a(y) \wedge \delta) \text{ OR } \psi = \neg\exists y(a(y) \wedge \delta)$$

Now we apply the preservation of action points lemma on the formula $a(y) \wedge \delta$ to get a finite number of intervals $\pi_1, \pi_2, \ldots, \pi_m$. For each such interval, we construct a formula

$$\theta_i = \exists w_1 \exists w_2 \cdots \exists w_n (\bigwedge_{i=1}^{n} a(w_i) \wedge \exists x(\pi(x) \wedge \pi_i(x, w_1, \ldots, w_n)) \\ \wedge \forall x((\pi(x) \wedge \pi_i(x, w_1, \ldots, w_n)) \supset \psi)).$$

We define $\theta = \bigvee_{i=1}^{m} \theta_i$.

To prove the lemma, let $\psi = \exists y(a(y) \wedge \delta)$. Now pick any timed word σ such that σ strongly satisfies φ, i.e. there is an x- restricted assignment $x = t_1$ for ψ w.r.t σ, such that $\sigma, x = t_1 \models \pi(x) \wedge \exists y(a(y) \wedge \delta)$. Applying our preservation of action points lemma on the formula $a(y) \wedge \delta$, we get an $i_0 \in \{1, 2, \ldots, m\}$. We will now show that $\sigma \models \theta_{i_0}$. To show this, we have to show that there is an assignment $\mathcal{W} = \{w_1 = b_1, \ldots, w_n = b_n\}$ such that

$$\sigma, \mathcal{W} \models (\bigwedge_{i=1}^{i=n} a(w_i) \wedge \exists x(\pi(x) \wedge \pi_{i_0}(x, w_1, \ldots, w_n)) \\ \wedge \forall x((\pi(x) \wedge \pi_{i_0}(x, w_1, \ldots, w_n)) \supset \psi)).$$

The preservation of action points lemma gives us an assignment $\mathcal{W}$ which preserves the action points in the interval δ. We will use the same assignment here. $\sigma, \mathcal{W} \models a(w_j)$ holds for each j by the preservation of action points lemma. By our hypothesis, $\sigma, x = t_1 \models \pi(x)$ and by the preservation of action points lemma we also have $\sigma, x = t_1, \mathcal{W} \models \pi_{i_0}$. Hence, we also have $\sigma, \mathcal{W} \models \exists x(\pi \wedge \pi_{i_0})$. Also, observe that from the preservation of action points lemma, we get that the choice of $\mathcal{W}$ depends only on σ and the assignment $x = t_1$.

Now all that remains is to show that $\sigma, \mathcal{W} \models \forall x((\pi(x) \wedge \pi_{i_0}(x, w_1, \ldots, w_n)) \supset \psi)$. Pick any assignment $x = t_1'$ of the variable x such that $\sigma, x = t_1', \mathcal{W} \models \pi(x) \wedge \pi_{i_0}(x, w_1, \ldots, w_n)$. We need to show that $\sigma, x = t_1' \models \psi$. By our assumption, we have that $\sigma, x = t_1 \models \pi(x) \wedge \exists y(a(y) \wedge \delta)$. Hence, there is an assigment $y = t_0$ such that $\sigma, x = t_1, y = t_0 \models a(y) \wedge \delta$. By the preservation of action points lemma, we get that $\sigma, x = t_1', y = t_0 \models a(y) \wedge \delta$. Hence, we get that $\sigma, x = t_1' \models \psi$.

Hence, we get that $\sigma \models \theta_{i_0}$ and hence $\sigma \models \theta$. Observe that, in the process we also showed that $\sigma \models \theta_{i_0}$ with an assignment $\mathcal{W}$ such that $x = t_1, \mathcal{W} \models \pi_{i_0}$ and the choice of $\mathcal{W}$ depended only on σ and the assignment $x = t_1$ for x.

One can similarly argue that the lemma holds if $\psi = \neg\exists y(a(y) \wedge \delta)$.

Inductive Step: Let us assume that we have proved the lemma for quantifier depth of $1, 2, \ldots, n-1$. Now, consider a formula $\varphi = \exists x(\pi(x) \wedge \psi)$ where ψ is of quantifier depth n. We will split this proof into two cases based on whether ψ is a formula in $\exists$ normal form, or negation of a formula $\exists$ normal form. We will show the lemma for the first case, and the second case can be shown similarly.

Case 1 (Positive Case) : $\psi = \exists y(a(y) \wedge \delta \wedge \psi')$.

We will first prove it assuming ψ' is of the form ν or $\neg\nu$ where ν is in $\exists$ normal form of quantifier depth $n-1$. Later, we will give the construction for when ψ' is a conjunction of formulas of the form ν or $\neg\nu$, where ν is in $\exists$ Normal form, and the proof of the lemma is similar.

Now consider the formula $\varphi' = \exists x(\pi(x) \wedge \psi')$. By our induction hypothesis, there exists a formula $\theta' = \bigvee_{j=1}^{k} \theta_j'$ such that the lemma holds, and each θ_j' is of the form:

$$\begin{aligned}\theta_j' = \exists w_1', w_2', \ldots, w_n'(&\bigwedge_{i=1}^{n} a(w_i') \wedge \exists x(\pi(x) \wedge \pi_j'(x, w_1', \ldots, w_n'))\\ &\wedge \forall x((\pi(x) \wedge \pi_j'(x, w_1', \ldots, w_n')) \supset \psi')).\end{aligned}$$

We apply the preservation of action points lemma to the formula $a(y) \wedge \delta$ to get a finite number of intervals $\pi_1, \ldots, \pi_m$. We now construct $m \times k$ formulas denoted by θ_{ij} where $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, k\}$ as follows:

$$\begin{aligned}\theta_{ij} = \exists w_1, w_2, \ldots, w_l \exists w_1', w_2', \ldots, w_n'(&\bigwedge_{i=1}^{l} a(w_i) \bigwedge_{i=1}^{n} a(w_i')\\ &\wedge \exists x(\pi(x) \wedge \pi_i(x, w_1, \ldots, w_l) \wedge \pi_j'(x, w_1', \ldots, w_n'))\\ &\wedge \forall x((\pi(x) \wedge \pi_i(x, w_1, \ldots, w_l) \wedge \pi_j'(x, w_1', \ldots, w_n')) \supset \psi)).\end{aligned}$$

Now define $\theta = \bigvee_{i=1}^{m} \bigvee_{j=1}^{k} \theta_{ij}$.

Pick any timed word σ such that σ strongly satisfies φ. We need to show that $\sigma \models \theta$. Observe that

$$\begin{aligned}\sigma \models \varphi &\implies \sigma \models \exists x(\pi(x) \wedge \exists y(a(y) \wedge \delta \wedge \psi')\\ &\implies \sigma, x = t_1, y = t_0 \models \pi(x) \wedge a(y) \wedge \delta \wedge \psi'.\end{aligned}$$

Hence, $\sigma, y = t_0 \models \exists x(\pi(x) \wedge \psi')$ i.e. σ strongly satisfies φ' with the x-restricted assignment $x = t_1$ for φ' w.r.t σ. Now by the induction hypothesis, $\sigma, y = t_0 \models \theta' \implies \sigma, y = t_0 \models \theta'_{j_0}$ for some j_0. Furthermore, $\sigma, y = t_0 \models \theta'_{j_0}$ with an assignmnet $\mathcal{W}'$ such that $x = t_1, \mathcal{W}' \models \pi'_{j_0}$ and the assignment $\mathcal{W}'$ depends only on σ and the assignment $x = t_1$.

Applying the preservation of action points lemma on the formula $a(y) \wedge \delta$ gives us a $i_0 \in \{1, \ldots, m\}$. We will now show that $\sigma \models \theta_{i_0 j_0}$. Recall that

$$\begin{aligned}\theta_{i_0 j_0} = \exists w_1, w_2, \ldots, w_l \exists w'_1, w'_2, \ldots, w'_n (\bigwedge_{i=1}^{l} a(w_i) \bigwedge_{i=1}^{n} a(w'_i)\\ \wedge \exists x(\pi(x) \wedge \pi_{i_0}(x, w_1, \ldots, w_l) \wedge \pi'_{j_0}(x, w'_1, \ldots, w'_n))\\ \wedge \forall x((\pi(x) \wedge \pi_{i_0}(x, w_1, \ldots, w_l) \wedge \pi'_{j_0}(x, w'_1, \ldots, w'_n)) \supset \psi)).\end{aligned}$$

We need to come up with an assignment $\mathcal{W}$ for the variables $w_1, \ldots, w_l$ and an assignment $\mathcal{W}'$ for the variables $w'_1, \ldots, w'_n$. The preservation of action points lemma on the formula $a(y) \wedge \delta$ using the timed word σ and the assignment $x = t_1$ for x gives us the assignment $\mathcal{W}$. Also, observe $\sigma, y = t_0 \models \theta'_{j_0}$. Hence,

$$\begin{aligned}\sigma, y = t_0 \models \exists w'_1, w'_2, \ldots, w'_n (\bigwedge_{i=1}^{n} a(w'_i) \wedge \exists x(\pi(x) \wedge \pi'_{j_0}(x, w'_1, \ldots, w'_n))\\ \wedge \forall x((\pi(x) \wedge \pi'_{j_0}(x, w'_1, \ldots, w'_n)) \supset \psi')).\end{aligned}$$

This gives us a valuation $\mathcal{W}'$ such that

$$\begin{aligned}\sigma, y = t_0, \mathcal{W}' \models (\bigwedge_{i=1}^{n} a(w'_i) \wedge \exists x(\pi(x) \wedge \pi'_{j_0}(x, w'_1, \ldots, w'_n))\\ \wedge \forall x((\pi(x) \wedge \pi'_{j_0}(x, w'_1, \ldots, w'_n)) \supset \psi')).\end{aligned}$$

Hence, we have $\sigma, \mathcal{W}, \mathcal{W}' \models \bigwedge_{i=1}^{l} a(w_i) \bigwedge_{j=1}^{n} a(w'_j)$. From the preservation of action points lemma, we know that, $\sigma, x = t_1, \mathcal{W} \models \pi_{i_0}(x, w_1, \ldots, w_l)$ and the choice of $\mathcal{W}$ depends only on σ and the assignment $x = t_1$. From the induction hypothesis, we know: $\sigma, x = t_1, \mathcal{W}' \models \pi(x) \wedge \pi'_{j_0}(x, w'_1, \ldots, w'_n)$. Combining these two together, we get, $\sigma, \mathcal{W}, \mathcal{W}' \models \exists x(\pi(x) \wedge \pi_{i_0} \wedge \pi'_{j_0})$.

All that remains is to show that $\sigma, \mathcal{W}, \mathcal{W}' \models \forall x((\pi \wedge \pi_{i_0} \wedge \pi'_{j_0}) \supset \psi)$.

To do this, pick any assignment $x = t'_1$ such that $\sigma, x = t'_1, \mathcal{W}, \mathcal{W}' \models \pi \wedge \pi_{i_0} \wedge \pi'_{j_0}$. Recall that $\psi = \exists y(a(y) \wedge \delta \wedge \psi')$. Hence, we need to show:

$$\sigma, x = t'_1, \mathcal{W}, \mathcal{W}' \models \exists y(a(y) \wedge \delta \wedge \psi').$$

From the preservation of action points lemma, we know that since $\sigma, x = t_1, y = t_0 \models a(y) \wedge \delta$, and since $\sigma, x = t'_1, \mathcal{W} \models \pi_{i_0}$, we have:

$$\sigma, x = t'_1, y = t_0, \mathcal{W}, \mathcal{W}' \models a(y) \wedge \delta.$$

We also already have that

$$\begin{aligned}
&\sigma, y = t_0, \mathcal{W}' \models (\bigwedge_{i=1}^{n} a(w'_i) \wedge \exists x(\pi(x) \wedge \pi'_{j_0}(x, w'_1, \ldots, w'_n)) \\
&\qquad\qquad\qquad \wedge \forall x((\pi(x) \wedge \pi'_{j_0}(x, w'_1, \ldots, w'_n)) \supset \psi')) \\
\implies &\sigma, y = t_0, \mathcal{W}' \models \forall x((\pi(x) \wedge \pi'_{j_0}(x, w'_1, \ldots, w'_n)) \supset \psi').
\end{aligned}$$

Now $\sigma, x = t'_1, \mathcal{W}, \mathcal{W}' \models \pi \wedge \pi'_{j_0}$. Hence, we get $\sigma, x = t'_1, y = t_0, \mathcal{W}, \mathcal{W}' \models \psi'$, which gives us $\sigma, x = t'_1, \mathcal{W}, \mathcal{W}' \models \exists y(a(y) \wedge \delta \wedge \psi')$. This is what we needed to show. Hence we are done.

Now consider $\psi = \exists y(a(y) \wedge \delta \wedge \psi')$ where $\psi' = \psi'_1 \wedge \psi'_2 \wedge \cdots \wedge \psi'$ and each ψ'_i is either a formula in $\exists$ normal form, or the negation of a formula in $\exists$ normal form. Consider the formulas: $\varphi'_i = \exists x(\pi \wedge \psi'_i)$ for $i = 1, 2, \ldots, n$. For each $i = 1, 2, \ldots, n$, we can apply the induction hypothesis to get a disjunction $\theta'_i = \bigvee_{j=1}^{k_i} \theta'_{ij}$ such that the lemma holds and where each θ'_{ij} is of the form:

$$\begin{aligned}
\theta'_{ij} = \exists w'_1, w'_2, \ldots, w'_n (&\bigwedge_{i=1}^{n} a(w'_i) \wedge \exists x(\pi(x) \wedge \pi'_{ij}(x, w'_1, \ldots, w'_n)) \\
&\wedge \forall x((\pi(x) \wedge \pi'_{ij}(x, w'_1, \ldots, w'_n)) \supset \psi'_i)).
\end{aligned}$$

We apply the preservation of action points lemma to the formula $a(y) \wedge \delta$ to get a finite number of intervals $\pi_1, \ldots, \pi_m$. We now construct $m \times k_1 \times k_2 \times \cdots \times k_n$ formulas denoted by $\theta_{ij_1j_2\ldots j_n}$ where $i \in \{1, 2, \ldots, m\}$ and $j_i \in \{1, 2, \ldots, k_i\}$ as follows:

$$\begin{aligned}
\theta_{ij_1j_2\ldots j_n} = \exists w_1, w_2, \ldots, w_l (&\bigwedge_{i=1}^{l} a(w_i) \wedge \exists x(\pi(x) \wedge \pi_i \wedge \pi'_{1j_1} \wedge \pi'_{2j_2} \wedge \cdots \wedge \pi'_{nj_n}) \\
&\wedge \forall x(\pi(x) \wedge \pi_i \wedge \pi'_{1j_1} \wedge \pi'_{2j_2} \wedge \cdots \wedge \pi'_{nj_n}) \supset \psi)).
\end{aligned}$$

Now define $\theta = \bigvee_{i=1}^{i=m} \bigvee_{j_1=1}^{j_1=k_1} \cdots \bigvee_{j_n=1}^{j_n=k_n} \theta_{ij_1j_2\ldots j_n}$.

It can be shown similar to the above that the theorem holds with this construction. ◀