

# More Dominantly Truthful Multi-Task Peer Prediction with a Finite Number of Tasks

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## Abstract

In the setting where we ask participants multiple similar possibly subjective multi-choice questions (e.g. Do you like Bulbasaur? Y/N; do you like Squirtle? Y/N), peer prediction aims to design mechanisms that encourage honest feedback without verification. A series of works have successfully designed multi-task peer prediction mechanisms where reporting truthfully is better than any other strategy (dominantly truthful), while they require an infinite number of tasks. A recent work proposes the first multi-task peer prediction mechanism, Determinant Mutual Information (DMI)-Mechanism, where not only is dominantly truthful but also works for a finite number of tasks (practical).

However, the existence of other practical dominantly-truthful multi-task peer prediction mechanisms remains to be an open question. This work answers the above question by providing

- a new family of information-monotone information measures: volume mutual information (VMI), where DMI is a special case;
- a new family of practical dominantly-truthful multi-task peer prediction mechanisms, VMI-Mechanisms.

To illustrate the importance of VMI-Mechanisms, we also provide a tractable effort incentive optimization goal. We show that DMI-Mechanism may not be optimal but we can construct a sequence of VMI-Mechanisms that are approximately optimal.

The main technical highlight in this paper is a novel geometric information measure, *Volume Mutual Information*, that is based on a simple idea: we can measure an object  $A$ 's information amount by the number of objects that is less informative than  $A$ . Different densities over the object lead to different information measures. This also gives Determinant Mutual Information a simple geometric interpretation.

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## 1 Introduction

Human evaluation is a commonly used measure when we lack an objective standard. For example, the internet company sometimes uses human evaluation to evaluate the online product's quality (e.g. app, online platform). However, eliciting high-quality feedback from the human evaluators can be tricky when they are asked to provide *subjective* judgment. There is no way to verify their subjective opinions. Paying these evaluators only for the agreement will discourage valuable feedback from the minority. Peer prediction (i.e. information elicitation without verification) [13], aims to design mechanisms that encourage honest subjective feedback from the user, even she is in the minority. In the setting where two users, say Alice and Bob, are asked to rate several similar products (e.g. restaurants), the peer prediction reward system will take their feedbacks as input and return them proper



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rewards. We want the reward system to be dominantly truthful. That is, for each user (who can belong to a minority group), regardless of other people's behaviors, she will obtain the highest amount of expected reward when she tells the truth and she will be paid the lowest in expectation if she reports some garbage feedback like five stars for all products.

To design dominantly truthful reward systems, Kong and Schoenebeck [10] propose an information-theoretic framework, Mutual Information Paradigm (MIP), to reduce the above mechanism design problem to the design of proper information measure. When the rating tasks are similar, we can assume that Alice and Bob's feedback for these tasks are i.i.d. samples of random variables  $\hat{X}_A, \hat{X}_B$ . MIP pays Alice and Bob the mutual information between  $\hat{X}_A, \hat{X}_B$  in expectation. The mutual information measure should be information-monotone. That is, any data-processing method performed on the random variables will decrease the mutual information. When MIP pays an information-monotone mutual information, the strategic behavior of Alice or Bob will decrease their expected payments since the strategy is a data-processing method. Thus, to design a dominantly truthful mechanism, it is sufficient to design an information measure which 1) is information-monotone; 2) can be estimated unbiasedly with a certain amount of samples.

The original Shannon mutual information satisfies the monotonicity property. However, it cannot be estimated unbiasedly with a finite number of samples thus cannot be used to construct the reward system that works for a finite number of tasks. A recent work [8] solves this issue by proposing a new mutual information measure, Determinant Mutual Information (DMI). Its corresponded mechanism, DMI-Mechanism, is dominantly truthful with only a finite number of tasks. The trick is that DMI's square has a polynomial format and the polynomial mutual information can be estimated unbiasedly with a finite number of tasks. DMI-Mechanism shows the existence of the finite-number-task dominantly truthful mechanism. However, The existence of other<sup>1</sup> finite-number-task dominantly truthful mechanisms remains to be an open question.

This work answers the above question by providing

- a new family of information-monotone information measures: volume mutual information (VMI), where DMI is a special case;
- a new family of practical dominantly-truthful multi-task peer prediction mechanisms, VMI-Mechanisms.

The family of mechanisms is constructed via the new mutual information family. In detail, to obtain the above results, the paper first formally show that every degree  $d$  *polynomial mutual information* can be used to construct the dominantly truthful peer prediction mechanisms that work for  $\geq d$  tasks. Most previous information measures are based on *distance method*. The construction of these measures rely on proper distance measures. However, these distance measures based mutual information do not have a polynomial format. This work proposes a novel geometric information measure design framework, *volume method*, to construct a new mutual information family, VMI. Previously, the square DMI is the only known polynomial mutual information even in the binary case. VMI contains a family of new<sup>2</sup> polynomial mutual information. We use these new polynomial mutual information measures to construct the new dominantly truthful peer prediction mechanisms that work for a finite number of tasks. To illustrate this new mutual information family better, we also provide a geometric

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<sup>1</sup> Other mechanisms means that these mechanisms are not simple transformations (e.g. affine transformation) of the DMI-Mechanism or based on a mutual information which is a polynomial of DMI (e.g.  $\text{DMI}^4 + \text{DMI}^2$ ).

<sup>2</sup> A polynomial mutual information is new if it is not a polynomial of DMI (e.g.  $\text{DMI}^4 + \text{DMI}^2$ ).

visualization in the binary case. The visualization provides a deeper understanding of the existed and new mutual information. For example, although the noise decreases the mutual information, the visualization shows that the original Shannon mutual information punishes the two-sided noises more than DMI, and punish the one-sided noises less than DMI.

Though this work is mainly motivated for answering the above open question, the volume mutual information is the main technical highlight of this work. The idea behind VMI is simple and natural. Given a pair of random variables  $X, Y$ , mutual information measure takes  $X$  and  $Y$ 's joint distribution as input and outputs their mutual information. Here  $(X', Y)$  is less informative than  $(X, Y)$  if we can perform an operation on  $X$  to obtain  $X'$  and this operation is independent of  $Y$ . A mutual information measure is information-monotone if the mutual information between  $X'$  and  $Y$  is less than that between  $X$  and  $Y$ . VMI measures how informative a distribution is by measuring the volume of distributions that is less informative than it. That is, the volume mutual information between  $X$  and  $Y$  is defined as follows:

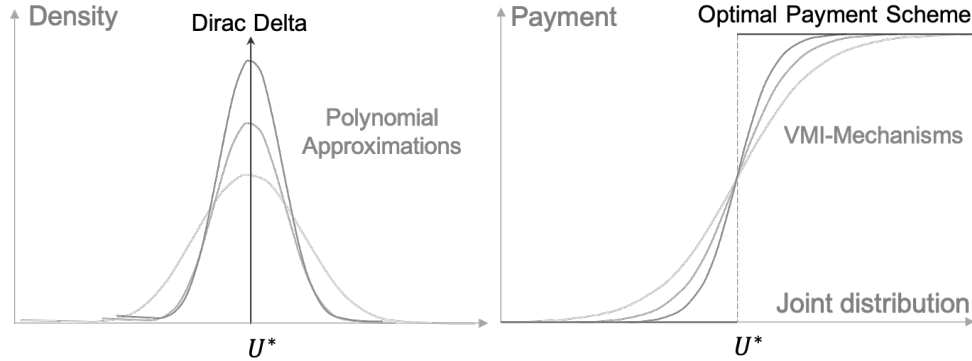
$$\text{VMI}(X; Y) := \text{Volume}(\{(X'; Y) | (X'; Y) \preceq (X; Y)\}).$$

Like other mutual information, volume mutual information operates on  $X$  and  $Y$ 's joint distribution. By assigning different densities to the space of joint distributions, we can obtain different formulas of volume mutual information with different properties. In particular, when the density function is a polynomial of the elements in joint distribution, we can obtain a family of polynomial volume mutual information as well.

Given a family of practical mechanisms, we have an optimization space. We then provide a tractable optimization goal and optimize over this family. If the participants do not need any effort to perform the tasks, we will focus on incentivizing the participants to tell the truth after they receive the signals. In this case, there is no need to construct more dominantly truthful, practical mechanisms. Thus, we consider the setting where participants require efforts to perform the tasks. In this setting, we want the participants not only to be honest after they have the signals but also to spend a certain amount of effort in obtaining the signals. We assume that the task requester has value for the elicited answers' distribution. We aim to maximize the requester's utility, which is defined as her value minus her payments for the participants. This work's analysis focuses on the setting where there are two participants, Alice and Bob.

It's left to optimize over the new VMI-Mechanisms. One way is to directly optimize over the new family. Another way is to optimize over all possible dominantly truthful mechanisms first. Then we can approximate the optimal mechanism (may not be practical) by a sequence of practical, dominantly truthful VMI-Mechanisms. It turns out the second way is easier for this problem. First, we observe that the optimal dominantly truthful payment scheme is a threshold payment scheme: there is a threshold joint distribution  $U^*$  and if Alice and Bob's reports' joint distribution is more informative than  $U^*$ , they will get a fixed amount of payments, otherwise, they get nothing. This payment scheme only works for an infinite number of tasks where we can perfectly estimate Alice and Bob's reports' joint distribution. However, there exists a sequence of practical VMI-Mechanisms that approximate the optimal threshold payment scheme. The idea is that the threshold payment scheme is a special VMI-Mechanism if we allow the density function to be a Dirac delta function on  $U^*$ . To construct a sequence of practical VMI-Mechanisms to approximate the threshold payment scheme, we use a sequence of polynomials to approximate the Dirac delta function. In the literature of proper scoring rules, there is a beta family of scoring rules [1, 12] which can be

used to approximate a threshold scoring rule, “misclassification” scoring. We are inspired to pick the multivariate Beta (Dirichlet) family to design a parametric family of VMI and use this family to approximate the optimal threshold payment scheme.



■ **Figure 1 An illustration of optimizing multi-task peer prediction mechanism:** the above figure illustrates the conceptual idea of optimizing the multi-task peer prediction mechanism. For ease of illustration, we draw the space of the joint distributions as a line though, in fact, it is not. The optimal payment scheme is a threshold function. A VMI-Mechanism corresponds to a density function over the space of the joint distributions. The Dirac delta density leads to the threshold payment scheme. We use a sequence of polynomial densities to approximate the Dirac delta density and use those densities to construct corresponding VMI-Mechanisms. Then we obtain a sequence of practical and dominantly truthful VMI-Mechanisms that approximate the optimal payment scheme.

Thus, we not only contribute a new family of practical dominantly truthful multi-task peer prediction mechanisms, but also provide a tractable effort incentive optimization goal. We show that under this goal, DMI-Mechanism may not be optimal but we can use our new family to construct a sequence of approximately optimal practical dominantly truthful mechanisms.

## 1.1 Related Work

Miller et al. [13] start the literature of peer prediction by considering the setting where the participants are asked a single question (e.g. do you like this restaurant or not?). They design a reward system where truth-telling is a strict equilibrium. However, this original peer prediction work requires the knowledge of the common prior over the participants. Prelec [16] proposes the Bayesian Truth Serum that removes this prior knowledge requirement by asking the participants to additionally report their forecasts for other people (e.g. what percentage of your peers like this restaurant?). However, Bayesian Truth Serum requires an infinite number of participants. Moreover, the additional forecast report requires additional efforts from the participants and makes the mechanism *non-minimal*. Radanovic and Faltings [18] extend Bayesian Truth Serum to a slightly different setting which involves sensors, but still requires a large number of agents. A series of works (e.g. [17, 4, 23, 9]) study how to remove the requirement for a large number of participants, while their mechanisms are non-minimal. Frongillo and Witkowski [5] focus on the design of minimal mechanisms where the participants do not need to report additional forecasts. However, when participants are only assigned a single task, they point out that minimal mechanisms require constraints on the participants’ belief model, i.e., are not *prior-independent*.

Dasgupta and Ghosh [3] start to consider the setting where the participants are assigned multiple similar tasks, the multi-task setting. In contrast to the single-task setting, the multi-task setting enables the design of both *prior-independent* and *minimal* mechanisms. In the multi-task setting, Radanovic et al. [19] use the distribution of reported answers from similar tasks as the prior probability of possible answers, while their mechanism requires the estimation of prior probability from a large number of tasks or participants. Kamble et al. [7] propose a mechanism where the participants can perform only a single task though the total number of tasks is large. However, this mechanism is not dominantly truthful and makes truth-telling only better than any symmetric equilibrium where all participants perform the same strategy. Shnayder et al. [21], Kong and Schoenebeck [10], Liu et al. [11] focus on the setting where there are a small number of participants and show that the dominantly truthful multi-task peer prediction mechanism exists. Kong and Schoenebeck [10] also provide a general information-theoretic framework for the design of the dominantly truthful peer prediction mechanisms. However, they all require the participants to perform an infinite number of tasks. Kong [8] addresses this issue by proposing the first dominantly truthful mechanism, DMI-Mechanism, which is prior-independent, minimal, and works for two participants and a finite number of tasks (practical). This mechanism is constructed by a new information measure, Determinant Mutual Information (DMI) whose square has a polynomial format. However, the existence of other practical dominantly truthful mechanisms remains to be an open question. This work answers the above question by providing a family of practical dominantly truthful peer prediction mechanisms, as well as a new family of information-monotone mutual information: volume mutual information (VMI), where DMI is a special case.

Regarding optimization in information elicitation, Neyman et al. [14], Hartline et al. [6], Zermeno [24], Merkle and Steyvers [12], Osband [15] focus on optimizing over proper scoring rules. Unlike this work, in the setting of scoring rules, the ground truth will be revealed later and the participants report only once. Cai et al. [2] consider the setting where workers are asked to report a data point and aim to find the optimal statistical estimator with the best effort incentives. We consider a very different setting. Frongillo and Witkowski [5] optimize over single-task peer prediction mechanisms where their mechanisms are not dominantly truthful. In contrast, we focus on the multi-task peer prediction setting where ground truth does not exist and the participants will perform multiple tasks. Moreover, we optimize over dominantly truthful, and practical mechanisms.

## 1.2 Multi-task Peer Prediction and Mutual Information

In this section, we will show how to employ information-monotone mutual information measures to design dominantly truthful mechanisms. We will also connect polynomial mutual information to the practical mechanism. Then we can reduce the design of the dominantly truthful and practical mechanism to information-monotone polynomial mutual information.

### Multi-task Peer Prediction

We focus on the setting where there are two participants, Alice and Bob, and a task requester. Alice and Bob are assigned  $T$  a priori similar tasks. For each task  $t$ , after performing the task, each participant  $i = A, B$  will receive a private signal  $c_i^t \in \mathcal{C}$  where  $\mathcal{C}$  is a size  $C$  choice set. For binary questions,  $C = 2$ . By assuming the tasks are a priori similar, the participants' honest signals  $\{(c_A^t, c_B^t)\}_t$  are  $T$  i.i.d. samples from random variables  $(X_A, X_B)$  whose distribution is denoted by  $U_{A,B}$ .  $U_{A,B}$  can be seen as a  $C \times C$  matrix where  $U_{A,B}(c_A, c_B)$  is the probability that  $(X_A, X_B) = (c_A, c_B)$ . A multi-task peer prediction mechanism will take all participants' reports  $\{(c_A^t, c_B^t)\}_{t=1}^T$  as input and output their corresponding payments  $p_A, p_B$ .

### Report Strategy Model

Alice may lie and her *strategy*  $S_A^t$  for each task  $t$  can be seen as a  $C \times C$  stochastic matrix where  $S_A^t(c_A^t, c_A^t)$  is the probability she reports  $\hat{c}_A^t$  given that she receives  $c_A^t$ . We follow Kong [8] and assume that every participant plays the consistent strategy for all tasks. That is, there exists  $S_A$  such that  $\forall t, S_A^t = S_A$ . We model Bob analogously. With this assumption, not only the participants' honest signals are i.i.d. samples, but also their reported signals are i.i.d. samples from random variables  $(\hat{X}_A, \hat{X}_B)$  whose distribution is denoted by  $\hat{U}_{A,B}$ . A strategy  $S$  is uninformative if it is independent of private signals, i.e.,  $S(\hat{c}, c) = S(\hat{c}, c')$  for all  $c, c', \hat{c} \in C$ .

► **Definition 1** (Dominantly truthful). *A multi-task peer prediction mechanism is dominantly truthful if, for all participants, truthful report strategy maximizes her expected payment regardless of other people's strategies; and if she believes other participants tell the truth, the truthful report strategy will be strictly better than uninformative report strategies.*

The second requirement guarantees that the flat payment mechanism is not dominantly truthful. With the above report strategy model, for a dominantly truthful mechanism where Alice's expected payment is represented as  $\mathcal{P}_A(\hat{U}_{A,B})$  and Bob's expected payment is represented as  $\mathcal{P}_B(\hat{U}_{A,B})$ , we have  $\forall S_A, S_B, U_{A,B}$ ,

$$\mathcal{P}_A(S_A U_{A,B} S_B^\top) \leq \mathcal{P}_A(U_{A,B} S_B^\top) \quad \mathcal{P}_B(S_A U_{A,B} S_B^\top) \leq \mathcal{P}_B(S_A U_{A,B}).$$

Kong and Schoenebeck [10] introduce an information-theoretic framework, Mutual Information Paradigm (MIP), for the design of dominantly truthful multi-task peer prediction mechanisms. MIP pays each participant the mutual information between her report and her peer's report. Once the mutual information is information-monotone, each participant will be incentivized to tell the truth to avoid the loss of information. We start to formally define information-monotonicity.

Let  $U_{X,Y}$  be a joint distribution over two random variables  $X$  and  $Y$ . We want to design an information measure MI that takes  $U_{X,Y}$  as input and outputs a non-negative real number, which reflects the amount of information contained in  $X$  that is related to  $Y$ . We also want MI to be *information-monotone*: when  $X'$  is "less informative" than  $X$  with respect to  $Y$ ,  $\text{MI}(U_{X',Y})$  should be less than  $\text{MI}(U_{X,Y})$ . Typically, the literature also writes  $\text{MI}(U_{X,Y})$  as  $\text{MI}(X; Y)$ . The following definition is the formal definition of information-monotonicity.

► **Definition 2** (Information-monotonicity). [20] *MI is information-monotone if for every two random variables  $X, Y$ , when  $X'$  is less informative than  $X$  with respect to  $Y$ , i.e.,  $X'$  is independent of  $Y$  conditioning  $X$ ,*

$$\text{MI}(X'; Y) \leq \text{MI}(X; Y).$$

Mutual information requires the distribution as input while we only have samples. However, since the participants are assumed to be the expected payment maximizer, the unbiased estimator is sufficient.

### Unbiased estimator of mutual information

Given a mutual information MI,  $\text{UBE}^{\text{MI}}$  is an unbiased estimator of MI with  $\geq r$  sample if for every two random variables  $(X, Y)$ , when  $\{(x_t, y_t)\}_{t=1}^T$  are  $T \geq r$  independent samples of  $(X, Y)$ ,

$$\mathbb{E}[\text{UBE}^{\text{MI}}(\{(x_t, y_t)\}_{t=1}^T)] = \text{MI}(X; Y).$$

### Mutual Information Paradigm(UBE<sup>MI</sup>)

Alice and Bob are assigned  $T \geq r$  a priori similar tasks in independent random orders. The participants finish the tasks without any communication.

**Report** For each task  $t$ , Alice privately receives  $c_A^t$  and reports  $\hat{c}_A^t$  and Bob is analogous.

**Payment** Alice's payment is  $p_A := \text{UBE}^{\text{MI}}(\{\hat{c}_A^t, \hat{c}_B^t\}_{t=1}^T)$  where  $\text{UBE}^{\text{MI}}$  is an unbiased estimator of an information-monotone MI that works for  $\geq r$  samples. Bob is analogous.

We say agents' prior is informative for MI if the mutual information tween their truthful reports are positive, i.e.,  $\text{MI}(X_A; X_B) > 0$ . This assumption is required to guarantee the second property of dominant truthfulness.

► **Lemma 3.** *When MI is information-monotone, non-negative, and vanishes for independent random variables, if agents' prior is informative with respect to MI, then the mutual information paradigm  $\text{UBE}^{\text{MI}}$  is dominantly truthful.*

**Proof.** In expectation, Alice's payment is  $\text{MI}(\hat{X}_A; \hat{X}_B)$  which will be maximized if she tells the truth. If agents' prior is informative with respect to MI and Alice believes Bob tells the truth, Alice's expected payment when she tells the truth will be  $\geq \text{MI}(X_A; X_B) > 0$ . If she reports uninformative signals, her expected payment will be zero since MI vanishes for independent random variables. Thus, the second property of dominant truthfulness is also satisfied. ◀

To design a practical dominantly truthful mechanism, the unbiased estimator needs to work for only a finite number of samples. We will show that once the mutual information is a degree  $d$  polynomial, it has an unbiased estimator that works for  $\geq d$  samples. Currently, the only example of polynomial mutual information is DMI's square.

► **Definition 4** (Polynomial Mutual Information). *MI is a polynomial mutual information when  $\text{MI}(X; Y)$  a multivariate polynomial of the entries of  $U_{X,Y}$ .*

► **Definition 5** (Determinant based Mutual Information (DMI) [8]).

$$\text{DMI}(X; Y) := |\det(U_{X,Y})|.$$

DMI is not a polynomial mutual information while DMI's square is. For example, in the binary case for every joint distribution matrix  $U_{X,Y} = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$ ,  $\text{DMI}(X; Y) = |u_{00}u_{11} - u_{10}u_{01}|$  is not a polynomial while  $\text{DMI}^2(X; Y) = (u_{00}u_{11} - u_{10}u_{01})^2$  is a polynomial.

► **Lemma 6.** *Every degree  $d$  polynomial mutual information MI has an unbiased estimator  $\text{UBE}^{\text{MI}}$  for  $T \geq d$  samples.*

**Proof.** Every degree  $d$  polynomial mutual information MI can be written as the sum of terms of format  $\Pr[X = c_1, Y = c'_1] * \Pr[X = c_2, Y = c'_2] * \dots * \Pr[X = c_k, Y = c'_k], k \leq d$ .

For each term  $\Pr[X = c_1, Y = c'_1] * \Pr[X = c_2, Y = c'_2] * \dots * \Pr[X = c_k, Y = c'_k], k \leq d$ , when we have  $k$  independent samples  $(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)$  of  $X, Y$ ,  $\prod_{i=1}^k \mathbb{1}(x_i = c_i, y_i = c'_i)$  is an unbiased estimator. Thus, since  $k \leq d$ ,  $T \geq d$  independent samples is sufficient to construct an unbiased estimator of each term as well as the sum of these terms MI. ◀

The above lemma shows that every degree  $d$  polynomial mutual information corresponds to a dominantly truthful peer prediction mechanism that works for  $\geq d$  tasks. For example, DMI's square is a degree  $2C$  polynomial. DMI-Mechanism [8] is constructed via an unbiased estimator of DMI's square and requires  $\geq 2C$  tasks.

## 2 Volume Mutual Information

This section will introduce the volume method and apply the volume method to obtain a new family of information-monotone mutual information measure, Volume Mutual Information (VMI), which can be polynomials.

### 2.1 Volume Method

Given a partially ordered set (poset)  $(L, \preceq)$ , we define the *lower set* of  $\ell$ 's as  $\downarrow \ell := \{\ell' \mid \ell' \in L, \ell' \preceq \ell\}$ . In discrete case, volume method measures each element by the size of its lower set. In continuous case, we need a monotone measure  $\mu$  and integral  $\int d\mu$  on  $L$ . That is,  $\mu$  assigns higher volume to bigger set and for two integrable real-valued functions  $f \leq g$  on  $X$ ,  $\int_X f d\mu \leq \int_X g d\mu$ . We defer the basic definitions for measure and integral to full version.

We assume that all lower sets are measurable with  $\mu$ . Since the higher-order element has a larger lower set, the volume of each element's lower set

$$V(\ell) := \text{Vol}(\downarrow \ell) := \mu(\downarrow \ell)$$

is a natural monotone function with respect to the partial order. More generally, we define a weighted version:

► **Definition 7** (Volume function). *Given a poset  $(L, \preceq)$  with a monotone measure  $\mu$  and a monotone integral  $\int d\mu$  on  $L$ , for every integrable non-negative density function  $w : L \mapsto \mathbb{R}^+$ , we define the volume function that is associated with  $w$  as*

$$V^w(\ell) := \text{Vol}^w(\downarrow \ell) := \int_{\downarrow \ell} w(x) d\mu(x).$$

When  $w(x) = 1$  everywhere,  $V^w(\ell) = V(\ell)$ .

► **Lemma 8.** *The volume function  $V^w : L \mapsto \mathbb{R}^+$  is a non-negative monotone function.*

The above lemma shows that  $V^w$  extends a partial order to a total order.

**Proof.** When  $\ell' \preceq \ell$ , since  $\preceq$  is transitive,  $\downarrow \ell' \subset \downarrow \ell$ . Due to the fact that the measure and the integral are monotone,  $V^w(\ell)$  is also monotone. ◀

### 2.2 Information-monotone Partial Order

To apply the volume method to the design of mutual information, we first use information-monotonicity to define a partial order among the joint distributions.  $U_{X',Y} \preceq U_{X,Y}$  iff  $X'$  is less informative than  $X$  with respect to  $Y$ , i.e,  $X'$  is independent of  $Y$  conditioning  $X$ . We will show that this is equivalent to the following definition.

► **Definition 9** ( $(L, \preceq)$  for MI). *We define domain  $L$  as the set of all possible  $C \times C$  joint distribution matrices.  $U' \preceq U$  if there exists a column-stochastic<sup>3</sup> matrix  $T$  such that  $U' = TU$ .*

<sup>3</sup> A matrix  $T$  is a column-stochastic matrix iff every entry of  $T$  is non-negative and every column of  $T$  sums to 1.



► **Example 10.**

$$\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} U \cong^4 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} U \preceq \begin{bmatrix} .5 & 0 \\ .5 & 1 \end{bmatrix} U \preceq U.$$

The first equality holds since  $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$  and  $\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ .

The second partial order is valid since  $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} .5 & 0 \\ .5 & 1 \end{bmatrix}$ .

The following lemma shows that designing information-monotone mutual information is equivalent to designing a monotone function on  $(L, \preceq)$ .

► **Lemma 11.** *MI is information-monotone if and only if MI is a monotone function on  $(L, \preceq)$ .*

**Proof.** We first show the  $\Leftarrow$  direction. when  $X'$  is less informative than  $X$  with respect to  $Y$ , i.e.,  $X'$  is independent of  $Y$  conditioning  $X$ ,

$$U_{X',Y}(x', y) = \Pr[X' = x', Y = y] = \sum_x \Pr[X' = x'|X = x] \Pr[X = x, Y = y].$$

Thus,  $U_{X',Y} = U_{X'|X}U_{X,Y}$ . Since  $U_{X'|X}$  is a column-stochastic matrix,  $U_{X',Y} \preceq U_{X,Y}$ . When MI is a monotone function on  $(L, \preceq)$ , MI is information-monotone.

To show the opposite direction, we start from the situation that MI is information-monotone. For every  $U$ , for every column-stochastic matrix  $T$ , we only need to show there exists  $X, X', Y$  such that  $X'$  is less informative than  $X$  and  $U_{X,Y} = U$  and  $U_{X',Y} = TU$ . We can construct such  $X, X', Y$  by setting  $\Pr[X = x, X' = x', Y = y] = U_{X,Y}(x, y)U_{X'|X}(x', x)$  for every  $x, x', y$ . Here  $U_{X,Y}(x, y) := U(x, y)$  and  $U_{X'|X}(x', x) := T(x', x)$ . It's easy to see that  $U_{X',Y} = TU$  and  $X'$  is less informative than  $X$  for  $Y$ .

Thus,  $\text{MI}(TU) = \text{MI}(X'; Y) \leq \text{MI}(X; Y) = \text{MI}(U)$ . The inequality follows from the fact that MI is information-monotone. Therefore, MI is also monotone on the poset and the  $\Rightarrow$  direction is also valid. ◀

## 2.3 Constructing Volume Mutual Information

This section will apply the volume method to obtain a new family of monotone mutual information measures, Volume Mutual Information (VMI). We have already defined the poset. Thus, to apply the volume method, we only need to pick the measure and integral.

We will use Hausdorff measure [22]. Intuitively, to provide a measure for any triangle's area on  $\mathbb{R}^2$ , the 2-dimensional Lebesgue measure  $\mathcal{L}^2$  works. However,  $\mathcal{L}^2$  will assign zero measure to any curve in  $\mathbb{R}^2$ . To provide a measure for a curve's length in  $\mathbb{R}^2$ , we can use the Hausdorff measure  $\mathcal{H}^1$ . We defer more introduction about the basic measure theory to the full version.

► **Definition 12** ( $(L, \preceq, \mu, f)$  for MI). *We define domain  $L$  as the set of all possible  $C \times C$  joint distribution matrices.  $U' \preceq U$  if there exists a column-stochastic matrix  $T$  such that  $U' = TU$ . We vectorize matrices and transform  $L$  to space in  $\mathbb{R}^{C^2}$ . We pick  $\mu$  as the  $C(C-1)$ -dimensional<sup>5</sup> Hausdorff measure  $\mathcal{H}^{C(C-1)}$ .*

<sup>5</sup> Though  $L$  is a subset of a  $C^2$  dimensional space, in the later sections, we will see the lower set has at most  $C(C-1)$  dimension.

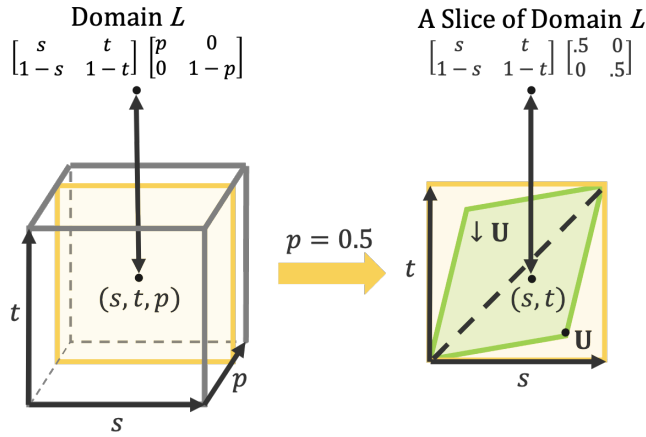
► **Example 13** ( $(L, \preceq, \mu, f)$  in binary case). The following observation allows us to visually illustrate  $(L, \preceq, \mu, f)$  for binary case in Figure 2.

► **Observation 14.** In binary case, there is an one to one mapping from  $[0, 1]^3$  to  $L$ . In fact,

$$L = \left\{ \begin{bmatrix} s & t \\ 1-s & 1-t \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix} \mid s, t, p \in [0, 1] \right\}.$$

Fixing  $p$ , there is an one to one mapping from  $[0, 1]^2$  to  $\downarrow U_p$  where  $U_p = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}$ .  $\downarrow U_p$  is the space of all joint distribution matrices whose column sum is  $(p, 1-p)$  and  $L = \{\downarrow U_p \mid p \in [0, 1]\}$ .

The proof is deferred to the full version.



■ **Figure 2 Visual illustration  $(L, \preceq, \mu, f)$  in binary case:**

- Domain  $L$ : there exists a one to one mapping from the domain  $L$  to a unit cube  $[0, 1]^3$ . Thus, we visualize  $L$  as a unit cube. The right square represents a slice of  $L$ ,  $\downarrow U_{.5}$ , the space of all joint distribution matrices whose column sum is  $(.5, .5)$ .
- Lower set  $\downarrow U$ : for each element  $U$ , all  $U' \preceq U$  constitute a parallelogram (the light green area) whose endpoints are  $\{U, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} U, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} U, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} U\}$ . This parallelogram is also called  $U$ 's lower set.
- Uninformative set: when  $s = t$  (the black dashed line), the distribution represents independent  $X$  and  $Y$ . In this case, the mutual information should be zero. We call the set of these independent distributions the uninformative set.
- Measure  $\mu$ : since the lower set is always on a 2-dimensional space, we use the 2-dimensional Hausdorff measure  $\mathcal{H}^2$  to measure the area of the parallelogram in  $\mathbb{R}^3$ .

► **Definition 15** (Volume Mutual Information  $\text{VMI}^w$ ). Given an integrable non-negative density function  $w$ , we define the Volume Mutual Information as

$$\text{VMI}^w(X; Y) := V^w(U_{X,Y}) = \text{Vol}^w(\downarrow U_{X,Y}) = \int_{\downarrow U_{X,Y}} w(x) d\mathcal{H}^{C(C-1)}(x).$$

Aided by programming, we can obtain the explicit formula of VMI (Example 20). The choice of density functions affects the property of VMI. Theoretically, we will show that uniform density leads to DMI and polynomial density obtains polynomial VMI (Theorem 16),

which leads to a family of practical dominantly truthful peer prediction mechanisms (Corollary 17). Numerically, we will show the influence of density visually by three concrete binary VMI (Example 20). To state the theorem formally, we first give a formal definition for polynomial mutual information.

► **Theorem 16.** *VMI<sup>w</sup> is an information-monotone mutual information. VMI<sup>w</sup> is also non-negative and when  $X$  and  $Y$  are independent,  $VMI^w(X; Y) = 0$ . Moreover, Uniform density with the uniform density,  $VMI(X; Y) \propto DMI(X; Y)^{C-1}$ ;*

*Polynomial density when the density function  $w$  is a non-negative degree  $d_w$  polynomial, when  $C$  is an odd number,  $VMI^D$  is a degree  $d_w + C(C - 1)$  information-monotone polynomial mutual information and when  $C$  is an even number,  $DMI * VMI^D$  is a degree  $d_w + C^2$  information-monotone polynomial mutual information.  $(VMI^D)^2$  is a degree  $2(d_w + C(C - 1))$  information-monotone polynomial mutual information.*

Every degree  $d$  polynomial monotone mutual information directly induce a dominantly truthful multi-task peer prediction mechanism that works for  $\geq d$  tasks (Lemma 6).

► **Corollary 17.** *There exists a family of practical, dominantly truthful and prior-independent multi-task peer prediction mechanisms.*

**Proof of Corollary 17.** Theorem 16 shows the existence of a family of polynomial mutual information. Lemma 6 shows that each degree  $d$  polynomial mutual information MI has an unbiased estimator with  $\geq d$  samples. Lemma 3 shows that when agents' prior is informative for MI, we can use the above unbiased estimator to construct a dominantly truthful peer prediction mechanism that works for  $\geq d$  tasks. ◀

We have proved that polynomial VMI can be used to construct practical mechanisms. In full version 2.4, we will also provide a concrete example for VMI-Mechanism in the binary case.

### Proof outline for Theorem 16

The fact that  $VMI^w$  is information-monotone follows directly from Lemma 8 and Lemma 11. We will apply the area formula to prove the other parts. With the uniform density, to show that  $VMI(X; Y) \propto DMI(X; Y)^{C-1}$ , we only need to show the original volume of the lower set is proportional to  $DMI(X; Y)^{C-1}$ . We will construct a proper affine mapping from  $\mathbb{R}^{C(C-1)}$  to  $L$  and directly apply the area formula to show this result. To show the last part of this theorem, we will write down the integration explicitly and then analyze it. We defer the full proof to the full version.

## 2.4 Visualization of Binary Volume Mutual Information

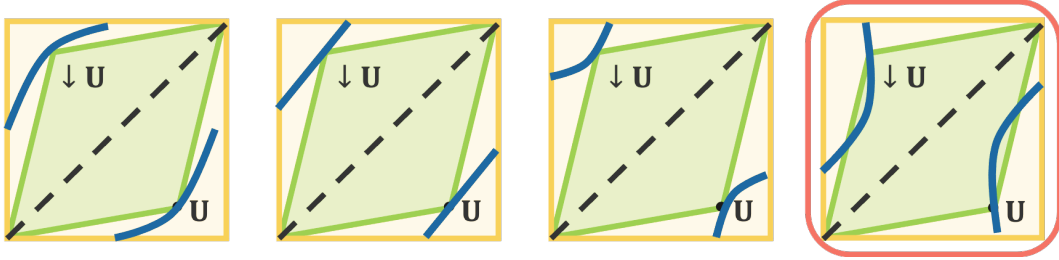
This section will provide a visualization method for all binary mutual information. By using this visualization method, we visualize three new VMIs for three styles of densities (mountain, plain, basin). We additionally visualize two existed mutual information measures in full version 2.4.

► **Definition 18** (Contour plots of binary MIs). *In binary case, the mutual information can be seen as a function with 3 variables:*

$$MI^{3d}(s, t, p; MI) := MI\left(\begin{bmatrix} s & t \\ 1-s & 1-t \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}\right).$$

To visualize the contour plot in a 2 dimensional space, we fix  $p = p_0$  and draw the contours of  $MI^{2d}(s, t, p_0, MI) := MI^{3d}(s, t, p_0; MI)$  on slice  $p = p_0$ .

Figure 3 illustrates the contours for information-monotone MIs and a MI that is not information-monotone.



■ **Figure 3 Information-monotone MI vs Un-information-monotone MI:** the first three figures illustrate the contours of different information-monotone MIs. In these figures, the contours on each element  $U$  (the blue lines) must always contain  $U$ 's lower set (the green parallelogram). The last figure (with a red frame) illustrates the contours of a MI which is not information-monotone.

We first visualize multiple commonly used MIs and compare their contours in the same square slice.

### Visualization of Commonly Used Mutual Information

We will visualize two existed commonly used mutual information measures in this section. These measures are designed by a distance-based approach. For two random variables  $X$  and  $Y$ ,  $U_Y$  represents the prior distribution over  $Y$  when we have no information. That is  $U_Y(y) = \Pr[Y = y]$ .  $U_{Y|x}$  denotes the posterior distribution  $Y$ , i.e.  $U_{Y|x}(y) = \Pr[Y = y|X = x]$  when we have information  $X = x$ . When  $X$  and  $Y$  are independent, knowing  $X$  will not change our belief for  $Y$ , i.e.,  $U_{Y|x}$  equals  $U_Y$ . When  $X$  and  $Y$  are highly correlated, knowing  $X$  changes the belief for  $Y$  a lot, i.e.,  $U_{Y|x}$  is quite different from  $U_Y$ . Intuitively, we can use the “distance” between the informative prediction  $U_{Y|x}$  and the uninformative prediction  $U_Y$  to represent the mutual information between  $X$  and  $Y$ . The distance measure should be picked carefully to satisfy information-monotonicity. Two different distance families,  $f$ -divergence  $D_f(\cdot, \cdot)$  and Bregman-divergence  $D_{PS}(\cdot, \cdot)$ , can induce two families of information-monotone mutual information measures [10]. We list these measures here.

- $f$  Mutual Information ( $FMI^f$ ):  $E_{x \leftarrow U_X} D_f(U_{Y|x}, U_Y)$
- Bregman Mutual Information ( $BMI^{PS}$ ):  $E_{x \leftarrow U_X} D_{PS}(U_{Y|x}, U_Y)$

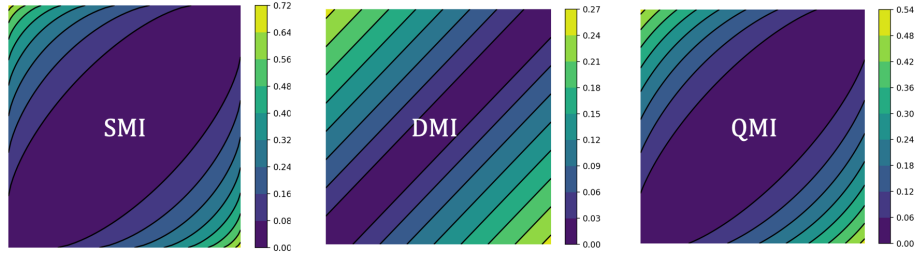
We then give two special cases of the above families. The commonly used KL-divergence belongs to both of the families and induces the classic Shannon mutual information. The commonly used scoring rule, the quadratic scoring rule, induces the quadratic mutual information.

- Shannon Mutual Information (SMI):  $E_{x \leftarrow U_X} D_{KL}(U_{Y|x}, U_Y)$
- Quadratic Mutual Information (QMI):  $E_{x \leftarrow U_X} \|U_{Y|x} - U_Y\|^2$

We visualize SMI and QMI, as well as DMI in Figure 4.

### Visualization of Binary Volume Information

We use the results of Lemma 19 and employ the computer to compute the indefinite integration and obtain the explicit formula of  $VMI^w$  in the binary case.



■ **Figure 4** Contours of SMI, DMI, QMI on slice  $p = .5$ : DMI has the parallel lines “|||” as contours and both SMI and QMI have shapes like “()”. Compared with “|||”, This “()” shape contour will punish two-sided noise (far from the square frame’s boundary) more and one-sided noise (on the boundary of the square frame) less.

► **Lemma 19.** *In binary case,*

$$\begin{aligned} \text{VMI}^w(U) &= 2|\det(U)| \int_{s=0}^1 \int_{t=0}^1 w\left(\begin{bmatrix} s & t \\ 1-s & 1-t \end{bmatrix} U\right) ds dt \\ &= 2|u_{00}u_{11} - u_{10}u_{01}| \int_{s=0}^1 \int_{t=0}^1 w\left(\begin{bmatrix} s & t \\ 1-s & 1-t \end{bmatrix} \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}\right) ds dt \end{aligned}$$

We defer the proof to full version.

► **Example 20.** Here we provide three concrete examples to show how the choice of density will affect the corresponding volume mutual information.

We pick the  $p_0 = .5$  slice to illustrate the 2-dimensional contour of the VMIs, which is the contour of  $\text{MI}^{2d}(s, t; .5, \text{VMI}^w)$ . We will also draw the heatmap of the density function. In the  $p_0 = .5$  slice, in the new coordinates, the density function changes to  $w^{2d}(s, t) := w\left(\begin{bmatrix} s & t \\ 1-s & 1-t \end{bmatrix} \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}\right)$ .

1. **Mountain**  $w\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 16abcd$ ,  $w^{2d}(s, t) = s(1-s)t(1-t)$ :

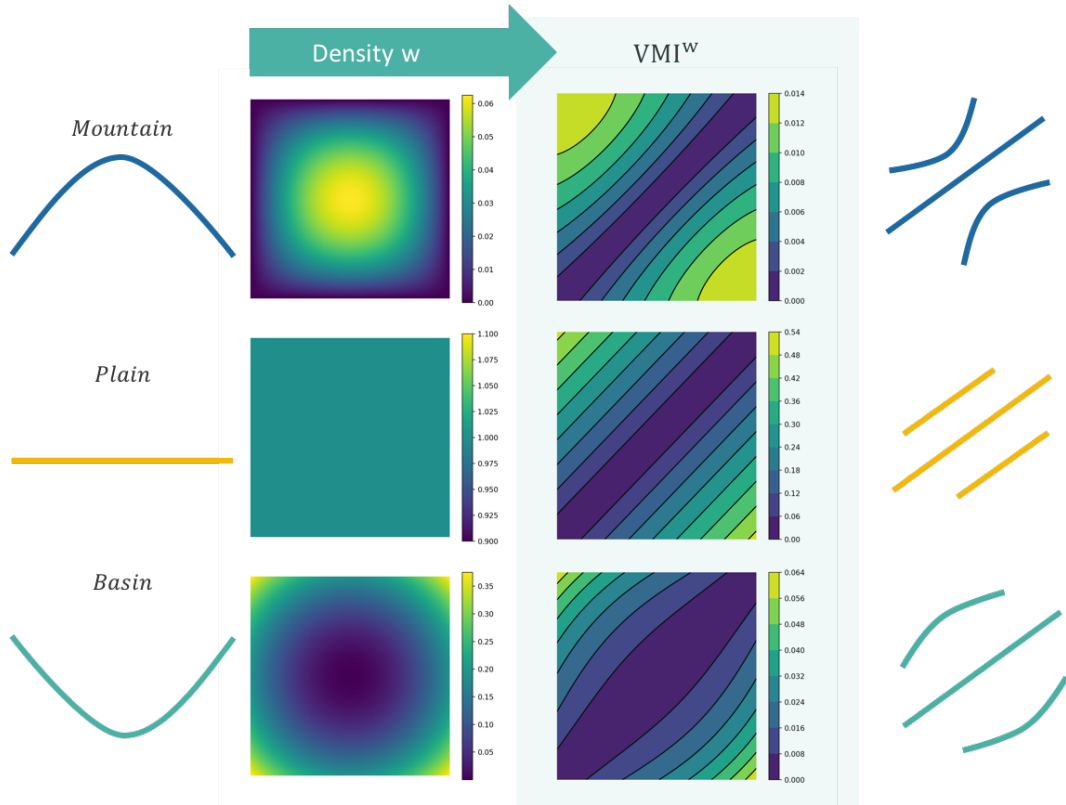
This density function is called “Mountain” since the center has a higher density than its surroundings. The highest density will be obtained when  $s = t = .5$ .

$$\begin{aligned} \text{VMI}^w(U) &= 2|\det(U)| \left( \frac{8u_{00}^2}{15}u_{01}^2 + \frac{4u_{01}}{3}u_{00}^2u_{11} + \frac{4u_{00}^2}{9}u_{11}^2 \right. \\ &\quad \left. + \frac{4u_{00}}{3}u_{01}^2u_{10} + \frac{40u_{00}}{9}u_{01}u_{10}u_{11} + \right. \\ &\quad \left. \frac{4u_{00}}{3}u_{10}u_{11}^2 + \frac{4u_{01}^2}{9}u_{10}^2 + \frac{4u_{01}}{3}u_{10}^2u_{11} + \frac{8u_{10}^2}{15}u_{11}^2 \right) \end{aligned}$$

2. **Plain**  $w\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 1$ ,  $w^{2d}(s, t) = 1$ :

$$\text{VMI}^w(U) = 2|\det(U)|$$

3. **Basin**  $w\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 3((a - .25)^2 + (b - .25)^2)$ ,  $w^{2d}(s, t) = \frac{3}{4}((s - .5)^2 + (t - .5)^2)$ :



■ **Figure 5** From density function  $w$  to  $VMI^w$ : the left column shows the heatmaps of the density functions  $w$  and the right column shows the contours of their corresponding  $VMI^w$ s. The “plain” shape density has uniform density everywhere. Its corresponding VMI, DMI, has the parallel lines “||” as contours. The “mountain” shape density has the highest density in the center. This will lead to a VMI whose contour is like “) (“. Compared with “||”, This “) (“ shape contour will punish one-sided noise (e.g. say “like” when “like”, say “hate” w.p.  $\frac{1}{2}$  when “hate” ) more. The “basin” shape density has the lowest density in the center. This will lead to a VMI whose contour is like “( )”. Compared with “||”, This “( )” shape contour will punish two-sided noise more.

This density function is called “Basin” since the center has a lower density than its surroundings. The lowest density will be obtained when  $s = t = .5$ .

$$VMI^w(U) = 2|\det(U)|(u_{00}^2 + 1.5u_{00}u_{10} + u_{01}^2 + 1.5u_{01}u_{11} + u_{10}^2 + u_{11}^2 - 0.375)$$

The visualizations of  $w$  and  $VMI^w$  are presented in Figure 5.

The above example also provides three concrete polynomial mutual information by multiplying  $|\det(U)|$  to each of them. The plain one corresponds to DMI’s square while the mountain and basin density provide two new polynomial mutual information for the binary case, which leads to two new practical dominantly truthful peer prediction mechanisms.

### Visualization of a New Practical Dominantly Truthful Mechanism

We have proved that polynomial VMI can be used to construct practical mechanism. Here we will also provide a concrete example in the binary case. We use a new polynomial binary mutual information  $VMI^*$  to construct a new peer prediction mechanism in the binary case.

Our results work for non-binary case, this example uses the binary case for ease of illustration. We pick the “Mountain” case (Example 20) and multiply  $\det(U)$  to obtain a new polynomial binary mutual information  $\text{VMI}^*$ .

$$\begin{aligned} \text{VMI}^*(X; Y) = & 2(u_{00}u_{11} - u_{01}u_{10})^2 \left( \frac{8u_{00}^2}{15}u_{01}^2 + \frac{4u_{01}}{3}u_{00}^2u_{11} + \frac{4u_{00}^2}{9}u_{11}^2 \right. \\ & + \frac{4u_{00}}{3}u_{01}^2u_{10} + \frac{40u_{00}}{9}u_{01}u_{10}u_{11} + \\ & \left. \frac{4u_{00}}{3}u_{10}u_{11}^2 + \frac{4u_{01}^2}{9}u_{10}^2 + \frac{4u_{01}}{3}u_{10}^2u_{11} + \frac{8u_{10}^2}{15}u_{11}^2 \right) \end{aligned}$$

where  $U = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$  is the joint distribution matrix of  $X, Y$ .

It’s hard to tell that  $\text{VMI}^*$  satisfies the information-monotonicity from the above formula while  $\text{VMI}^*$ ’s contour plot (Figure 6) intuitively shows the monotonicity. Section 2.4 shows that  $\text{VMI}^*$  is information-monotone from its construction. With  $\text{VMI}^*$ ’s formula, we can construct a new constant-round dominantly truthful mechanism in the binary case by paying the participants the unbiased estimator of  $\text{VMI}^*$ . Previously, DMI-Mechanism is the only known constant-round dominantly truthful mechanism.

### VMI\*-Mechanism

$n$  participants are assigned  $T \geq 8$  a priori similar tasks. The participants finish the tasks without any communication.

**Report** For each task  $t$ , each participant  $i$  privately receives  $c_i^t$  and reports  $\hat{c}_i^t$ .

**Payment** For every two agents  $i \neq j \in [n]$ , we arbitrarily pick 8 tasks and  $E_t(c, c')$  is a binary indicator event such that  $E_t(c, c') = 1$  if for task  $t$ , agent  $i$ ’s answer is  $c$  and agent  $j$ ’s answer is  $c'$ . Otherwise,  $E_t(c, c') = 0$ . We define

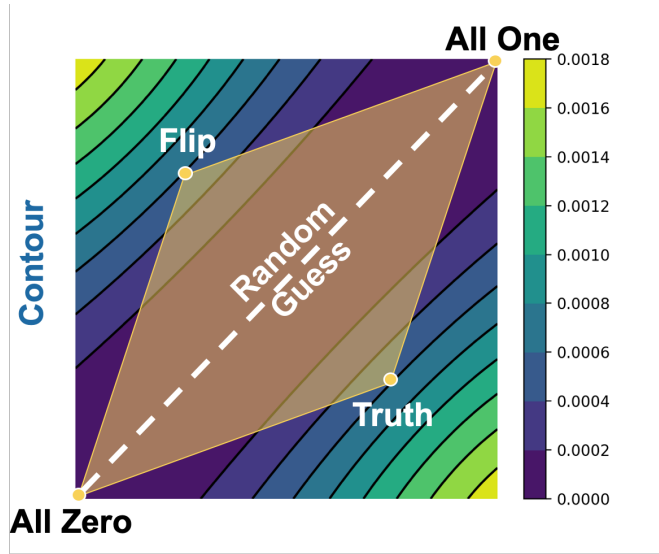
$$\begin{aligned} p_{ij} := & 2(E_1(0,0)E_2(1,1) - E_1(0,1)E_2(1,0))(E_3(0,0)E_4(1,1) - E_3(0,1)E_4(1,0)) \\ & \left( \frac{8E_5(0,0)E_6(0,0)}{15}E_7(0,1)E_8(0,1) + \frac{4E_5(0,1)}{3}E_6(0,0)E_7(0,0)E_8(1,1) \right. \\ & + \frac{4E_5(0,0)E_6(0,0)}{9}E_7(1,1)E_8(1,1) + \frac{4E_5(0,0)}{3}E_6(0,1)E_7(0,1)E_8(1,0) \\ & + \frac{40E_5(0,0)}{9}E_6(0,1)E_7(1,0)E_8(1,1) + \frac{4E_5(0,0)}{3}E_6(1,0)E_7(1,1)E_8(1,1) \\ & + \frac{4E_5(0,1)E_6(0,1)}{9}E_7(1,0)E_8(1,0) + \frac{4E_5(0,1)}{3}E_6(1,0)E_7(1,0)E_8(1,1) \\ & \left. + \frac{8E_5(1,0)E_6(1,0)}{15}E_7(1,1)E_8(1,1) \right) \end{aligned}$$

Agent  $i$ ’s payment is  $p_i := \sum_{j \neq i \in [n]} p_{ij}$

The above mechanism is a special mutual information paradigm by using  $\text{VMI}^*$ ’s unbiased estimator. According to Lemma 3,  $\text{VMI}^*$ -Mechanism is dominantly truthful, prior-independent and works for  $\geq 8$  tasks.

## 3 Optimizing Multi-task Peer Prediction

Finally this section will discuss the optimization of multi-task peer prediction and use VMI to construct the optimal multi-task peer prediction. We start by introducing the optimization goal. The dominant truthfulness guarantees that truth-telling is the best report strategy,



■ **Figure 6** Illustration for VMI\*-Mechanism: Alice and Bob participate in the mechanism. Fixing Bob’s strategy, when  $U$  is the joint distribution over Bob and honest Alice’s reports, Alice’s strategy  $\mathbf{S}$  corresponds to joint distribution  $\mathbf{S}U$ . We draw the contours of VMI\* on the slice on  $U$  and visualize Alice’s strategy simultaneously. All strategies consist of a light yellow parallelogram with four pure strategies as vertices: truth-telling, always flipping the answer, always answering zero, always answering one. From the plot, when Alice tells the truth or always flips her answer, she will be paid the highest. When Alice reports uninformative answer like always saying zero/one or random guessing without looking at the questions, she will be paid zero, i.e., the lowest.

given that the participants receive the signals, while it may not give the participants incentive to spend a sufficient amount of effort to perform the tasks. Most previous work’s analysis focuses on the setting where the participants do not need to invest any effort to obtain the signals (e.g. Do you like Panda Express). In this case, dominant truthfulness is sufficient. However, for a certain amount of tasks (e.g. online product evaluation, art evaluation), the participants need to invest effort. We will introduce an effort strategy model such that we can properly define the mechanism design goal about incentivizing efforts.

### Effort Strategy Model

We assume that when Alice and Bob spend full efforts, the joint distribution over their signals is  $U_G$ . Alice can pick an effort strategy that leads to an intrinsic noise  $N_A \in \mathbb{R}^{C \times C}$  for the signal she observes. That is,  $N_A(c', c)$  is the probability that her full effort’s signal is  $c$ , while she observes signal  $c'$ . Alice’s effort is modeled as a function of her intrinsic noise  $N_A \in \mathbb{R}^{C \times C}$ ,  $e_A(N_A)$ .

The requester’s expected *value* for the elicited answers is a function of the underlying joint distribution over Alice and Bob’s answers,  $v(\hat{U}_{A,B})$ . In our model, since both  $U_{A,B}$  and  $\hat{U}_{A,B}$  can be seen as  $C \times C$  matrices, we can represent  $\hat{U}_{A,B}$  as follows.

$$\hat{U}_{A,B} = S_A U_{A,B} S_B^\top = S_A N_A U_G N_B^\top S_B^\top.$$

We will make natural monotonicity and continuity assumptions for the value and effort functions. Intuitively, more noisy intrinsic noise requires less effort and leads to less value to the task requester.



Formally, we assume that the effort functions and value functions and information-monotone in the sense that

$$\forall U' \preceq U, v(U') \leq v(U), v(U'^{\top}) \leq v(U^{\top});$$

$$\forall N' \preceq N, e_A(N') \leq e_A(N), e_B(N') \leq e_B(N)$$

which implies that post-processing the data does not require any effort or increase the value. We additionally assume that the value/effort function is continuous and both Alice and Bob pick their effort strategies from a finite discrete set.

We will optimize over dominantly truthful and practical mechanisms. Thus, once Alice and Bob determine their effort strategies, they will truthfully report their signals. Therefore, we can use  $U_{A,B} = N_A U_G N_B^{\top}$  instead of  $\hat{U}_{A,B}$ . Then Alice's expected payment is a function of  $U_{A,B}$  and denoted by  $\mathcal{P}_A(U_{A,B})$ . We model Bob analogously.

► **Example 21.** Alice and Bob are assigned multiple similar quality evaluation task. Alice has three possible effort strategies which lead to the following intrinsic noises:

$$N_A^0(\text{bad}, \text{bad}) = .5, N_A^0(\text{bad}, \text{good}) = .5, e_A(N_A^0) = 0 \quad (\text{full noise})$$

$$N_A^1(\text{bad}, \text{bad}) = 1, N_A^1(\text{bad}, \text{good}) = .4, e_A(N_A^1) = 1 \quad (\text{one-sided noise})$$

$$N_A^2(\text{bad}, \text{bad}) = .8, N_A^2(\text{bad}, \text{good}) = .2, e_A(N_A^2) = 10 \quad (\text{two-sided noise})$$

and Bob has two possible effort strategies which lead to intrinsic noises  $N_B^0 = N_A^0, e_B(N_B^0) = 0, N_B^1 = N_A^1, e_B(N_B^1) = 1$ .

Regarding the value of the requester, when either Alice or Bob's signal is fully noisy, the requester's value will be zero. For other cases,

$$v(N_A^1 U_G N_B^{1\top}) = 15, v(N_A^2 U_G N_B^{1\top}) = 50$$

### Optimization Goal

The optimization problem is

$$\begin{aligned} \max_{\mathcal{P}_A, \mathcal{P}_B} \quad & v(U_{A,B}) - \mathcal{P}_A(U_{A,B}) - \mathcal{P}_B(U_{A,B}) \quad (\text{maximize the requester's expected utility}) \\ \text{s.t.} \quad & U_{A,B} = N_A U_G N_B^{\top} \\ & N_A \in \arg \max_{N'_A} \mathcal{P}_A(N'_A U_G N_B^{\top}) - e_A(N'_A) \quad ((N_A, N_B) \text{ consists of an equilibrium}) \\ & N_B \in \arg \max_{N'_B} \mathcal{P}_B(N_A U_G N'_B^{\top}) - e_B(N'_B) \end{aligned}$$

If there are multiple equilibria  $(N_A, N_B)$ , Alice and Bob will choose the equilibrium that maximizes  $\min(\text{Alice's expected utility}, \text{Bob's expected utility})$ . If there are multiple equilibria that maximize their min expected utility, we will maximize the lower bound of the requester's utility over those equilibria.

### DMI is not optimal

In this example,  $N_A^1 = \begin{bmatrix} 1 & .4 \\ 0 & .6 \end{bmatrix}$  and  $N_A^2 = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix}$  have the same determinant, thus, DMI-Mechanism must reward Alice the same amount of payment no matter Alice pick the one-sided noise effort or two-sided noise effort. Then as long as the expected payment is greater than 1, Alice must pick the one-sided noise since it requires much less effort. However,

the requester values the other choice, the two-sided one, much more even if the requester should pay more. Later we will show, unlike DMI-mechanism which is less pleasant to the requester in this setting, a series of VMI-mechanisms can approximately make the requester obtain the optimal utility.

### Modeling discussion

This optimization goal requires the knowledge of  $U_G$  and the cost of different effort strategies. Note that  $U_G$  does not represent the full knowledge. For example, the requester knows that about 10% products are bad thus  $U_G = \begin{bmatrix} 10\% & 0 \\ 0 & 90\% \end{bmatrix}$ . However, the requester does not know which products are bad, thus she still need to elicit information from the crowds. The cost of different effort strategies represents the requester's estimation for the task difficulty. For example, for some tasks it may be easy to get a 80% accurate answer but very difficult to get a 90% accurate answer. Though this optimization goal requires a certain prior knowledge, we believe this gives the first step for effort incentive optimization over practical multi-task peer prediction mechanisms.

We will optimize over all possible  $\mathcal{P}_A, \mathcal{P}_B$  which are Alice and Bob's *expected* payments under *dominantly truthful and practical* mechanisms. That is why the above formula does not involve Alice and Bob's report strategies. After we find a family of dominantly truthful and practical mechanisms, we can directly optimize the above goal over the family. Another way is to first optimize over all possible dominantly truthful  $\mathcal{P}_A, \mathcal{P}_B$ , even if there does not exist a practical mechanism which pays  $\mathcal{P}_A, \mathcal{P}_B$  in expectation<sup>7</sup>. Then we can use a sequence of practical mechanisms to approximate the optimal dominantly truthful mechanism. It turns out the second approach is much easier in our setting.

**Step 1 Practical VMI-Mechanisms:** Generalize DMI-Mechanism to a family of dominantly truthful and practical mechanisms, VMI-Mechanisms;

**Step 1.1 Mechanism design  $\Rightarrow$  Mutual information design:** Reduce the design of dominantly truthful and practical mechanisms to the design of polynomial information-monotone mutual information measure;

**Step 1.2 VMI construction:** Construct information-monotone Volume Mutual Information (VMI) and show that we can obtain polynomial VMI by assigning distribution space a polynomial density;

**Step 2 Optimal threshold payment:** Optimize over all possible dominantly truthful  $\mathcal{P}_A, \mathcal{P}_B$  and show that the optimal payment function is a threshold function;

**Step 3 Approximating threshold payment via VMI-Mechanisms:** Show that the optimal threshold payment corresponds to a special VMI with Dirac delta density; use a sequence of polynomial densities to approximate the Dirac delta density and finally construct corresponding VMI-Mechanisms.

We have finished the first step. The remaining parts are deferred to the full version.

## 4 Conclusion and Discussion

We provide a novel construction of a new family of mutual information measures, volume mutual information (VMI). We then construct a family of dominantly truthful and practical multi-task peer prediction mechanisms, VMI-Mechanisms. Moreover, we provide a tractable

<sup>7</sup> In other words, we can implement such  $\mathcal{P}_A, \mathcal{P}_B$  only if we have the perfect estimation of  $\hat{U}_{A,B}$  from infinite number of tasks.

effort incentive optimization goal for multi-task peer prediction. We show that with this goal, the optimal payment scheme is the threshold payment scheme and there always exists a sequence of dominantly truthful and practical multi-task peer prediction mechanisms, VMI-Mechanisms, that are approximately optimal.

Though the construction of approximately optimal VMI-Mechanisms requires us to perfectly know the optimal threshold, we believe this work provides the first step for optimization over dominantly truthful and practical multi-task peer prediction mechanisms. One important future direction is to relax the modeling assumption for optimization. For example, when we do not perfectly know the threshold, we can use proper densities (e.g. a smaller  $\alpha$  with more uncertainty) to obtain a more robust mechanism. The approximation gradually increases the requirement for the number of tasks. When given the constraint for the number of tasks, another future direction is to use a computer-aided approach to optimize over VMI-Mechanisms directly. Moreover, we provide a visualization that eases the understanding of mutual information measures. Additionally, this visualization naturally leads to a visual way to fully classify all monotone mutual information in the binary case by the shape of contours. We hope this visualization in binary can also provide insights for the non-binary case.

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