Dynamic Cantor Derivative Logic

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Abstract

Topological semantics for modal logic based on the Cantor derivative operator gives rise to derivative logics, also referred to as d-logics. Unlike logics based on the topological closure operator, d-logics have not previously been studied in the framework of dynamical systems, which are pairs (X, f) consisting of a topological space X equipped with a continuous function $f\colon X \to X$.

We introduce the logics **wK4C**, **K4C** and **GLC** and show that they all have the finite Kripke model property and are sound and complete with respect to the d-semantics in this dynamical setting. In particular, we prove that **wK4C** is the d-logic of all dynamic topological systems, **K4C** is the d-logic of all T_D dynamic topological systems, and **GLC** is the d-logic of all dynamic topological systems based on a scattered space. We also prove a general result for the case where f is a homeomorphism, which in particular yields soundness and completeness for the corresponding systems **wK4H**, **K4H** and **GLH**.

The main contribution of this work is the foundation of a general proof method for finite model property and completeness of dynamic topological d-logics. Furthermore, our result for **GLC** constitutes the first step towards a proof of completeness for the trimodal topo-temporal language with respect to a finite axiomatisation – something known to be impossible over the class of all spaces.

2012 ACM Subject Classification Theory of computation → Modal and temporal logics

Keywords and phrases dynamic topological logic, Cantor derivative, temporal logic, modal logic

Digital Object Identifier 10.4230/LIPIcs.CSL.2022.19

Related Version *Full Version*: https://arxiv.org/abs/2107.10349

1 Introduction

Dynamic (topological) systems are mathematical models of processes that may be iterated indefinitely. Formally, they are defined as pairs $\langle \mathfrak{X}, f\rangle$ consisting of a topological space $\mathfrak{X} = \langle X, \tau\rangle$ and a continuous function $f\colon X \to X$; the intuition is that points in the space $\mathfrak{X}$ "move" along their orbit, $x, f(x), f^2(x), \ldots$ which usually simulates changes in time. *Dynamic topological logic* (**DTL**) combines modal logic and its topological semantics with linear temporal logic (see Pnueli [23]) in order to reason about dynamical systems in a decidable framework.

Due to their rather broad definition, dynamical systems are routinely used in many pure and applied sciences, including computer science. To cite some recent examples, in data-driven dynamical systems, data-related problems may be solved through data-oriented research in dynamical systems as suggested by Brunton and Kutz [4]. Weinan [6] proposes a dynamic theoretic approach to machine learning where dynamical systems are used to model nonlinear functions employed in machine learning. Lin and Antsaklis's [20] hybrid dynamical systems have been at the centre of research in control theory, artificial intelligence and computer-aided verification. Mortveit and Reidys's [22] sequential dynamical systems generalise aspects of systems such as cellular automata, and also provide a framework through which we can study dynamical processes in graphs. Another example of dynamical systems

30th EACSL Annual Conference on Computer Science Logic (CSL 2022).
Editors: Florin Manea and Alex Simpson; Article No. 19; pp. 19:1–19:17
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

and computer science can be found in the form of linear dynamical systems, i.e. systems with dynamics given by a linear transformation. Examples of such systems in computer science include Markov chains, linear recurrence sequences and linear differential equations. Moreover, there are known strong connections between dynamical systems and algorithms. This may be found for example in the work of Hanrot, Pujol and Stehlé [15], and in the work of Chu [5].

Such applications raise a need for effective formal reasoning about topological dynamics. Here, we may take a cue from modal logic and its topological semantics. The study of the latter dates back to McKinsey and Tarski [21], who proved that the modal logic **S4** is complete for a wide class of spaces, including the real line. Artemov, Davoren and Nerode [1] extended **S4** with a "next" operator in the spirit of **LTL**, producing the logic **S4C**. They proved that this logic is sound and complete with respect to the class of all dynamic topological systems. The system **S4C** was enriched with the "henceforth" tense by Kremer and Mints, who dubbed the new logic *dynamic topological logic* (**DTL**). Later, Konev et al. [16] showed that **DTL** is undecidable, and Fernández-Duque [11] showed that it is not finitely axiomatisable on the class of all dynamic topological spaces.

The aforementioned work on dynamic topological logic interprets the modal operator $\Diamond$ as a closure operator. However, McKinsey and Tarski had already contemplated semantics that are instead based on the Cantor derivative [21]: the *Cantor derivative* of a set A, usually denoted by $d(A)$, is the set of points x such that x is in the closure of $A \setminus \{x\}$ (see Section 2). This interpretation is often called *d-semantics* and the resulting logics are called *d-logics.* These logics were first studied in detail by Esakia, who showed that the d-logic **wK4** is sound and complete with respect to the class of all topological spaces [8]. It is well-known that semantics based on the Cantor derivative are more expressive than semantics based on the topological closure. For example, consider the property of a space $\mathfrak{X}$ being *dense-in-itself*, meaning that $\mathfrak{X}$ has no isolated points (see Section 3.2). The property of being dense-in-itself cannot be expressed in terms of the closure operator, but it *can* be expressed in topological d-semantics by the formula $\Diamond\top$.

Logics based on the Cantor derivative appear to be a natural choice for reasoning about dynamical systems. However, there are no established results of completeness for such logics in the setting of dynamical systems, i.e. when a topological space is equipped with a continuous function. Our goal is to prove the finite Kripke model property, completeness and decidability of logics with the Cantor derivative operator and the "next" operator $\bigcirc$ over some prominent classes of dynamical systems: namely, those based on arbitrary spaces, on T_D spaces (spaces validating the 4 axiom $\Box p \to \Box\Box p$) and on *scattered spaces* (see Section 3.2 for definitions). The reason for considering scattered spaces is to circumvent the lack of finite axiomatisability of **DTL** by restricting to a suitable subclass of all dynamical systems. In the study of dynamical systems and topological modal logic, one often works with *dense-in-themselves* spaces. This is a sensible consideration when modelling physical spaces, as Euclidean spaces are dense-in-themselves. However, as we will see in Section 3.2, some technical issues that arise when studying **DTL** over the class of all spaces disappear when restricting our attention to scattered spaces, which in contrast have many isolated points. Further, we consider dynamical systems where f is a *homeomorphism,* i.e. where f^{-1} is also a continuous function. Such dynamical systems are called *invertible.*

The basic dynamic d-logic we consider is **wK4C**, which consists of **wK4** and the temporal axioms for the continuous function f. In addition, we investigate two extensions of **wK4C**: **K4C** and **GLC**. As we will see, **K4C** is the d-logic of all T_D dynamical systems, and **GLC** is the d-logic of all dynamical systems based on a scattered space. Unlike the generic logic of the trimodal topo-temporal language $\mathcal{L}^{\circ *}_{\Diamond}$, we conjecture that a complete finite axiomatisation for **GLC**, extended with axioms for the "henceforth" operator, will not require changes to the trimodal language. This logic is of special interest to us as it would allow for the first finite axiomatisation and completeness results for a logic based on the trimodal topo-temporal language.

Outline. This paper is structured as follows: in Section 2 we give the required definitions and notations necessary to understand the paper. In Section 3 we provide some background on prior work on the topic of dynamic topological logics. Moreover, we motivate our interest in **GLC**, the most unusual logic we work with.

In Section 4 we present the canonical model, and in Section 5 we construct a "finitary" accessibility relation on it. Both are then used in Section 6 in order to develop a proof technique that, given the right modifications, would work for many d-logics above **wK4C**. In particular, we use it to prove the finite model property, soundness and completeness for the d-logics **wK4C**, **K4C** and **GLC**, with respect to the appropriate classes of Kripke models.

In Section 7 we prove topological d-completeness of **K4C**, **wK4C** and **GLC** with respect to the appropriate classes of dynamical systems. In Section 8 we present logics for systems with homeomorphisms and provide a general completeness result which, in particular, applies to the d-logics **wK4H**, **K4H** and **GLH**. Finally, in Section 9 we provide some concluding remarks.

2 Preliminaries

In this section we review some basic notions required for understanding this paper. We work with the general setting of *derivative spaces*, in order to unify the topological and Kripke semantics of our logics.

▶ **Definition 1** (topological space). *A topological space is a pair $\mathfrak{X} = \langle X, \tau \rangle$, where X is a set and τ is a subset of $\wp(X)$ that satisfies the following conditions:*
- $X, \varnothing \in \tau$;
- *if* $U, V \in \tau$, *then* $U \cap V \in \tau$;
- *if* $\mathcal{U} \subseteq \tau$, *then* $\bigcup \mathcal{U} \in \tau$.

The elements of τ are called open sets, and we say that τ forms a topology on X. A complement of an open set is called a closed *set.*

The main operation on topological spaces we are interested in is the *Cantor derivative.*

▶ **Definition 2** (Cantor derivative). *Let $\mathfrak{X} = \langle X, \tau \rangle$ be a topological space. Given $S \subseteq X$, the* Cantor derivative *of S is the set $d(S)$ of all limit points of S, i.e. $x \in d(S) \iff \forall U \in \tau$ s.t. $x \in U$, $(U \cap S) \backslash \{x\} \neq \varnothing$. We may write $d(S)$ or dS indistinctly.*

When working with more than one topological space, we will often denote the Cantor derivative of the topological space $\langle X, \tau \rangle$ by d_τ. Given subsets $A, B \subseteq X$, it is not difficult to verify that $d = d_\tau$ satisfies the following properties:
1. $d(\varnothing) = \varnothing$;
2. $d(A \cup B) = d(A) \cup d(B)$;
3. $dd(A) \subseteq A \cup d(A)$.

In fact, these conditions lead to a more general notion of *derivative spaces:*[1]

[1] Derivative spaces are a special case of *derivative algebras* introduced by Esakia [9], where $\wp(X)$ is replaced by an arbitrary Boolean algebra.

▶ **Definition 3.** *A* derivative space *is a pair $\langle X, \rho\rangle$, where X is a set and $\rho\colon \wp(X) \to \wp(X)$ is a map satisfying properties 1-3 (with ρ in place of d).*

Accordingly, if $\mathfrak{X} = \langle X, \tau\rangle$ is a topological space and d_τ is the Cantor derivative on $\mathfrak{X}$, then $\langle X, d_\tau\rangle$ is a derivative space. However, there are other examples of derivative spaces. The standard topological closure may be defined by $c(A) = A \cup d_\tau(A)$. Then, $\langle X, c\rangle$ is also a derivative space, which satisfies the additional property $A \subseteq c(A)$ (and, *a fortiori,* $cc(A) = c(A)$); we call such derivative spaces *closure spaces.* For convenience, we denote the closure of the topological space $\langle X, \tau\rangle$ by c_τ.

Another example of derivative spaces comes from weakly transitive Kripke frames. For the sake of succinctness, we call these frames *derivative frames.* Below and throughout the text, we write $\exists x \sqsupset y\ \varphi$ instead of $\exists x(y \sqsubset x \wedge \varphi)$, and adopt a similar convention for the universal quantifier and other relational symbols.

▶ **Definition 4.** *A* derivative frame *is a pair $\mathfrak{F} = \langle W, \sqsubset\rangle$ where W is a non-empty set and $\sqsubset$ is a* weakly transitive *relation on W, meaning that $w \sqsubset v \sqsubset u$ implies that $w \sqsubseteq u$, where $\sqsubseteq$ is the reflexive closure of $\sqsubset$.*

We chose the notation $\sqsubset$ because it is suggestive of a transitive relation, but remains ambiguous regarding reflexivity, as there may be irreflexive and reflexive points. Note that $\sqsubset$ is weakly transitive iff $\sqsubseteq$ is transitive. Given $A \subseteq W$, we define $\downarrow_\sqsubset$ as a map $\downarrow_\sqsubset\colon \wp(W) \to \wp(W)$ such that

$$\downarrow_\sqsubset(A) = \{w \in W : \exists v \sqsupset w(v \in A)\}.$$

The following is then readily verified:

▶ **Lemma 5.** *If $\langle W, \sqsubset\rangle$ is a derivative frame, then $\langle W, \downarrow_\sqsubset\rangle$ is a derivative space.*

There is a connection between derivative frames and topological spaces. Given a derivative frame $\langle W, \sqsubset\rangle$, we define a topology $\tau_\sqsubset$ on W such that $U \in \tau_\sqsubset$ iff U is upwards closed under $\sqsubset$, in the sense that if $w \in U$ and $v \sqsupset w$ then $v \in U$. Topologies of this form are *Aleksandroff topologies.* The following is well known and easily verified.

▶ **Lemma 6.** *Let $\langle W, \sqsubset\rangle$ be a derivative frame and $\tau = \tau_\sqsubset$. Then, $d_\tau = \downarrow_\sqsubset$ iff $\sqsubset$ is irreflexive and $c_\tau = \downarrow_\sqsubset$ iff $\sqsubset$ is reflexive.*

Dynamical systems consist of a topological space equipped with a continuous function. Recall that if $\langle X, \tau\rangle$ and $\langle Y, \upsilon\rangle$ are topological spaces and $f\colon X \to Y$, then f is *continuous* if whenever $U \in \upsilon$, it follows that $f^{-1}(U) \in \tau$. The function f is *open* if $f(V)$ is open whenever V is open, and f is a *homeomorphism* if f is continuous, open and bijective. It is well known (and not hard to check) that f is continuous iff $c_\tau f^{-1}(A) \subseteq f^{-1} c_\upsilon(A)$ for any $A \subseteq Y$. By unfolding the definition of the closure operator, this becomes $f^{-1}(A) \cup d_\tau f^{-1}(A) \subseteq f^{-1}(A) \cup f^{-1} d_\upsilon(A)$, or equivalently, $d_\tau f^{-1}(A) \subseteq f^{-1}(A) \cup f^{-1} d_\upsilon(A)$. We thus arrive at the following definition.

▶ **Definition 7.** *Let $\langle X, \rho_X\rangle$ and $\langle Y, \rho_Y\rangle$ be derivative spaces. We say that $f\colon X \to Y$ is* continuous *if for all $A \subseteq Y$, $\rho_X f^{-1}(A) \subseteq f^{-1}(A) \cup f^{-1}\rho_Y(A)$. We say that f is a* homeomorphism *if it is bijective and $\rho_X f^{-1}(A) = f^{-1}\rho_Y(A)$.*

It is worth checking that these definitions coincide with their standard topological counterparts.

▶ **Lemma 8.** *If $\langle X, \tau\rangle$, $\langle Y, \upsilon\rangle$ are topological spaces with Cantor derivatives d_τ and d_υ respectively, and $f\colon X \to Y$, then*
1. *if f is continuous as a function between topological spaces, it is continuous as a function between derivative spaces, and*
2. *if f is a homeomorphism as a function between topological spaces, it is a homeomorphism as a function between derivative spaces.*

We are particularly interested in the case where $X = Y$, which leads to the notion of *dynamic derivative system.*

▶ **Definition 9.** *A* dynamic derivative system *is a triple $\mathfrak{S} = \langle X, \rho, f\rangle$, where $\langle X, \rho\rangle$ is a derivative space and $f\colon X \to X$ is continuous. If f is a homeomorphism, we say that $\mathfrak{S}$ is* invertible.

If $\mathfrak{S} = \langle X, \rho, f\rangle$ is such that $\rho = d_\tau$ for some topology τ, we say that $\mathfrak{S}$ is a *dynamic topological system* and identify it with the triple $\langle X, \tau, f\rangle$. If $\rho = {\downarrow_\sqsubset}$ for some weakly transitive relation $\sqsubset$, we say that $\mathfrak{S}$ is a *dynamic Kripke frame* and identify it with the triple $\langle X, \sqsubset, f\rangle$.

It will be convenient to characterise dynamic Kripke frames in terms of the relation $\sqsubset$.

▶ **Definition 10** (monotonicity and weak monotonicity)**.** *Let $\langle W, \sqsubset\rangle$ be a derivative frame. A function $f\colon W \to W$ is* monotonic *if $w \sqsubset v$ implies $f(w) \sqsubset f(v)$, and* weakly monotonic *if $w \sqsubset v$ implies $f(w) \sqsubseteq f(v)$.*

The function f is persistent *if it is a bijection and for all $w, v \in W$, $w \sqsubset v$ if and only if $f(w) \sqsubset f(v)$. We say that a Kripke frame is* invertible *if it is equipped with a persistent function.*

▶ **Lemma 11.** *If $\langle W, \sqsubset\rangle$ is a derivative frame and $f\colon W \to W$, then*
1. *if f is weakly monotonic then it is continuous with respect to ${\downarrow_\sqsubset}$, and*
2. *if f is persistent then it is a homeomorphism with respect to ${\downarrow_\sqsubset}$.*

Our goal is to reason about various classes of dynamic derivative systems using the logical framework defined in the next section.

3 Dynamic Topological Logics

In this section we discuss dynamic topological logic in the general setting of dynamic derivative systems. Given a non-empty set PV of propositional variables, the language $\mathcal{L}^{\circ}_{\Diamond}$ is defined recursively as follows:

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Diamond\varphi \mid \bigcirc\varphi,$$

where $p \in \mathsf{PV}$. It consists of the Boolean connectives $\wedge$ and $\neg$, the temporal modality $\bigcirc$, and the modality $\Diamond$ for the derivative operator with its dual $\Box := \neg\Diamond\neg$. The interior modality may be defined by $\boxdot\varphi := \varphi \wedge \Box\varphi$.

▶ **Definition 12** (semantics)**.** *A* dynamic derivative model *(DDM) is a quadruple $\mathfrak{M} = \langle X, \rho, f, \nu\rangle$ where $\langle X, \rho, f\rangle$ is a dynamic derivative system and $\nu : \mathsf{PV} \to \wp(X)$ is a valuation function assigning a subset of X to each propositional letter in PV. Given $\varphi \in \mathcal{L}^{\circ}_{\Diamond}$, we define the truth set $\|\varphi\| \subseteq X$ of φ inductively as follows:*
- $\|p\| = \nu(p)$;
- $\|\neg\varphi\| = X \setminus \|\varphi\|$;
- $\|\varphi \wedge \psi\| = \|\varphi\| \cap \|\psi\|$;
- $\|\Diamond\varphi\| = \rho(\|\varphi\|)$;
- $\|\bigcirc\varphi\| = f^{-1}(\|\varphi\|)$.

We write $\mathfrak{M}, x \models \varphi$ if $x \in \|\varphi\|$, and $\mathfrak{M} \models \varphi$ if $\|\varphi\| = X$. We may write $\|\cdot\|_{\mathfrak{M}}$ or $\|\cdot\|_\nu$ instead of $\|\cdot\|$ when working with more than one model or valuation.

We define other connectives (e.g. $\vee, \rightarrow$) as abbreviations in the usual way. The fragment of $\mathcal{L}^{\circ}_{\Diamond}$ that includes only $\Diamond$ will be denoted by $\mathcal{L}_{\Diamond}$. Since our definition of the semantics applies to any derivative space and a general operator ρ, we need not differentiate in our results between d-logics, logics based on closure semantics and logics based on relational semantics. Instead, we indicate the specific class of derivative spaces to which the result applies.

In order to keep with the familiar axioms of modal logic, it is convenient to discuss the semantics of $\Box$. Accordingly, we define the dual of the derivative, called the *co-derivative.*

▶ **Definition 13** (co-derivative). *Let $\langle X, \rho\rangle$ be a derivative space. For each $S \subseteq X$ we define $\hat{\rho}(S) := X \backslash \rho(X \backslash S)$ to be the* co-derivative *of S.*

The co-derivative satisfies the following properties, where $A, B \subseteq X$:

1. $\hat{\rho}(X) = X$;
2. $A \cap \hat{\rho}(A) \subseteq \hat{\rho}\hat{\rho}(A)$;
3. $\hat{\rho}(A \cap B) = \hat{\rho}(A) \cap \hat{\rho}(B)$.

It can readily be checked that for any dynamic derivative model $\langle X, \rho, f, \nu\rangle$ and any formula φ, $\|\Box\varphi\| = \hat{\rho}(\|\varphi\|)$. The co-derivative can be used to define the standard *interior* of a set, given by $i(A) = A \cap \hat{\rho}(A)$ for each $A \subseteq X$. This implies that $U \subseteq \hat{\rho}(U)$ for each open set U, but not necessarily $\hat{\rho}(U) \subseteq U$. Next, we discuss the systems of axioms that are of interest to us. Let us list the axiom schemes and rules that we will consider in this paper:

Taut := All propositional tautologies

$\mathrm{K} := \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

$\mathrm{T} := \Box\varphi \rightarrow \varphi$

$\mathrm{w4} := \varphi \wedge \Box\varphi \rightarrow \Box\Box\varphi$

$\mathrm{L} := \Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$

$4 := \Box\varphi \rightarrow \Box\Box\varphi$

$\mathrm{Next}_{\neg} := \neg\bigcirc\varphi \leftrightarrow \bigcirc\neg\varphi$

$\mathrm{Next}_{\wedge} := \bigcirc(\varphi \wedge \psi) \leftrightarrow \bigcirc\varphi \wedge \bigcirc\psi$

$\mathrm{C} := \bigcirc\varphi \wedge \bigcirc\Box\varphi \rightarrow \Box\bigcirc\varphi$

$\mathrm{H} := \Box\bigcirc\varphi \leftrightarrow \bigcirc\Box\varphi$

$\mathrm{MP} := \dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

$\mathrm{Nec}_{\Box} := \dfrac{\varphi}{\Box\varphi}$

$\mathrm{Nec}_{\bigcirc} := \dfrac{\varphi}{\bigcirc\varphi}$

The "base modal logic" over $\mathcal{L}_{\Diamond}$ is given by

$$\mathbf{K} := \mathrm{Taut} + \mathrm{K} + \mathrm{MP} + \mathrm{Nec}_{\Box},$$

but we are mostly interested in proper extensions of $\mathbf{K}$. Let Λ, Λ' be logics over languages $\mathcal{L}$ and $\mathcal{L}'$. We say that Λ extends Λ' if $\mathcal{L}' \subseteq \mathcal{L}$ and all the axioms and rules of Λ' are derivable in Λ. A logic over $\mathcal{L}_{\Diamond}$ is *normal* if it extends $\mathbf{K}$. If Λ is a logic and φ is a formula, $\Lambda + \varphi$ is the least extension of Λ which contains every substitution instance of φ as an axiom.

We then define $\mathbf{wK4} := \mathbf{K} + \mathrm{w4}$, $\mathbf{K4} := \mathbf{K} + 4$, $\mathbf{S4} := \mathbf{K4} + \mathrm{T}$ and $\mathbf{GL} := \mathbf{K4} + \mathrm{L}$. These logics are well known and characterise certain classes of topological spaces and Kripke frames which we review below. In addition, for a logic Λ over $\mathcal{L}_{\Diamond}$, $\Lambda\mathbf{F}$ is the logic over $\mathcal{L}^{\circ}_{\Diamond}$ given by[2]

$$\Lambda\mathbf{F} := \Lambda + \mathrm{Next}_{\neg} + \mathrm{Next}_{\wedge} + \mathrm{Nec}_{\bigcirc}.$$

This simply adds axioms of linear temporal logic to Λ, which hold whenever $\bigcirc$ is interpreted using a function. Finally, we define $\Lambda\mathbf{C} := \Lambda\mathbf{F} + \mathrm{C}$ and $\Lambda\mathbf{H} := \Lambda\mathbf{F} + \mathrm{H}$, which as we will see correspond to derivative spaces with a continuous function or a homeomorphism respectively. The following is well known and dates back to McKinsey and Tarski [21].

[2] Logics of the form $\Lambda\mathbf{F}$ correspond to dynamical systems with a possibly discontinuous function. We will not discuss discontinuous systems in this paper; see [1] for more information.

▶ **Theorem 14.** **S4** *is the logic of all topological closure spaces, the logic of all transitive, reflexive derivative frames, and the logic of the real line with the standard closure.*

The logic **K4** includes the axiom $\Box p \to \Box\Box p$, which is not valid over the class of all topological spaces. The class of spaces satisfying this axiom is denoted by T_D, defined as the class of spaces in which every singleton is the result of an intersection between an open set and a closed set. Moreover, Esakia showed that this is the logic of transitive derivative frames [9].

▶ **Theorem 15.** **K4** *is the logic of all T_D topological derivative spaces, as well as the logic of all transitive derivative frames.*

Many familiar topological spaces, including Euclidean spaces, satisfy the T_D property, making **K4** central in the study of topological modal logic. A somewhat more unusual class of spaces, which is nevertheless of particular interest to us, is the class of *scattered spaces.*

▶ **Definition 16** (scattered space)**.** *A topological space $\langle X, \tau\rangle$ is* scattered *if for every $S \subseteq X$, $S \subseteq d(S)$ implies $S = \varnothing$.*

This is equivalent to the more common definition of a scattered space where a topological space is called scattered if every non-empty subset has an isolated point. Scattered spaces are closely related to converse well-founded relations. Below, recall that $\langle W, \sqsubset\rangle$ is *converse well-founded* if there is no infinite sequence $w_0 \sqsubset w_1 \sqsubset \ldots$ of elements in W.

▶ **Lemma 17.** *If $\langle W, \sqsubset\rangle$ is an irreflexive frame, then $\langle W, \tau_\sqsubset\rangle$ is scattered iff $\sqsubset$ is converse well-founded.*

▶ **Theorem 18** (Simmons [24] and Esakia [7])**.** **GL** *is the logic of all scattered topological derivative spaces, as well as the logic of all converse well-founded derivative frames and the logic of all finite, transitive, irreflexive derivative frames.*

Aside from its topological interpretation, the logic **GL** is of particular interest as it is also the logic of provability in Peano arithmetic, as was shown by Boolos [3]. Meanwhile, logics with the C and H axioms correspond to classes of dynamical systems.

▶ **Lemma 19.**

1. *If Λ is sound for a class of derivative spaces Ω, then Λ**C** is sound for the class of dynamic derivative systems $\langle X, \rho, f\rangle$, where $\langle X, \rho\rangle \in \Omega$ and f is continuous.*
2. *If Λ is sound for a class of derivative spaces Ω, then Λ**H** is sound for the class of dynamic derivative systems $\langle X, \rho, f\rangle$, where $\langle X, \rho\rangle \in \Omega$ and f is a homeomorphism.*

The above lemma is easy to verify from the definitions of continuous functions and homeomorphisms in the context of derivative spaces (Definition 7).

3.1 Prior Work

The study of dynamic topological logic originates with Artemov, Davoren and Nerode, who observed that it is possible to reason about dynamical systems within modal logic. They introduced the logic **S4C** and proved that it is decidable, as well as sound and complete for the class of all dynamic closure systems (i.e. dynamic derivative systems based on a closure space). Kremer and Mints [18] considered the logic **S4H**, and also showed it to be sound and complete for the class of dynamic closure systems where f is a homeomorphism.

The latter also suggested adding the "henceforth" operator, $*$, from Pnueli's linear temporal logic (**LTL**) [23], leading to the language we denote by $\mathcal{L}^{\circ *}_{\Diamond}$. The resulting trimodal system was named *dynamic topological logic* (**DTL**). Kremer and Mints offered an axiomatisation for **DTL**, but Fernández-Duque proved that it is incomplete; in fact, **DTL** is not finitely axiomatisable [11]. Fernández-Duque also showed that **DTL** enjoys a natural axiomatisation when extended with the *tangled closure* [13]. In contrast, Konev et al. established that **DTL** over the class of dynamical systems with a homeomorphism is non-axiomatisable [17].

3.2 The tangled closure on scattered spaces

Our interest in considering the class of scattered spaces within dynamic topological logic is motivated by results of Fernández-Duque [13]. He showed that the set of valid formulas of $\mathcal{L}^{\circ *}_{\Diamond}$ over the class of all dynamic closure systems is not finitely axiomatisable. Nevertheless, he found a natural (yet infinite) axiomatisation by introducing the *tangled closure* and adding it to the language of **DTL** [10]. Here, we use the more general *tangled derivative*, as defined by Goldblatt and Hodkinson [14].

▶ **Definition 20** (tangled derivative). *Let $\langle X, d\rangle$ be a derivative space and let $\mathcal{S} \subseteq \wp(X)$. Given $A \subseteq X$, we say that $\mathcal{S}$ is tangled in A if for all $S \in \mathcal{S}$, $A \subseteq d(S \cap A)$. We define the* tangled derivative *of $\mathcal{S}$ as*

$$\mathcal{S}^* := \bigcup\{A \subseteq X : \mathcal{S} \text{ is tangled in } A\}.$$

The *tangled closure* is then the special case of the tangled derivative where d is a closure operator. Fernández-Duque's axiomatisation is based on the extended language $\mathcal{L}^{\circ *}_{\Diamond\!\!\!\Diamond}$. This language is obtained by extending $\mathcal{L}^{\circ *}_{\Diamond}$ with the following operation.

▶ **Definition 21** (tangled language). *We define $\mathcal{L}^{\circ *}_{\Diamond\!\!\!\Diamond}$ by extending the recursive definition of $\mathcal{L}^{\circ *}_{\Diamond}$ in such a way that if $\varphi_1, \ldots, \varphi_n \in \mathcal{L}^{\circ *}_{\Diamond\!\!\!\Diamond}$, then $\Diamond\!\!\!\Diamond\{\varphi_1, \ldots, \varphi_n\} \in \mathcal{L}^{\circ *}_{\Diamond\!\!\!\Diamond}$. The semantic clauses are then extended so that on any model $\mathfrak{M}$,*

$$\|\Diamond\!\!\!\Diamond\{\varphi_1, \ldots, \varphi_n\}\| = \{\|\varphi_1\|, \ldots, \|\varphi_n\|\}^*.$$

The logic **DGL** is an extension of **GLC** that includes the temporal operator $*$. Unlike the complete axiomatisation of **DTL** that requires the tangled operator, in the case of **DGL**, we should be able to avoid this and use the original spatial operator $\Diamond$ alone. This is due to the following:

▶ **Theorem 22.** *Let $\mathfrak{X} = \langle X, \tau\rangle$ be a scattered space and $\{\varphi_1, \ldots, \varphi_n\}$ a set of formulas. Then*

$$\Diamond\!\!\!\Diamond\{\varphi_1, \ldots, \varphi_n\} \equiv \bot.$$

This leads to the conjecture that the axiomatic system of Kremer and Mints [18], combined with **GL**, will lead to a finite axiomatisation for **DGL**. While such a result requires techniques beyond the scope of the present work, the completeness proof we present here for **GLC** is an important first step. Before proving topological completeness for this and the other logics we have mentioned, we show that they are complete and have the finite model property for their respective classes of dynamic derivative frames.

4 The Canonical Model

The first step in our Kripke completeness proof will be a fairly standard canonical model construction. A *maximal Λ-consistent set* (Λ-MCS) w is a set of formulas that is Λ-consistent, i.e. $w \nvdash_\Lambda \bot$, and every set of formulas that properly contains it is Λ-inconsistent.

Given a logic Λ over $\mathcal{L}^{\circ}_{\Diamond}$, let $\mathfrak{M}^{\Lambda}_{\mathrm{c}} = \langle W_{\mathrm{c}}, \sqsubset_{\mathrm{c}}, g_{\mathrm{c}}, \nu_{\mathrm{c}} \rangle$ be the canonical model for Λ, where

1. W_{c} is the set of all Λ-MCSs;
2. $w \sqsubset_{\mathrm{c}} v$ iff for all formulas φ, if $\Box\varphi \in w$, then $\varphi \in v$;
3. $g_{\mathrm{c}}(w) = \{\varphi : \bigcirc\varphi \in w\}$;
4. $\nu_{\mathrm{c}}(p) = \{w : p \in w\}$.

It can easily be verified that **wK4C** defines the class of all weakly transitive Kripke models with a weakly monotonic map. Moreover, **K4C** defines the class of all transitive Kripke models with a weakly monotonic map. We call these models **wK4C** models and **K4C** models respectively.

▶ **Lemma 23.** *If Λ extends* **wK4C***, then the canonical model for Λ is a* **wK4C** *model. If Λ extends* **K4C***, then the canonical model of Λ is a* **K4C** *model.*

It is a well-known fact that the transitivity axiom $\Box p \to \Box\Box p$ is derivable in **GL** (see [25]). Therefore, **GLC** extends the system **K4C**. The proofs of the following two lemmas are standard and can be found for example in [2].

▶ **Lemma 24** (existence lemma). *Let Λ be a normal modal logic and let $\mathfrak{M}^{\Lambda}_{\mathrm{c}} = \langle W_{\mathrm{c}}, \sqsubset_{\mathrm{c}}, g_{\mathrm{c}}, \nu_{\mathrm{c}} \rangle$. Then, for every $w \in W_{\mathrm{c}}$ and every formula φ in Λ, if $\Diamond\varphi \in w$ then there exists a point $v \in W_{\mathrm{c}}$ such that $w \sqsubset_{\mathrm{c}} v$ and $\varphi \in v$.*

▶ **Lemma 25** (truth lemma). *Let Λ be a normal modal logic. For every $w \in W_{\mathrm{c}}$ and every formula φ in Λ,*

$$\mathfrak{M}^{\Lambda}_{\mathrm{c}}, w \models \varphi \text{ iff } \varphi \in w.$$

▶ **Corollary 26.** *The logic* **wK4C** *is sound and complete with respect to the class of all weakly monotonic dynamic derivative frames, and* **K4C** *is sound and complete with respect to the class of all weakly monotonic, transitive dynamic derivative frames.*

5 A finitary accessibility relation

One key ingredient in our finite model property proof will be the construction of a "finitary" accessibility relation $\sqsubset_{\Phi}$ on the canonical model. This accessibility relation will have the property that each point has finitely many successors, yet the existence lemma will hold for formulas in a prescribed finite set Φ.

We define *the $\sqsubset_{\mathrm{c}}$-cluster $C(w)$* for each point $w \in W_{\mathrm{c}}$ as

$$C(w) = \{w\} \cup \{v : w \sqsubset_{\mathrm{c}} v \sqsubset_{\mathrm{c}} w\}.$$

▶ **Definition 27** (φ-final set). *A set w is said to be a φ-final set (or point) if w is an MCS, $\varphi \in w$, and whenever $w \sqsubset_{\mathrm{c}} v$ and $\varphi \in v$, it follows that $v \in C(w)$.*

Let us write $\sqsubseteq_{\mathrm{c}}$ for the reflexive closure of $\sqsubset_{\mathrm{c}}$. It will be convenient to characterise $\sqsubseteq_{\mathrm{c}}$ in the canonical model syntactically. Recall that $\boxdot\varphi := \varphi \wedge \Box\varphi$.

▶ **Lemma 28.** *If Λ extends* **wK4C** *and $w, v \in W_{\mathrm{c}}$, then $w \sqsubseteq_{\mathrm{c}} v$ if and only if whenever $\boxdot\varphi \in w$, it follows that $\boxdot\varphi \in v$.*

We are now ready to prove the main result of this section regarding the existence of the finitary relation $\sqsubset_{\Phi}$.

▶ **Lemma 29.** *Let Λ extend* **wK4C** *and Φ be a finite set of formulas closed under subformulas. There is an auxiliary relation $\sqsubset_\Phi$ on the canonical model of Λ such that:*

(i) *$\sqsubset_\Phi$ is a subset of $\sqsubset_c$;*

(ii) *For each $w \in W$, the set $\sqsubset_\Phi(w)$ is finite;*

(iii) *If $\Diamond\varphi \in w \cap \Phi$, then there exists $v \in W$ with $w \sqsubset_\Phi v$ and $\varphi \in v$;*

(iv) *If $w \sqsubset_c v \sqsubset_c w$ then $\sqsubset_\Phi (w) \subseteq \sqsubseteq_\Phi (v)$;*

(v) *$\sqsubset_\Phi$ is weakly transitive. Moreover, if Λ extends* **K4C** *then $\sqsubset_\Phi$ is transitive, and if Λ extends* **GLC** *then $\sqsubset_\Phi$ is irreflexive.*

6 Stories and Φ-morphisms

In this subsection we show that the logics **wK4C**, **K4C** and **GLC** have the finite model property by constructing finite models and truth preserving maps from these models to the canonical model.

If $\sqsubset$ is a weakly transitive relation on A, $\langle A, \sqsubset\rangle$ is tree-like if whenever $a \sqsubseteq c$ and $b \sqsubseteq c$, it follows that $a \sqsubseteq b$ or $b \sqsubseteq a$. We will use labelled tree-like structures called *moments* to record the "static" information at a point; that is, the structure involving $\sqsubset$, but not f.

▶ **Definition 30** (moment). *A* Λ-moment *is a structure $\mathfrak{m} = \langle |\mathfrak{m}|, \sqsubset_\mathfrak{m}, \nu_\mathfrak{m}, r_\mathfrak{m}\rangle$, where $\langle |\mathfrak{m}|, \sqsubset_\mathfrak{m}\rangle$ is a finite tree-like Λ frame with a root $r_\mathfrak{m}$, and $\nu_\mathfrak{m}$ is a valuation on $|\mathfrak{m}|$.*

In order to also record "dynamic" information, i.e. information involving the transition function, we will stack up several moments together to form a "story". Below, $\bigsqcup$ denotes a disjoint union.

▶ **Definition 31** (story). *A* story (with duration I) *is a structure $\mathfrak{S} = \langle |\mathfrak{S}|, \sqsubset_\mathfrak{S}, f_\mathfrak{S}, \nu_\mathfrak{S}, r_\mathfrak{S}\rangle$ such that there are $I < \omega$, moments $\mathfrak{S}_i = \langle |\mathfrak{S}_i|, \sqsubset_i, \nu_i, r_i\rangle$ for each $i \leq I$, and functions $(f_i)_{i<I}$ such that:*

1. *$|\mathfrak{S}| = \bigsqcup_{i\leq I} |\mathfrak{S}_i|$;*
2. *$\sqsubset_\mathfrak{S} = \bigsqcup_{i\leq I} \sqsubset_i$;*
3. *$\nu_\mathfrak{S}(p) = \bigsqcup_{i\leq I} \nu_i(p)$ for each variable p;*
4. *$r_\mathfrak{S} = r_0$;*
5. *$f_\mathfrak{S} = \mathrm{Id}_I \cup \bigsqcup_{i<I} f_i$ with $f_i \colon |\mathfrak{S}_i| \to |\mathfrak{S}_{i+1}|$ being a weakly monotonic map such that $f_i(r_i) = r_{i+1}$ for all $i < I$ (we say that f_i is* root preserving*), and Id_I is the identity on $|\mathfrak{S}_I|$.*

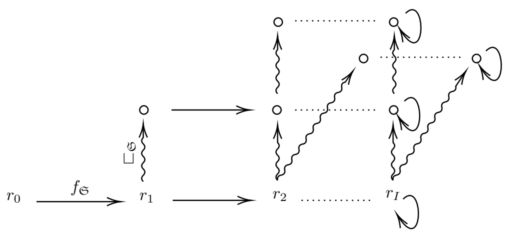

Figure 1 An example of a **GL**-story. The squiggly arrows represent the relation $\sqsubset_\mathfrak{S}$ while the straight arrows represent the function $f_\mathfrak{S}$. Each vertical slice represents a **GL**-moment. In the case of other types of stories, we may also have clusters besides singletons.

We often omit the subindices $\mathfrak{m}$ or $\mathfrak{S}$ when this does not lead to confusion. We may also assign different notations to the components of a moment, so that if we write $\mathfrak{m} = \langle W, \sqsubset, \nu, x\rangle$, it is understood that $W = |\mathfrak{m}|$, $\sqsubset = \sqsubset_\mathfrak{m}$, etc.

Recall that a *p-morphism* between Kripke models is a type of map that preserves validity. It can be defined in the context of dynamic derivative frames as follows:

▶ **Definition 32** (dynamic p-morphism). *Let $\mathfrak{M} = \langle W_\mathfrak{M}, \sqsubset_\mathfrak{M}, g_\mathfrak{M}\rangle$ and $\mathfrak{N} = \langle W_\mathfrak{N}, \sqsubset_\mathfrak{N}, g_\mathfrak{N}\rangle$ be dynamic derivative frames. Let $\pi\colon W_\mathfrak{M} \to W_\mathfrak{N}$. We say that π is a* dynamic p-morphism *if $w \sqsubset_\mathfrak{M} v$ implies that $\pi(w) \sqsubset_\mathfrak{N} \pi(v)$, $\pi(w) \sqsubset_\mathfrak{N} u$ implies that there is $v \sqsupset_\mathfrak{M} w$ with $\pi(v) = u$, and $\pi \circ g_\mathfrak{M} = g_\mathfrak{N} \circ \pi$.*

It is then standard that if $\pi\colon W_\mathfrak{M} \to W_\mathfrak{N}$ is a surjective, dynamic p-morphism, then any formula valid on $\mathfrak{M}$ is also valid on $\mathfrak{N}$. However, our relation $\sqsubset_\Phi$ will allow us to weaken these conditions and still obtain maps that preserve the truth of (some) formulas.

▶ **Definition 33** (Φ-morphism). *Fix a logic Λ and let $\mathfrak{M}^\Lambda_\mathrm{c} = \langle W_\mathrm{c}, \sqsubset_\mathrm{c}, g_\mathrm{c}, \nu_\mathrm{c}\rangle$ and $\mathfrak{S}$ be a story of duration I. A map $\pi : |\mathfrak{S}| \to W_\mathrm{c}$ is called a* dynamic Φ-morphism *if for all $x \in |\mathfrak{S}|$ the following conditions are satisfied:*

1. *$x \in \nu_\mathfrak{S}(p) \iff p \in \pi(x)$;*
2. *If $x \in |\mathfrak{S}_i|$ for some $i < I$, then $g_\mathrm{c}(\pi(x)) = \pi(f_\mathfrak{S}(x))$;*
3. *If $x \sqsubset_\mathfrak{S} y$ then $\pi(x) \sqsubset_\mathrm{c} \pi(y)$;*
4. *If $\pi(x) \sqsubset_\Phi v$ for some $v \in W_\mathrm{c}$, then there exists $y \in |\mathfrak{S}|$ such that $x \sqsubset_\mathfrak{S} y$ and $v = \pi(y)$.*

If we drop condition 2, we say that π is a Φ-morphism.

We now show that a dynamic Φ-morphism π preserves the truth of formulas of suitable $\bigcirc$-depth, where the latter is defined as usual in terms of nested occurrences of $\bigcirc$ in a formula φ.

▶ **Lemma 34** (truth preservation). *Let $\mathfrak{S}$ be a story of duration I and $x \in |\mathfrak{S}_0|$. Let π be a dynamic Φ-morphism to the canonical model of some normal logic Λ over $\mathcal{L}^\circ_\Diamond$. Suppose that $\varphi \in \Phi$ is a formula of $\bigcirc$-depth at most I. Then $\varphi \in \pi(x)$ iff $x \in \|\varphi\|_\mathfrak{S}$.*

We will next demonstrate that for every point w in the canonical model, there exists a suitable moment $\mathfrak{m}$ and a Φ-morphism mapping $\mathfrak{m}$ to w. In order to do this, we define a procedure for constructing new moments from smaller ones.

▶ **Definition 35** (moment construction). *Let $\Lambda \in \{\mathbf{wK4}, \mathbf{K4}, \mathbf{GL}\}$ and $C' \cup \{x\} \subseteq C(x)$ for some x in the canonical model $\mathfrak{M}^\Lambda_\mathrm{c}$. Let $\vec{\mathfrak{a}} = \langle \mathfrak{a}_m\rangle_{m<N}$ be a sequence of moments. We define a structure $\mathfrak{n} = \binom{\vec{\mathfrak{a}}}{C'}_x$ as follows:*

1. $|\mathfrak{n}| = C' \sqcup \bigsqcup_{m<N} |\mathfrak{a}_m|$;
2. *$y \sqsubset_\mathfrak{n} z$ if either*
   - *$y, z \in C'$, $\Lambda \neq \mathbf{GL}$ and $y \sqsubset_\mathrm{c} z$,*
   - *$y \in C'$ and $z \in |\mathfrak{a}_m|$ for some m, or*
   - *$y, z \in |\mathfrak{a}^m|$ and $y \sqsubset_{\mathfrak{a}_m} z$ for some m;*
3. *$\nu_\mathfrak{n}(p) = \{x \in C' : x \in \nu_\mathrm{c}(p)\} \sqcup \bigsqcup_{m<N} \nu_{\mathfrak{a}_m}(p)$;*
4. *$r_\mathfrak{n} = x$.*

▶ **Lemma 36.** *Given $\Lambda \in \{\mathbf{wK4C}, \mathbf{K4C}, \mathbf{GLC}\}$, for all $w \in W_\mathrm{c}$ there exists a Λ-moment $\mathfrak{m}$ and a Φ-morphism $\pi\colon |\mathfrak{m}| \to W_\mathrm{c}$ such that $\pi(r_\mathfrak{m}) = w$.*

Proof sketch. We prove the stronger claim that there is a moment $\mathfrak{m}$ and a map $\pi\colon |\mathfrak{m}| \to W_{\rm c}$ that is a p-morphism on the structure $(W_{\rm c}, \sqsubset_\Phi)$. We will say that π is a *p-morphism with respect to* $\sqsubset_\Phi$. Let $\sqsubset^1_\Phi$ be the strict $\sqsubset_\Phi$ successor, i.e. $w \sqsubset^1_\Phi v$ iff $w \sqsubset_\Phi v$ and $\neg(v \sqsubset_\Phi w)$. Since $\sqsubset^1_\Phi$ is converse well-founded, we can assume inductively that for each v such that $w \sqsubset^1_\Phi v$, there is a moment $\mathfrak{m}_v$ and a p-morphism $\pi_v : |\mathfrak{m}_v| \to W_{\rm c}$ with respect to $\sqsubset_\Phi$ that maps the root of $\mathfrak{m}_v$ to v. Accordingly, we define a moment

$$\mathfrak{m} = \begin{pmatrix} \{\mathfrak{m}_v : v \sqsupset^1_\Phi w\} \\ C_\Phi(w) \end{pmatrix}_w .$$

It is not difficult to verify that $\mathfrak{m}$ thus defined is a Λ-moment.

Next we define a map $\pi : |\mathfrak{m}| \to W_{\rm c}$ as

$$\pi(x) = \begin{cases} x & \text{if } x \in C_\Phi(w), \\ \pi_v(x) & \text{if } x \in |\mathfrak{m}_v|. \end{cases}$$

The map π can be shown to be a p-morphism for $\sqsubset_\Phi$. Hence $\mathfrak{m}$ is a Λ-moment and π is a Φ-morphism such that $\pi(r_\mathfrak{m}) = w$, as required. ◀

We next define the notions of *pre-Φ-morphism* and *quotient moment* that will be essential for the rest of the proof.

▶ **Definition 37** (pre-Φ-morphism, Λ-bottom). *Let $\mathfrak{m}$ be a moment and $\pi : |\mathfrak{m}| \to W_{\rm c}$. We say that $x \in |\mathfrak{m}|$ is* at the Λ-bottom *for $\Lambda \in \{\mathbf{wK4}, \mathbf{K4}, \mathbf{GL}\}$ if*
- *$\Lambda \neq \mathbf{GL}$ and $\pi(x) \in C(\pi(r_\mathfrak{m}))$, or*
- *$\Lambda = \mathbf{GL}$ and for all $y \sqsubset_\mathfrak{m} x$, $\pi(y) = \pi(x)$.*

We will refer to "Λ-bottom" simply as "bottom" when this does not lead to confusion.

We say that $\pi : |\mathfrak{m}| \to W_{\rm c}$ is a pre-Φ-morphism *if it fulfils conditions 1 and 4 of a Φ-morphism (Definition 33), and $x \sqsubset_\mathfrak{m} y$ implies that either $\pi(x) \sqsubset_{\rm c} \pi(y)$ or x, y are at the bottom.*

▶ **Definition 38** (quotient moment). *Let $\Lambda \in \{\mathbf{wK4C}, \mathbf{K4C}, \mathbf{GLC}\}$. Let $\mathfrak{m}$ be a Λ-moment and $\pi : |\mathfrak{m}| \to W_{\rm c}$ be a pre-Φ-morphism. We define $x \sim y$ if either $x = y$, or x, y are at the bottom and $\pi(x) = \pi(y)$. Given $y \in |\mathfrak{m}|$ we set $[y] = \{z : z \sim y\}$.*

The quotient moment $\mathfrak{m}/\pi$ *of $\mathfrak{m}$ and its respective map $[\pi] : |\mathfrak{m}/\pi| \to W_{\rm c}$ are defined as follows, where $x, y \in |\mathfrak{m}|$:*

(a) $|\mathfrak{m}/\pi| = \{[x] : x \in |\mathfrak{m}|\}$;
(b) $[x] \sqsubset_{\mathfrak{m}/\pi} [y]$ *iff one of the following conditions is satisfied:*
- *x, y are at the bottom, $\pi(x) \sqsubset_{\rm c} \pi(y)$, and $\Lambda \neq \mathbf{GLC}$;*
- *x is at the bottom and y is not at the bottom;*
- *x, y are not at the bottom and $x \sqsubset_\mathfrak{m} y$;*

(c) $\nu_{\mathfrak{m}/\pi}(p) = \{[x] : x \in \nu_\mathfrak{m}(p)\}$;
(d) $r_{\mathfrak{m}/\pi} = [r_\mathfrak{m}]$;
(e) $[\pi]([x]) = \pi(x)$.

The quotient moment and its respective map hold some essential properties for the derivation of this section's main result. The idea is that by constructing a Λ-moment m with an associated pre-Φ-morphism π, we get that $\mathfrak{m}/\pi$ is still a Λ-moment, but now $[\pi]$ is a proper Φ-morphism. The next few results make this intuition precise.

▶ **Lemma 39.** *For $\Lambda \in \{\mathbf{wK4}, \mathbf{K4}, \mathbf{GL}\}$, if $\mathfrak{m}$ is any Λ-moment and $\pi : |\mathfrak{m}| \to W_{\rm c}$ is a pre-Φ-morphism, then $\mathfrak{m}/\pi$ is a Λ-moment and $[\cdot]\colon |\mathfrak{m}| \to |\mathfrak{m}/\pi|$ is weakly monotonic and root-preserving.*

▶ **Proposition 40.** *If* $\mathfrak{m}$ *is a moment and* $\pi \colon |\mathfrak{m}| \to W_c$ *is a pre-*Φ*-morphism, then the map* $[\pi]$ *is a well-defined* Φ*-morphism.*

Using these properties, we can prove the existence of an appropriate story that maps to the canonical model. This is based on the following useful lemma:

▶ **Lemma 41.** *Fix* $\Lambda \in \{\mathbf{wK4C}, \mathbf{K4C}, \mathbf{GLC}\}$ *and let* $\mathfrak{M}_c^\Lambda = \langle W_c, \sqsubset_c, g_c, \nu_c \rangle$. *Let* $\mathfrak{m}$ *be a* Λ*-moment and suppose that there exists a* Φ*-morphism* $\pi \colon |\mathfrak{m}| \to W_c$. *Then, there exists a moment* $\mathfrak{n}$*, a weakly monotonic map* $f \colon |\mathfrak{m}| \to |\mathfrak{n}|$*, and a* Φ*-morphism* $\rho \colon |\hat{\mathfrak{n}}| \to W_c$ *such that* $g_c \circ \pi = \rho \circ f$.

Proof. We proceed by induction on the height of $\mathfrak{m}$. Let C be the cluster of $r_\mathfrak{m}$ and let $\vec{\mathfrak{a}} = \langle \mathfrak{a}_n \rangle_{n<N}$ be the generated sub-models of the immediate strict successors of $r_\mathfrak{m}$; note that each $\mathfrak{a}_n$ is itself a moment of smaller height. By the induction hypothesis, there exist moments $\langle \mathfrak{a}'_n \rangle_{n<N}$, root-preserving, weakly monotonic maps $f_n \colon |\mathfrak{a}_n| \to |\mathfrak{a}'_n|$, and Φ-morphism $\rho_n \colon |\mathfrak{a}'_n| \to W_c$ such that $g_c \circ \pi_n = \rho_n \circ f_n$. Moreover, for each $v \sqsupset_\Phi g_c(r_\mathfrak{m})$, by Lemma 36 there are $\mathfrak{b}_v$ and a Φ-morphism $\rho_v \colon |\mathfrak{b}_v| \to W_c$ mapping the root of $\mathfrak{b}_v$ to v. Let $D = g_c\pi(C) \cup C_\Phi(g_c(x))$, and let $\hat{\mathfrak{n}} = \binom{\vec{\mathfrak{a}} * \vec{\mathfrak{b}}}{D}_{g_c(w)}$. It is not difficult to verify that the maps ρ_v can be used to define a pre-Φ-morphism $\hat{\rho}$ from $\hat{\mathfrak{n}}$ to W_c. Let

$$\hat{f}(w) = \begin{cases} g_c(w) & \text{if } w \in C, \\ f_n(w) & \text{if } w \in |\mathfrak{a}_n|. \end{cases}$$

It is easy to see that $\hat{f} \colon |\mathfrak{m}| \to |\hat{\mathfrak{n}}|$ is weakly monotonic and satisfies $g_c \circ \pi = \hat{\rho} \circ \hat{f}$. Setting $\mathfrak{n} = \hat{\mathfrak{n}}/\hat{\rho}$, $f = [\hat{f}]$ and $\rho = [\hat{\rho}]$, Proposition 40 implies that $\mathfrak{n}$, f and ρ have the desired properties. ◀

▶ **Proposition 42.** *Fix* $\Lambda \in \{\mathbf{wK4C}, \mathbf{K4C}, \mathbf{GLC}\}$. *Given* $I < \omega$ *and* $w \in W_c$*, there is a story* $\mathfrak{S}$ *of duration* I *and a dynamic* Φ*-morphism* $\pi \colon |\mathfrak{S}| \to W_c$ *with* $w = \pi(r_\mathfrak{S})$.

Proof. Proceed by induction on I. For $I = 0$, this is essentially Lemma 36. Otherwise, by the induction hypothesis, assume that a story $\hat{\mathfrak{S}}$ of depth I and dynamic p-morphism $\hat{\pi}$ exist. By Lemma 41, there is a moment $\mathfrak{S}_{I+1}$, map $f_I \colon |\mathfrak{S}_I| \to |\mathfrak{S}_{I+1}|$, and Φ-morphism $\pi_{I+1} \colon |\mathfrak{S}_{I+1}| \to W_c$ commuting with f_I. We define $\mathfrak{S}$ by adding $\mathfrak{S}_{I+1}$ to $\hat{\mathfrak{S}}$ in order to obtain the desired story. ◀

It follows that any satisfiable formula is also satisfiable on a finite story, hence satisfiable on a finite model, yielding the main result of this section.

▶ **Theorem 43.** *The logics* **wK4C**, **K4C** *and* **GLC** *are sound and complete for their respective class of finite dynamic* Λ*-frames.*

7 Topological d-completeness

In this section we establish completeness results for classes of dynamic topological systems with continuous functions. We begin with the logic **GLC**, simply because the topological d-completeness for **GLC** is almost immediate, given that a **GLC** model is already a dynamic derivative model based on a scattered space via the standard up-set topology.

▶ **Theorem 44.** **GLC** *is the d-logic of all dynamic topological systems based on a scattered space, and enjoys the finite model property for this class.*

In order to prove topological d-completeness for **wK4C**, we first provide a definition and some generalisations of known results. We use similar constructions as in e.g. [9].

▶ **Definition 45.** *Let $\mathfrak{F} = \langle W, \sqsubset, g\rangle$ be a* **wK4C***-frame and let W^{i} and W^{r} be the sets of irreflexive and reflexive points respectively. We define a new frame $\mathfrak{F}_\oplus = \langle W_\oplus, \sqsubset_\oplus, g_\oplus\rangle$, where*
1. $W_\oplus = (W^{\mathrm{i}} \times \{0\}) \cup (W^{\mathrm{r}} \times \{0,1\})$;
2. $(w,i) \sqsubset_\oplus (v,j)$ *iff* $w \sqsubset v$ *and* $(w,i) \neq (v,j)$;
3. $g_\oplus(w,i) = (g(w), 0)$.

The following is standard [9] and easily verified.

▶ **Proposition 46.** *If $\mathfrak{F} = \langle W, \sqsubset, g\rangle$ is any dynamic derivative frame, then $\mathfrak{F}_\oplus$ is an irreflexive dynamic derivative frame and $\pi \colon W_\oplus \to W$ given by $\pi(w,i) = w$ is a surjective, dynamic p-morphism.*

Proposition 46 allows us to obtain topological completeness from Theorem 43, as Lemma 6 tells us that irreflexive Kripke frames are essentially Cantor derivative spaces. We thus obtain the following:

▶ **Theorem 47.** **wK4C** *is the d-logic of all dynamic topological systems, and enjoys the finite model property for this class.*

Finally we turn our attention to **K4C**. Unlike the other two logics, **K4C** (or even **K4**) does not have the topological finite model property, despite having the Kripke finite model property. In the case of Aleksandroff spaces, the class T_D is easy to describe.

▶ **Lemma 48.** *The Aleksandroff space of a* **K4***-frame $\mathfrak{F} = \langle W, \sqsubset\rangle$ is T_D if and only if $\sqsubset$ is antisymmetric, in the sense that $w \sqsubset v \sqsubset w$ implies $w = v$.*

If we moreover want the Kripke and the d-semantics to coincide on $\mathfrak{F}$, we need $\sqsubset$ to be irreflexive. Thus we wish to "unwind" $\mathfrak{F}$ to get rid of all non-trivial clusters. The following construction achieves this.

▶ **Definition 49.** *Let $\mathfrak{F} = \langle W, \sqsubset, g\rangle$ be a dynamic* **K4** *frame. We define a new frame $\vec{\mathfrak{F}} = \langle \vec{W}, \vec{\sqsubset}, \vec{g}\rangle$, where*
- *$\vec{W}$ is the set of all finite sequences $(w_0, \ldots, w_n)$, where $w_i \sqsubset w_{i+1}$ for all $i < n$;*
- *$\mathbf{w} \mathrel{\vec{\sqsubset}} \mathbf{v}$ iff $\mathbf{w}$ is a strict initial segment of $\mathbf{v}$;*
- *$\vec{g}(w_0, \ldots, w_n)$ is the subsequence of $(g(w_0), \ldots, g(w_n))$, obtained by deleting every entry that is equal to its immediate predecessor.*

We moreover define a map $\pi \colon \vec{W} \to W$, where $\pi(w_0, \ldots, w_n) = w_n$.

In the definition of $\vec{g}$, note that g is only weakly monotonic, so it may be that, for instance, $g(w_0) = g(w_1)$; in this case, we include only one copy of $g(w_0)$ to ensure that $\vec{g}(\mathbf{w}) \in \vec{W}$.

▶ **Proposition 50.** *If $\mathfrak{F} = \langle W, \sqsubset, g\rangle$ is any dynamic* **K4** *frame, then $\vec{\mathfrak{F}}$ is an antisymmetric and irreflexive dynamic* **K4** *frame. Moreover, $\pi \colon \vec{W} \to W$ is a surjective, dynamic p-morphism.*

Similar constructions have already appeared in e.g. [16]. From this we obtain the following:

▶ **Theorem 51.** **K4C** *is the d-logic of the class of all dynamic topological systems based on a T_D space, as well as the d-logic of the class of all dynamic topological systems based on an Aleksandroff T_D space.*

8 Invertible Systems

Recall that if $\langle X, \tau\rangle$ is a topological space and $f\colon X \to X$ is a function, then f is a *homeomorphism* if it is a bijection and both f and f^{-1} are continuous.

Unlike in the continuous case, for logics with homeomorphisms, every formula is equivalent to a formula where all occurrences of $\bigcirc$ are applied to atoms. More formally, we say that a formula is in *$\bigcirc$-normal form* if it is of the form $\varphi(\bigcirc^{k_0}p_0, \ldots, \bigcirc^{k_n}p_n)$, where $\varphi(p_0, \ldots, p_n)$ does not contain $\bigcirc$, the p_i's are variables, and the k_i's are natural numbers. It is readily observed that the axiom H allows us to "push" all instances of $\bigcirc$ to the propositional level. Thus we obtain the following useful representation lemma:

▶ **Lemma 52.** *For every logic Λ and every formula φ, there is a formula φ' in $\bigcirc$-normal form such that $\Lambda\mathbf{H} \vdash \varphi \leftrightarrow \varphi'$.*

As we shall see shortly, $\Lambda\mathbf{H}$ inherits completeness and the finite model property almost immediately from Λ. If $\mathfrak{X} = \langle X, \rho_X\rangle$ and $\mathfrak{Y} = \langle Y, \rho_Y\rangle$ are derivative spaces, then the *topological sum* is defined as $\mathfrak{X} \oplus \mathfrak{Y} = (X \oplus Y, \rho_X \oplus \rho_Y)$, where $X \oplus Y$ is the disjoint union of the two sets, and $\rho_X \oplus \rho_Y$ is given by $(\rho_X \oplus \rho_Y)A = \rho_X(A \cap X) \cup \rho_Y(A \cap Y)$. We say that a class Ω of derivative spaces is *closed under sums* if whenever $\mathfrak{A}, \mathfrak{B} \in \Omega$, it follows that $\mathfrak{A} \oplus \mathfrak{B} \in \Omega$.

Given a derivative space $\mathfrak{A} = \langle A, \rho\rangle$, we may write $\mathfrak{A}^n$ instead of $\mathfrak{A} \oplus \mathfrak{A} \oplus \ldots \oplus \mathfrak{A}$ (n times), and if $\mathfrak{A}$ has domain A, we may identify the domain of $\mathfrak{A}^n$ with $A^n = A \times \{0, \ldots, n-1\}$. It should be clear that if $\mathfrak{A} \in \Omega$ and Ω is closed under sums, then $\mathfrak{A}^n \in \Omega$. Let $(m)_n$ denote the remainder of m modulo n. We then define a dynamical structure $\mathfrak{A}^{(n)} = (\mathfrak{A}^n, f)$, where $f\colon A^n \to A^n$ is given by $f(w, i) = (w, (i+1)_n)$.

▶ **Lemma 53.** *Let Ω be a class of derivative spaces closed under sums. Then, if $\mathfrak{A} \in \Omega$, it follows that $\mathfrak{A}^{(n)}$ is an invertible dynamic derivative system based on an element of Ω.*

For our proof of completeness, we need the notion of *extended valuation.*

▶ **Definition 54.** *Let $\mathfrak{A} = \langle A, \rho\rangle$ be a derivative space. Let $\mathsf{PV}^{\bigcirc}$ be the set of all expressions $\bigcirc^i p$, where $i \in \mathbb{N}$ and p is a propositional variable. An* extended valuation *on $\mathfrak{A}$ is a relation $\nu \subseteq \mathsf{PV}^{\bigcirc} \times A$.*

If ν is an extended valuation on $\mathfrak{A}$, we define a valuation on $\mathfrak{A}^{(n)}$ so that for any variable p and $(w, i) \in A^n$, $(w, i) \in \nu^{(n)}(p)$ if and only if $w \in \nu(\bigcirc^i p)$.

▶ **Lemma 55.** *Let $\mathfrak{A} = \langle A, \rho\rangle$ be any derivative space and ν be any extended valuation on $\mathfrak{A}$. Let $w \in A$, $i < n$, and φ be any formula in $\bigcirc$-normal form which has $\bigcirc$-depth less than $n - i$. Then, $\langle \mathfrak{A}^{(n)}, \nu^{(n)}\rangle, (w, i) \models \varphi$ if and only if $\langle \mathfrak{A}, \nu\rangle, w \models \varphi$.*

It then readily follows that φ is consistent if and only if φ' is consistent, which implies that φ' is satisfiable on Ω. This is equivalent to φ being satisfiable on the class of invertible systems based on Ω. From this, we obtain the following general result.

▶ **Theorem 56.** *Let Λ be complete for a class Ω of derivative spaces. Then, if Ω is closed under sums, it follows that $\Lambda\mathbf{H}$ is complete for the class of invertible systems based on Ω.*

▶ **Corollary 57.**

1. $\mathbf{wK4H}$ *is sound and complete for*
   - **a.** *The class of all finite invertible dynamic* $\mathbf{wK4}$ *frames.*
   - **b.** *The class of all finite invertible dynamic topological systems with Cantor derivative.*

2. **K4H** *is sound and complete for*
   a. *The class of all finite invertible dynamic* **K4** *frames.*
   b. *The class of all invertible, T_D dynamic topological systems with Cantor derivative.*
3. **GLH** *is sound and complete for*
   a. *The class of all finite invertible dynamic* **GL** *frames.*
   b. *The class of all finite invertible, scattered dynamic topological systems with Cantor derivative.*

9 Conclusion

We have recast dynamic topological logic in the more general setting of derivative spaces and established the seminal results for the $\{\Diamond, \bigcirc\}$-fragment for the variants of the standard logics with the Cantor derivative in place of the topological closure. Semantics on the Cantor derivative give rise to a richer family of modal logics than their counterparts based on closure. This is evident in the distinction between e.g. the logics **wKC** and **K4C**, both of which collapse to **S4C** when replaced by their closure-based counterparts. This line of research goes hand in hand with recent trends that consider the Cantor derivative as the basis of topological semantics [12, 19].

There are many natural problems that remain open. The logic **S4C** is complete for the Euclidean plane. In the context of d-semantics, the logic of the Euclidean plane is a strict extension of **K4**, given that punctured neighbourhoods are *connected* in the sense that they cannot be split into two disjoint, non-empty open sets. Thus one should not expect **K4C** to be complete for the plane. This raises the question: what is the dynamic d-logic of Euclidean spaces in general, and of the plane in particular?

The d-semantics also poses new lines of inquiry with respect to the class of functions considered. We have discussed continuous functions and homeomorphisms. Artemov et al. [1] also considered arbitrary functions, and we expect that the techniques used by them could be modified without much issue for d-semantics. On the other hand, in the setting of closure-based logics, the logic of spaces with continuous, open maps that are not necessarily bijective coincides with the logic of spaces with a homeomorphism. This is no longer true in the d-semantics setting, as the validity of the H axiom requires injectivity. Along these lines, the d-logic of *immersions* (i.e. continuous, injective functions) would validate the original continuity axiom $\bigcirc\Box p \to \Box\bigcirc p$. Thus there are several classes of dynamical systems whose closure-based logics coincide, but are split by d-semantics.

Finally, there is the issue of extending our language to the trimodal language with the "henceforth" operator. It is possible that the d-logic of all dynamic topological systems may be axiomatised using the tangled derivative, much as the tangled closure was used to provide an axiomatisation of the closure-based **DTL**. However, given that the tangled derivative is made trivial on scattered spaces, we conjecture that the trimodal d-logic based on this class will enjoy a natural, finite axiomatisation. The work presented here is an important first step towards proving this.

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