Structural Properties of the First-Order Transduction Quasiorder

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---- **Abstract** ----

Logical transductions provide a very useful tool to encode classes of structures inside other classes of structures. In this paper we study first-order (FO) transductions and the quasiorder they induce on infinite classes of finite graphs. Surprisingly, this quasiorder is very complex, though shaped by the locality properties of first-order logic. This contrasts with the conjectured simplicity of the monadic second order (MSO) transduction quasiorder. We first establish a local normal form for FO transductions, which is of independent interest. Then we prove that the quotient partial order is a bounded distributive join-semilattice, and that the subposet of *additive* classes is also a bounded distributive join-semilattice. The FO transduction quasiorder has a great expressive power, and many well studied class properties can be defined using it. We apply these structural properties to prove, among other results, that FO transductions of the class of paths are exactly perturbations of classes with bounded bandwidth, that the local variants of monadic stability and monadic dependence are equivalent to their (standard) non-local versions, and that the classes with pathwidth at most k, for $k \geq 1$ form a strict hierarchy in the FO transduction quasiorder.

1 Introduction and statement of results

Transductions provide a model theoretical tool to encode relational structures (or classes of relational structures) inside other (classes of) relational structures. Transductions naturally induce a quasiorder, that is, a reflexive and transitive binary relation, on classes of relational structures. We study here the first-order (FO) and monadic second-order (MSO) transduction quasiorders $\sqsubseteq_{FO}$ and $\sqsubseteq_{MSO}$ on infinite classes of finite graphs. These quasiorders are very different and both have a sound combinatorial and model theoretic relevance, as we will outline below. To foster the further discussion, let us (slightly informally) introduce the concept of transductions. Formal definitions will be given in Section 2.

A *transduction* T (on graphs) is the composition of a copying operation, a coloring operation, and a simple interpretation. The *copying operation* C_k maps a graph G to the graph $\mathsf{C}_k(G)$ obtained by taking k disjoint copies of G and making all the copies of a single vertex adjacent; the *coloring operation* maps a graph G to the set $\Gamma(G)$ of all possible colorings of G; a *simple interpretation* I maps a colored graph G^+ to a graph H, whose vertex set (resp. edge set) is a definable subset of $V(G^+)$ (resp. of $V(G^+) \times V(G^+)$). In this way, the transduction T maps a graph G to a set $\mathsf{T}(G)$ of graphs defined as $\mathsf{T}(G) \coloneqq \mathsf{I} \circ \Gamma \circ \mathsf{C}_k(G) = \{\mathsf{I}(H^+) : H^+ \in \Gamma(\mathsf{C}_k(G))\}$. This naturally extends to a graph class[1] $\mathscr{C}$ by $\mathsf{T}(\mathscr{C}) \coloneqq \bigcup_{G \in \mathscr{C}} \mathsf{T}(G)$.

We say that a class $\mathscr{C}$ *is a transduction of* a class $\mathscr{D}$ if there exists a transduction T with $\mathscr{C} \subseteq \mathsf{T}(\mathscr{D})$, and we denote this by $\mathscr{C} \sqsubseteq \mathscr{D}$. We write $\mathscr{C} \equiv \mathscr{D}$ for $\mathscr{C} \sqsubseteq \mathscr{D}$ and $\mathscr{D} \sqsubseteq \mathscr{C}$, $\mathscr{C} \sqsubset \mathscr{D}$ for $\mathscr{C} \sqsubseteq \mathscr{D}$ and $\mathscr{C} \not\equiv \mathscr{D}$, and $\mathscr{C} \lhd \mathscr{D}$ for the property that $(\mathscr{C}, \mathscr{D})$ is a *cover*, that is, that $\mathscr{C} \sqsubset \mathscr{D}$ and there is no class $\mathscr{F}$ with $\mathscr{C} \sqsubset \mathscr{F} \sqsubset \mathscr{D}$. For a logic $\mathcal{L}$ we write $\mathscr{C} \sqsubseteq_{\mathcal{L}} \mathscr{D}$ to stress that the simple interpretation of the transduction uses $\mathcal{L}$-formulas.

For most commonly studied logics $\mathcal{L}$ transductions compose and in this case $\sqsubseteq_{\mathcal{L}}$ is a quasiorder. We study here mainly the first-order (FO) and monadic second-order (MSO) transduction quasiorders $\sqsubseteq_{\mathrm{FO}}$ and $\sqsubseteq_{\mathrm{MSO}}$. As with the colorings all vertex subsets become definable, it follows that we can restrict our attention to infinite *hereditary* classes, that is, infinite classes that are closed under taking induced subgraphs.

MSO transductions are basically understood. Let us write $\mathcal{E}$ for the class of edgeless graphs, $\mathcal{T}_n$ for the class of forests of depth n (where the depth of a (rooted) tree is the maximum number of vertices on a root-leaf path, hence $\mathcal{T}_1 = \mathcal{E}$), $\mathcal{P}$ for the class of all paths, $\mathcal{T}$ for the class of all trees and $\mathcal{G}$ for the class of all graphs. The MSO transduction quasiorder is conjectured to be simply the chain $\mathcal{E} \lhd_{\mathrm{MSO}} \mathcal{T}_2 \lhd_{\mathrm{MSO}} \ldots \lhd_{\mathrm{MSO}} \mathcal{T}_n \lhd_{\mathrm{MSO}} \ldots \sqsubseteq_{\mathrm{MSO}} \mathcal{P} \lhd_{\mathrm{MSO}} \mathcal{T} \lhd_{\mathrm{MSO}} \mathcal{G}$ [2].

In a combinatorial setting this hierarchy has a very concrete meaning and it was investigated using the following notions: a class $\mathscr{C}$ has *bounded shrubdepth* if $\mathscr{C} \sqsubseteq_{\mathrm{MSO}} \mathcal{T}_n$ for some n; $\mathscr{C}$ has *bounded linear cliquewidth* if $\mathscr{C} \sqsubseteq_{\mathrm{MSO}} \mathcal{P}$; $\mathscr{C}$ has *bounded cliquewidth* if $\mathscr{C} \sqsubseteq_{\mathrm{MSO}} \mathcal{T}$. These definitions very nicely illustrate the treelike structure of graphs from the above mentioned classes from a logical point of view, which is combinatorially captured by the existence of treelike decompositions with certain properties. It is still open whether the MSO transduction quasiorder is as shown above [2, Open Problem 9.3], though the initial fragment $\mathcal{E} \lhd_{\mathrm{MSO}} \mathcal{T}_2 \lhd_{\mathrm{MSO}} \mathcal{T}_3 \lhd_{\mathrm{MSO}} \ldots \lhd_{\mathrm{MSO}} \mathcal{T}_n$ has been proved to be as stated in [8]. Thus we are essentially left with the following three questions: Does $\mathscr{C} \sqsubseteq_{\mathrm{MSO}} \mathcal{P}$ imply $(\exists n)\ \mathscr{C} \sqsubseteq_{\mathrm{MSO}} \mathcal{T}_n$? This is equivalent to the question whether one can transduce with MSO arbitrary long paths from any class of unbounded shrubdepth (see [11] for a proof of the CMSO version). Is the pair $(\mathcal{P}, \mathcal{T})$ a cover? Is the pair $(\mathcal{T}, \mathcal{G})$ a cover? This last question is related to a famous conjecture of Seese [18] and the CMSO version has been proved in [4].

As in the MSO case, the FO transduction quasiorder allows to draw important algorithmic and structural dividing lines. For instance MSO collapses to FO on classes of bounded shrubdepth [8]. Classes of bounded shrubdepth are also characterized as being FO transductions of classes of trees of bounded depth [9]. FO transductions give alternative characterizations of other graph class properties mentioned above: a class $\mathscr{C}$ has bounded linear cliquewidth if and only if $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathcal{H}$, where $\mathcal{H}$ denotes the class of *half-graphs* (bipartite graphs with vertex set $\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\}$ and edge set $\{a_i b_j\ :\ 1 \le i \le j \le n\}$ for some n) [3], and bounded cliquewidth if and only if $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathcal{TP}$, where $\mathcal{TP}$ denotes the class of *trivially perfect graphs* (comparability graphs of rooted forests) [3]. Also, it follows from [1] that FO transductions

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[1] By a class we always mean a set of finite graphs, where we identify isomorphic graphs.
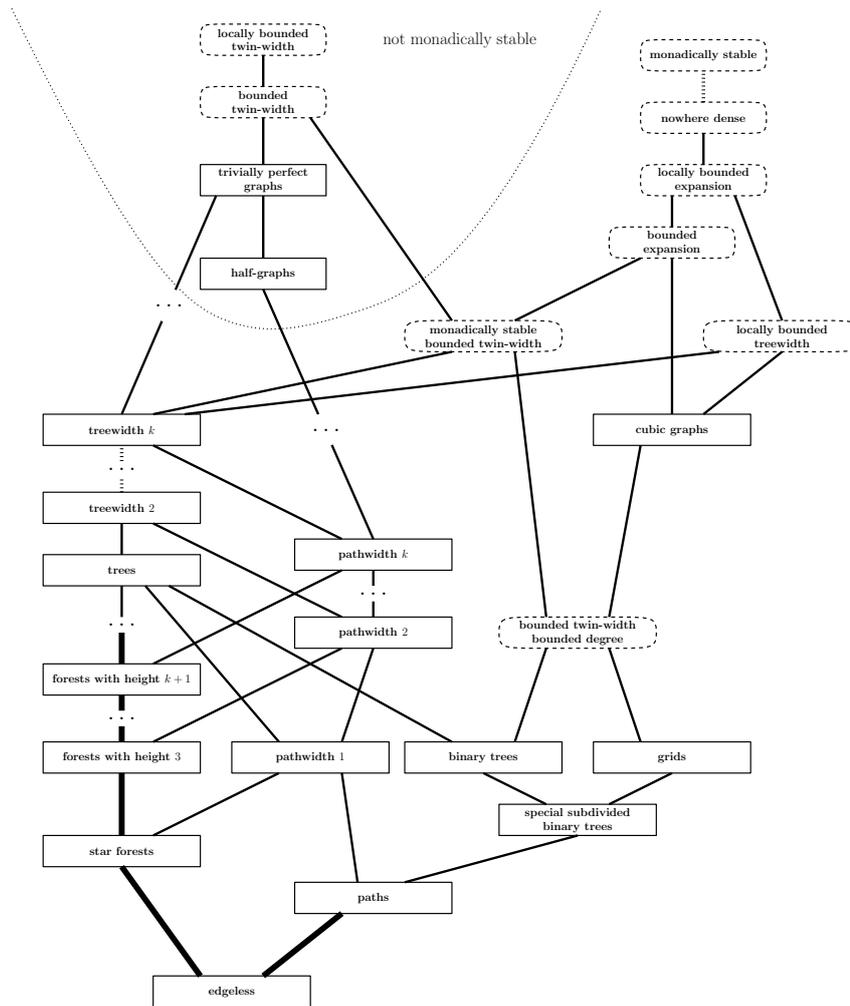
Figure 1 Partial outline of the FO transduction quasiorder. The special subdivided binary trees are those subdivisions of binary trees that are subgraphs of the grid. Dashed boxes correspond to families of not necessarily transduction equivalent graph classes sharing a common property. Fat lines correspond to covers, normal lines correspond to strict containment $\sqsubset$, dotted lines correspond to containment (with a possible collapse). Some parts of Figure 1 will be refined in Figure 2.

allow to give an alternative characterizations of classical model theoretical properties: A class $\mathscr{C}$ is *monadically stable* if $\mathscr{C} \not\sqsupseteq_{\mathrm{FO}} \mathcal{H}$ and *monadically dependent* if $\mathscr{C} \not\sqsupseteq_{\mathrm{FO}} \mathcal{G}$. We further call a class $\mathscr{C}$ *monadically straight* if $\mathscr{C} \not\sqsupseteq_{\mathrm{FO}} \mathcal{TP}$. To the best of our knowledge this property has not been studied in the literature but seems to play a key role in the study of FO transductions.

The FO transduction quasiorder has not been studied in detail previously and it turns out that it is much more complicated than the MSO transduction quasiorder. This is outlined in Figure 1, and it is the goal of this paper to explore this quasiorder.

We are motivated by three aspects of the $\sqsubseteq_{\mathrm{FO}}$ quasiorder that have been specifically considered in the past and appeared to be highly non-trivial. The first aspect is the conjectured property that every class that cannot FO transduce paths has bounded shrubdepth (hence is an FO transduction of a class of bounded height trees). The second aspect that was studied in detail concerns the chain formed by classes with bounded pathwidth, which is

eventually covered by the class of half-graphs. This is related to the fact that in the FO transduction quasiorder there is no class between the classes with bounded pathwidth and the class $\mathcal{H}$ of half-graphs [15, 16]. The third aspect concerns the chain of classes with bounded treewidth, which is eventually covered by the class of trivially perfect graphs. This is related to the fact that if $\mathcal{H} \not\sqsubseteq_{\mathrm{FO}} \mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathcal{TP}$ (that is, $\mathscr{C}$ is a monadically stable class with bounded cliquewidth), then $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathcal{TW}_n$ for some n, where $\mathcal{TW}_n$ denotes the class of graphs with treewidth at most n [14].

In this paper, we establish the three kinds of results and show that despite its complexity the FO transduction quasiorder is strongly structured.

A local normal form for transductions

In Section 3.1 we introduce a normal form for FO transductions that captures the local character of first-order logic, by proving that every FO transduction can be written as the composition of a copying operation, a transduction that connects only vertices at a bounded distance, and a perturbation, which is a sequence of subset complementations (Theorem 2). In Section 4 we give two applications of this normal form. We first characterize the equivalence class of the class of paths in the FO transduction quasiorder (Theorem 8). Then, we prove that the local versions of monadic stability, monadic straightness, and monadic dependence are equivalent to the non-local versions (Theorem 9). This result is of independent interest and may be relevant e.g. for locality based FO model-checking on these classes.

Structural properties of the transduction quasiorder

In Section 5 we prove that the partial orders obtained as the quotient of the transduction quasiorder and the non-copying transduction quasiorder are bounded distributive join-semilattices (Theorem 14) and discuss some of their properties. In particular we prove that every class closed under disjoint union is join-irreducible. Recall that a partial order $(X, \leq)$ is a *join semi-lattice* if for all $x, y \in X$ there exists a least upper bound $x \vee y$ of $\{x, y\}$, called the join of x and y. It is *distributive* if, for all $a, b, x \in X$ with $x \leq a \vee b$ there exist $x_1 \leq a$, $x_2 \leq b$, with $x = x_1 \vee x_2$. An element $x \in X$ is *join-irreducible* if x is not the join of two incomparable elements. Then we consider the subposets induced by *additive* classes, which are the classes equivalent to the class of disjoint unions of pairs of graphs in the class. We prove that these subposets are also bounded distributive join-semilattices (Theorem 23), but with a different join. We discuss some properties of these subposets and in particular prove that every class closed under disjoint union and equivalent to its subclass of connected graphs is join-irreducible.

The transduction quasiorder on some classes

In Section 6 we focus on the transduction quasiorders on the class of paths, the class of trees, classes of bounded height trees, classes with bounded pathwidth, classes with bounded treewidth, and derivatives. In particular we prove that classes with bounded pathwidth form a strict hierarchy (Theorem 30). This result was the main motivation for this study, and we conjecture that a similar statement holds with treewidth. This would be a consequence of the conjecture that the class of all graphs with treewidth at most n is incomparable with the class of all graphs with pathwidth at most $n + 1$, for every positive integer n.

2 Preliminaries and basic properties of transductions

We assume familiarity with first-order logic and graph theory and refer e.g. to [5, 10] for background and for all undefined notation. The vertex set of a graph G is denoted as $V(G)$ and its edge set $E(G)$. All graphs considered in this paper are finite. The *complement* of a graph G is the graph $\overline{G}$ with the same vertex set, in which two vertices are adjacent if they are not adjacent in G. The disjoint union of two graphs G and H is denoted as $G \cup H$, and their *complete join* $\overline{\overline{G} \cup \overline{H}}$ as $G + H$. We denote by K_t the complete graph on t vertices. Hence, $G + K_1$ is obtained from G by adding a new vertex, called an *apex*, that is connected to all vertices of G. For a class $\mathscr{C}$ of graphs we denote by $\mathscr{C} + K_1$ the class obtained from $\mathscr{C}$ by adding an apex to each graph of $\mathscr{C}$. The *lexicographic product* $G \bullet H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if either u is adjacent to u' in G or $u = u'$ and v is adjacent to v' in H. The *pathwidth* $\mathrm{pw}(G)$ of a graph G is equal to one less than the smallest clique number of an interval graph that contains G as a subgraph, that is, $\mathrm{pw}(G) = \min\{\omega(H) - 1 :$ for an interval graph H with $H \supseteq G\}$. The *treewidth* $\mathrm{tw}(G)$ of a graph G is equal to one less than the smallest clique number of a chordal graph that contains G as a subgraph, that is, $\mathrm{tw}(G) = \min\{\omega(H) - 1 :$ for a chordal graph H with $H \supseteq G\}$. We write G^k for the *k-th power* of G (which has the same vertex set as G and two vertices are connected if their distance is at most k in G). The *bandwidth* of a graph G is $\mathrm{bw}(G) = \min\{\ell :$ for $P \in \mathcal{P}$ with $P^\ell \supseteq G, \}$.

In this paper we consider either graphs or Σ-*expanded graphs*, that is, graphs with additional unary relations in Σ (for a set Σ of unary relation symbols). We usually denote graphs by $G, H, \dots$ and Σ-expanded graphs by $G^+, H^+, G^*, H^*, \dots$, but sometimes we will use $G, H, \dots$ for Σ-expanded graphs as well. We shall often use the term "colored graph" instead of Σ-expanded graph. In formulas, the adjacency relation will be denoted as $E(x, y)$. For each non-negative integer r we can write a formula $\delta_{\leq r}(x, y)$ such that for every graph G and all $u, v \in V(G)$ we have $G \models \delta_{\leq r}(u, v)$ if and only if the distance between u and v in G is at most r. For improved readability we write $\mathrm{dist}(x, y) \leq r$ for $\delta_{\leq r}(x, y)$. The *open neighborhood* $N^G(v)$ of a vertex v is the set of neighbors of v. For $U \subseteq V(G)$ we write $B_r^G(U)$ for the subgraph of G induced by the vertices at distance at most r from some vertex of U. For the sake of simplicity we use for balls of radius r the notation $B_r^G(v)$ instead of $B_r^G(\{v\})$ and, if G is clear from the context, we drop the superscript G. For a class $\mathscr{C}$ and an integer r, we denote by $\mathcal{B}_r^{\mathscr{C}}$ the class of all the balls of radius r of graphs in $\mathscr{C}$: $\mathcal{B}_r^{\mathscr{C}} = \{B_r^G(v) \mid G \in \mathscr{C}$ and $v \in V(G)\}$. For a formula $\varphi(x_1, \dots, x_k)$ and a graph (or a Σ-expanded graph) G we define

$$\varphi(G) := \{(v_1, \dots, v_k) \in V(G)^k \; : \; G \models \varphi(v_1, \dots, v_k)\}.$$

For a positive integer k, the *k-copy operation* C_k maps a graph G to the graph $\mathsf{C}_k(G)$ consisting of k copies of G where the copies of each vertex are made adjacent (that is, the copies of each vertex induce a clique and there are no other edges between the copies of G). Note that for $k = 1$, C_k maps each graph G to itself. (Thus C_1 is the identity mapping.)

For a set Σ of unary relations, the *coloring operation* Γ_Σ maps a graph G to the set $\Gamma_\Sigma(G)$ of all its Σ-expansions.

A *simple interpretation* I of graphs in Σ-expanded graphs is a pair $(\nu(x), \eta(x, y))$ consisting of two formulas (in the first-order language of Σ-expanded graphs), where η is symmetric and anti-reflexive (i.e. $\models \eta(x, y) \leftrightarrow \eta(y, x)$ and $\models \eta(x, y) \rightarrow \neg(x = y)$). If G^+ is a Σ-expanded graph, then $H = \mathsf{I}(G^+)$ is the graph with vertex set $V(H) = \nu(G^+)$ and edge set $E(H) = \eta(G) \cap \nu(G)^2$.

A *transduction* T is the composition $\mathsf{I} \circ \Gamma_\Sigma \circ \mathsf{C}_k$ of a copy operation C_k, a coloring operation Γ_Σ, and a simple interpretation I of graphs in Σ-expanded graphs. In other words, for every graph G we have $\mathsf{T}(G) = \{\mathsf{I}(H^+) : H \in \Gamma_\Sigma(\mathsf{C}_k(G))\}$. A transduction T is *non-copying* if it is the composition of a coloring operation and a simple interpretation, that is if it can written as $\mathsf{I} \circ \Gamma_\Sigma \circ \mathsf{C}_1 \, (= \mathsf{I} \circ \Gamma_\Sigma)$. We say that a transduction T' *subsumes* a transduction T if for every graph G we have $\mathsf{T}'(G) \supseteq \mathsf{T}(G)$. We denote by $\mathsf{T}' \geq \mathsf{T}$ the property that T' subsumes T.

For a class $\mathscr{D}$ and a transduction T we define $\mathsf{T}(\mathscr{D}) = \bigcup_{G \in \mathscr{D}} \mathsf{T}(G)$ and we say that a class $\mathscr{C}$ is a T-*transduction* of $\mathscr{D}$ if $\mathscr{C} \subseteq \mathsf{T}(\mathscr{D})$. We also say that T *encodes* $\mathscr{C}$ in $\mathscr{D}$. A class $\mathscr{C}$ of graphs is a *(non-copying) transduction* of a class $\mathscr{D}$ of graphs if it is a T-transduction of $\mathscr{D}$ for some (non-copying) transduction T. We denote by $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathscr{D}$ (resp. $\mathscr{C} \sqsubseteq_{\mathrm{FO}}^\circ \mathscr{D}$) the property that the class $\mathscr{C}$ is an FO transduction (resp. a non-copying FO transduction) of the class $\mathscr{D}$. It is easily checked that the composition of two (non-copying) transductions is a (non-copying) transduction (see, for instance [7]). Thus the relations $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathscr{D}$ and $\mathscr{C} \sqsubseteq_{\mathrm{FO}}^\circ \mathscr{D}$ are quasiorders on classes of graphs Intuitively, if $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathscr{D}$, then $\mathscr{C}$ is at most as complex as $\mathscr{D}$. Equivalences for $\sqsubseteq_{\mathrm{FO}}$ and $\sqsubseteq_{\mathrm{FO}}^0$ are defined naturally.

We say that a class $\mathscr{C}$ *does not need copying* if for every integer k the class $\mathsf{C}_k(\mathscr{C})$ is a non-copying transduction of $\mathscr{C}$. For example, as a matching cannot be transduced from an edgeless graph without copying, the class of edgeless graphs needs copying. To the opposite, the reader can easily check that the class of paths does not need copying.

We take time for some observations.

▶ **Observation 1.** *If $\mathscr{C}$ does not need copying and $\mathscr{C} \equiv_{\mathrm{FO}} \mathscr{D}$, then $\mathscr{D}$ does not need copying.*

Proof. This follows from the fact that every class $\mathscr{C}$ is a non-copying transduction of $\mathsf{C}_k(\mathscr{C})$. ◀

▶ **Observation 2.** *A class $\mathscr{C}$ does not need copying if and only if $\mathsf{C}_2(\mathscr{C}) \equiv_{\mathrm{FO}}^\circ \mathscr{C}$.*

Proof. It is easily checked that for every positive integer k there is a non-copying transduction T_k such that $\mathsf{T}_k \circ \mathsf{C}_k \circ \mathsf{C}_2$ subsumes C_{2k}. Assume $\mathsf{C}_2(\mathscr{C}) \equiv_{\mathrm{FO}}^\circ \mathscr{C}$. Then if $\mathsf{C}_k(\mathscr{C}) \sqsubseteq_{\mathrm{FO}}^\circ \mathscr{C}$ we deduce from $\mathsf{C}_2(\mathscr{C}) \sqsubseteq_{\mathrm{FO}}^\circ \mathscr{C}$ that $\mathsf{C}_{2k}(\mathscr{C}) \sqsubseteq_{\mathrm{FO}}^\circ \mathscr{C}$. By induction we get $\mathsf{C}_k(\mathscr{C}) \equiv_{\mathrm{FO}}^\circ \mathscr{C}$ for every positive integer k. ◀

▶ **Observation 3.** *A class $\mathscr{D}$ does not need copying if and only if for every class $\mathscr{C}$ we have $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathscr{D}$ if and only if $\mathscr{C} \sqsubseteq_{\mathrm{FO}}^\circ \mathscr{D}$.*

▶ **Observation 4.** *If a class $\mathscr{C}$ is closed under adding pendant vertices (that is, if $G \in \mathscr{C}$ and $v \in V(G)$, then G', which is obtained from G by adding a new vertex adjacent only to v, is also in $\mathscr{C}$) then $\mathscr{C}$ does not need copying.*

A *subset complementation* transduction is defined by the quantifier-free interpretation on a Σ-expansion (with $\Sigma = \{M\}$) by $\eta(x, y) := (x \neq y) \wedge \neg\big(E(x, y) \leftrightarrow (M(x) \wedge M(y))\big)$. In other words, the subset complementation transduction complements the adjacency inside the subset of the vertex set defined by M. We denote by $\oplus M$ the subset complementation defined by the unary relation M. A *perturbation* is a composition of (a bounded number of) subset complementations. Let r be a non-negative integer. A formula $\varphi(x_1, \ldots, x_k)$ is *r-local* if for every (Σ-expanded) graph G and all $v_1, \ldots, v_k \in V(G)$ we have $G \models \varphi(v_1, \ldots, v_k) \iff B_r^G(\{v_1, \ldots, v_k\}) \models \varphi(v_1, \ldots, v_k)$. An r-local formula $\varphi(x_1, \ldots, x_k)$ is *strongly r-local* if $\models \varphi(x_1, \ldots, x_k) \to \mathrm{dist}(x_i, x_j) \leq r$ for all $1 \leq i < j \leq k$ (see [13]).

▶ **Lemma 1** (Gaifman's Locality Theorem [6]). *Every formula $\varphi(x_1, \ldots, x_m)$ is equivalent to a Boolean combination of t-local formulas and so-called* basic local sentences *of the form*

$$\exists x_1 \ldots \exists x_k \big(\bigwedge_{1 \leq i \leq k} \chi(x_i) \wedge \bigwedge_{1 \leq i < j \leq k} \mathrm{dist}(x_i, x_j) > 2r \big) \quad \text{(where χ is r-local)}.$$

Furthermore, if the quantifier-rank of φ is q, then $r \leq 7^{q-1}$, $t \leq 7^{q-1}/2$, and $k \leq q + m$.

We call a transduction T *immersive* if it is non-copying and the formulas in the interpretation associated to T are strongly local.

3 Local properties of FO transductions

3.1 A local normal form

We now establish a normal form for first-order transductions that captures the local character of first-order logic and further study the properties of immersive transductions. The normal form is based on Gaifman's Locality Theorem and uses only *strongly local formulas*, while the basic-local sentences are handled by subset complementations. This normal form will be one of the main tools to establish results in the paper.

▶ **Theorem 2.** *Every non-copying transduction T is subsumed by the composition of an immersive transduction $\mathsf{T}_{\mathrm{imm}}$ and a perturbation P, that is $\mathsf{T} \leq \mathsf{P} \circ \mathsf{T}_{\mathrm{imm}}$.*

Consequently, every transduction T is subsumed by the composition of a copying operation C, an immersive transduction $\mathsf{T}_{\mathrm{imm}}$ and a perturbation P, that is $\mathsf{T} \leq \mathsf{P} \circ \mathsf{T}_{\mathrm{imm}} \circ \mathsf{C}$.

Proof. Let $\mathsf{T} = \mathsf{I}_\mathsf{T} \circ \Gamma_{\Sigma_\mathsf{T}}$ be a non-copying transduction. Without loss of generality, we may assume that the interpretation I_T defines the domain directly from the Σ_T-expansion. Then the only non-trivial part of the interpretation is the adjacency relation, which is defined by a symmetric and anti-reflexive formula $\eta(x, y)$. We shall prove that the transduction T is subsumed by the composition of an immersive transduction T_ψ and a perturbation P.

We define Σ_{T_ψ} as the disjoint union of Σ_T and a set $\Sigma_\psi = \{T_i \mid 1 \leq i \leq n_1\}$ for some integer n_1 we shall specify later and let $\Sigma_\mathsf{P} = \{Z_j \mid 1 \leq j \leq n_2\}$ for some integer n_2 we shall also specify later. Let q be the quantifier rank of $\eta(x, y)$. According to Lemma 1, η is logically equivalent to a formula in Gaifman normal form, that is, to a Boolean combination of t-local formulas and basic-local sentences $\theta_1, \ldots, \theta_{n_1}$. To each θ_i we associate a unary predicate $T_i \in \Sigma_\psi$. We consider the formula $\widetilde{\eta}(x, y)$ obtained from the Gaifman normal form of $\eta(x, y)$ by replacing the sentence θ_i by the atomic formula $T_i(x)$. Note that $\widetilde{\eta}$ is t-local.

Under the assumption that $\mathrm{dist}(x, y) > 2t$ every t-local formula $\chi(x, y)$ is equivalent to $\chi_1(x) \wedge \chi_2(y)$ for t-local formulas $\chi_1(x)$ and $\chi_2(y)$. Furthermore, t-local formulas are closed under boolean combinations. By bringing $\widetilde{\eta}$ into disjunctive normal form and grouping conjuncts appropriately, it follows that under the assumption $\mathrm{dist}(x, y) > 2t$ the formula $\widetilde{\eta}$ is equivalent to a formula $\widetilde{\varphi}(x, y)$ of the form $\bigvee_{(i,j) \in \mathcal{F}} \zeta_i(x) \wedge \zeta_j(y)$, where $\mathcal{F} \subseteq [n_2] \times [n_2]$ for some integer n_2 and the formulas ζ_i $(1 \leq i \leq n_2)$ are t-local. By considering appropriate boolean combinations (or, for those familiar with model theory, by assuming that the ζ_i define local types) we may assume that $\models \forall x \bigwedge_{i \neq j} \neg(\zeta_i(x) \wedge \zeta_j(x))$, that is, every element of a graph satisfies at most one of the ζ_i. Note also that $\mathcal{F}$ is symmetric as η (hence $\widetilde{\eta}$ and $\widetilde{\varphi}$) are symmetric.

We define $\psi(x, y) := \neg(\widetilde{\eta}(x, y) \leftrightarrow \widetilde{\varphi}(x, y)) \wedge (\mathrm{dist}(x, y) \leq 2t)$, which is $2t$-strongly local, and we define $\mathsf{I}_{\mathsf{T}_\psi}$ as the interpretation of graphs in Σ_{T_ψ}-structures by using the same definitions as in I_T for the domain, then defining the adjacency relation by $\psi(x, y)$. To

each formula ζ_i we associate a unary predicate $Z_i \in \Sigma_{\mathsf{P}}$. We define the perturbation P as the sequence of subset complementations $\oplus Z_i$ (for $(i, i) \in \mathcal{F}$) and of $\oplus Z_i \oplus Z_j \oplus (Z_i \cup Z_j)$ (for $(i, j) \in \mathcal{F}$ and $i < j$). Denote by $\varphi(x, y)$ the formula defining the edges in the interpretation I_{P}. Note that when the Z_i are pairwise disjoint, then P complements exactly the edges of Z_i or between Z_i and Z_j, respectively. The operation $\oplus(Z_i \cup Z_j)$ complements all edges between Z_i and Z_j, but also inside Z_i and Z_j, which is undone by $\oplus Z_j$ and $\oplus Z_i$.

Now assume that a graph H is a T-transduction of a graph G, and let G^+ be a Σ_{T}-expansion of G such that $H = I_{\mathsf{T}}(G^+)$. We define the Σ_ψ-expansion G^* of G^+ (which is thus a Σ_{T_ψ}-expansion of G) by defining, for each $i \in [n_1]$, $T_i(G^*) = V(G)$ if $G^+ \models \theta_i$ and $T_i(G^*) = \emptyset$ otherwise. Let $K = I_{\mathsf{T}_\psi}(G^*)$. We define the Σ_{P}-expansion K^+ of K by defining, for each $j \in [n_2]$, $Z_j(K^+) = \zeta_j(G^+)$. By the assumption that $\models \forall x \bigwedge_{i \neq j} \neg(\zeta_i(x) \wedge \zeta_j(x))$ the Z_j are pairwise disjoint. Now, when $\mathrm{dist}(x, y) > 2t$ there is no edge between x and y in K, hence φ on K^+ is equivalent to $\widetilde{\varphi}$ on G^*, which in turn in this case is equivalent to $\widetilde{\eta}(x, y)$ on G^*. On the other hand, when $\mathrm{dist}(x, y) \leq 2t$, then the perturbation is applied to the edges defined by $\neg(\widetilde{\eta}(x, y) \leftrightarrow \widetilde{\varphi}(x, y))$, which yields exactly the edges defined by $\widetilde{\eta}$ on G^*. Thus we have $\eta(G^+) = \widetilde{\eta}(G^*) = \varphi(K^+)$, hence $I_{\mathsf{P}}(K^+) = H$.

It follows that the transduction T is subsumed by the composition of the immersive transduction T_ψ and a sequence of subset complementations, the perturbation P. ◀

▶ **Corollary 3.** *For every immersive transduction T and every perturbation P, there exist immersive transduction T' and a perturbation P', such that $\mathsf{P}' \circ \mathsf{T}'$ subsumes $\mathsf{T} \circ \mathsf{P}$.*

3.2 Immersive transductions

Intuitively, copying operations and perturbations are simple operations. The main complexity of a transduction is captured by its immersive part. The strongly local character of immersive transductions is the key tool in our further analysis. It will be very useful to give another (seemingly) weaker property for the existence of an immersive transduction in another class, which is the existence of a transduction that does not shrink the distances too much, as we prove now.

▶ **Lemma 4.** *Assume there is a non-copying transduction T encoding $\mathscr{C}$ in $\mathscr{D}$ with associated interpretation I and an $\epsilon > 0$ with the property that for every $H \in \mathscr{C}$ and $G \in \mathscr{D}$ with $H \in \mathsf{T}(G)$ we have $\mathrm{dist}_H(u, v) \geq \epsilon \, \mathrm{dist}_G(u, v)$ (for all $u, v \in V(H)$). Then there exists an immersive transduction encoding $\mathscr{C}$ in $\mathscr{D}$ that subsumes T.*

Proof. Let $\mathsf{T} = I \circ \Gamma_\Sigma$ with $I = (\nu(x), \eta(x, y))$. By Gaifman's locality theorem, there is a set $\Sigma' \supseteq \Sigma$ of unary relations and a formula $\varphi(x, y)$, such that for every Σ-expanded graph G^+ there is a Σ'-expansion G^* of G^+ with $G^* \models \varphi(x, y)$ if and only if $G^+ \models \eta(x, y)$, where φ is t-local for some t (as in the proof of Theorem 2). We further define a new mark M and let $I' = (M(x), \varphi(x, y) \wedge \mathrm{dist}(x, y) \leq 1/\epsilon)$. The transduction $\mathsf{T}' = I' \circ \Gamma_{\Sigma' \cup \{M\}}$ is immersive and subsumes the transduction T. ◀

Recall that $G + K_1$ is obtained from G by adding a new vertex, called an *apex*, that is connected to all vertices of G. Of course, by adding an apex we shrink all distances in G. The next lemma shows that when we can transduce $\mathscr{C} + K_1$ in a class $\mathscr{F}$ with an immersive transduction, then we can in fact transduce $\mathscr{C}$ in the local balls of $\mathscr{F}$.

▶ **Lemma 5.** *Let $\mathscr{C}, \mathscr{F}$ be graph classes, and let T be an immersive transduction encoding a class $\mathscr{D}$ in $\mathscr{F}$ with $\mathscr{D} \supseteq \{G + K_1 \mid G \in \mathscr{C}\}$. Then there exists an integer r such that $\mathscr{C} \sqsubseteq_{\mathrm{FO}}^\circ \mathcal{B}_r^{\mathscr{F}}$.*

Proof. Let $\mathsf{T} = \mathsf{I} \circ \Gamma_\Sigma$ be an immersive transduction encoding $\mathscr{D}$ in $\mathscr{F}$. For every graph $G \in \mathscr{C}$ there exists a graph $F \in \mathscr{F}$ such that $G + K_1 = \mathsf{I}(F^+)$, where F^+ is a Σ-expansion of F. Let v be the apex of $G + K_1$. By the strong locality of I we get $\mathsf{I}(F^+) = \mathsf{I}(B_r^{F^+}(v))$ for some fixed r depending only on T. Let U be a transduction allowing to take an induced subgraph, then G can be encoded in the class $\mathcal{B}_r^{\mathscr{C}}$ by the non-copying transduction $\mathsf{U} \circ \mathsf{T}$. ◀

Finally, we show that when transducing an additive class $\mathscr{C}$ in a class $\mathscr{D}$, then we do not need perturbations at all.

▶ **Lemma 6.** *Let $\mathscr{C}$ be an additive class with $\mathscr{C} \sqsubseteq_{\mathrm{FO}}^{\circ} \mathscr{D}$. Then there exists an immersive transduction encoding $\mathscr{C}$ in $\mathscr{D}$.*

Proof. According to Theorem 2, the transduction of $\mathscr{C}$ in $\mathscr{D}$ is subsumed by the composition of an immersive transduction T (with associated interpretation $\mathsf{I} = (\nu, \eta)$) and a perturbation (with associated interpretation I_P). As η is strongly local there exists r such that for all $G \in \mathscr{D}$ and Σ_T-expansions G^+ and all $u, v \in \mathsf{I}(G^+)$ we have $\mathrm{dist}_{\mathsf{I}(G^+)}(u, v) \geq \mathrm{dist}_G(u, v)/r$. Let c be the number of unary relations used in the perturbation. Let H be a graph in $\mathscr{C}$, let $n > 3 \cdot c^{|H|}$ and let $K = nH$ (n disjoint copies of H). By assumption there exists an expansion G^+ of a graph G in $\mathscr{D}$ with $K = \mathsf{I}_P \circ \mathsf{I}(G^+)$. By the choice of n, at least 3 copies H_1, H_2, and H_3 of H in K satisfy the same unary predicates at the same vertices. For $a \in \{1, 2, 3\}$ and $v \in V(H_1)$, we denote by $\tau_a(v)$ the vertex of H_a corresponding to the vertex v of H_1 ($\tau_1(v)$ being the vertex v itself). Let u, v be adjacent vertices of H_1. Assume that u and v have distance greater than r in G. Then u and v are made adjacent in K by the perturbation P (the edge cannot have been created by η as it is strongly r-local). As $\tau_a(u)$ is not adjacent with $\tau_b(v)$ for $b \neq a$ there must be paths of length at most r linking $\tau_a(u)$ with $\tau_b(v)$ in G for $a \neq b$ (the interpretation I must have introduced an edge that the perturbation removed again). This however implies that there is a path of length at most $3r$ between u and v in G (going from u to $\tau_2(v)$ to $\tau_3(u)$ to v). It follows that for all $u, v \in V(K)$ we have $\mathrm{dist}_K(u, v) \geq \mathrm{dist}_G(u, v)/(3r)$. Hence the transduction obtained by composing T with the extraction of the induced subgraph H_1 implies the existence of an immersive transduction of $\mathscr{C}$ in $\mathscr{D}$, according to Lemma 4. ◀

▶ **Corollary 7** (Elimination of the perturbation). *Let $\mathscr{C}$ be an additive class with $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathscr{D}$. Then there exists a copy operation C and an immersive transduction $\mathsf{T}_{\mathrm{imm}}$ such that $\mathsf{T}_{\mathrm{imm}} \circ \mathsf{C}$ is a transduction encoding $\mathscr{C}$ in $\mathscr{D}$.*

4 Some applications of the local normal form

4.1 Transductions in paths

▶ **Theorem 8.** *A class $\mathscr{C}$ is FO transduction equivalent to the class of paths if and only if it is a perturbation of a class with bounded bandwidth that contains graphs with arbitrarily large connected components.*

Proof. Assume T is a transduction of $\mathscr{C}$ in $\mathcal{P}$. According to Theorem 2, $\mathsf{T} \leq \mathsf{P} \circ \mathsf{T}_{\mathrm{imm}} \circ \mathsf{C}_k$, where $k \geq 1$, $\mathsf{T}_{\mathrm{imm}}$ is immersive, and P is a perturbation. Observe first that $\mathsf{C}_k(\mathcal{P})$ is included in the class of all subgraphs of the $(k+1)$-power of paths. By the strong locality property of immersive transductions, every class obtained from $\mathcal{P}$ by the composition of a copy operation and an immersive transduction has its image included in the class of all the subgraphs of the ℓ-power of paths, for some integer ℓ depending only on the transduction, hence, in a class of bounded bandwidth. Conversely, assume that $\mathscr{C}$ is a perturbation of a

class $\mathscr{D}$ containing graphs with bandwidth at most ℓ that contains graphs with arbitrarily large connected components. Then $\mathscr{D}$ is a subclass of the monotone closure (containing all subgraphs of the class) of the class $\mathcal{P}^\ell$ of ℓ-powers of paths, which has bounded star chromatic number. We show in the full version of this paper [17] that we can obtain the monotone closure of a class with bounded star chromatic number as a transduction. By this result and the observation that taking the ℓ-power is obviously a transduction, we get that $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathcal{P}$. To see that vice versa $\mathcal{P} \sqsubseteq_{\mathrm{FO}} \mathscr{C}$ observe that we can first undo the perturbation by carrying out the edge complementations in reverse order. Then we have arbitrarily large connected components, which in a graph of bounded bandwidth have unbounded diameter. From this we can transduce arbitrarily long paths by extracting an induced subgraph.   ◀

4.2   Local monadically stable, straight, and dependent classes

A class $\mathscr{C}$ is *locally monadically dependent* if, for every integer r, the class $\mathcal{B}_r^{\mathscr{C}}$ is monadically dependent; a class $\mathscr{C}$ is *locally monadically stable* if, for every integer r, the class $\mathcal{B}_r^{\mathscr{C}}$ is monadically stable; a class $\mathscr{C}$ is *locally monadically straight* if, for every integer r, the class $\mathcal{B}_r^{\mathscr{C}}$ is monadically straight.

▶ **Theorem 9.** *For a class $\mathscr{C}$ of graphs we have the following equivalences:*

1. *$\mathscr{C}$ is locally monadically dependent if and only if $\mathscr{C}$ is monadically dependent;*

2. *$\mathscr{C}$ is locally monadically straight if and only if $\mathscr{C}$ is monadically straight;*

3. *$\mathscr{C}$ is locally monadically stable if and only if $\mathscr{C}$ is monadically stable.*

Proof. The proof will follow from the following claim.

▷ Claim 10.   Let $\mathscr{C}$ be a class such that the class $\mathscr{C}' = \{n(G + K_1) \mid n \in \mathbb{N}, G \in \mathscr{C}\}$ is a transduction of $\mathscr{C}$. Then, for every class $\mathscr{D}$ we have $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathscr{D}$ if and only if there exists some integer r with $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathcal{B}_r^{\mathscr{D}}$.

Proof. Obviously, if there exists some integer r with $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathcal{B}_r^{\mathscr{D}}$, then $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathscr{D}$. Now assume $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathscr{D}$. As $\mathscr{C}' \equiv_{\mathrm{FO}} \mathscr{C}$ we prove as in Lemma 6 that there is a transduction of $\mathscr{C}'$ in $\mathscr{D}$ that is the composition of a copy operation C and an immersive transduction T. Let $\mathscr{D}' = \mathsf{C}(\mathscr{D})$. According to Lemma 5, there is an integer r such that $\mathscr{C} \sqsubseteq_{\mathrm{FO}}^{\circ} \mathcal{B}_r^{\mathscr{D}'}$ thus, as $\mathcal{B}_r^{\mathscr{D}'} = \mathsf{C}(\mathcal{B}_r^{\mathscr{D}})$, we have $\mathscr{C} \sqsubseteq_{\mathrm{FO}} \mathcal{B}_r^{\mathscr{D}}$.   ◁

The class $\{n(G + K_1) \mid n \in \mathbb{N}, G \in \mathcal{G}\}$ is obviously a transduction of $\mathcal{G}$. Hence, according to Claim 10, a class $\mathscr{C}$ is locally monadically dependent if and only if it is monadically dependent. The class $\{n(G + K_1) \mid n \in \mathbb{N}, G \in \mathcal{TP}\}$ is a transduction of $\mathcal{TP}$. Hence, according to Claim 10, a class $\mathscr{C}$ is locally monadically straight if and only if it is monadically straight. The class $\{n(G + K_1) \mid n \in \mathbb{N}, G \in \mathcal{H}\}$ is a transduction of $\mathcal{H}$. Hence, according to Claim 10, a class $\mathscr{C}$ is locally monadically stable if and only if it is monadically stable.   ◀

▶ **Example 11.** Although the class of unit interval graphs has unbounded clique-width, every proper hereditary subclass of unit interval graphs has bounded clique-width [12]. This is in particular the case for the class of unit interval graphs with bounded radius. As classes with bounded clique-width are monadically dependent, the class of unit interval graphs is locally monadically dependent, hence monadically dependent.

5   Structural properties of the transduction quasiorders

Many properties will be similar when considering $\sqsubseteq_{\mathrm{FO}}$ and $\sqsubseteq_{\mathrm{FO}}^{\circ}$. To avoid unnecessary repetitions of the statements and arguments, we shall use the notations $\sqsubseteq, \sqsubset, \equiv$ to denote either $\sqsubseteq_{\mathrm{FO}}^{\circ}, \sqsubset_{\mathrm{FO}}^{\circ}, \equiv_{\mathrm{FO}}^{\circ}$ or $\sqsubseteq_{\mathrm{FO}}, \sqsubset_{\mathrm{FO}}, \equiv_{\mathrm{FO}}$.

For two classes $\mathscr{C}_1$ and $\mathscr{C}_2$ define $\mathscr{C}_1 + \mathscr{C}_2 = \{G_1 \cup G_2 : G_1 \in \mathscr{C}_1, G_2 \in \mathscr{C}_2\}$. A class $\mathscr{C}$ is *additive* if $\mathscr{C} + \mathscr{C} \equiv \mathscr{C}$. For instance, every class closed under disjoint union is additive, while the class of all stars and all paths is not additive. Note that if $\mathscr{C}_1$ and $\mathscr{C}_2$ are additive then $\mathscr{C}_1 + \mathscr{C}_2$ is also additive. We further say that a class $\mathscr{C}$ is *essentially connected* if it is equivalent to the subclass $\mathrm{Conn}(\mathscr{C})$ of all its connected graphs.

In this section we will consider the quasiorders $\sqsubseteq_{\mathrm{FO}}^{\circ}$ and $\sqsubseteq_{\mathrm{FO}}$, as well as their restrictions to additive classes of graphs. Let $\mathfrak{C}$ be the collection of all graph classes, and let $\mathfrak{A}$ be the collection of all additive graph classes. While speaking about these quasiorders, we will implicitly consider their quotient by the equivalence relation $\equiv$, which are partial orders. For instance, when we say that $(\mathfrak{C}, \sqsubseteq)$ is a join-semilattice, we mean that $(\mathfrak{C}/\equiv, \sqsubseteq)$ is a join semilattice. The symbol $\lhd$ will always been used with reference to $(\mathfrak{C}, \sqsubseteq)$, $\mathscr{C} \lhd \mathscr{D}$ expressing that there exist no class $\mathscr{F}$ with $\mathscr{C} \sqsubset \mathscr{F} \sqsubset \mathscr{D}$. When we shall consider covers in $(\mathfrak{A}, \sqsubseteq)$ we will say explicitly that $(\mathscr{C}, \mathscr{D})$ is a cover in $(\mathfrak{A}, \sqsubseteq)$, expressing that there exists no additive class $\mathscr{F}$ with $\mathscr{C} \sqsubset \mathscr{F} \sqsubset \mathscr{D}$.

5.1   The transduction semilattices $(\mathfrak{C}, \sqsubseteq_{\mathbf{FO}}^{\circ})$ and $(\mathfrak{C}, \sqsubseteq_{\mathbf{FO}})$

The aim of this section is to prove that $(\mathfrak{C}, \sqsubseteq_{\mathrm{FO}}^{\circ})$ and $(\mathfrak{C}, \sqsubseteq_{\mathrm{FO}})$ are distributive join-semilattices and to state some of their properties.

▶ **Lemma 12.** *If $\mathscr{D} \sqsubseteq \mathscr{C}_1 \cup \mathscr{C}_2$, then there is a partition $\mathscr{D}_1 \cup \mathscr{D}_2$ of $\mathscr{D}$ with $\mathscr{D}_1 \sqsubseteq \mathscr{C}_1$ and $\mathscr{D}_2 \sqsubseteq \mathscr{C}_2$. If $\mathscr{D}$ is additive, then $\mathscr{D} \sqsubseteq \mathscr{C}_1 \cup \mathscr{C}_2 \iff \mathscr{D} \sqsubseteq \mathscr{C}_1$ or $\mathscr{D} \sqsubseteq \mathscr{C}_2$.*

Proof. The first statement is straightforward. We now prove the second statement. For an integer n, let G_n be the disjoint union of all the graphs in $\mathscr{D}$ with at most n vertices.

Assume $\mathscr{D}$ is additive and $\mathscr{D} \sqsubseteq \mathscr{C}_1 \cup \mathscr{C}_2$. According to the first statement, there exists a partition $\mathscr{D}_1, \mathscr{D}_2$ of $\mathscr{D}$ with $\mathscr{D}_1 \sqsubseteq \mathscr{C}_1$ and $\mathscr{D}_2 \sqsubseteq \mathscr{C}_2$. For $G \in \mathscr{D}$ define $\mathscr{S}(G) = \{H \cup G : H \in \mathscr{D}\}$. Note that $\mathscr{S}(G) \subseteq \mathscr{D} + \mathscr{D}$. Let $\mathscr{D}' = \mathscr{D} + \mathscr{D}$. As $\mathscr{D}' \sqsubseteq \mathscr{D}_1 \cup \mathscr{D}_2$ there exists a partition $\mathscr{D}_1', \mathscr{D}_2'$ of $\mathscr{D}'$ with $\mathscr{D}_1' \sqsubseteq \mathscr{D}_1$ and $\mathscr{D}_2' \sqsubseteq \mathscr{D}_2$. If, for every $G \in \mathscr{D}$ we have $\mathscr{S}(G) \cap \mathscr{D}_1' \neq \emptyset$ then $\mathscr{D} \sqsubseteq \mathscr{D}_1'$ (by the generic transduction extracting an induced subgraph) thus $\mathscr{D} \equiv \mathscr{D}_1 \sqsubseteq \mathscr{C}_1$. Similarly, if for every $G \in \mathscr{C}$ we have $\mathscr{S}(G) \cap \mathscr{D}_2' \neq \emptyset$ then $\mathscr{D} \sqsubseteq \mathscr{C}_2$. Assume for contradiction that there exist $G_1, G_2 \in \mathscr{D}$ with $\mathscr{S}(G_i) \cap \mathscr{D}_i' = \emptyset$. Then $G_1 \cup G_2$ belongs neither to $\mathscr{D}_1'$ nor to $\mathscr{D}_2'$, contradicting the assumption that $\mathscr{D}_1', \mathscr{D}_2'$ is a partition of $\mathscr{D}' = \mathscr{D} + \mathscr{D}$.   ◀

▶ **Lemma 13.** *If $\mathscr{C}_1$ and $\mathscr{C}_2$ are incomparable, then $\mathscr{C}_1 \cup \mathscr{C}_2$ is not equivalent to an additive class. In particular, $\mathscr{C}_1 \cup \mathscr{C}_2 \not\equiv \mathscr{C}_1 + \mathscr{C}_2$.*

Proof. We prove by contradiction that $\mathscr{C}_1 \cup \mathscr{C}_2$ is not equivalent to an additive class. Assume that we have $\mathscr{D} \sqsubseteq \mathscr{C}_1 \cup \mathscr{C}_2$, where $\mathscr{D}$ is additive. According to Lemma 12 we have $\mathscr{D} \sqsubseteq \mathscr{C}_1$ or $\mathscr{D} \sqsubseteq \mathscr{C}_2$ thus if $\mathscr{C}_1 \cup \mathscr{C}_2 \sqsubseteq \mathscr{D}$, then $\mathscr{C}_2 \sqsubseteq \mathscr{C}_1$ or $\mathscr{C}_1 \sqsubseteq \mathscr{C}_2$, contradicting the hypothesis that $\mathscr{C}_1$ and $\mathscr{C}_2$ are incomparable.   ◀

▶ **Theorem 14.** *The quasiorder $(\mathfrak{C}, \sqsubseteq)$ is a distributive join-semilattice, where the join of $\mathscr{C}_1$ and $\mathscr{C}_2$ is $\mathscr{C}_1 \cup \mathscr{C}_2$. In this quasiorder, additive classes are join-irreducible. This quasiorder has a minimum $\mathcal{E}$ and a maximum $\mathcal{G}$.*

Proof. Of course we have $\mathscr{C}_1 \sqsubseteq \mathscr{C}_1 \cup \mathscr{C}_2$ and $\mathscr{C}_2 \sqsubseteq \mathscr{C}_1 \cup \mathscr{C}_2$. Now assume $\mathscr{D}$ is such that $\mathscr{C}_1 \sqsubseteq \mathscr{D}$ and $\mathscr{C}_2 \sqsubseteq \mathscr{D}$. Let T_1 and T_2 be transductions encoding $\mathscr{C}_1$ and $\mathscr{C}_2$ in $\mathscr{D}$, with associated interpretations $\mathsf{I}_1 = (\nu_1, \eta_1)$ and $\mathsf{I}_2 = (\nu_1, \eta_1)$. By relabeling the colors, we can assume that the set Σ_1 of unary relations used by I_1 is disjoint from the set Σ_2 of unary relations used by I_2. Without loss of generality, we have $\mathsf{T}_1 = \mathsf{I}_1 \circ \Gamma_{\Sigma_1} \circ \mathsf{C}$ and $\mathsf{T}_2 = \mathsf{I}_2 \circ \Gamma_{\Sigma_2} \circ \mathsf{C}$, where C is a copying operation if $\sqsubseteq$ is $\sqsubseteq_{\mathrm{FO}}$, or the identity mapping if $\sqsubseteq$ is $\sqsubseteq_{\mathrm{FO}}^{\circ}$. Let M be a new unary relation. We define the interpretation $\mathsf{I} = (\nu, \eta)$ by $\nu := \big((\exists v\; M(v)) \wedge \nu_1\big) \vee \big(\neg(\exists v\; M(v)) \wedge \nu_2\big)$ and $\eta := \big((\exists v\; M(v)) \wedge \eta_1\big) \vee \big(\neg(\exists v\; M(v)) \wedge \eta_2\big)$. Let $G \in \mathscr{C}_1 \cup \mathscr{C}_2$. If $G \in \mathscr{C}_1$, then there exists a coloring H^+ of $H \in \mathsf{C}(\mathscr{D})$ with $G = \mathsf{I}_1(H^+)$. We define H^* as the expansion of H^+ where all vertices also belong to the unary relation M. Then $G = \mathsf{I}(H^*)$. Otherwise, if $G \in \mathscr{C}_2$, then there exists a coloring H^+ of $H \in \mathsf{C}(\mathscr{D})$ with $G = \mathsf{I}_2(H^+)$ thus $G = \mathsf{I}(H^+)$. As we did not introduce new copying transductions we deduce $\mathscr{C}_1 \cup \mathscr{C}_2 \sqsubseteq \mathscr{D}$. It follows that $(\mathfrak{C}, \sqsubseteq)$ is a join semi-lattice, which is distributive according to Lemma 12.

That additive classes are join-irreducible follows from Lemma 13. ◄

We now state an easy lemma on covers in distributive join-semilattices.

▶ **Lemma 15.** *Let $(X, \leq)$ be a distributive join-semilattice (with join $\vee$). If $a \lhd b$ and $b \not\leq a \vee c$, then $a \vee c \lhd b \vee c$.*

Proof. Assume $a \vee c \leq x \leq b \vee c$. As $(X, \leq)$ is distributive there exist $b' \leq b$ and $c' \leq c$ with $x = b' \vee c'$. Thus $a \leq a \vee b' \leq b$. As $a \lhd b$, either $a = a \vee b'$ (thus $b' \leq a$) and thus $x = a \vee c$, or $a \vee b' = b$ and then $b \vee c \leq a \vee b' \vee c \leq a \vee x \vee c = x \leq b \vee c$ thus $x = b \vee c$. Hence either $a \vee c = b \vee c$ (which would contradict $b \not\leq a \vee c$), or $a \vee c \lhd b \vee c$. ◄

▶ **Corollary 16.** *If $\mathscr{C}_1 \lhd \mathscr{C}_2$ and $\mathscr{C}_2 \not\sqsubseteq \mathscr{C}_1 \cup \mathscr{D}$, then $\mathscr{C}_1 \cup \mathscr{D} \lhd \mathscr{C}_2 \cup \mathscr{D}$.*

▶ **Corollary 17.** *If $\mathscr{C}_1 \lhd \mathscr{C}_2$, $\mathscr{C}_1 \sqsubseteq \mathscr{D}$, and $\mathscr{C}_2$ and $\mathscr{D}$ are incomparable, then $\mathscr{D} \lhd \mathscr{D} \cup \mathscr{C}_2$.*

Proof. As $\mathscr{C}_2 \not\sqsubseteq \mathscr{D}$ and $\mathscr{C}_1 \sqsubseteq \mathscr{D}$ we have $\mathscr{C}_2 \not\sqsubseteq \mathscr{D} \cup \mathscr{C}_1$. ◄

▶ **Corollary 18.** *If $\mathscr{C}_1 \lhd \mathscr{C}_2$, $\mathscr{C}_2$ and $\mathscr{D}$ are incomparable and $\mathscr{C}_2$ is additive, then $\mathscr{C}_1 \cup \mathscr{D} \lhd \mathscr{C}_2 \cup \mathscr{D}$.*

5.2 The transduction semilattices $(\mathfrak{A}, \sqsubseteq_{\mathrm{FO}}^{\circ})$ and $(\mathfrak{A}, \sqsubseteq_{\mathrm{FO}})$

The aim of this section is to prove that $(\mathfrak{A}, \sqsubseteq_{\mathrm{FO}}^{\circ})$ and $(\mathfrak{A}, \sqsubseteq_{\mathrm{FO}})$ are distributive join-semilattices and to state some of their properties.

▶ **Lemma 19.** *If $\mathscr{C}_1$ and $\mathscr{C}_2$ are incomparable, then $\mathscr{C}_1 + \mathscr{C}_2$ is not essentially connected.*

Proof. We prove by contradiction that $\mathscr{C}_1 + \mathscr{C}_2$ is not essentially connected. It is immediate that $\mathrm{Conn}(\mathscr{C}_1 + \mathscr{C}_2) \subseteq \mathscr{C}_1 \cup \mathscr{C}_2$. So if $\mathscr{C}_1 + \mathscr{C}_2$ is essentially connected, then $\mathscr{C}_1 \cup \mathscr{C}_2$ and $\mathscr{C}_1 + \mathscr{C}_2$ are equivalent, contradicting Lemma 13. ◄

▶ **Lemma 20.** *A class $\mathscr{D}$ is additive if and only if for all classes $\mathscr{C}_1, \mathscr{C}_2$ we have*

$$\mathscr{C}_1 + \mathscr{C}_2 \sqsubseteq \mathscr{D} \quad \Longleftrightarrow \quad \mathscr{C}_1 \sqsubseteq \mathscr{D} \text{ and } \mathscr{C}_2 \sqsubseteq \mathscr{D}.$$

Proof. Assume $\mathscr{D}$ is additive. If $\mathscr{C}_1 + \mathscr{C}_2 \sqsubseteq \mathscr{D}$, then $\mathscr{C}_1 \cup \mathscr{C}_2 \sqsubseteq \mathscr{D}$ thus $\mathscr{C}_1 \sqsubseteq \mathscr{D}$ and $\mathscr{C}_2 \sqsubseteq \mathscr{D}$. Conversely, assume $\mathscr{C}_1 \sqsubseteq \mathscr{D}$ and $\mathscr{C}_2 \sqsubseteq \mathscr{D}$. Then $\mathscr{C}_1 \cup \mathscr{C}_2 \sqsubseteq \mathscr{D}$. Let $\mathsf{T} = \mathsf{I} \circ \Gamma_\Sigma \circ \mathsf{C}$ be a transduction such that $\mathscr{C}_1 \cup \mathscr{C}_2 \subseteq \mathsf{T}(\mathscr{D})$, where $\mathsf{I} = (M(x), \varphi(x, y))$ with $M \in \Sigma$, and where C is either a copying operation if $\sqsubseteq$ is $\sqsubseteq_{\mathrm{FO}}$, or the identity mapping if $\sqsubseteq$ is $\sqsubseteq_{\mathrm{FO}}^{\circ}$. Let Σ' be

the signature obtained from Σ by adding two unary predicates $A(x)$ and $B(x)$. We define $\varphi_A(x,y)$ (resp. $\varphi_B(x,y)$) by replacing in $\varphi(x,y)$ the predicate M by the predicate A (resp. by the predicate B). Let $\varphi'(x,y) = \big(A(x) \wedge A(y) \wedge \varphi_A(x,y)\big) \vee \big(B(x) \wedge B(y) \wedge \varphi_B(x,y)\big)$, let $\mathsf{I}' = (A(x) \vee B(x), \varphi'(x,y))$, and let $\mathsf{T} = \mathsf{I}' \circ \Gamma_{\Sigma'} \circ \mathsf{C}$. Then it is easily checked that $\mathscr{C}_1 + \mathscr{C}_2 \subseteq \mathsf{T}'(\mathscr{D})$. Conversely, assume that for all classes $\mathscr{C}_1, \mathscr{C}_2$ we have $\mathscr{C}_1 + \mathscr{C}_2 \sqsubseteq \mathscr{D} \iff \mathscr{C}_1 \sqsubseteq \mathscr{D}$ and $\mathscr{C}_2 \sqsubseteq \mathscr{D}$. Then (by choosing $\mathscr{C}_1 = \mathscr{C}_2 = \mathscr{D}$) we deduce $\mathscr{D} + \mathscr{D} \equiv \mathscr{D}$. ◄

▶ **Lemma 21.** *Assume $\mathscr{D}$ is additive and $\mathscr{C}_1$ and $\mathscr{C}_2$ are incomparable. If $\mathscr{D} \sqsubseteq \mathscr{C}_1 + \mathscr{C}_2$, then there exist classes $\mathscr{D}_1$ and $\mathscr{D}_2$ such that $\mathscr{D} \equiv \mathscr{D}_1 + \mathscr{D}_2$, $\mathscr{D}_1 \sqsubseteq \mathscr{C}_1$ and $\mathscr{D}_2 \sqsubseteq \mathscr{C}_2$. Moreover, if $\mathscr{C}_1$ and $\mathscr{C}_2$ are additive we can require that $\mathscr{D}_1$ and $\mathscr{D}_2$ are additive.*

Proof. According to Corollary 7 there exists a copy operation C (which reduces to the identity if $\sqsubseteq$ is $\sqsubseteq_{\mathrm{FO}}^\circ$) and an immersive transduction $\mathsf{T}_{\mathrm{imm}}$ such that $\mathscr{D} \subseteq \mathsf{T}_{\mathrm{imm}} \circ \mathsf{C}(\mathscr{C}_1 + \mathscr{C}_2)$. Let I be the interpretation part of $\mathsf{T}_{\mathrm{imm}}$. Let $G \in \mathscr{D}$ and let H^+ be a coloring of $H = \mathsf{C}(K)$, with $K \in \mathscr{C}_1 + \mathscr{C}_2$ and $G = \mathsf{I}(H^+)$. As $\mathsf{T}_{\mathrm{imm}}$ is immersive, each connected component of G comes from a connected component of H^+ hence from a connected component of K. By grouping the connected components used in K by their origin ($\mathscr{C}_1$ or $\mathscr{C}_2$) we get that G is the disjoint union of $G_1 \in \mathsf{T}_{\mathrm{imm}} \circ C(K_1)$ and $G_2 \in \mathsf{T}_{\mathrm{imm}} \circ C(K_2)$, where $K_1 \in \mathscr{C}_1$ and $K_2 \in \mathscr{C}_2$. So $\mathscr{D} \sqsubseteq \mathscr{D}_1 + \mathscr{D}_2$, where $\mathscr{D}_1 \sqsubseteq \mathscr{C}_1$ and $\mathscr{D}_2 \sqsubseteq \mathscr{C}_2$. Moreover, as obviously $\mathscr{D}_1 \sqsubseteq \mathscr{D}$ and $\mathscr{D}_2 \sqsubseteq \mathscr{D}$ we derive from Lemma 20 that we have $\mathscr{D}_1 + \mathscr{D}_2 \sqsubseteq \mathscr{D}$. Hence $\mathscr{D} \equiv \mathscr{D}_1 + \mathscr{D}_2$. For $i = 1, 2$, if $\mathscr{C}_i$ is additive, then we can assume that $\mathscr{D}_i$ is also additive. ◄

▶ **Corollary 22.** *If $\mathscr{D}$ is additive and essentially connected, then*

$$\mathscr{D} \sqsubseteq \mathscr{C}_1 + \mathscr{C}_2 \iff \mathscr{D} \sqsubseteq \mathscr{C}_1 \text{ or } \mathscr{D} \sqsubseteq \mathscr{C}_2.$$

Proof. According to Lemma 21 there exist $\mathscr{D}_1, \mathscr{D}_2$ with $\mathscr{D} \equiv \mathscr{D}_1 + \mathscr{D}_2$, $\mathscr{D}_1 \sqsubseteq \mathscr{C}_1$ and $\mathscr{D}_2 \sqsubseteq \mathscr{C}_2$. However, as $\mathscr{D}$ is essentially connected, $\mathscr{D}_1$ and $\mathscr{D}_2$ cannot be incomparable. Thus $\mathscr{D} \sqsubseteq \mathscr{C}_1$ or $\mathscr{D} \sqsubseteq \mathscr{C}_2$. ◄

▶ **Theorem 23.** *The quasiorder $(\mathfrak{A}, \sqsubseteq)$ is a distributive join-semilattice, where the join of $\mathscr{C}_1$ and $\mathscr{C}_2$ is $\mathscr{C}_1 + \mathscr{C}_2$. In this quasiorder, essentially connected (additive) classes are join-irreducible. This quasiorder has a minimum $\mathcal{E}$ and a maximum $\mathcal{G}$.*

Proof. That $(\mathfrak{A}, \sqsubseteq)$ is a join-semilattice follows from Lemma 20; that it is distributive follows from Lemma 21. The last statement follows from Lemma 19. ◄

▶ **Corollary 24.** *Assume that $\mathscr{C}_1$ and $\mathscr{C}_2$ are incomparable and additive, $\mathscr{D}$ is additive and essentially connected, $\mathscr{C}_1 \sqsubseteq \mathscr{D}$, and $\mathscr{C}_2 \sqsubseteq \mathscr{D}$. Then we have*

$$\mathscr{C}_1 \cup \mathscr{C}_2 \sqsubset \mathscr{C}_1 + \mathscr{C}_2 \sqsubset \mathscr{D}.$$

▶ **Corollary 25.** *Assume that $\mathscr{C}_1$ and $\mathscr{C}_2$ are incomparable and additive, $\mathscr{D}$ is additive and essentially connected, $\mathscr{D}$ is incomparable with $\mathscr{C}_1$ and $\mathscr{C}_2 \sqsubset \mathscr{D}$. Then $\mathscr{C}_1 + \mathscr{C}_2$ is incomparable with $\mathscr{D}$.*

Proof. Assume for contradiction that $\mathscr{C}_1 + \mathscr{C}_2 \sqsubseteq \mathscr{D}$. According to Theorem 23, we have $\mathscr{C}_1 \sqsubseteq \mathscr{D}$, contradicting the assumption that $\mathscr{D}$ is incomparable with $\mathscr{C}_1$.

Assume for contradiction that $\mathscr{D} \sqsubseteq \mathscr{C}_1 + \mathscr{C}_2$. According to Theorem 23 there exists (by distributivity) classes $\mathscr{D}_1$ and $\mathscr{D}_2$ with $\mathscr{D}_1 \sqsubseteq \mathscr{C}_1$, $\mathscr{D}_2 \sqsubseteq \mathscr{C}_2$, and $\mathscr{D} = \mathscr{D}_1 + \mathscr{D}_2$. As $\mathscr{D}$ is essentially connected, according to Theorem 23 it is join-irreducible. Thus $\mathscr{D}_1$ and $\mathscr{D}_2$ are comparable. Thus $\mathscr{D} \subseteq \mathscr{C}_1$ or $\mathscr{D} \subseteq \mathscr{C}_2$. The first case does not hold as $\mathscr{D}$ is incomparable with $\mathscr{C}_1$, and the second case does not hold as $\mathscr{C}_2 \sqsubset \mathscr{D}$. ◄

Using the distributive join-semillatice structure of $(\mathfrak{A}, \sqsubseteq)$, the following corollaries follow from Lemma 15.

▶ **Corollary 26.** *In the poset $(\mathfrak{A}, \sqsubseteq)$, if $(\mathscr{C}_1, \mathscr{C}_2)$ is a cover and $\mathscr{C}_2 \not\sqsubseteq \mathscr{C}_1 + \mathscr{D}$ then $(\mathscr{C}_1 + \mathscr{D}, \mathscr{C}_2 + \mathscr{D})$ is a cover.*

▶ **Corollary 27.** *If $(\mathscr{C}_1, \mathscr{C}_2)$ is a cover of $(\mathfrak{A}, \sqsubseteq)$, $\mathscr{C}_1 \sqsubseteq \mathscr{D}$, and $\mathscr{C}_2$ and $\mathscr{D}$ are incomparable, then $(\mathscr{D}, \mathscr{D} + \mathscr{C}_2)$ is a cover of $(\mathfrak{A}, \sqsubseteq)$.*

▶ **Corollary 28.** *If $(\mathscr{C}_1, \mathscr{C}_2)$ is a cover of $(\mathfrak{A}, \sqsubseteq)$, $\mathscr{C}_2$ and $\mathscr{D}$ are incomparable, and $\mathscr{C}_2$ is essentially connected, then $(\mathscr{D}, \mathscr{D} + \mathscr{C}_2)$ is a cover of $(\mathfrak{A}, \sqsubseteq)$.*

6   The transduction quasiorder on some classes

In this section we consider the poset $(\mathfrak{C}, \sqsubseteq_{\mathrm{FO}})$. We focus on the structure of the partial order in the region of classes with bounded tree-width. A schematic view of the structure of $(\mathfrak{C}, \sqsubseteq_{\mathrm{FO}})$ on classes with tree-width at most 2 is shown Figure 2

Recall that since MSO collapses to FO on colored trees of bounded depth we have the following chain of covers $\mathcal{E} \lhd_{\mathrm{FO}} \mathcal{T}_2 \lhd_{\mathrm{FO}} \mathcal{T}_3 \lhd_{\mathrm{FO}} \dots$. We first prove that parallel to this chain we have a chain of covers $\mathcal{E} \lhd_{\mathrm{FO}} \mathcal{P} \lhd_{\mathrm{FO}} \mathcal{P} \cup \mathcal{T}_2 \lhd_{\mathrm{FO}} \mathcal{P} \cup \mathcal{T}_3 \lhd_{\mathrm{FO}} \dots$ and for all $n \geq 1$ we have $\mathcal{T}_n \lhd_{\mathrm{FO}} \mathcal{P} \cup \mathcal{T}_n$.

▶ **Theorem 29** (see [17] for the proof)**.** *We have $\mathcal{E} \lhd_{\mathrm{FO}} \mathcal{P}$ and, for every $n \geq 1$, the chain of covers*

$$(\mathcal{P} + \mathcal{T}_n) \lhd_{\mathrm{FO}} (\mathcal{P} + \mathcal{T}_n) \cup \mathcal{T}_{n+1} \lhd_{\mathrm{FO}} (\mathcal{P} + \mathcal{T}_n) \cup \mathcal{T}_{n+2} \lhd_{\mathrm{FO}} \dots$$

In particular, for $n = 1$ we get $\mathcal{P} \lhd_{\mathrm{FO}} \mathcal{P} \cup \mathcal{T}_2 \lhd_{\mathrm{FO}} \mathcal{P} \cup \mathcal{T}_3 \lhd_{\mathrm{FO}} \dots$
Moreover, for all $n \geq 1$ we have $\mathcal{T}_n \lhd_{\mathrm{FO}} \mathcal{P} \cup \mathcal{T}_n$.

The difficult part of the next theorem is to prove that $\mathcal{T}_{n+2} \not\sqsubseteq_{\mathrm{FO}} \mathcal{PW}_n$. We use that the class $\mathcal{T}_{n+2}$ is additive, which by Corollary 7 implies that we can eliminate perturbations and focus on immersive transductions. This allows us to consider host graphs in $\mathcal{PW}_n$ that have bounded radius, where we can find a small set of vertices whose removal decreases the pathwidth. We encode the adjacency to these vertices by colors and proceed by induction.

▶ **Theorem 30** (see [17] for the proof)**.** *For $n \geq 1$ we have $\mathcal{T}_{n+1} \sqsubseteq_{\mathrm{FO}} \mathcal{PW}_n$ but $\mathcal{T}_{n+2} \not\sqsubseteq_{\mathrm{FO}} \mathcal{PW}_n$. Consequently, for $m > n \geq 1$ we have*

$$\mathcal{T}_m + \mathcal{PW}_n \lhd_{\mathrm{FO}} (\mathcal{T}_m + \mathcal{PW}_n) \cup \mathcal{T}_{m+1} \lhd_{\mathrm{FO}} (\mathcal{T}_m + \mathcal{PW}_n) \cup \mathcal{T}_{m+2} \lhd_{\mathrm{FO}} \dots$$

$$\mathcal{T}_m + \mathcal{PW}_n \lhd_{\mathrm{FO}} (\mathcal{T}_m + \mathcal{PW}_n) \cup \mathcal{T}_{m+1} \sqsubseteq_{\mathrm{FO}} \mathcal{T}_{m+1} + \mathcal{PW}_n.$$

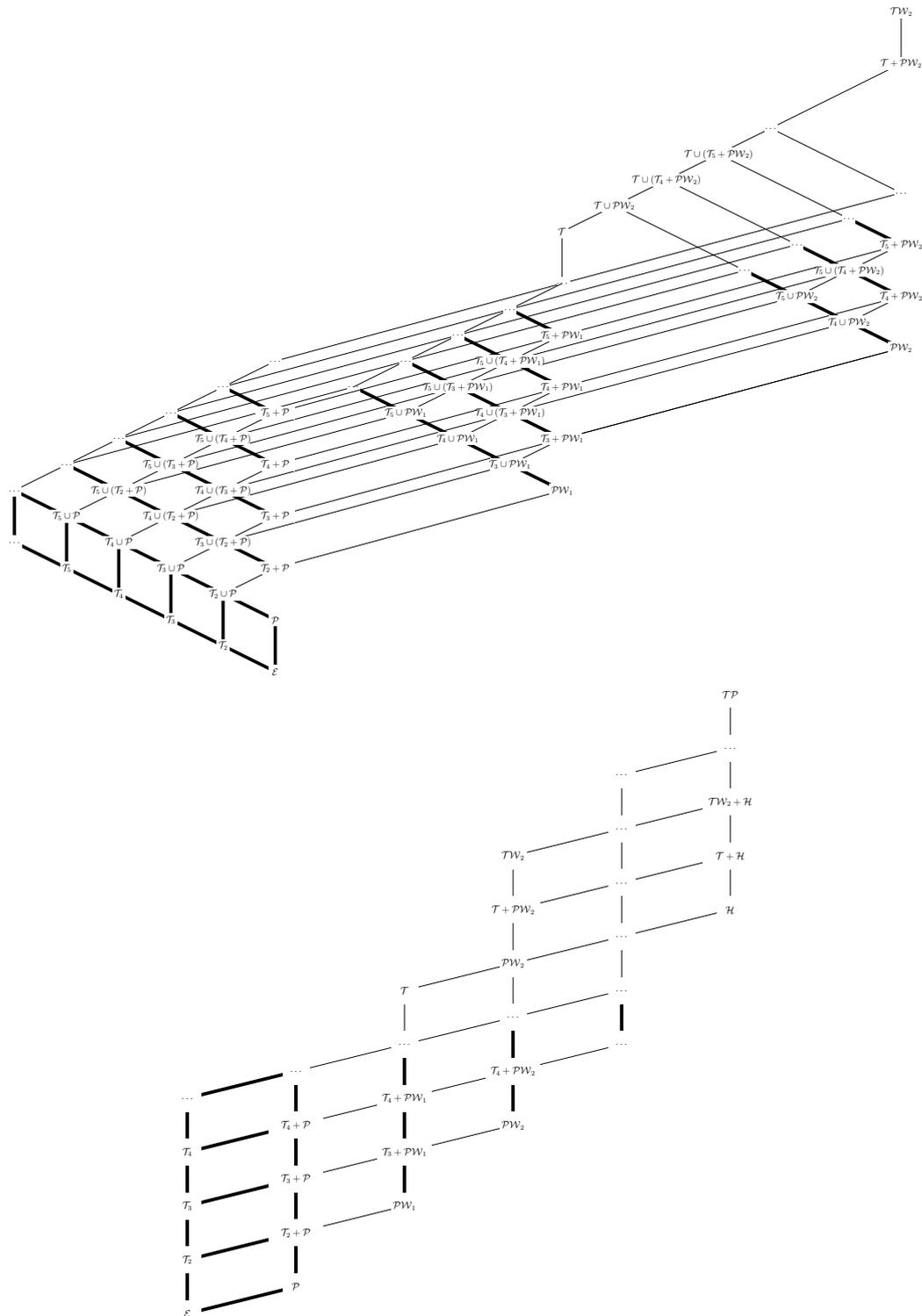*In particular, fixing $m = n + 1$ we get that for $n \geq 1$ we have*

$$\mathcal{PW}_n \lhd_{\mathrm{FO}} \mathcal{PW}_n \cup \mathcal{T}_{n+2} \lhd_{\mathrm{FO}} \mathcal{PW}_n \cup \mathcal{T}_{n+3} \lhd_{\mathrm{FO}} \dots \sqsubseteq_{\mathrm{FO}} \mathcal{PW}_n \cup \mathcal{T}$$

$$\mathcal{PW}_n \lhd_{\mathrm{FO}} \mathcal{PW}_n \cup \mathcal{T}_{n+2} \sqsubseteq_{\mathrm{FO}} \mathcal{T}_{n+2} + \mathcal{PW}_n \sqsubseteq_{\mathrm{FO}} \mathcal{PW}_{n+1}.$$

▶ **Theorem 31** (see [17] for the proof)**.** *For $m > n \geq 2$, $\mathcal{T}_m + \mathcal{PW}_n$ is incomparable with $\mathcal{T}$. Consequently, we have*

$$\mathcal{T} \sqsubseteq_{\mathrm{FO}} \mathcal{T} \cup \mathcal{PW}_2 \sqsubseteq_{\mathrm{FO}} \mathcal{T} \cup (\mathcal{T}_4 + \mathcal{PW}_2) \sqsubseteq_{\mathrm{FO}} \dots \sqsubseteq_{\mathrm{FO}} \mathcal{T} + \mathcal{PW}_2 \sqsubseteq_{\mathrm{FO}} \mathcal{TW}_2.$$

With the above results in hand we obtain for $(\mathfrak{C}, \sqsubseteq_{\mathrm{FO}})$ and $(\mathfrak{A}, \sqsubseteq_{\mathrm{FO}})$ the structures sketched in Figure 2.

Figure 2 A fragment of $(\mathfrak{C}, \sqsubseteq_{\mathrm{FO}})$ (top) and a fragment of $(\mathfrak{A}, \sqsubseteq_{\mathrm{FO}})$ (bottom). Thick edges are covers.

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