



Asynchronous Rumor Spreading in Dynamic Graphs

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Abstract

We study asynchronous rumor spreading algorithm in dynamic and static graphs. In the asynchronous rumor spreading, for a given underlying graph, each node is equipped with an exponential time clock of rate 1. When a node's clock ticks, the node calls a random neighbor in order to exchange a rumor, if at least one of them knows it. Assuming a single node knows a rumor, we apply a differential equation-based technique to obtain an upper bound for the *spread time* of the algorithm in general dynamic graphs, which is the first time when all nodes get informed with high probability. In particular, we derive an upper bound for the spread time of the algorithm in a discrete version of a geometric mobile network, introduced by Clementi et al. [7]. In this model, a set of n agents independently performs random walks on a $\sqrt{n} \times \sqrt{n}$ plane and every two agents are able to communicate if they are within Euclidean distance at most R , where $f(n)\sqrt{\log n} \leq R \leq \sqrt{n}$ and $f(n)$ is a slowly growing function in n . Here, we show that the algorithm spreads a rumor through the network in $\mathcal{O}(\log n + \sqrt{n}/R)$ time, with high probability. Although we only show an upper bound the spread time of the algorithm in a 2 dimensional space, the framework can be also applied for geometric mobile networks defined over higher dimensional space and other random dynamic evolving networks such as stationary edge-Markovian model. Besides these synchronous and discrete dynamic models, we also consider the spreading time in dynamical Erdős-Rényi graphs.

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1 Introduction

Randomized rumor spreading algorithms are important primitives for information dissemination through a network. The standard randomized rumor spreading proceeds in succeeding rounds. In each round, every node in the network calls a random neighbor and they possibly exchange the rumor, if at least one of them knows it. Demers et al. [10] first introduced the algorithm to consistently distribute an update in a network of databases. Feige et al. [12] observed that the algorithm is scalable in terms of network size, and robust against the node/link failure and thus it has been applied in a wide range of distributed settings (e.g., see [3, 16]). The *spread time* is a well-studied parameter associated with the rumor spreading algorithms which is the first time when all nodes have been informed with high probability. The spread time of the algorithm has been studied on various network topologies [2, 11]. Moreover, it has been shown that the spread time of the algorithm in any static n -node network is at most $\mathcal{O}(\log n/\Phi)$, where Φ denotes the graph conductance [5]. In many distributed networks such as peer-to-peer, social and ad hoc networks, agents may not act in a synchronized manner. Therefore, Boyd et al. [3] proposed the asynchronous randomized rumor spreading algorithm, where each node has its own clock and contacts a



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random neighbor in order to exchange the rumor according to arrival times of its Poisson process with rate 1. The algorithm and its variations have been further studied in static networks [1, 13, 20, 23].

Information spreading in dynamic graphs has been a fundamental question and the subject of a large body of works (e.g., see [8] and references therein). For instance, the spread time of various algorithms has been studied in popular dynamic evolving graphs whose evolution is governed by a stochastic process such as *geometric mobile* [18, 21], *edge-Markovian* [6], and *node-Markovian* dynamic graphs [8]. Deterministic and adversarial settings have also been considered in [14, 22]. A dynamic evolving graph, denoted by $\mathcal{G} = \{G^{(t)}\}_{t=0}^{\infty}$, is usually referred to as a sequence of graphs with same set of nodes, but the edge set may change over discrete time $t = 0, 1, \dots$. It has gained popularity as it models a wide range of real-world networks including the wireless communication, mobile, and peer-to-peer networks.

1.1 Related Works

Kowalski and Caro [17] considered the asynchronous rumour spreading on general graphs and introduced a graphical quantity based on degree distribution of nodes that are incident to edges in any cut set. By applying the quantity they derive upper bounds for the spread time. Also, Panagiotou and Spiedel [20] rigorously analyzed the spread time of the asynchronous algorithm in Erdős-Rényi graphs $G(n, p)$ with $p = \omega(\log n/n)$. In order to show their results, they presented a large deviation inequality for the sum of a particular set of exponential random variables, which cannot be generalized for every graph.

Giakkoupis et al. [13] applied coupling techniques and established an interesting relation between synchronous and asynchronous rumor spreading algorithms. Let G be a given n -node static network and assume that $T_s(G)$ and $T_a(G)$ are the spread time of synchronous and the standard asynchronous rumor spreading algorithms on G , respectively. They showed that $T_a(G) = \mathcal{O}(T_s(G) + \log n)$. Moreover, they derived an upper bound for $\frac{T_s(G)}{T_a(G)}$, which is $n^{1/2}(\log n)^{\mathcal{O}(1)}$. Giakkoupis et al. [14] considered the spread time of the synchronous rumor spreading in dynamic evolving graphs. They showed that the rumor propagates through the graph whenever $\sum_t \{\Phi(G^{(t)}) \cdot D\} = \Omega(\log n)$, where $D = \max_u \delta_u / \Delta_u$, Δ_u and δ_u are the upper and lower bounds for degree of node u over time, respectively, and the maximum is taken over all nodes.

Pourmiri and Mans [22] established a similar upper bound for the spread time of asynchronous rumor spreading in dynamic graphs that is the first time when $\sum_t \{\Phi(G^{(t)}) \cdot \rho(G^{(t)})\} = \Omega(\log n)$, where $\rho(G^{(t)})$ is called the *graph diligence*. The graph diligence presents a more refined version of parameter D and $\rho(G^{(t)}) \geq \delta(t) / \Delta(t)$, where $\Delta(t)$ and $\delta(t)$ denote the maximum and minimum degree of $G^{(t)}$, respectively. Moreover, they present a family of dynamic graphs for which the upper bounds is tight up to a $o((\log n)^2)$ factor.

The aforementioned results have shown that besides the graph conductance, variation of degree sequence in a dynamic graph directly affect the spread time.

1.2 Our Main Results

We focus on asynchronous rumor spreading in dynamic and static graphs and present a general technique to obtain an upper bound for the spread time. The upper bounds are based on a differential equation taking into account the expansion properties of various subset of nodes, the maximum and the minimum degree of nodes. The methods have two advantages; (i) In contrast to existing method, this technique can be extended to settings where graphical parameters continuously or discretely change over time. (ii) It provides an alternative way

to compute the spread time by defining a Poisson random variable that is stochastically dominated by the size of informed nodes. The latter allows us to apply a concentration result for Poisson random variables and obtain a lower bound for the size of informed nodes at any time.

For every n -vertex graph $G = (V, E)$ and $1 \leq x < n$, *conductance function*, denoted by $\Phi(x)$, measures the expansion property of any subset of nodes of size at most x in G , which is defined as follows

$$\Phi(x) = \min_{\substack{S \subset V(G) \\ 1 \leq |S| \leq x}} \frac{|E(S, \bar{S})|}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}},$$

where $\text{vol}(S) = \sum_{u \in S} d_u$ and $E(S, \bar{S})$ denotes the set of edges crossing S and its complement \bar{S} (i.e., $V \setminus S$). It is easy to see that the standard graph conductance can be rewritten as $\Phi(G) = \Phi(n/2)$. Lovász and Kannan [19] introduced the concept of conductance function and showed that a lazy random walk on a connected graph with n nodes converges to its stationary distribution within at most $\int_{1/n}^{1/2} dt / (t\Phi(nt)^2)$ time. Somewhat analogous to this result we estimate the number of informed nodes up to time t by a Poisson-distributed random variable with rate $\Lambda(t)$, where $d\Lambda(t)/dt$ satisfies a differential equation (see Lemma 2.9).

Using the differential equation presented at Lemma 2.9, we derive upper bounds for the spread time of the asynchronous rumor spreading in a general dynamic evolving graph, geometric mobile, and dynamical Erdős-Rényi graphs. A dynamic evolving graph is a sequence of n -node graphs, $G^{(1)}, G^{(2)}, \dots$, where they all have the same set of nodes but set of edges changes over time $t = 1, \dots$

► **Theorem 1.1.** *Suppose that $\mathcal{G} = \{G^{(t)}\}_{t=1}^{\infty}$ denote a dynamic evolving graph whose nodes' degrees range over interval $[\delta, \Delta]$. Also, assume that graph exposed at any time t , $G^{(t)}$, has conductance at least Φ . Then, with high probability,*

$$T(\mathcal{G}) = \mathcal{O}(\Delta \log n / (\delta \Phi)),$$

where $T(\mathcal{G})$ denote the first time when all nodes get informed.

► **Remark.** It turns out that the upper bound tight up to a $o((\log n)^2)$. In fact, there exists a dynamic evolving graph with $\Delta/\delta = \Theta(\sqrt{n})$ and $\Phi = \Theta(\log \log n / \log n)$ for which the rumor spreads in $\Omega(\sqrt{n} / \log n)$ time. For more details see [22, Theorem 1.2].

Geometric Mobile Network

The *geometric mobile stationary network*, introduced by Clementi et al. [7], is a discrete version of *random walk mobility* model, where nodes represent radio stations in a wireless communication system [4]. For some small number $\epsilon > 0$, initially, n agents are randomly distributed on nodes of a $\sqrt{n}/\epsilon \times \sqrt{n}/\epsilon$ 2-dimensional grid, embedded on a $\sqrt{n} \times \sqrt{n}$ square plane. For a given parameter $r > 0$, in each time step $t = 1, \dots$, each agent independently and uniformly at randomly moves to a node whose Euclidean distance from its current location is at most r . Given this random process, in each time step t , we define network $G^{(t)}$ whose vertex set is the set of all agents and there is an edge between any two agents in $G^{(t)}$ if their Euclidean distance is at most R , where $f(n)\sqrt{\log n} \leq R \leq \sqrt{n}$ in the plane, and $f(n)$ is a slowly growing function in n . The model is denoted by $\mathcal{M}(n, R) = \{G^{(t)}\}_{t=0}^{\infty}$ and it is assumed that the agents are initially distributed according to the stationary distribution of the random walk on the grid.

► **Theorem 1.2.** *Suppose that $\mathcal{M}(n, R) = \{G^{(t)}\}_{t=0}^{\infty}$ is a geometric mobile network with $f(n)\sqrt{\log n} \leq R \leq \sqrt{n}/2$, where $f(n)$ is a slowly growing function. Also, assume that, initially, a node of $G^{(0)}$ is aware of a rumor. Then, with high probability, the asynchronous rumor spreading algorithm propagates the rumor in $\mathcal{O}(\sqrt{n}/R + \log n)$ time.*

Interestingly, the upper bound has the same magnitude as the spread time of *flooding* in the network [7]. The flooding is a simple variant of the synchronous rumor spreading algorithm where each informed node pushes the rumor to all of its neighbors. For sufficiently large $R = \Theta(\sqrt{n})$, $\mathcal{M}(n, R)$ is almost fully connected and the theorem gives an upper bound of $\mathcal{O}(\log n)$ for the algorithm, which is tight for the asynchronous rumor spreading in any fully connected network [3]. The proof technique of the theorem can be also applied for geometric mobile network defined over higher dimensional space.

Dynamical Erdős-Rényi Graph

Hägström, Peres and Steif [15] introduced dynamical percolation graph by adding a time dynamics to the well-known percolation model. The model, initially, starts with a fixed underlying graph G whose edges have been associated with a Poisson clock of rate μ . When an edge's clock ticks, then the edge is activated (opened) with probability p and deactivated (closed) with probability $1 - p$. Later, Sousi and Thomas [24] studied a setting where the underlying graph is an n -node complete graph, $\mu = o((\log n)^{-6}/n)$ and $p = c/n$, where $c > 1$ is a constant. The dynamic graph is called dynamical Erdős-Rényi graph $ER(n, p, \mu)$ modeling a sparse dynamic graph whose edges get updated, slowly. They studied the mixing properties of a random walk on the graph and show that the random walk mixes in $\frac{\log n}{\mu}(1 + o(1))$ time.

► **Theorem 1.3.** *Suppose that for some constant $c > 1$, $p = c/n$ and $\mu = o((\log n)^{-6}/n)$. Also, assume that initially a rumor is injected to a node of $ER(n, \mu, p)$. Then, with probability $1 - o(1)$, the rumor propagates through the $ER(n, \mu, p)$ within $\mathcal{O}((\log n)^2/\mu)$ time.*

A natural question would be to investigate the relation between the mixing and spread time in dynamic percolation graphs.

Outline

In Section 2 we present useful definitions and some preliminaries. We prove Theorems 1.1, 1.2, in Sections 3 and 4, respectively. Also, we give a proof sketch for Theorem 1.3 in Section 5.

2 Notations and Preliminaries

In this section we first define notations and some useful preliminaries. Throughout this paper, n denotes the number of nodes in the dynamic or static graph and \log stands for the logarithm to the base of e . We say an event, say \mathcal{E}_n , holds with high probability, if $\Pr[\mathcal{E}_n] \geq 1 - n^{-c}$, for some constant $c > 1$. For the sake of brevity we use *w.h.p.* to denote with high probability. Now, let us formally present some definitions.

► **Definition 2.1 (Conductance function).** *Let $G = (V, E)$ be an n -vertex simple graph. Then, for every $1 \leq x < n$, conductance function is defined as*

$$\Phi(x) = \min_{\substack{S \subset V(G) \\ 1 \leq |S| \leq x}} \frac{|E(S, \bar{S})|}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}, \quad (1)$$

where $E(S, \bar{S})$ is the set of edges crossing S and its complement and $\text{vol}(S) = \sum_{u \in S} d_u$.

► **Definition 2.2** (Asynchronous rumor spreading). *Suppose G is a graph whose nodes are associated with an exponential time clock of rate 1. Also, assume that initially a rumor is injected to a node of G . Each node contacts a random neighbor according to the arrival times of its Poisson process with rate 1. When they contact each other, they may learn the rumor, if at least one of them knows it. Also, we define the spread time as the first time when all nodes get informed with high probability and we use $T(G)$ to denote the spread time.*

► **Definition 2.3** (Non-homogeneous Poisson process). *Suppose that for every $\tau \geq 0$ there is a Poisson process with rate $\lambda(\tau) \geq 0$. Then, $\mathbf{P} = \{\lambda(\tau) : \tau \geq 0\}$ is called a non-homogeneous Poisson or counting process. Also, let $N(\tau)$ denote the number of occurrences made by process during $[0, \tau]$.*

► **Definition 2.4** (Stochastic Dominance). *We say random variable X stochastically dominates random variable Y , if for any arbitrary number a , we have that*

$$\Pr[X \leq a] \leq \Pr[Y \leq a].$$

We will now present a well-known theorem regarding non-homogeneous Poisson process.

► **Theorem 2.5** ([9, Chapter 2]). *Suppose that $\mathbf{P} = \{\lambda(\tau) : \tau \geq 0\}$ is a non-homogeneous Poisson process. Also assume that $\lambda(\tau) : [0, \infty) \rightarrow [0, \infty)$ is an integrable function. Then, for every $0 \leq a \leq b$, $N(b) - N(a)$ has a Poisson distribution with rate*

$$\Lambda = \int_a^b \lambda(\tau) d\tau.$$

For more information about non-homogeneous Poisson processes we refer the interested reader to [9]. We now present a large deviation bound for Poisson random variables, whose proof is based on the moment generating function of the Poisson random variables.

► **Theorem 2.6.** *Suppose that X denote a Poisson random variable with rate Λ . Then we have that*

$$\Pr[|X - \Lambda| \geq \eta] \leq 2 \cdot e^{\frac{-\eta^2}{2(\Lambda + \eta)}}.$$

Towards studying distribution of $T(G)$, we divide the asynchronous algorithm in n states where each state $1 \leq j \leq n$ stands for the situation where we have j informed nodes. For every $j = 1, \dots, n-1$, define $t_j(G)$ to be the waiting time for the algorithm to jump from the j -th state to the $(j+1)$ -st one. Clearly, we have that

$$T(G) = \sum_{j=1}^{n-1} t_j(G).$$

► **Lemma 2.7.** *Suppose that $G = (V, E)$ denote an n -node graph. Also, assume that, initially, rumor is injected to a node of G . Then, for every $2 \leq j \leq n-1$, conditional on the first j informed nodes, say I_j , $t_j(G)$ is an exponential random variable with rate*

$$\beta_j(G) = \sum_{\{u,v\} \in E(I_j, U_j)} \left\{ \frac{1}{d(u)} + \frac{1}{d(v)} \right\},$$

where $E(I_j, U_j)$ is the set of edges crossing I_j and its complement U_j (set of non-informed nodes). Moreover $t_j(G)$ is independent of t_{j-1}, \dots, t_1 .

Proof. Notice that for every pair of vertices $\{u, v\} \in E$, u and v contact each other with Poisson rate $1/d(u) + 1/d(v)$, where $d(u)$ and $d(v)$ are degrees of u and v , respectively. Let I_j denotes the set of first j informed nodes. Then the $(j + 1)$ -st node gets informed with Poisson rate $\beta_j(G)$, which is

$$\beta_j(G) = \sum_{\{u,v\} \in E(I_j, U_j)} \left\{ \frac{1}{d(u)} + \frac{1}{d(v)} \right\},$$

As soon as the j -th node gets informed and I_j gets determined, by memory-less property of exponential distribution, $t_j(G)$ is an exponentially distributed random variable which is independent of $t_1(G), \dots, t_{j-1}(G)$. \blacktriangleleft

► **Lemma 2.8.** *Suppose that, for some n , $A = (\alpha_1, \dots, \alpha_n), B = (\beta_1, \dots, \beta_n) \in \mathbb{R}_+^n$ be two arbitrary vectors where for every $1 \leq j \leq n$, $\alpha_j \leq \beta_j$. Also, let $\mathbf{P}(A)$ and $\mathbf{P}(B)$ denote non-homogeneous Poisson process for which, $j = 1, \dots, n$, the j -th event happens with rate α_j and β_j , respectively. Then, during a given time interval, the number of events generated by $\mathbf{P}(B)$ stochastically dominates the number of events generated by $\mathbf{P}(A)$.*

The proof is given in Appendix A.

2.1 Some Useful Lemmas

► **Lemma 2.9.** *Suppose that $\mathcal{G} = \{G^{(t)}\}_{t=1}^{\infty}$ denotes an evolving dynamic graph whose nodes' degree range over $[\delta, \Delta]$. Also, let $\Phi(x)$, $1 \leq x \leq n$, be a lower bound for the conductance function of any graph $G^{(t)} \in \mathcal{G}$. Now, assume that initially a rumor is injected to an arbitrary node and the asynchronous algorithm starts propagating the rumor. Then, the number of informed nodes up to time t stochastically dominates a Poisson distribution with rate $\Lambda(t)$ satisfying at*

$$\Lambda'(t) = 2 \cdot (\delta/\Delta) \cdot \Phi(\min\{n - C(t), C(t)\}) \min\{C(t), n - C(t)\},$$

where $C(t)$ counts the number of events happened by a Poisson distribution with rate $\Lambda(t)$. In particular, $\Phi(x)$ can be replaced with any function $F(x) \leq \Phi(x)$ where $1 \leq x \leq n/2$.

Proof. Let β_j denotes the Poisson rate at which the $(j + 1)$ -st node gets informed. Also let I_j and U_j denote the set of first j informed and $n - j$ uninformed nodes, respectively. At any time t , by Lemma 2.7, for every $1 \leq j \leq n - 1$, we get that

$$\beta_j = \sum_{\{u,v\} \in E_t(I_j, U_j)} \left\{ \frac{1}{d_t(u)} + \frac{1}{d_t(v)} \right\} \geq \frac{2|E_t(I_j, U_j)|}{\Delta},$$

where $d_t(u)$ and $d_t(v)$ denote the degree of u and v at time t . Also, the last inequality follows from $1/d(u) + 1/d(v) \geq 2/\Delta$. Note that $d_t(u)$ and $d_t(v)$ are not zero as they are incident to edge $\{u, v\}$ crossing cut $E_t(I_j, U_j)$. By the lemma statement and the definition of conductance function, we have that

$$\begin{aligned} |E_t(I_j, U_j)| &\geq \Phi(\min\{j, n - j\})(\min\{\text{vol}(I_j), \text{vol}(U_j)\}) \\ &\geq \Phi(\min\{j, n - j\}) \cdot (\min\{j, n - j\}) \cdot \delta, \end{aligned}$$

where $\text{vol}(I_j)$ and $\text{vol}(U_j)$ denote the volume of sets I_j and U_j in $G^{(t)}$ and hence lower bounded by $I_j \cdot \delta$ and $U_j \cdot \delta$, respectively. Notice that one can replace $\Phi(x)$ by any $F(x) \leq \Phi(x)$ and the lower bound still holds. Therefore, for every $1 \leq j \leq n - 1$,

$$\beta_j \geq 2(\delta/\Delta)\Phi(\min\{j, n - j\}) \cdot \min\{j, n - j\} \quad (2)$$

Define a non-homogeneous Poisson process $\mathbf{P} = \{\lambda(t) : \tau \in [0, \infty)\}$, where we have

$$\lambda(t) = \begin{cases} (2\delta/\Delta)\Phi(C(t))C(t) & \text{if } C(t) \leq n/2, \\ (2\delta/\Delta)\Phi(n - C(t))[n - C(t)] & n/2 < C(t) < n, \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

where $C(t)$ counts the number of informed nodes during $[0, t]$. Lemma 2.8 and Inequality (2) together show that the number of informed nodes stochastically dominates the number of events happened by \mathbf{P} . Moreover, by Theorem 2.5 during any interval $[0, t]$, $C(t)$ has a Poisson distribution with rate $\Lambda(t) = \int_0^t \lambda(z)dz$. Applying the fundamental theorem of the calculus yields that, for $1 \leq C(t) \leq n$,

$$\Lambda'(t) = (2\delta/\Delta)\Phi(\min\{n - C(t), C(t)\}) \min\{n - C(t), C(t)\}. \quad (4)$$

◀

The following useful lemma helps to approximate $C(t)$ and apply Lemma 2.9.

► **Lemma 2.10.** *Suppose that $(\log n)^{1.3} \leq \Lambda(t) \leq n - \sqrt{n(\log n)^{1.3}}$. Then for any constant $0 < \varepsilon < 1$, we have that*

$$\Lambda'(t) \geq (1 - \varepsilon)(2\delta/\Delta) \cdot \Phi(\min\{\Lambda(t), n - \Lambda(t)\}(1 + \varepsilon)) \cdot \min\{\Lambda(t), n - \Lambda(t)\}.$$

In particular, $\Phi(x)$ can be replaced by any function $F(x) \leq \Phi(x)$, $1 \leq x < n$.

Proof. Recall that for every $t > 0$, $C(t)$ counts the number of events made by a Poisson distribution with rate $\Lambda(t)$ during time interval $[0, t]$. Let us set $\eta = \sqrt{\Lambda(t)} \cdot (\log n)^{1.1}$ and apply a concentration result (e.g., Theorem 2.6) for $C(t)$ and obtain an estimation for $C(t)$ as follows.

$$\Pr [|C(t) - \Lambda(t)| \geq \eta] \leq 2 \cdot e^{\frac{-\eta^2}{2(\Lambda(t) + \eta)}} = 2e^{\frac{-\Lambda(t)(\log n)^{1.1}}{4\Lambda(t)}} \leq 2e^{-(\log n)^{1.1}/4} = n^{-\omega(1)}. \quad (5)$$

By (5) we have that with probability $1 - n^{-\omega(1)}$,

$$\Lambda(t) - \eta \leq C(t) \leq \Lambda(t) + \eta$$

and hence,

$$n - \Lambda(t) - \eta \leq n - C(t) \leq n - \Lambda(t) + \eta.$$

Combing the both inequalities implies that with high probability

$$\min\{n - \Lambda(t), \Lambda(t)\} - \eta \leq \min\{n - C(t), C(t)\} \leq \min\{n - \Lambda(t), \Lambda(t)\} + \eta.$$

Note that if $(\log n)^{1.3} \leq \Lambda(t) \leq n/2$, then we have that

$$\eta = \sqrt{\Lambda(t)(\log n)^{1.1}} \leq \frac{\Lambda(t)}{(\log n)^{.1}}.$$

Moreover, if $n/2 \leq \Lambda(t) \leq n - \sqrt{n(\log n)^{1.3}}$, then $n - \Lambda(t) \geq \sqrt{n(\log n)^{1.3}}$ and

$$\eta = \sqrt{\Lambda(t)(\log n)^{1.1}} \leq \frac{\sqrt{n(\log n)^{1.3}}}{(\log n)^{.1}} \leq \frac{n - \Lambda(t)}{(\log n)^{.1}}$$

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Using the above two inequalities implies that $\eta \leq \min\{\Lambda(t), n - \Lambda(t)\}/(\log n)^{.1}$. Thus,

$$\begin{aligned} \min\{n - \Lambda(t), \Lambda(t)\} \left(1 - \frac{1}{(\log n)^{.1}}\right) &\leq \min\{n - C(t), C(t)\} \\ &\leq \min\{n - \Lambda(t), \Lambda(t)\} \left(1 + \frac{1}{(\log n)^{.1}}\right) \end{aligned} \quad (6)$$

Let us apply (6) in the differential equation presented at Lemma 2.9 implies that

$$\Lambda'(t) \geq 2 \cdot (\delta/\Delta) \cdot \Phi \left(\min\{n - C(t), C(t)\} \left(1 + \frac{1}{(\log n)^{.1}}\right) \right) \min\{C(t), n - C(t)\} \left(1 - \frac{1}{(\log n)^{.1}}\right).$$

We replace $1/(\log n)^{.1}$ by any constant $0 < \varepsilon < 1$ and the statement follows. \blacktriangleleft

3 Spread Time and Graph Conductance

This section is devoted to the proof of Theorem 1.1 presenting an upper bound for the spread time in terms of graph conductance.

Proof of Theorem 1.1. By Lemma 2.9 we have that the number of informed nodes stochastically dominates a Poisson distribution with rate $\Lambda(t)$ satisfying

$$\Lambda'(t) = (2\delta/\Delta) \cdot \Phi \cdot \min\{C(t), n - C(t)\}, \quad (7)$$

where $C(t)$ denotes the number of events happened by a non-homogenous process $\{\lambda(t) : t \geq 0\}$ and $\lambda(t) = (2\delta/\Delta) \cdot \Phi \cdot \min\{C(t), n - C(t)\}$. Therefore, the number of informed nodes during any interval $[0, \tau]$ stochastically dominates a Poisson-distributed random variable with rate $\int_0^\tau \lambda(t) dt$. Moreover, using a large deviation result for Poisson-distributed random variables (e.g., Theorem 2.6), the number of events is concentrated around $\int_0^\tau \lambda(t) dt$. Therefore, by computing $\int_0^\tau \lambda(t) dt$ one can obtain a lower bound for the number of informed nodes during time interval $[0, \tau]$, with high probability, and an upper bound for the spread time, consequently. To do so according to $C(t)$, we study the process in three consecutive phases.

■ **Initial phase.** This phase starts with $C(t) = 1$ and ends when $C(t)$ exceeds $\log n$. Let T_{init} be the time when the phase ends. By Theorem 2.5, $C(t)$ is a Poisson random variable with rate $\Lambda(t) = \int_0^t \lambda(z) dz$.

$$\Lambda(t) = \int_0^t \lambda(z) dz \geq \int_0^t (2\delta/\Delta) \Phi dz = (2\delta/\Delta) \cdot \Phi \cdot t \quad (8)$$

Then, by setting $t = 4\Delta \log n / (\delta\Phi)$, we conclude that $\lambda(t) \geq 8 \log n$. Using a large deviation bound (see e.g., Theorem 2.6) we get that

$$\Pr \left[C(t) \leq \Lambda(t) - \sqrt{\Lambda(t) 8 \log n} \right] \leq \exp\{-8 \log n / 4\} = n^{-2}.$$

Therefore, with high probability, $T_{init} \leq \frac{4\Delta \log n}{\delta\Phi}$.

■ **Middle Phase.** This phase starts with $C(t) = \log n$ and ends when $C(t)$ exceeds $n/2$. Let T_{mid} be the first time when the phase ends. Also, define t_0, t_1, t_2 to be the first times that we have $\Lambda(t_0) = (\log n)^{1.3}$, $\Lambda(t_1) = n/2$ and $\Lambda(t_2) = 2n/3$, respectively. In this phase for every $t \in [T_{init}, T_{mid}]$, we have that

$$\Lambda(t) = \int_0^t \lambda(z) dz \geq \int_{T_{init}}^t (2\delta/\Delta) \cdot \Phi \cdot (\log n) dz = (2\delta/\Delta) \cdot \Phi \cdot (\log n) \cdot (t - T_{init}).$$

This implies that there exists

$$t_0 \leq T_{init} + \frac{\Delta(\log n)^3}{2\delta\Phi} \leq \mathcal{O}\left(\frac{\Delta \log n}{\delta\Phi}\right) \quad (9)$$

for which we have $\Lambda(t_0) = (\log n)^{1.3}$. Applying Lemma 2.10 and using the fact that $\Phi \leq \Phi(x)$, $1 \leq x < n$, we have that for every $(\log n)^{1.3} \leq \Lambda(t) \leq n - \sqrt{(\log n)^{1.3}n}$,

$$\frac{\Lambda'(t)}{\min\{\Lambda(\tau), n - \Lambda(\tau)\}} \geq (2(1 - \varepsilon)\delta/\Delta)\Phi,$$

where $0 < \varepsilon < 1$ is an arbitrary constant. Taking the integral from both sides, on interval $[t_0, t]$, and setting appropriate integral constants we get that

$$\Lambda(t) \geq \begin{cases} \exp\left\{\frac{2(1-\varepsilon)\delta\Phi}{\Delta}(t - t_0) + (1.3) \log \log n\right\} & (\log)^{1.3} \leq \Lambda(t) \leq n/2 \\ n - \exp\left\{\frac{-2\delta(1-\varepsilon)\Phi}{\Delta}(t - t_1) + \log(n/2)\right\} & n/2 \leq \Lambda(t) \leq n - \sqrt{n(\log n)^{1.3}} \end{cases}$$

where t_1 is the first time when $\Lambda(t_1) = n/2$. From the first row in the above piecewise function we conclude that there exist

$$t_1 \leq \frac{\Delta \log(n/2)}{2(1 - \varepsilon)\delta\Phi} + t_0 = \mathcal{O}\left(\frac{\Delta \log n}{\delta\Phi}\right),$$

where the last equality follows from (9). Considering the second row and the previous equality we deduce that there exists

$$t_2 = \mathcal{O}\left(\frac{\Delta \log n}{2(1 - \varepsilon)\delta\Phi}\right) + t_1 = \mathcal{O}\left(\frac{\Delta \log n}{\delta\Phi}\right)$$

with $\Lambda(t_2) \geq 2n/3$. Since $C(t)$ has Poisson rate $\Lambda(t)$, by using a large deviation inequality we get that

$$\Pr[C(t_2) \leq n/2] \leq \Pr\left[C(t_2) \leq \Lambda(t_2) - \log n \sqrt{\Lambda(t_2)}\right] \leq n^{-\omega(1)}.$$

Therefore, w.h.p., $T_{mid} \leq t_2 = \mathcal{O}\left(\frac{\Delta \log n}{\delta\Phi}\right)$.

- **Final Phase.** This phase starts with $C(t) = n/2$ and ends when $C(t) = n$. Let T_{final} denote the time when the phase ends. Notice that by definition of \mathbf{P} , the process is symmetric in $C(t)$, as $\lambda(t)$ is proportional to $1 \leq \min\{C(t), n - C(t)\} \leq n/2$. Considering the time interval for which the process starts at $C(t) = \lceil n/2 \rceil$ and ends at $C(t) = n$. The length of this interval has the same distribution as the time that \mathbf{P} requires to start from $C(t) = 1$ and reach to the $C(t) = \lfloor n/2 \rfloor$. Therefore, from the previous phases we have that, with high probability,

$$T_{final} - T_{mid} = \mathcal{O}\left(\frac{\Delta \log n}{\delta\Phi}\right).$$

The number of informed nodes up to time t , $I(t)$, stochastically dominates $C(t)$. Thus

$$\Pr[I(T_{final}) < n] \leq \Pr[C(T_{final}) < n] = n^{-\omega(1)}$$

Therefore, with high probability $T(G) = \mathcal{O}\left(\frac{\Delta \log n}{\delta\Phi}\right)$. ◀

4 Geometric Mobile Networks

In this section, we prove Theorem 1.2 presenting an upper bound for the spread time of the asynchronous rumor spreading in a geometric mobile network introduced by Clementi et al. [7]. Let us first present the following lemma regarding some useful properties of the dynamic graph whose proof is deferred to Appendix B.

► **Lemma 4.1.** *Suppose that $\mathcal{M}(n, R) = \{G^{(t)}\}_{t=0}^{\infty}$, is a geometric mobile network with $f(n)\sqrt{\log n} \leq R < \sqrt{n}$, where $f(n)$ is a slowly growing function in n . Then, with probability $1 - n^{-\omega(1)}$, for every $1 \leq t \leq n^3$, the followings hold:*

1. *For every node u , $d_u(t) = \Theta(R^2)$, where $d_u(t)$ is the degree of node u in $G^{(t)}$.*
2. *There exists a constant $a > 0$ such that conductance function $G^{(t)}$ satisfies*

$$\Phi(x) \geq \begin{cases} a & 1 \leq x \leq R^2, \\ a \frac{R}{\sqrt{\min\{x, n-x\}}} & R^2 < x \leq n-1. \end{cases}$$

Proof of Theorem 1.2. Similar to the proof of Theorem 1.1, applying Lemma 2.9 results that the number of informed nodes stochastically dominates a Poisson distribution with rate $\Lambda(t)$ satisfying

$$\Lambda'(t) = (2\delta/\Delta) \cdot \Phi(\min\{C(t), n - C(t)\}) \cdot \min\{C(t), n - C(t)\},$$

where $C(t)$ denotes the number of events happened by a non-homogenous process $\{\lambda(t) : t \geq 0\}$ and $\lambda(t) = (2\delta/\Delta) \cdot \Phi(\min\{C(t), n - C(t)\}) \cdot \min\{C(t), n - C(t)\}$. Therefore the number of informed nodes during any interval $[0, \tau]$ stochastically dominates a Poisson-distributed random variable with rate $\int_0^\tau \lambda(t) dt$. On the other hand, a large deviation bound (e.g., see Theorem 2.6) for the Poisson-distributed random variable shows that the number of events is concentrated around $\int_0^\tau \lambda(t) dt$, with high probability. Therefore, one can obtain a lower bound for the number informed nodes during time interval $[0, \tau]$ by approximating $\int_0^\tau \lambda(t) \cdot dt$. In what follows, by a case analysis according to $C(t)$ we estimate $\int_0^\tau \lambda(t) \cdot dt$ and apply the large deviation bound to obtain an upper bound for the spread time, with high probability.

By Lemma 4.1, we observe that, w.h.p., at any time step t , $1 \leq t \leq n^3$, $G^{(t)}$ is almost regular. Thus the minimum degree of $G^{(t)}$ over its maximum degree is a constant and we have $\delta/\Delta = b = \Theta(1)$. The lemma also gives a lower bound for the conductance function $\Phi(x)$. Conditioning on the mentioned properties about $G^{(t)}$'s, $1 \leq t \leq n^3$ that hold with probability $1 - n^{-\omega(1)}$, one can apply Lemma 2.9 and conclude that

$$\Lambda'(t) = \begin{cases} c_1 C(t) & 1 \leq C(t) \leq \min\{R^2, n/2\} \\ c_1 \sqrt{\min\{C(t), n - C(t)\}} & \min\{R^2, n/2\} < C(t) \leq n-1, \end{cases} \quad (10)$$

where $R^2 < n/2$, $c_1 = 2 \cdot a \cdot b$ is a constant and a appeared in Lemma 4.1. Moreover, applying Lemma 2.10 implies that for every constant $0 < \varepsilon_0 < 1$ we have that

$$\Lambda'(t) \geq \frac{1 - \varepsilon_0}{\sqrt{1 + \varepsilon_0}} \begin{cases} c_1 \Lambda(t) & (\log n)^{1.1} \leq \Lambda(t) \leq \max\{R^2, (\log n)^{1.1}\}, \\ c_1 R \sqrt{\min\{\Lambda(t), n - \Lambda(t)\}} & \max\{R^2, (\log n)^{1.1}\} < \Lambda(t) \leq n-1, \end{cases}$$

In order to have a simpler form we set $1 - \varepsilon = \frac{1 - \varepsilon_0}{\sqrt{1 + \varepsilon_0}}$. Therefore, if $\max\{R^2, (\log n)^{1.1}\} < \Lambda(t) \leq n-1$, then we have that

$$\frac{\Lambda'(t)}{\sqrt{\min\{\Lambda(t), n - \Lambda(t)\}}} \geq (1 - \varepsilon) c_1 R. \quad (11)$$

Note that the number of informed nodes up to time t stochastically dominates $C(t)$. Therefore, in what follows we analyze $C(t)$ in three consecutive phases.

- **Initial phase.** This phase starts with $C(t) = 1$ and finishes when $C(t)$ exceed R^2 . Let T_{init} be the first time when this phase ends. By (10) we get that $\Lambda'(t) = c_1 C(t)$. Therefore, if $C(t) = j$, then the $(j + 1)$ -st event happens with Poisson rate at least $c_1 \cdot j$. Let s_j denote the waiting time for moving from $C(t) = j$ to $C(t) = j + 1$. Also, let $m = \lfloor R^2 \rfloor$ and define $S_m = \sum_{j=1}^{m-1} s_j$. Since s_j has an exponential distribution with rate $c_1 j$, we conclude that

$$\mathbf{E}[s_j] = \mathbf{E}[\mathbf{E}[s_j | C(t) = j]] \leq \mathbf{E}\left[\frac{1}{c_1 j}\right] = \frac{1}{c_1 j}$$

By linearity of expectation we get that

$$\mathbf{E}[S_m] = \sum_{j=1}^m \mathbf{E}[s_j] \leq \sum_{j=1}^m \frac{1}{c_0 j} = H_m / c_1 = (\log m) / c_0 + \mathcal{O}(1)$$

where H_m is the m -th harmonic number and we have that $H_m = \log m + \mathcal{O}(1)$. By a large deviation inequality for this particular set of exponential random variables (e.g, see Lemma A.1) we can show that, for every $c \geq 0$, $\Pr[S_m \geq \log m + c \log n] \leq n^{-c}$. Therefore, with high probability,

$$T_{init} \leq (\log m) / c_1 + c \log n = \mathcal{O}(\log n). \tag{12}$$

- **Middle phase.** This phase starts with $C(t) = R^2$ and ends when $C(t)$ exceeds $n/2$. Let T_{mid} be the first time when this phase ends. Define t_0, t_1, t_2 to be the first times that we have $\Lambda(t_0) = \max\{R^2, (\log n)^{1.3}\}$, $\Lambda(t_1) = n/2$ and $\lambda(t_2) = 2n/3$. By (10) we have that for every $R^2 \leq C(t) \leq n/2$,

$$\Lambda(t_0) = \int_0^{t_0} c_1 R \sqrt{C(t)} dt \geq c_1 R^2 (t_0 - T_{init}),$$

where the lower bound follows from the fact that $C(t) \geq R^2$. The presented lower bound implies that there exists t_0 such that $\Lambda(t_0) = \max\{R^2, (\log n)^{1.3}\}$. Moreover, we have that $R^2 \geq (\log n)$ and hence

$$t_0 \leq T_{init} + (\log n)^{.3} = \mathcal{O}(\log n). \tag{13}$$

Using the fact that for every $t \geq t_0$, $\Lambda(t) \geq \max\{R^2, (\log n)^{1.3}\}$, we integrate from both sides of (11) and we have that if $t_0 \leq t \leq t_1$ we have that $\max\{R^2, (\log n)^{1.3}\} \leq \Lambda(t) \leq n/2$. Thus,

$$\int_{t_0}^{t_1} \frac{\Lambda'(t) dt}{\sqrt{\Lambda(t)}} = 2\sqrt{\Lambda(t_1)} - 2\sqrt{\Lambda(t_0)} \geq \int_{t_0}^{t_1} (1 - \varepsilon) c_1 R dt = (1 - \varepsilon) c_1 R (t_1 - t_0).$$

If $t_1 \leq t \leq t_2$, then we have that $n/2 \leq \Lambda(t) \leq 2n/3$ and hence

$$\int_{t_1}^{t_2} \frac{\Lambda'(t) dt}{\sqrt{n - \Lambda(t)}} = 2\sqrt{n - \Lambda(t_1)} - 2\sqrt{n - \Lambda(t_2)} \geq \int_{t_1}^{t_2} (1 - \varepsilon) c_1 R dt = (1 - \varepsilon) c_1 R (t_2 - t_1)$$

Recall that we have defined $\Lambda(t_0) = \max\{R^2, (\log n)^{1.3}\}$, $\Lambda(t_1) = n/2$ and $\lambda(t_2) = 2n/3$. Considering the above lower bounds, one can observe that $t_1 - t_0 = \mathcal{O}(\sqrt{n}/R)$ and $t_2 - t_1 = \mathcal{O}(\sqrt{n}/R)$. Therefore,

$$t_2 = (t_2 - t_1) + (t_1 - t_0) + t_0 = \mathcal{O}(\sqrt{n}/R) + t_0 = \mathcal{O}(\sqrt{n}/R) + \mathcal{O}(\log n),$$

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where the equality follows from (13). Since $C(t_2)$ is distributed as a Poisson random variable with rate $\Lambda(t_2) = 2n/3$, by using Theorem 2.6 we have that

$$\Pr [C(t_2) < n/2] \leq \Pr [|C(t_2) - 2n/3| \geq n/6] = n^{-\omega(1)}.$$

Then, w.h.p. $T_{mid} \leq t_2 = \mathcal{O}(\sqrt{n}/R + \log n)$.

- **Final phase.** This phase starts with $C(t) = n/2$ and ends when $C(t) = n$. Let T_{final} be the time when the phase ends. The analysis makes use of the fact that Poisson process with rate $\Lambda(t)$ is symmetric in $C(t)$. Therefore, similar to the the final phase in the proof of Theorem 1.1, we have that w.h.p $T_{final} - T_{mid} = \mathcal{O}(\sqrt{n}/R) + \log n$ and hence $T_{final} = \mathcal{O}(\sqrt{n}/R + \log n)$.

Using the fact that the number of informed node up to time t , say $I(t)$, dominates $C(t)$ we have that

$$\Pr [C(T_{final}) < n] \leq \Pr [I(T_{final}) < n]$$

Therefore, w.h.p. the spread time in $\mathcal{M}(n, R)$ is bounded by $T_{final} = \mathcal{O}(\sqrt{n}/R + \log n)$. ◀

5 Dynamical Erdős-Rényi Graphs

In this section we provide a proof sketch for Theorem 1.3 which presents an upper bound for the spread time in a dynamical Erdős-Rényi graph $ER(n, p, \mu)$. Before that we need some properties of the graph that have been shown in [24]. Recall $ER(n, p, \mu)$ is a continuous Markov chain whose stationary distribution is a random graph distributed as Erdős-Rényi graph $G(n, p)$.

► **Definition 5.1** ([24, Definition 2.2]). *For a specified constant c , we say that graph $G = (V, E)$ is good and we write $G \in \mathcal{H}(c)$, if G has a unique connected component \mathcal{C} with $|\mathcal{C}| \geq c \cdot \log n$, and we call it the giant component and satisfies the following properties.*

- *Size.* We have $|\mathcal{C}| \geq c \cdot n$.
- *Maximum degree.* The maximum degree of \mathcal{C} is at most $c \log n$.
- *Number of edges.* There are at most cn edges in \mathcal{C}
- *Expansion properties.* We have that $\Phi_{\mathcal{C}} \geq c(\log n)^{-2}$, where $\Phi_{\mathcal{C}}$ is the conductance of the giant component.

► **Proposition 5.2** ([24, Proposition 2.3]). *For any graph G sampled from the stationary distribution of dynamical Erdős-Rényi process. Then, we have that $\Pr [G \notin \mathcal{H}(c)] = \mathcal{O}(n^{-9})$.*

Proof of Theorem 1.3. We analyze the algorithm in three consecutive phases. For every $\tau > 0$, let $\mathcal{G}(\tau)$ be the dynamical Erdős-Rényi graph at time τ . Also, let T_{mix} denote the mixing time of the dynamical Erdős-Rényi graph, which is $T_{mix} = (2 + o(1)) \frac{\log n}{\mu}$ (e.g., see [24]).

- **Informing a node of the giant component.** This phase starts with one single informed node and ends when a node of giant component knows the rumor. Let T_1 be the first time when this phase ends. For each $k = 1, \dots$, define $\tau_k = k \cdot T_{mix}$. By Proposition 5.2, $\mathcal{G}(\tau_k) \in \mathcal{H}(c)$ and hence it has a unique giant component of size at least $c \cdot n$. Therefore, at time τ_k , the informed node is not included in the giant component with probability at least $(1 - c)^k$. Hence, we deduce that with probability at least $1 - 1/n$ after at most $(\log n/c)T_{mix} = \frac{(\log n)^2}{c\mu}(2 + o(1))$ time this phase ends.

- **Informing all nodes within the giant component.** This phase starts with a single informed node that belongs to the giant component and ends when all nodes in the giant component get informed. Let T_2 denotes the time this phase requires to finish. By Proposition 5.2, the giant has at most cn edges. Thus, the first edge in the component gets updated with Poisson rate $cn\mu = \mathcal{O}((\log n)^{-6})$. This implies that with probability $1 - o(1)$, during time $[T_1, T_1 + (\log n)^5]$, the component remains connected and will be the same during the time interval. Applying Theorem 1.1 implies that rumor spreads through the giant component within $\mathcal{O}\left(\frac{\Delta \log n}{\delta \Phi}\right)$ time. By Proposition 5.2, with probability $1 - \mathcal{O}(n^{-9})$, the giant component is a good graph and hence we have that $\delta = 1$, $\Delta = \mathcal{O}(\log n)$ and $\Phi_C = \Omega((\log n)^{-2})$. Thus, with probability $1 - o(1)$, the rumor spreads through the giant component within $T_2 - T_1 = \mathcal{O}((\log n)^4)$ time.
- **Informing rest of the nodes.** This phase starts with at least cn informed nodes, which is the size of giant component. Let $\mathbf{U}(\tau)$ and $\mathbf{I}(\tau)$ be the set of informed and non-informed nodes up to time τ . Let T_3 denote the time when this phase ends. For each $k = 1, \dots$, define $\tau_k = T_2 + k \cdot T_{mix}$ and suppose that $u \in \mathbf{U}(\tau_k)$. It is worth mentioning that definition of τ_k allows us to have a new sample of $G(n, p)$ and for each τ_k , $\mathcal{G}(\tau_k)$ is distributed as $G(n, p)$. For every $k = 1, \dots$, define random variables $X_u(k)$ to be the number of informed nodes that are adjacent to u at time τ_k . Also $|\mathbf{I}(\tau_k)| \geq c \cdot n$ is non-decreasing function in time. Therefore, $X_u(k)$ dominates binomial random variable $X \sim \text{Bin}(cn, p)$. Then, for every $0 < \theta < 1$, by Zygmund-Paley inequality we have that

$$\begin{aligned} \Pr[X_u(k) > \theta cnp] &\geq \Pr[X_u(k) > \theta cnp] \\ &\geq (1 - \theta)^2 \frac{(cnp)^2}{cnp(1 - p) + (cnp)^2} \\ &\geq (1 - \theta)^2 \frac{cnp}{1 + cnp}. \end{aligned}$$

Setting $\theta = 1/2$ and using the fact that $p = \Theta(1/n)$, we conclude that

$$\Pr[X_u(k) > 0] \geq \frac{cnp}{4(1 + cnp)} = c_1,$$

where c_1 is a constant. Therefore, with probability at least c_1 , for every $k = 1, \dots$, $u \in \mathbf{U}(\tau_k)$ is connected to some informed node. Fixing arbitrary $u \in \mathbf{U}(\tau_1)$, the probability that at time τ_k , u is not connected to some informed node is at least $(1 - c_1)^k$. Thus, by setting $k = 2 \log n / c_1$, with probability at most n^{-2} , u is not connected. By union bound over all non-informed nodes, we conclude that, with probability $1 - 1/n$, every non-informed node is being connected to some informed one before time τ_k . Provided $u \in \mathbf{U}(\tau)$ has an informed neighbor, say v , they share the rumor with rate $1/d(u) + 1/d(v)$. Notice that for every k , $\mathcal{G}(\tau_k) \sim G(n, p)$ and hence its maximum degree is at most $\mathcal{O}(\log n)$. Therefore, u and v communicates with rate at least $\Omega(1/\log n)$. So with high probability during a period of length at most $(\log n)^3$, u gets informed from its neighbor v , which follows from a concentration result for a Poisson random variable of rate $\Omega(1/\log n) \cdot (\log n)^3 \geq (\log n)^{3/2}$.

Note that edge $\{u, v\}$ may disappear with rate μ , however, during a time interval of length $(\log n)^3$, the edge disappear with probability $\mu(\log n)^3 = o(1)$. Therefore, with probability $1 - o(1)$ after at most $T_2 + (2 \log n / c_1)T_{mix} + \mathcal{O}((\log n)^3)$ time every node gets informed. From the first and second phases $T_2 = T_1 + \mathcal{O}((\log n)^4) = \mathcal{O}(\log n T_{mix})$ and hence, with probability $1 - o(1)$, the rumor spreads in $\mathcal{O}((\log n)^2 / \mu)$ time. ◀

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A Missing Proofs of Section 2

Proof of Lemma 2.8. For very $j = 1, \dots, n - 1$, define s_j and t_j to be exponentially distributed random variables with rates α_j and β_j , respectively. Define $X = (s_1, \dots, s_n)$ and $Y = (t_1, \dots, t_n)$. Toward proving the stochastic dominance, we couple X and Y , first by revealing Y , and then, for every $1 \leq j \leq n - 1$, set $s_j = \beta_j t_j / \alpha_j$. Now, for every $j = 1, \dots, n$, and any positive number x we have that

$$\begin{aligned} \Pr[s_j \geq x] &= \Pr\left[\frac{\alpha_j}{\beta_j} s_j \geq \frac{\alpha_j}{\beta_j} x\right] = \Pr\left[t_j > \frac{\alpha_j}{\beta_j} x\right] \\ &= \exp\{-\beta_j(\alpha_j/\beta_j)x\} = \exp\{-\alpha_j x\}. \end{aligned}$$

Therefore, $X = (s_1, \dots, s_n)$ are exponentially distributed according to A . Also, $\beta_j/\alpha_j \geq 1$ and hence, for every $1 \leq m \leq n - 1$, $t_j \leq s_j$ and we get that

$$\sum_{j=1}^m t_j \leq \sum_{j=1}^m s_j.$$

This implies that, Poisson process $\mathbf{P}(A)$ requires at least as much time as $\mathbf{P}(B)$ requires to generate m events. For any given t , let $\mathbf{N}_A(t)$ and $\mathbf{N}_B(t)$ denote the number of events

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happened by $\mathbf{P}(A)$ and $\mathbf{P}(B)$ during $[0, t]$, respectively. Then we have that $\mathbf{N}_A(t) \leq \mathbf{N}_B(t)$. Thus, for every positive integer k , we get that

$$\Pr[\mathbf{N}_B(t) \leq k] \leq \Pr[\mathbf{N}_A(t) \leq k],$$

which proves the lemma. \blacktriangleleft

We will combine the Lemmas 9 and 10 from [20] and get the following concentration bound for a particular set of exponential random variables.

► **Lemma A.1** (Lemma 9 and 10. [20]). *Let $f(m, j)$ be a deterministic sequence such that for $1 \leq j < m$, $\mathbf{E}[s_j | I_j]^{-1} \geq f(m, j)$ and $f(m, j) = \Theta(\min\{j, m - j\})$. Moreover, let $\{t_j^+\}_{j=1}^{m-1}$ be a sequences of independent random variables, where t_j^+ is exponentially distributed with parameter $f(m, j)$. Also let $T^+ = \sum_{j=1}^{m-1} t_j^+$, and $S_m = \sum_{j=1}^{m-1} s_j$. Then, we have*

$$\text{for } 0 < \lambda < \min_{j \in [m-1]} f(m, j), \quad \mathbf{E}[e^{\lambda S_m}] \leq \exp\{\lambda \mathbf{E}[T^+] + O(1)\}.$$

This implies that, for every $z > 0$,

$$\Pr[S_{m-1} \geq z] \leq \exp\{\lambda \mathbf{E}[T^+] + O(1) - \lambda z\}$$

B Properties of Geometric Mobile Networks

This section is devoted to the proof of Lemma 4.1. Geometric mobile model is a dynamic evolving network $\mathcal{M}(n, R) = \{G^{(t)}\}_{t=0}^{\infty}$ contains a set of n agents, denoted by $[n]$. Each agent independently performs nearest neighbor random walks on $H = (L_{n,\epsilon}, E)$ and there is an edge between two agents if their Euclidean distance is at most R . Recall that

$$L_{n,\epsilon} = \{(k \cdot \epsilon, l \cdot \epsilon) : k, l \in \mathbb{N}, k, l \leq \sqrt{n}/\epsilon\},$$

and

$$E = \{\{x, y\} : x, y \in L_{n,\epsilon}, d(x, y) \leq r\}.$$

It is easy to see that for every $x \in L_{n,\epsilon}$ there are $\Theta(r^2)$ locations at distance at most r from x and hence H is almost regular (i.e., the ratio of the maximum and minimum degree is at most a constant). Since H is connected and almost regular, a Markov chain defined by the random walk is ergodic and converges to an almost uniform stationary distribution over $L_{n,\epsilon}$, say π . Notice that by an almost uniform we mean that for every location $x, y \in L_{n,\epsilon}$, $\pi(x)/\pi(y) = \Theta(1)$. Since n agents perform random walks on H , we will have an Markov chain with state space

$$\underbrace{L_{n,\epsilon} \times L_{n,\epsilon} \times \dots \times L_{n,\epsilon}}_{n \text{ times}}.$$

Then, by a basic property of ergodic and finite Markov chains, at any time $t \geq 1$, each agent is located at location $x \in L_{n,\epsilon}$ with probability $\pi(x)$. Before proving Lemma 4.1, we first present some useful lemmas. Note that we use node or agent but they have the same meaning.

► **Lemma B.1.** *Let A denote a arbitrary 2-dimensional grid with m' nodes and S be an arbitrary set of nodes in A , with size at most $m'/2$. Then, there exists a constant $c > 0$ such that $N(S) \geq c' \sqrt{|S|}$, where $N(S)$ is the number of nodes that have at least one neighbor in S .*

The proof can be found in [7, Theorem 4.1].

► **Lemma B.2.** *Let M be an $m \times m$ grid embedded on $\sqrt{n} \times \sqrt{n}$ square plane, where $m = \sqrt{5n}/R$. Then, with high probability, each cell of M contains $\Theta(R^2)$ agents.*

Proof. Each cell in M is an $R/\sqrt{5} \times R/\sqrt{5}$ square and contains $R^2/5\varepsilon^2$ nodes from $L_{n,\varepsilon}$. Let us fix some $1 \leq t \leq n^3$ and arbitrary cell C . Then, the location of each agent at time t has an almost uniform distribution π over $L_{n,\varepsilon}$. For every agent $u \in [n]$, define indicator random variable $I_{u,C}$ as follows:

$$I_{u,C} = \begin{cases} 1 & \text{if } u \text{ is located in cell } C, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\Pr[I_{u,C} = 1] = \sum_{x \in C \cap L_{n,\varepsilon}} \pi(x) = (R^2/5\varepsilon^2) \times \Theta(\varepsilon^2/n) = \Theta(R^2/5n).$$

Also let $Y = \sum_{u \in [n]} I_{u,C}$ to denote the number of agents at time t in cell C . By the linearity of expectation we have that $\mathbf{E}[Y] = \Theta(R^2)$. Applying a Chernoff bound, we conclude that

$$\Pr[|Y - \mathbf{E}[Y]| \geq \mathbf{E}[Y]/2] \leq e^{-\mathbf{E}[Y]/12} = n^{-\omega(1)}.$$

Therefore, with probability $n^{-\omega(1)}$, cell C does not contain $\Theta(R^2)$ agents at time t . An application of union bound over all time steps and cells implies that with probability $1 - n^{-\omega(1)}$, for every $1 \leq t \leq n^3$, each cell of M contains $\Theta(R^2)$ agents which completes the proof. ◀

► **Lemma B.3** (Restatement of Lemma 4.1). *Suppose that $\mathcal{M}(n, R) = \{G^{(t)}\}_{t=0}^{\infty}$, is a geometric mobile network with $f(n)\sqrt{\log n} \leq R \leq \sqrt{n}$, where $f(n)$ is a slowly growing function in n . Then, with probability $1 - n^{-\omega(1)}$, for every $1 \leq t \leq n^3$, the followings hold:*

1. *For every node u (agent), $d_u(t) = \Theta(R^2)$, where $d_u(t)$ is the degree of node u in $G^{(t)}$.*
2. *There exists constant $a > 0$ such that conductance function $G^{(t)}$ satisfies*

$$\Phi(x) \geq \begin{cases} a & 1 \leq x \leq R^2, \\ a \frac{R}{\sqrt{\min\{x, n-x\}}} & R^2 < x \leq n-1. \end{cases}$$

Proof of (1). Let us fix an arbitrary time step $1 \leq t \leq n^3$ and an arbitrary agent, say u , that is located at some $x \in L_{n,\varepsilon}$. Define

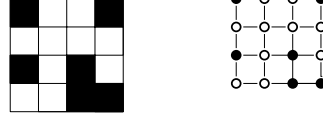
$$B(x) = \{y : y \in L_{n,\varepsilon}, d(x, y) \leq R\}.$$

It is not hard to see that for every x , $|B(x)| = \Theta((R/\varepsilon)^2)$. For every $y \in B(x)$ and $u \in [n]$, let us define the indicator random variable $I_{u,y}$ as follows:

$$I_{u,y} = \begin{cases} 1 & \text{if } u \text{ is located at } y, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $Y = \sum_{v \in [n] \setminus u} \sum_{y \in B(x)} I_{v,y}$ is the degree of agent u in $G^{(t)}$. Since every agent v has almost uniform distribution π over $L_{n,\varepsilon}$ and agents are independent from each other, we get that

$$\Pr[I_{v,y} = 1] = \pi(y) = \Theta(\varepsilon^2/n).$$



■ **Figure 1** The right grid is the dual of colored left grid, where colored faces are translated to colored vertices.

Thus, by linearity of expectation we have

$$\begin{aligned} \mathbf{E}[Y] &= \sum_{v \in [n] \setminus u} \sum_{y \in B(x)} \mathbf{E}[I_{v,y}] = \sum_{v \in [n] \setminus u} \sum_{y \in B(x)} \pi(y) \\ &= (n-1) \sum_{y \in B(x)} \pi(y) = (n-1)|B(x)|\Theta(\varepsilon^2/n) = \Theta(R^2) = \omega(\log n). \end{aligned}$$

$I_{v,y}$'s are mutually independent so we apply a Chernoff bound and conclude that

$$\Pr[|Y - \mathbf{E}[Y]| \geq \mathbf{E}[Y]/2] \leq e^{-\mathbf{E}[Y]/12} = n^{-\omega(1)}.$$

Note that the above inequality holds only for an arbitrary and fixed time step t and node u . By union bound over all n agents and n^3 time steps, we conclude that with probability $n^3 \times n \times n^{-\omega(1)}$ there is a time step s and agent w such that $d_w(s) \notin [\mathbf{E}[Y]/2, 3\mathbf{E}[Y]/2]$. Therefore, with probability $1 - n^{-\omega(1)}$, for every agent u and time step t , $d_u(t) = \Theta(R^2)$, which completes proof of (1).

Proof of (2). Suppose that $m = \sqrt{5n}/R$ and consider an $m \times m$ grid M embedded in a plane square of $\sqrt{n} \times \sqrt{n}$, whose cells are $\sqrt{R/5} \times \sqrt{R/5}$ squares. By Lemma B.2, with high probability, for every t , each cell of M contains $\Theta(R^2)$ agents. Fix an arbitrary set of agents (nodes), say S , with size $1 \leq s \leq n/2$. Also fix some time step t . Then, with respect to S and time t , we color cells of M as follows. Cell C becomes white, if at most $3/4$ agents in C are contained in S and black otherwise. As a result, each cell of M gets colored by either black or white at time step t . Let B and W denote the set of black and white cells in M . Now, let us consider the dual of the grid M , which is again a $(m-1) \times (m-1)$ grid. Notice that the vertex set of the dual graph is the interior faces of the primal and two vertices in the dual are connected if their corresponding faces (cells) are side by side (e.g. see Figure 1). By definition of $\mathcal{M}(n, R) = \{G^{(t)}\}_{t=0}^{\infty}$, every two agents located at any two side-by-side cells are connected by an edge, because their Euclidean distance is at most $\sqrt{4R^2/5 + R^2/5} = R$. According to the size of B we consider two cases:

■ $|B| < m^2/2$:

Let D be the set of vertices corresponding to cells of B in the dual graph (i.e. set of black nodes in the right grid of Fig. 1). Thus, $|D| < m^2/2$. By Lemma B.1, we have that $N(D) \geq c'\sqrt{|D|}$, for some constant c' . This implies that there are at least $c'\sqrt{|B|}$ white cells that are connected to cells of B . By the coloring rule, we deduce that if a black cell and a white cell are side by side, then at least $3/4$ agents from the black cell contained in S are connected to at least $1/4$ agents of the white cell contained in \bar{S} . Moreover, every agent of S contained in a white cell is connected to at least $1/4$ agents from the same cell, which are contained in \bar{S} . Remember that by Lemma B.2 each cell contains $\Theta(R^2)$ agents, w.h.p. Thus, we get that in $G^{(t)}$,

$$|E(S, \bar{S})| \geq c\sqrt{|B|}\Theta(R^2) + \sum_{C \in W} x(C)\Theta(R^2),$$

where $x(C)$ is the number of agents in S that are contained in white cell C . Moreover, $G^{(t)}$ is almost regular with degree $\Theta(R^2)$ and each cell contains at most $\Theta(R^2)$ agents. So we have

$$\text{vol}(S) \leq |B|\Theta(R^4) + \sum_{C \in W} x(C)\Theta(R^2).$$

Now, we may consider two cases $|B| = 0$ and $|B| > 0$. In first case, by two above inequalities we get

$$\frac{|\mathbb{E}(S, \bar{S})|}{\text{vol}(S)} = \Theta(1). \quad (14)$$

For the second case we get

$$\begin{aligned} \frac{|\mathbb{E}(S, \bar{S})|}{\text{vol}(S)} &\geq \frac{\sqrt{|B|}\Theta(R^4) + \sum_{C \in W} x(C)\Theta(R^2)}{|B|\Theta(R^4) + \sum_{C \in W} x(C)\Theta(R^2)} \geq \Theta\left(\frac{\sqrt{|B|}R^4}{|B|R^4}\right) \\ &= \Theta\left(\frac{1}{\sqrt{|B|}}\right) \geq \Theta\left(\sqrt{\frac{3R^2}{20|S|}}\right) = \Theta\left(\frac{R}{\sqrt{|S|}}\right), \end{aligned} \quad (15)$$

where the second inequality comes from the fact for every number $z > x > 0$ and arbitrary $x > 0$, we have that $\frac{x+y}{z+y} \geq \frac{x}{z}$. Also, the third one follows from $|B|\Theta(R^2)3/4 \leq |S|$, as $3/4$ agents in each black cell contained in S .

- $|B| \geq m^2/2$: In this case, we first observe that $|W| = \Theta(m^2)$. Toward a contradiction, we assume that

$$|W| = o(m^2) = o(n/R^2)$$

and hence white cells can have at most $|W|\Theta(R^2) = o(n)$ agents, by lemma B.2, each cell contains $\Theta(R^2)$ agents. On the other hand, by definition, black cells can accommodate at most $n/4$ agents from \bar{S} , which contradicts assumption that $|\bar{S}| \geq n/2$. So we have that $|W| = \Theta(m^2) = \Theta(|B|)$. Since $|W| + |B| = m^2$ we conclude that $|W| < m^2/2$. Again similar to the previous case, there are at least $c\sqrt{|W|}$ black cells, which are adjacent to white cells and we have

$$|\mathbb{E}(S, \bar{S})| \geq c\sqrt{|W|}\Theta(R^2) + \sum_{C \in W} x(C)\Theta(R^2).$$

Moreover,

$$\text{vol}(S) \leq |B|\Theta(R^4) + \sum_{C \in W} x(C)\Theta(R^2) = |W|\Theta(R^4) + \sum_{C \in W} x(C)\Theta(R^2),$$

where it follows from the fact that $|W| = \Theta(|B|)$. Similar to the previous case we will have,

$$\begin{aligned} \frac{|\mathbb{E}(S, S^c)|}{\text{vol}(S)} &\geq \frac{\sqrt{|W|}\Theta(R^4) + \sum_{C \in W} x(C)\Theta(R^2)}{|W|\Theta(R^4) + \sum_{C \in W} x(C)\Theta(R^2)} \\ &\geq \Theta\left(\frac{1}{\sqrt{|W|}}\right) = \Theta\left(\frac{1}{\sqrt{|B|}}\right) = \Theta\left(\sqrt{\frac{3R^2}{20|S|}}\right) = \Theta\left(\frac{R}{\sqrt{|S|}}\right). \end{aligned} \quad (16)$$

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From Inequalities (14), (15), and (16) we conclude that for every time step $1 \leq t \leq n^3$ and a set of agents of size at most $n/2$ in $G^{(t)}$, there exists constant $a > 0$ such that

$$\frac{|\mathbb{E}(S, \bar{S})|}{\text{vol}(S)} \geq \min \left\{ a \frac{R}{\sqrt{|S|}}, a \right\}. \quad (17)$$

For every subset of agents, say S , define

$$g(S) = \begin{cases} S & \text{if } |S| \leq n/2, \\ \bar{S} & \text{otherwise.} \end{cases}$$

Clearly, we have that $|g(S)| = \min\{|S|, n - |S|\} \leq n/2$ and $|\mathbb{E}(S, \bar{S})| = |\mathbb{E}(g(S), \overline{g(S)})|$ completing the proof. \blacktriangleleft