Extending the Reach of the Point-To-Set Principle

Jack H. Lutz
Department of Computer Science, Iowa State University, Ames, IA, USA

Neil Lutz
Computer Science Department, Swarthmore College, PA, USA

Elvira Mayordomo
Departamento de Informática e Ingeniería de Sistemas,
Instituto de Investigación en Ingeniería de Aragón, University of Zaragoza, Spain

Abstract

The *point-to-set principle* of J. Lutz and N. Lutz (2018) has recently enabled the theory of computing to be used to answer open questions about fractal geometry in Euclidean spaces $\mathbb{R}^n$. These are classical questions, meaning that their statements do not involve computation or related aspects of logic.

In this paper we extend the *reach* of the point-to-set principle from Euclidean spaces to arbitrary separable metric spaces X. We first extend two fractal dimensions – computability-theoretic versions of classical Hausdorff and packing dimensions that assign dimensions $\dim(x)$ and $\mathrm{Dim}(x)$ to *individual points* $x \in X$ – to arbitrary separable metric spaces and to arbitrary gauge families. Our first two main results then extend the point-to-set principle to arbitrary separable metric spaces and to a large class of gauge families.

We demonstrate the power of our extended point-to-set principle by using it to prove new theorems about classical fractal dimensions in hyperspaces. (For a concrete computational example, the stages $E_0, E_1, E_2, \ldots$ used to construct a self-similar fractal E in the plane are elements of the hyperspace of the plane, and they converge to E in the hyperspace.) Our third main result, proven via our extended point-to-set principle, states that, under a wide variety of gauge families, the classical packing dimension agrees with the classical upper Minkowski dimension on *all* hyperspaces of compact sets. We use this theorem to give, for all sets E that are analytic, i.e., $\mathbf{\Sigma}^1_1$, a tight bound on the packing dimension of the hyperspace of E in terms of the packing dimension of E itself.

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1 Introduction

It is rare for the theory of computing to be used to answer open mathematical questions – especially questions in continuous mathematics – whose statements do not involve computation or related aspects of logic.[1] The *point-to-set principle* [22], described below, has enabled several recent developments that do exactly this. This principle has been used to obtain strengthened lower bounds on the Hausdorff dimensions of generalized Furstenberg sets [27], extend the fractal intersection formula for Hausdorff dimension from Borel sets to arbitrary sets [25], and prove that Marstrand's projection theorem for Hausdorff dimension holds for any set E whose Hausdorff and packing dimensions coincide, whether or not E is analytic [26].[2] (See [5, 6, 23, 24] for reviews of these developments.) More recently, the point-to-set principle has been used to prove that $V = L$ implies that the maximal thin co-analytic set has Hausdorff dimension 1 [40] and that the Continuum Hypothesis implies that every $s \in (0, 1]$ is the Hausdorff dimension of a Hamel basis of the vector space $\mathbb{R}$ over the field $\mathbb{Q}$ [21]. These applications of the point-to-set principle all concern fractal geometry in Euclidean spaces $\mathbb{R}^n$.[3]

This paper extends the reach of the point-to-set principle beyond Euclidean spaces. To explain this, we first review the point-to-set principle to date. (All quantities defined in this intuitive discussion are defined precisely later in the paper.) The two best-behaved classical fractal dimensions, Hausdorff dimension and packing dimension, assign to every subset E of a Euclidean space $\mathbb{R}^n$ dimensions $\dim_{\mathrm{H}}(E)$ and $\dim_{\mathrm{P}}(E)$, respectively. When E is a "smooth" set that intuitively has some integral dimension between 0 and n, the Hausdorff and packing dimensions agree with this intuition, but more complex sets E may have any real-valued dimensions satisfying $0 \leq \dim_{\mathrm{H}}(E) \leq \dim_{\mathrm{P}}(E) \leq n$. Hausdorff and packing dimensions have many applications in information theory, dynamical systems, and other areas of science [2, 7, 14, 35].

Early in this century, algorithmic versions of Hausdorff and packing dimensions were developed to quantify the information densities of various types of data. The computational resources allotted to these algorithmic dimensions range from finite-state to computable enumerability and beyond, but the point-to-set principle concerns the computably enumerable algorithmic dimensions introduced in [20, 1].[4] These assign to each *individual point* x in a Euclidean space $\mathbb{R}^n$ an *algorithmic dimension* $\dim(x)$ and a *strong algorithmic dimension* $\mathrm{Dim}(x)$. The point-to-set principle of [22] is a complete characterization of the classical Hausdorff and packing dimensions in terms of oracle relativizations of these very non-classical dimensions of individual points. Specifically, the point-to-set principle says that, for every set E in a Euclidean space $\mathbb{R}^n$,

$$\dim_{\mathrm{H}}(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x) \tag{1.1}$$

[1] We use the adjective "classical" for theorems and questions whose statements do not involve computability or logic, regardless of when they were proven or formulated. A "classical" theorem can thus be very new.

[2] These very non-classical proofs of new classical theorems have provoked new work in the fractal geometry community. Orponen [34] has very recently used a discretized potential-theoretic method of Kaufman [16] and tools of Katz and Tao [15] to give a new, classical proof of the two main theorems of [26].

[3] Applications of the theory of computing – specifically Kolmogorov complexity – to discrete mathematics are more numerous and are surveyed in [19]. Other applications to continuous mathematics, not involving the point-to-set principle, include theorems in descriptive set theory [32, 12, 17], Riemannian moduli space [43], and Banach spaces [18].

[4] These have also been called "constructive" dimensions and "effective" dimensions by various authors.

and

$$\dim_{\mathrm{P}}(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^A(x), \tag{1.2}$$

where the dimensions on the right are relative to the oracle A. The point-to-set principle is so named because it enables one to use a lower bound on the relativized algorithmic dimension of a single, judiciously chosen *point* in a set E to prove a lower bound on the classical dimension of the *set* E.

The classical Hausdorff and packing dimensions work not only in Euclidean spaces, but in arbitrary metric spaces. In contrast, nearly all work on algorithmic dimensions to date (the exception being [29]) has been in Euclidean spaces or in spaces of infinite sequences over finite alphabets. Our objective here is to significantly reduce this gap by extending the theory of algorithmic dimensions, along with the point-to-set principle, to arbitrary separable metric spaces. (A metric space X is *separable* if it has a countable subset D that is *dense* in the sense that every point in X has points in D arbitrarily close to it.)

In parallel with extending algorithmic dimensions to separable metric spaces, we also extend them to arbitrary gauge families. It was already explicit in Hausdorff's original paper [8] that his dimension could be defined via various "lenses" that we now call *gauge functions*. In fact, one often uses, as we do here, a *gauge family* φ, which is a one-parameter family of gauge functions φ_s for $s \in (0, \infty)$. For each separable metric space X, each gauge family φ, and each set $E \subseteq X$, the classical φ*-gauged Hausdorff dimension* $\dim_{\mathrm{H}}^{\varphi}(E)$ and φ*-gauged packing dimension* $\dim_{\mathrm{P}}^{\varphi}(E)$ are thus well-defined. In this paper, for each separable metric space X, each gauge family φ, and each point $x \in X$, we define the φ*-gauged algorithmic dimension* $\dim^{\varphi}(x)$ and the φ*-gauged strong algorithmic dimension* $\mathrm{Dim}^{\varphi}(x)$ of the point x. We should mention here that there is a particular gauge family θ that gives the "un-gauged" dimensions in the sense that the identities $\dim_{\mathrm{H}}^{\theta}(E) = \dim_{\mathrm{H}}(E)$, $\dim_{\mathrm{P}}^{\theta}(E) = \dim_{\mathrm{P}}(E)$, $\dim^{\theta}(x) = \dim(x)$, and $\mathrm{Dim}^{\theta}(x) = \mathrm{Dim}(x)$ always hold.

Our first two main results (Theorems 4.1 and 4.2) extend the point-to-set principle to arbitrary separable metric spaces and a wide variety of gauge families, proving that, for every separable metric space X, every gauge family φ satisfying mild asymptotic constraints, and every set $E \subseteq X$,

$$\dim_{\mathrm{H}}^{\varphi}(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^{\varphi, A}(x) \tag{1.3}$$

and

$$\dim_{\mathrm{P}}^{\varphi}(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^{\varphi, A}(x). \tag{1.4}$$

Various nontrivial modifications to both machinery and proofs are necessary in getting from (1.1) and (1.2) to (1.3) and (1.4).

As an illustration of the power of our approach, we investigate the dimensions of hyperspaces. The *hyperspace* $\mathcal{K}(X)$ of a metric space X is the set of all nonempty compact subsets of X, equipped with the Hausdorff metric [44]. (For example, the "stages" $E_0, E_1, E_2, \ldots$ of a self-similar fractal $E \subseteq \mathbb{R}^n$ converge to E in the hyperspace $\mathbb{R}^n$.) The hyperspace of a separable metric space is itself a separable metric space, and the hyperspace is typically infinite-dimensional, even when the underlying metric space is finite-dimensional. One use of gauge families is reducing such infinite dimensions to enable quantitative comparisons. For example, McClure [30] defined, for each gauge family φ, a *jump* $\widetilde{\varphi}$ (our notation) that is also a gauge family, and he proved [31] for every *self-similar* subset E of a separable metric space X,

$$\dim_{\mathrm{H}}^{\widetilde{\theta}}(\mathcal{K}(E)) = \dim_{\mathrm{H}}(E),$$

where θ is the above-mentioned "un-gauged" gauge family.

Here we prove a *hyperspace dimension theorem* for the upper and lower Minkowski (i.e., box-counting) dimensions $\underline{\dim}_{\mathcal{M}}$ and $\overline{\dim}_{\mathcal{M}}$. This states that, for every separable metric space X, *every* gauge family φ, and *every* $E \subseteq X$,

$$\underline{\dim}_{\mathcal{M}}^{\widetilde{\varphi}}(\mathcal{K}(E)) = \underline{\dim}_{\mathcal{M}}^{\varphi}(E) \tag{1.5}$$

and

$$\overline{\dim}_{\mathcal{M}}^{\widetilde{\varphi}}(\mathcal{K}(E)) = \overline{\dim}_{\mathcal{M}}^{\varphi}(E). \tag{1.6}$$

We note that it is implicit in [30] that these identities hold for totally bounded sets E and gauge families φ satisfying a doubling condition.

Our third main result (Theorem 5.2) says that, for every separable metric space X, every "well-behaved" gauge family φ, and every compact set $E \subseteq X$,

$$\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E)) = \overline{\dim}_{\mathcal{M}}^{\widetilde{\varphi}}(\mathcal{K}(E)). \tag{1.7}$$

Our proof of this result makes essential use of (1.6) and the point-to-set principle (1.4).

Finally, we use the point-to-set principle (1.4), the identities (1.6) and (1.7), and some additional machinery to prove the *hyperspace packing dimension theorem* (Theorem 5.4), which says that, for every separable metric space X, every well-behaved gauge family φ, and every *analytic* (i.e., $\mathbf{\Sigma}_1^1$, an analog of NP that Sipser famously investigated [37, 38, 39]) set $E \subseteq X$,

$$\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E)) \geq \dim_{\mathrm{P}}^{\varphi}(E). \tag{1.8}$$

It is implicit in [30] that (1.8) holds for all σ-compact sets E.

At the time of this writing it is an open question whether there is an analogous hyperspace dimension theorem for Hausdorff dimension.

David Hilbert famously wrote the following [10].

> The final test of every new theory is its success in answering preexistent questions that the theory was not specifically created to answer.

The theory of algorithmic dimensions passed Hilbert's final test when the point-to-set principle gave us the results in the first paragraph of this introduction. We hope that the machinery developed here will lead to further such successes in the wider arena of separable metric spaces.

2 Gauged Classical Dimensions

We review the definitions of gauged Hausdorff, packing, and Minkowski dimensions. We refer the reader to [7, 28] for a complete introduction and motivation.

Let (X, ρ) be a metric space where ρ is the metric. (From now on we will omit ρ when referring to the space (X, ρ).) X is *separable* if there exists a countable set $D \subseteq X$ that is *dense* in X, meaning that for every $x \in X$ and $\delta > 0$, there is a $d \in D$ such that $\rho(x, d) < \delta$. The *diameter* of a set $E \subseteq X$ is $\mathrm{diam}(E) = \sup\{\rho(x, y) \mid x, y \in E\}$; notice that the diameter of a set can be infinite. A *cover* of $E \subseteq X$ is a collection $\mathcal{U} \subseteq \mathcal{P}(X)$ such that $E \subseteq \bigcup_{U \in \mathcal{U}} U$, and a δ-*cover* of E is a cover $\mathcal{U}$ of E such that $\mathrm{diam}(U) \leq \delta$ for all $U \in \mathcal{U}$.

▶ **Definition** (gauge functions and families). A *gauge function* is a continuous,[5] nondecreasing function from $[0,\infty)$ to $[0,\infty)$ that vanishes only at 0 [8, 36]. A *gauge family* is a one-parameter family $\varphi = \{\varphi_s \mid s \in (0,\infty)\}$ of gauge functions φ_s satisfying

$$\varphi_s(\delta) = o(\varphi_t(\delta)) \text{ as } \delta \to 0^+$$

whenever $s > t$.

The *canonical gauge family* is $\theta = \{\theta_s \mid s \in (0,\infty)\}$, defined by $\theta_s(\delta) = \delta^s$. "Un-gauged" or "ordinary" Hausdorff, packing, and Minkowski dimensions are special cases of the following definitions, using $\varphi = \theta$.

Some of our gauged dimension results will require the existence of a "precision family" for the gauge family.

▶ **Definition** (precision family). A *precision sequence* for a gauge function φ is a function $\alpha : \mathbb{N} \to \mathbb{Q}^+$ that vanishes as $r \to \infty$ and satisfies $\varphi(\alpha(r)) = O(\varphi(\alpha(r+1)))$ as $r \to \infty$. A *precision family* for a gauge family $\varphi = \{\varphi_s \mid s \in (0,\infty)\}$ is a one-parameter family $\alpha = \{\alpha_s \mid s \in (0,\infty)\}$ of precision sequences satisfying

$$\sum_{r\in\mathbb{N}} \frac{\varphi_t(\alpha_s(r))}{\varphi_s(\alpha_s(r))} < \infty$$

whenever $s < t$.

▶ **Observation 2.1.** *$\alpha_s(r) = 2^{-sr}$ is a precision family for the canonical gauge family θ.*

▶ **Definition** (gauged Hausdorff measure and dimension). For every metric space X, set $E \subseteq X$, and gauge function φ, the *φ-gauged Hausdorff measure* of E is

$$H^{\varphi}(E) = \lim_{\delta\to 0^+} \inf \left\{ \sum_{U\in\mathcal{U}} \varphi(\mathrm{diam}(U)) \;\middle|\; \mathcal{U} \text{ is a countable } \delta\text{-cover of } E \right\}.$$

For every gauge family $\varphi = \{\varphi_s \mid s \in (0,\infty)\}$, the *$\varphi$-gauged Hausdorff dimension* of E is

$$\dim^{\varphi}_{\mathrm{H}}(E) = \inf\{ s \in (0,\infty) \mid H^{\varphi_s}(E) = 0 \}.$$

▶ **Definition** (gauged packing measure and dimension). For every metric space X, set $E \subseteq X$, and $\delta \in (0,\infty)$, let $\mathcal{V}_\delta(E)$ be the set of all countable collections of disjoint open balls with centers in E and diameters at most δ. For every gauge function φ and $\delta > 0$, define the quantity

$$P^{\varphi}_{\delta}(E) = \sup_{\mathcal{U}\in\mathcal{V}_\delta(E)} \sum_{U\in\mathcal{U}} \varphi(\mathrm{diam}(U)).$$

Then the *φ-gauged packing pre-measure* of E is

$$P^{\varphi}_{0}(E) = \lim_{\delta\to 0^+} P^{\varphi}_{\delta}(E),$$

[5] Some authors require only that the function is right-continuous when working with Hausdorff dimension and left-continuous when working with packing dimension. Indeed, left continuity is sufficient for our hyperspace packing dimension theorem.

and the φ*-gauged packing measure* of E is

$$P^{\varphi}(E) = \inf\left\{ \sum_{U\in\mathcal{U}} P_0^{\varphi}(U) \,\middle|\, \mathcal{U} \text{ is a countable cover of } E \right\}.$$

For every gauge family $\varphi = \{\varphi_s \mid s \in (0,\infty)\}$, the φ*-gauged packing dimension* of E is

$$\dim_{\mathrm{P}}^{\varphi}(E) = \inf\{s \in (0,\infty) \mid P^{\varphi_s}(E) = 0\}.$$

▶ **Definition** (gauged Minkowski dimensions)**.** For every metric space X, $E \subseteq X$, and $\delta \in (0,\infty)$, let

$$N(E,\delta) = \min\left\{ |F| \,\middle|\, F \subseteq X \text{ and } E \subseteq \bigcup_{x\in F} B_\delta(x) \right\},$$

where $B_\delta(x)$ is the open ball of radius δ centered at x. Then for every gauge family $\varphi = \{\varphi_s\}_{s\in(0,\infty)}$ the φ*-gauged lower* and *upper Minkowski dimension* of E are

$$\underline{\dim}_{\mathcal{M}}^{\varphi}(E) = \inf\left\{ s \,\middle|\, \liminf_{\delta\to 0^+} N(E,\delta)\varphi_s(\delta) = 0 \right\}$$

and

$$\overline{\dim}_{\mathcal{M}}^{\varphi}(E) = \inf\left\{ s \,\middle|\, \limsup_{\delta\to 0^+} N(E,\delta)\varphi_s(\delta) = 0 \right\},$$

respectively.

When X is separable, it is sometimes useful to require that the balls covering E have centers in the countable dense set D. For all $E \subseteq X$ and $\delta \in (0,\infty)$, let

$$\hat{N}(E,\delta) = \min\left\{ |F| \,\middle|\, F \subseteq D \text{ and } E \subseteq \bigcup_{x\in F} B_\delta(x) \right\}.$$

▶ **Observation 2.2.** *If X is a separable metric space and $\varphi = \{\varphi_s\}_{s\in(0,\infty)}$ is a gauge family, then for all $E \subseteq X$,*

1. $\underline{\dim}_{\mathcal{M}}^{\varphi}(E) = \inf\left\{ s \,\middle|\, \liminf_{\delta\to 0^+} \hat{N}(E,\delta)\varphi_s(\delta) = 0 \right\}$.
2. $\overline{\dim}_{\mathcal{M}}^{\varphi}(E) = \inf\left\{ s \,\middle|\, \limsup_{\delta\to 0^+} \hat{N}(E,\delta)\varphi_s(\delta) = 0 \right\}$.

The following relationship between upper Minkowski dimension and packing dimension was previously known to hold for the canonical gauge family θ, a result that is essentially due to Tricot [42]. Our proof of this gauged generalization, is adapted from the presentation by Bishop and Peres [3] of the un-gauged proof.

▶ **Lemma 2.3** (generalizing Tricot [42])**.** *Let X be any metric space, $E \subseteq X$, and φ a gauge family.*

1. *If $\varphi_t(2\delta) = O(\varphi_s(\delta))$ as $\delta \to 0^+$ for all $s < t$, then*

$$\dim_{\mathrm{P}}^{\varphi}(E) \geq \inf\left\{ \sup_{i\in\mathbb{N}} \overline{\dim}_{\mathcal{M}}^{\varphi}(E_i) \,\middle|\, E \subseteq \bigcup_{i\in\mathbb{N}} E_i \right\}.$$

2. *If there is a precision family for φ, then*

$$\dim_{\mathrm{P}}^{\varphi}(E) \leq \inf\left\{ \sup_{i\in\mathbb{N}} \overline{\dim}_{\mathcal{M}}^{\varphi}(E_i) \,\middle|\, E \subseteq \bigcup_{i\in\mathbb{N}} E_i \right\}.$$

3 Gauged Algorithmic Dimensions

In this section we formulate algorithmic dimensions in arbitrary separable metric spaces and with arbitrary gauge families.

For the rest of this paper, let $X = (X, \rho)$ be a separable metric space, and fix a function $f : \{0,1\}^* \to X$ such that the set $D = \mathrm{range}(f)$ is dense in X. The metric space X is *computable* if there is a computable function $g : (\{0,1\}^*)^2 \times \mathbb{Q}^+ \to \mathbb{Q}$ that approximates ρ on D in the sense that, for all $v, w \in \{0,1\}^*$ and $\delta \in \mathbb{Q}^+$.

$$|g(v, w, \delta) - \rho(f(v), f(w))| \leq \delta.$$

Our results here hold for all separable metric spaces, whether or not they are computable, but our methods make explicit use of the function f.

Following standard practice [33, 4, 19], fix a universal oracle Turing machine U, and define the *(plain) Kolmogorov complexity* of a string $w \in \{0,1\}^*$ *relative to* an oracle $A \subseteq \mathbb{N}$ to be

$$\mathrm{C}^A(w) = \min\left\{|\pi| \,\middle|\, \pi \in \{0,1\}^* \text{ and } U^A(\pi) = w\right\},$$

i.e., the minimum number of bits required to cause U to output w when it has access to the oracle A. The *(plain) Kolmogorov complexity* of w is then $\mathrm{C}(w) = \mathrm{C}^{\emptyset}(w)$.

We define the *(plain) Kolmogorov complexity* of a point $q \in D$ to be

$$\mathrm{C}(q) = \min\{\mathrm{C}(w) \mid w \in \{0,1\}^* \text{ and } f(w) = q\},$$

noting that this depends on the enumeration f of D that we have fixed.

The *Kolmogorov complexity* of a point $x \in X$ at *precision* $\delta \in (0, \infty)$ is

$$\mathrm{C}_\delta(x) = \min\{\mathrm{C}(q) \mid q \in D \text{ and } \rho(q, x) < \delta\}.$$

The *algorithmic dimension* of a point $x \in X$ is

$$\dim(x) = \liminf_{\delta \to 0^+} \frac{\mathrm{C}_\delta(x)}{\log(1/\delta)}, \tag{3.1}$$

and the *strong algorithmic dimension* of x is

$$\mathrm{Dim}(x) = \limsup_{\delta \to 0^+} \frac{\mathrm{C}_\delta(x)}{\log(1/\delta)}. \tag{3.2}$$

These two dimensions[6] have been extensively investigated in the special cases where X is a Euclidean space $\mathbb{R}^n$ or a sequence space Σ^ω [23, 4].

Having generalized algorithmic dimensions to arbitrary separable metric spaces, we now generalize them to arbitrary gauge families.

Let $\varphi = \{\varphi_s \mid s \in (0, \infty)\}$ be a gauge family. Then, the *φ-gauged algorithmic dimension* of a point $x \in X$ is

$$\dim^\varphi(x) = \inf\left\{s \,\middle|\, \liminf_{\delta \to 0^+} 2^{\mathrm{C}_\delta(x)} \varphi_s(\delta) = 0\right\}, \tag{3.3}$$

[6] The definitions given here differ slightly from the standard formulation in which prefix Kolmogorov complexity is used instead of plain Kolmogorov complexity and the precision parameter δ belongs to $\{2^{-r} \mid r \in \mathbb{N}\}$. The present formulation is equivalent to the standard one for un-gaugued dimensions and facilitates our generalization to gauged algorithmic dimensions. In particular, plain Kolmogorov complexity is only needed to accommodate gauge functions φ in which the convergence of φ to 0 as $\delta \to 0^+$ is very slow.

and the *φ-gauged strong algorithmic dimension* of x is

$$\mathrm{Dim}^{\varphi}(x) = \inf \left\{ s \,\middle|\, \limsup_{\delta \to 0^+} 2^{\mathrm{C}_\delta(x)} \varphi_s(\delta) = 0 \right\}, \tag{3.4}$$

Gauged algorithmic dimensions $\dim^{\varphi}(x)$ have been investigated by Staiger [41] in the special case where X is a sequence space Σ^ω.

A routine inspection of (3.1)–(3.4) verifies the following.

▶ **Observation 3.1.** *For all $x \in X$, $\dim^{\theta}(x) = \dim(x)$ and $\mathrm{Dim}^{\theta}(x) = \mathrm{Dim}(x)$, where θ is the canonical gauge family given by $\theta_s(\delta) = \delta^s$.*

A specific investigation of algorithmic (or classical) dimensions might call for a particular gauge function or family for one of two reasons. First, many gauge functions may assign the same dimension to an object under consideration (because they converge to 0 at somewhat similar rates as $\delta \to 0^+$) but additional considerations may identify one of these as being the most precisely tuned to the phenomenon of interest. Finding such a gauge function is called finding the "exact dimension" of the object under investigation. This sort of calibration has been studied extensively for classical dimensions [7, 36] and by Staiger [41] for algorithmic dimension.

The second reason, and the reason of interest to us here, why specific investigations might call for particular gauge families is that a given gauge family φ may be so completely out of tune with the phenomenon under investigation that the φ-gauged dimensions of the objects of interest are either all minimum (all 0) or else all maximum (all the same dimension as the space X itself). In such a circumstance, a gauge family that converges to 0 more quickly or slowly than φ may yield more informative dimensions. Several such circumstances were investigated in a complexity-theoretic setting by Hitchcock, J. Lutz, and Mayordomo [11].

The following routine observation indicates the direction in which one adjusts a gauge family's convergence to 0 in order to adjust the resulting gauged dimensions upward or downward.

▶ **Observation 3.2.** *If φ and ψ are gauge families with $\varphi_s(\delta) = o(\psi_s(\delta))$ as $\delta \to 0^+$ for all $s \in (0, \infty)$, then, for all $x \in X$, $\dim^{\varphi}(x) \le \dim^{\psi}(x)$ and $\mathrm{Dim}^{\varphi}(x) \le \mathrm{Dim}^{\psi}(x)$.*

We now define an operation on gauge families that is implicit in earlier work [30] and is explicitly used in the results of Section 5.

▶ **Definition** (jump)**.** The *jump* of a gauge family φ is the family $\widetilde{\varphi}$ given $\widetilde{\varphi}_s(\delta) = 2^{-1/\varphi_s(\delta)}$.

▶ **Observation 3.3.** *The jump of a gauge family is a gauge family.*

We now note that the jump of a gauge family always converges to 0 more quickly than the original gauge family.

▶ **Lemma 3.4.** *For all gauge families φ and all $s \in (0, \infty)$, $\widetilde{\varphi}_s(\delta) = o(\varphi_s(\delta))$ as $\delta \to 0^+$.*

Observation 3.3 and Lemma 3.4 immediately imply the following.

▶ **Corollary 3.5.** *For all gauge families φ and all $x \in X$, $\dim^{\widetilde{\varphi}}(x) \le \dim^{\varphi}(x)$ and $\mathrm{Dim}^{\widetilde{\varphi}}(x) \le \mathrm{Dim}^{\varphi}(x)$.*

The definitions and results of this section relativize to arbitrary oracles $A \subseteq \mathbb{N}$ in the obvious manner, so the Kolmogorov complexities $\mathrm{C}^A(q)$ and $\mathrm{C}^A_\delta(x)$ and the dimensions $\dim^A(x)$, $\mathrm{Dim}^A(x)$, $\dim^{\varphi,A}(x)$, and $\mathrm{Dim}^{\varphi,A}(x)$ are all well-defined and behave as indicated.

▶ **Observation 3.6.** *For all gauge families φ, all $x \in X$, and all $s > 0$,*

$$\log\left(2^{\mathrm{C}_\delta(x)}\widetilde{\varphi}_s(\delta)\right) = \frac{\mathrm{C}_\delta(x)\varphi_s(\delta) - 1}{\varphi_s(\delta)}.$$

The $\widetilde{\varphi}$-gauged algorithmic dimensions admit the following characterizations, the second of which is used in the proof of our hyperspace packing dimension theorem.

▶ **Theorem 3.7.** *For all gauge families φ and all $x \in X$, the following identities hold.*

1. $\dim^{\widetilde{\varphi}}(x) = \inf\left\{ s \;\middle|\; \liminf_{\delta\to 0^+} \mathrm{C}_\delta(x)\varphi_s(\delta) = 0 \right\}$.
2. $\mathrm{Dim}^{\widetilde{\varphi}}(x) = \inf\left\{ s \;\middle|\; \limsup_{\delta\to 0^+} \mathrm{C}_\delta(x)\varphi_s(\delta) = 0 \right\}$.

4 The General Point-to-Set Principle

We now show that the point-to-set principle of J. Lutz and N. Lutz [22] holds in arbitrary separable metric spaces and for gauged dimensions. The proofs of these theorems are more delicate and involved than those in [22]. This is partially due to the fact that the metric spaces here need not be finite-dimensional, and to the weak restrictions we place on the gauge family.

▶ **Theorem 4.1** (general point-to-set principle for Hausdorff dimension)**.** *For every separable metric space X, every gauge family φ, and every set $E \subseteq X$,*

$$\dim_{\mathrm{H}}^{\varphi}(E) \geq \min_{A\subseteq\mathbb{N}} \sup_{x\in E} \dim^{\varphi,A}(x).$$

Equality holds if there is a precision family for φ.

▶ **Theorem 4.2** (general point-to-set principle for packing dimension)**.** *Let X be any separable metric space, $E \subseteq X$, and φ a gauge family.*

1. *If $\varphi_t(2\delta) = O(\varphi_s(\delta))$ and $\varphi_s(\delta) = O(1/\log\log(1/\delta))$ as $\delta \to 0^+$ for all $s < t$, then*

$$\dim_{\mathrm{P}}^{\varphi}(E) \geq \min_{A\subseteq\mathbb{N}} \sup_{x\in E} \mathrm{Dim}^{\varphi,A}(x).$$

2. *If there is a precision family for φ, then*

$$\dim_{\mathrm{P}}^{\varphi}(E) \leq \min_{A\subseteq\mathbb{N}} \sup_{x\in E} \mathrm{Dim}^{\varphi,A}(x).$$

Proof of Theorem 4.2.

1. Assume that $\varphi_t(2\delta) = O(\varphi_s(\delta))$ and $\varphi_s(\delta) = O(1/\log\log(1/\delta))$ hold for all $s < t$. It suffices to show that there exists $A \subseteq \mathbb{N}$ such that, for all $x \in E$,

$$\mathrm{Dim}^{\varphi,A}(x) \leq \dim_{\mathrm{P}}^{\varphi}(E). \tag{4.1}$$

Let $s > t > u > \dim_{\mathrm{P}}^{\varphi}(E)$. Since $u > \dim_{\mathrm{P}}^{\varphi}(E)$, Lemma 2.3 and our hypothesis on φ tell us that there is a cover $\{E_i\}_{i\in\mathbb{Z}^+}$ of E such that, for all $i \in \mathbb{Z}^+$,

$$\overline{\dim}_{\mathcal{M}}^{\varphi}(E_i) \leq u. \tag{4.2}$$

For each $i \in \mathbb{Z}^+$ and $\delta \in \mathbb{Q} \cap (0,1)$, let $F(i,\delta) \subseteq D$ satisfy

$$|F(i,\delta)| = \hat{N}(E_i,\delta)$$

and

$$E_i \subseteq \bigcup_{d \in F(i,\delta)} B_\delta(d).$$

Define $h : \mathbb{Z}^+ \times \mathbb{Q} \cap (0,1) \to (\{0,1\}^*)^*$ by

$$h(i,\delta) = \big(w_{i,\delta,1}, \ldots, w_{i,\delta,\hat{N}(E_i,\delta)}\big),$$

where, recalling that f is the function mapping bit strings onto the dense set D,

$$F(i,\delta) = \left\{ f\big(w_{i,\delta,1}\big), \ldots, f\big(w_{i,\delta,\hat{N}(E_i,\delta)}\big) \right\}.$$

Let A be an oracle encoding h.
To prove (4.1), let $x \in E$. It suffices to show that

$$\lim_{\delta \to 0^+} 2^{\mathrm{C}^A_\delta(x)} \varphi_s(\delta) = 0.$$

For this, let $\varepsilon > 0$. It suffices to show that, for all sufficiently small $\delta \in \mathbb{Q}^+$,

$$\mathrm{C}^A_\delta(x) < \log \frac{\varepsilon}{\varphi_s(\delta)}. \tag{4.3}$$

For each $\delta \in \mathbb{Q} \cap (0,1)$, let $r(\delta) = \lceil \log \frac{1}{\delta} \rceil$ and $\delta' = 2^{-r(\delta)}$, so that $\frac{\delta}{2} < \delta' \leq \delta$. Since $s > t$, our hypothesis on φ tells us that there is a constant $a > 0$ such that, for all sufficiently small $\delta \in \mathbb{Q}^+$,

$$\frac{1}{\varphi_t(\delta')} \leq \frac{a}{\varphi_s(2\delta')} \leq \frac{a}{\varphi_s(\delta)}. \tag{4.4}$$

Since $t > u$, (4.2) tells us that, for all $i \in \mathbb{N}$,

$$\lim_{\delta \to 0^+} \hat{N}(E_i,\delta) \varphi_t(\delta) = 0.$$

Hence, for all $i \in \mathbb{N}$ and all sufficiently small $\delta \in \mathbb{Q}^+$,

$$\hat{N}(E_i,\delta)\varphi_t(\delta) < \frac{\varepsilon}{2a}. \tag{4.5}$$

In particular, then, (4.4) and (4.5) tell us that, for all sufficiently small $\delta \in \mathbb{Q}^+$,

$$\hat{N}(E_i,\delta') \leq \frac{\varepsilon}{2a\varphi_t(\delta')} \leq \frac{\varepsilon}{2\varphi_s(\delta)}. \tag{4.6}$$

For each $i, k \in \mathbb{Z}^+$ and $\delta \in \mathbb{Q} \cap (0,1)$, let $\pi \in \{0,1\}^*$ be a string that encodes i, $r(\delta)$, and k, with

$$|\pi| = \log k + O(\log i + \log r(\delta)).$$

Let M be an oracle Turing machine that, with oracle A and program π, outputs the string $w_{i,\delta',k}$ that is the k^{th} component of $h(i,\delta')$ (if there is one), where $\delta' = 2^{-r(\delta)}$. Let c_M be an optimality constant for M.
To see that (4.3) holds, choose $i \in \mathbb{Z}^+$ such that $x \in E_i$, and let $\delta \in \mathbb{Q} \cap (0,1)$. Let $\delta' = 2^{-r(\delta)}$, and choose $k \in \big\{1, \ldots, \hat{N}(E_i,\delta')\big\}$ such that $x \in B_{\delta'}\left(f\big(w_{i,\delta',k}\big)\right)$. Then

$$f\big(w_{i,\delta',k}\big) \in D \cap B_{\delta'}(x) \subseteq D \cap B_\delta(x),$$

so (4.6) gives, for all sufficiently small $\delta \in \mathbb{Q}^+$,

$$\begin{aligned}
\mathrm{C}^A_\delta(x) &\leq \mathrm{C}^A\big(w_{i,\delta',k}\big)\\
&\leq \mathrm{C}^A_M\big(w_{i,\delta',k}\big) + c_M\\
&\leq |\pi| + c_M\\
&\leq \log k + c_M + O(\log i + \log r(\delta))\\
&\leq \log \hat{N}(E_i,\delta') + O(\log i + \log r(\delta))\\
&\leq \log \frac{\varepsilon}{2\varphi_s(\delta)} + O(\log i + \log r(\delta)).
\end{aligned}$$

Since i is a constant and, by our assumption, $\log r(\delta) \leq \log(\log(1/\delta)+1) = O(1/\varphi_t(\delta)) = o(1/\varphi_s(\delta))$, the second term vanishes as $\delta \to 0^+$, affirming (4.3).

2. Let $s > t > \sup_{x\in E} \mathrm{Dim}^{\varphi,A}(x)$. Then for each $x \in E$ and all sufficiently small $\delta \in \mathbb{Q}^+$, $\mathrm{C}^A_\delta(x) < \log(1/\varphi_t(\delta))$. For all $\delta \in \mathbb{Q}^+$, let

$$\mathcal{U}_\delta = \left\{B_\delta(f(w)) \;\middle|\; \mathrm{C}^A(w) \leq \log(1/\varphi_t(\delta))\right\},$$

and for each $i \in \mathbb{N}$, let

$$E_i = \{x \mid \forall \delta < 1/i,\ x \in \mathcal{U}_\delta\}.$$

Then $E \subseteq \bigcup_{i\in\mathbb{N}} E_i$. For each $\delta < 1/i$, $N(E_i,\delta) < 2/\varphi_t(\delta)$, so $N(E_i,\delta)\varphi_s(\delta) = o(1)$, and therefore $\overline{\dim}^\varphi_{\mathcal{M}}(E_i) \leq s$. Assuming that there is a precision family for φ, the result follows by Lemma 2.3. ◀

5 Hyperspace Dimension Theorems

This section presents our main theorems.

As before, let $X = (X,\rho)$ be a separable metric space. The *hyperspace* of X is the metric space $\mathcal{K}(X) = (\mathcal{K}(X),\rho_{\mathrm{H}})$, where $\mathcal{K}(X)$ is the set of all nonempty compact subsets of X and ρ_{H} is the *Hausdorff metric* [9] on $\mathcal{K}(X)$ defined by

$$\rho_{\mathrm{H}}(E,F) = \max\left\{\sup_{x\in E}\rho(x,F), \sup_{y\in F}\rho(E,y)\right\},$$

where $\rho(x,F) = \inf_{y\in F}\rho(x,y)$ and $\rho(E,y) = \inf_{x\in E}\rho(x,y)$.

Let $f : \{0,1\}^* \to X$ and $D = \mathrm{range}(f)$ be fixed as at the beginning of section 3, so that D is dense in X. Let $\mathcal{D}$ be the set of all nonempty, finite subsets of D. It is well known and easy to show that $\mathcal{D}$ is a countable dense subset of $\mathcal{K}(X)$, and it is routine to define from f a function $\widetilde{f} : \{0,1\}^* \to \mathcal{K}(X)$ such that $\mathrm{range}(\widetilde{f}) = \mathcal{D}$. Thus $\mathcal{K}(X)$ is a separable metric space, and the results in section 4 hold for $\mathcal{K}(X)$.

It is important to note the distinction between the classical Hausdorff and packing dimensions $\dim_{\mathrm{H}}(E)$ and $\dim_{\mathrm{P}}(E)$ of a nonempty compact subset E of X and the algorithmic dimensions $\dim(E)$ and $\mathrm{Dim}(E)$ of this same set when it is regarded as a point in $\mathcal{K}(X)$.

Our first hyperspace dimension theorem applies to lower and upper Minkowski dimensions. This theorem, which is proven using a counting argument, is very general, placing no restrictions on the gauge family φ or the separable metric space X.

▶ **Theorem 5.1** (hyperspace Minkowski dimension theorem)**.** *For every gauge family φ and every $E \subseteq X$,*

$$\underline{\dim}^{\widetilde{\varphi}}_{\mathcal{M}}(\mathcal{K}(E)) = \underline{\dim}^{\varphi}_{\mathcal{M}}(E) \quad \textit{and} \quad \overline{\dim}^{\widetilde{\varphi}}_{\mathcal{M}}(\mathcal{K}(E)) = \overline{\dim}^{\varphi}_{\mathcal{M}}(E).$$

Our third main result is the surprising fact that in a hyperspace, packing dimension and upper Minkowski dimension are equivalent for compact sets.

▶ **Theorem 5.2.** *For every separable metric space X, every compact set $E \subseteq X$, and every gauge family φ such that $\varphi_t(2\delta) = O(\varphi_s(\delta))$ and $\varphi_s(\delta) = O(1/\log\log(1/\delta))$ as $\delta \to 0^+$ for all $s < t$ and there is a precision family for φ,*

$$\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E)) = \overline{\dim}_{\mathcal{M}}^{\widetilde{\varphi}}(\mathcal{K}(E)).$$

The point-to-set principle is central to our proof of this theorem: We recursively construct a single compact set $L \subseteq E$ (i.e., a single point in the hyperspace $\mathcal{K}(E)$) that has high Kolmogorov complexity at infinitely many precisions, relative to an appropriate oracle A. We then invoke Theorem 3.7 to show that this L has high φ-gauged strong algorithmic dimension relative to A. By the point-to-set principle, then, $\mathcal{K}(E)$ has high packing dimension.

▶ **Observation 5.3.** *The conclusion of Theorem 5.2 does not hold for arbitrary sets E.*

Proof. Let $E = \{1/n : n \in \mathbb{N}\}$. Then $\overline{\dim}_{\mathcal{M}}^{\theta}(E) = 1/2$, but every compact subset of E is finite, so $\mathcal{K}(E)$ is countable and $\dim_{\mathrm{P}}^{\widetilde{\theta}}(\mathcal{K}(E)) = 0$. ◀

▶ **Theorem 5.4** (hyperspace packing dimension theorem)**.** *If X is a separable metric space, $E \subseteq X$ is an analytic set, and φ is a gauge family such that $\varphi_s(2\delta) = O(\varphi_s(\delta))$ and $\varphi_s(\delta) = O(1/\log\log(1/\delta))$ as $\delta \to 0^+$ for all $s \in (0,\infty)$ and there is a precision family for φ, then*

$$\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E)) \geq \dim_{\mathrm{P}}^{\varphi}(E).$$

Proof. For compact sets E, Theorem 5.2 and the hyperspace Minkowski dimension theorem (Theorem 5.1) imply $\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E)) = \overline{\dim}_{\mathcal{M}}^{\varphi}(E)$.

A result of Joyce and Preiss (Corollary 1 in [13]) states that every analytic set with positive (possibly infinite) gauged packing measure contains a compact subset with positive (finite) packing measure in the same gauge. It follows that if E is analytic, then for all $\varepsilon > 0$ there exists a compact subset $E_\varepsilon \subseteq E$ with $\dim_{\mathrm{P}}^{\varphi}(E_\varepsilon) \geq \dim_{\mathrm{P}}^{\varphi}(E) - \varepsilon$. Therefore

$$\begin{aligned}\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E_\varepsilon)) &= \overline{\dim}_{\mathcal{M}}^{\varphi}(E_\varepsilon)\\ &\geq \dim_{\mathrm{P}}^{\varphi}(E_\varepsilon)\\ &\geq \dim_{\mathrm{P}}^{\varphi}(E) - \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ completes the proof. ◀

6 Conclusion

Our results exhibit and amplify the power of the theory of computing to make unexpected contributions to other areas of the mathematical sciences. We hope and expect to see more such results in the near future.

We mention three open problems whose solutions may contribute to such progress. First, at the time of this writing, a hyperspace Hausdorff dimension theorem remains an open problem. The difficulty in adapting our approach to that problem is that in the proof of Theorem 5.2, the set L we construct is only guaranteed to have high complexity at infinitely many precisions. An analogous proof for Hausdorff dimension would require constructing a set L that has high complexity at all but finitely many precisions.

Second, it would be useful to identify classes of spaces in which Billingsley-type algorithmic dimensions – dimensions shaped by probability measures – can be formulated.

Finally, we do not at this time know how to characterize algorithmic dimensions in separable metric spaces in terms of martingales or more general gales. This is despite the fact that algorithmic dimensions were first formulated in these terms in sequence spaces [20, 1].

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