Optimal Oracles for Point-To-Set Principles

D. M. Stull ✉
Northwestern University, Evanston, IL, USA

Abstract

The point-to-set principle [14] characterizes the Hausdorff dimension of a subset $E \subseteq \mathbb{R}^n$ by the *effective* (or algorithmic) dimension of its individual points. This characterization has been used to prove several results in classical, i.e., without any computability requirements, analysis. Recent work has shown that algorithmic techniques can be fruitfully applied to Marstrand's projection theorem, a fundamental result in fractal geometry.

In this paper, we introduce an extension of point-to-set principle - the notion of *optimal oracles* for subsets $E \subseteq \mathbb{R}^n$. One of the primary motivations of this definition is that, if E has optimal oracles, then the conclusion of Marstrand's projection theorem holds for E. We show that every analytic set has optimal oracles. We also prove that if the Hausdorff and packing dimensions of E agree, then E has optimal oracles. Moreover, we show that the existence of sufficiently nice outer measures on E implies the existence of optimal Hausdorff oracles. In particular, the existence of exact gauge functions for a set E is sufficient for the existence of optimal Hausdorff oracles, and is therefore sufficient for Marstrand's theorem. Thus, the existence of optimal oracles extends the currently known sufficient conditions for Marstrand's theorem to hold.

Under certain assumptions, every set has optimal oracles. However, assuming the axiom of choice and the continuum hypothesis, we construct sets which do not have optimal oracles. This construction naturally leads to a generalization of Davies' theorem on projections.

2012 ACM Subject Classification Theory of computation

Keywords and phrases Algorithmic randomness, Kolmogorov complexity, geometric measure theory

Digital Object Identifier 10.4230/LIPIcs.STACS.2022.57

Acknowledgements I would like to thank Denis Hirschfeldt, Jack Lutz and Chris Porter for very valuable discussions and suggestions. I would also like to thank the participants of the recent AIM workshop on Algorithmic Randomness.

1 Introduction

Effective, i.e., algorithmic, dimensions were introduced [12, 1] to study the randomness of points in Euclidean space. The effective dimension, $\dim(x)$ and effective strong dimension, $\text{Dim}(x)$, are real values which measure the asymptotic density of information of an *individual point* x. The connection between effective dimensions and the classical Hausdorff and packing dimension is given by the point-to-set principle of J. Lutz and N. Lutz [14]: For any $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x), \text{ and} \tag{1}$$

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \text{Dim}^A(x)\,. \tag{2}$$

Call an oracle A satisfying (1) a *Hausdorff oracle* for E. Similarly, we call an oracle A satisfying (2) a *packing oracle* for E. Thus, the point-to-set principle shows that the classical notion of Hausdorff or packing dimension is completely characterized by the effective dimension of its individual points, relative to a Hausdorff or packing oracle, respectively.

Recent work as shown that algorithmic dimensions are not only useful in effective settings, but, via the point-to-set principle, can be used to solve problems in geometric measure theory [15, 17, 18, 19, 30]. It is important to note that the point-to-set principle allows one to use algorithmic techniques to prove theorems whose statements have seemingly nothing to do with computability theory. In this paper, we focus on the connection between algorithmic dimension and Marstrand's projection theorem.

Marstrand, in his landmark paper [21], was the first to study how the dimension of a set is changed when projected onto a line. He showed that, for any *analytic* set $E \in \mathbb{R}^2$, for almost every angle $\theta \in [0, \pi)$,

$$\dim_H(p_\theta\, E) = \min\{\dim_H(E), 1\}, \tag{3}$$

where $p_\theta(x, y) = x\cos\theta + y\sin\theta$[1]. The study of projections has since become a central theme in fractal geometry (see [8] or [25] for a more detailed survey of this development).

Marstrand's theorem begs the question of whether the analytic requirement on E can be dropped. It is known that, without further conditions, it cannot. Davies [5] showed that, assuming the axiom of choice and the continuum hypothesis, there are non-analytic sets for which Marstrands conclusion fails. However, the problem of classifying the sets for which Marstrands theorem does hold is still open. Recently, Lutz and Stull [20] used the point-to-set principle to prove that the projection theorem holds for sets for which the Hausdorff and packing dimensions agree[2]. This expanded the reach of Marstrand's theorem, as this assumption is incomparable with analyticity.

In this paper, we give the broadest known sufficient condition (which makes essential use of computability theory) for Marstrand's theorem. In particular, we introduce the notion of *optimal Hausdorff oracles* for a set $E \subseteq \mathbb{R}^n$. We prove that Marstrand's theorem holds for every set E which has optimal Hausdorff oracles.

An optimal Hausdorff oracle for a set E is a Hausdorff oracle which minimizes the algorithmic complexity of "most"[3] points in E. It is not immediately clear that any set E has optimal oracles. Nevertheless, we show that two natural classes of sets $E \subseteq \mathbb{R}^n$ do have optimal oracles.

We show that every analytic, and therefore Borel, set has optimal oracles. We also prove that every set whose Hausdorff and packing dimensions agree has optimal Hausdorff oracles. Thus, we show that the existence of optimal oracles encapsulates the known conditions sufficient for Marstrand's theorem to hold. Moreover, we show that the existence of sufficiently nice outer measures on E implies the existence of optimal Hausdorff oracles. In particular, the existence of exact gauge functions (Section 2.1) for a set E is sufficient for the existence of optimal Hausdorff oracles for E, and is therefore sufficient for Marstrand's theorem. Thus, the existence of optimal Hausdorff oracles is weaker than the previously known conditions for Marstrand's theorem to hold.

We also show that the notion of optimal oracles gives insight to sets for which Marstrand's theorem does *not* hold. Assuming the axiom of choice and the continuum hypothesis, we construct sets which do not have optimal oracles. This construction, with minor adjustments, proves a generalization of Davies' theorem proving the existence of sets for which (3) does not hold. In addition, the inherently algorithmic aspect of the construction might be useful for proving set-theoretic properties of exceptional sets for Marstrand's theorem.

---

[1] This result was later generalized to $\mathbb{R}^n$, for arbitrary n, as well as extended to hyperspaces of dimension m, for any $1 \leq m \leq n$ (see e.g. [22, 23, 24]).
[2] Orponen [29] has recently given another proof of Lutz and Stull's result using more classical tools.
[3] By most, we mean a subset of E of the same Hausdorff dimension as E

Finally, we define optimal *packing* oracles for a set. We show that every analytic set E has optimal packing oracles. We also show that every E whose Hausdorff and packing dimensions agree have optimal packing oracles. Assuming the axiom of choice and the continuum hypothesis, we show that there are sets with optimal packing oracles without optimal Hausdorff oracles (and vice-versa).

The structure of the paper is as follows. In Section 2.1 we review the concepts of measure theory needed, and the (classical) definition of Hausdorff dimension. In Section 2.2 we review algorithmic information theory, including the formal definitions of effective dimensions. We then introduce and study the notion of optimal oracles in Section 3. In particular, we give a general condition for the existence of optimal oracles in Section 3.1. We use this condition to prove that analytic sets have optimal oracles in Section 3.2. We conclude in Section 3.3 with an example, assuming the axiom of choice and the continuum hypothesis, of a set without optimal oracles. The connection between Marstrands projection theorem and optimal oracles is explored in Section 4. In this section, we prove that Marstrands theorem holds for every set with optimal oracles. In Section 4.1, we use the construction of a set without optimal oracles to give a new, algorithmic, proof of Davies' theorem. Finally, in Sectino 5, we define and investigate the notion of optimal packing oracles.

2 Preliminaries

2.1 Outer Measures and Classical Dimension

A set function $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ is called an *outer measure* on $\mathbb{R}^n$ if

1. $\mu(\emptyset) = 0$,
2. if $A \subseteq B$ then $\mu(A) \leq \mu(B)$, and
3. for any sequence $A_1, A_2, \ldots$ of subsets,

$$\mu(\bigcup_i A_i) \leq \sum_i \mu(A_i).$$

If μ is an outer measure, we say that a subset A is μ-*measurable* if

$$\mu(A \cap B) + \mu(B - A) = \mu(B),$$

for every subset $B \subseteq \mathbb{R}^n$.

An outer measure μ is called a *metric outer measure* if every Borel subset is μ-measurable and

$$\mu(A \cup B) = \mu(A) + \mu(B),$$

for every pair of subsets A, B which have positive Hausdorff distance. That is,

$$\inf\{\|x - y\| \mid x \in A, y \in B\} > 0.$$

An important example of a metric outer measure is the s-dimensional Hausdorff measure. For every $E \subseteq [0, 1)^n$, define the s-dimensional Hausdorff content at precision r by

$$h^s_r(E) = \inf\left\{\sum_i d(Q_i)^s \mid \bigcup_i Q_i \text{ covers } E \text{ and } d(Q_i) \leq 2^{-r}\right\},$$

where $d(Q)$ is the diameter of ball Q. We define the s-dimensional Hausdorff measure of E by

$$\mathcal{H}^s(E) = \lim_{r\to\infty} h^s_r(E).$$

▶ Remark 1. It is well-known that $\mathcal{H}^s$ is a metric outer measure for every s.

The *Hausdorff dimension* of a set E is then defined by

$$\dim_H(E) = \inf_s\{\mathcal{H}^s(E) = \infty\} = \sup_s\{\mathcal{H}^s(E) = 0\}.$$

Another important metric outer measure, which gives rise to the packing dimension of a set, is the s-dimensional packing measure. For every $E \subseteq [0,1)^n$, define the s-dimensional packing pre-measure by

$$p^s(E) = \limsup_{\delta\to 0}\left\{\sum_{i\in\mathbb{N}} d(B_i)^s \mid \{B_i\} \text{ is a set of disjoint balls and } B_i \in C(E,\delta)\right\},$$

where $C(E,\delta)$ is the set of all closed balls with diameter at most δ with centers in E. We define the s-dimensional packing measure of E by

$$\mathcal{P}^s(E) = inf\left\{\sum_j p^s(E_j) \mid E \subseteq \bigcup E_j\right\},$$

where the infimum is taken over all countable covers of E. For every s, the s-dimensional packing measure is a metric outer measure.

The *packing dimension* of a set E is then defined by

$$\dim_P(E) = \inf_s\{\mathcal{P}^s(E) = 0\} = \sup_s\{\mathcal{P}^s(E) = \infty\}.$$

In order to prove that every analytic set has optimal oracles, we will make use of the following facts of geometric measure theory (see, e.g., [7], [2]).

▶ **Theorem 1.** *The following are true.*

1. *Suppose $E \subseteq \mathbb{R}^n$ is compact and satisfies $\mathcal{H}^s(E) > 0$. Then there is a compact subset $F \subseteq E$ such that $0 < \mathcal{H}^s(F) < \infty$.*
2. *Every analytic set $E \subseteq \mathbb{R}^n$ has a Σ^0_2 subset $F \subseteq E$ such that $\dim_H(F) = \dim_H(E)$.*
3. *Suppose $E \subseteq \mathbb{R}^n$ is compact and satisfies $\mathcal{P}^s(E) > 0$. Then there is a compact subset $F \subseteq E$ such that $0 < \mathcal{P}^s(F) < \infty$.*
4. *Every analytic set $E \subseteq \mathbb{R}^n$ has a Σ^0_2 subset $F \subseteq E$ such that $\dim_P(F) = \dim_P(E)$.*

It is possible to generalize the definition of Hausdorff measure using gauge functions. A function $\phi : [0,\infty) \to [0,\infty)$ is a *gauge function* if ϕ is monotonically increasing, strictly increasing for $t > 0$ and continuous. If ϕ is a gauge, define the ϕ-Hausdorff content at precision r by

$$h^\phi_r(E) = \inf\left\{\sum_i \phi(d(Q_i)) \mid \bigcup_i Q_i \text{ covers } E \text{ and } d(Q_i) \leq 2^{-r}\right\},$$

where $d(Q)$ is the diameter of ball Q. We define the ϕ-Hausdorff measure of E by

$$\mathcal{H}^\phi(E) = \lim_{r\to\infty} h^\phi_r(E).$$

Thus we recover the s-dimensional Hausdorff measure when $\phi(t) = t^s$.

Gauged Hausdorff measures give fine-grained information about the size of a set. There are sets E which Hausdorff dimension s, but $\mathcal{H}^s(E) = 0$ or $\mathcal{H}^s(E) = \infty$. However, it is sometimes possible to find an appropriate gauge so that $0 < \mathcal{H}^\phi(E) < \infty$. When $0 < \mathcal{H}^\phi(E) < \infty$, we say that ϕ is an *exact gauge for E*.

▶ Example. For almost every Brownian path X in $\mathbb{R}^2$, $\mathcal{H}^2(X) = 0$, but $0 < \mathcal{H}^\phi(X) < \infty$, where $\phi(t) = t^2 \log \frac{1}{t} \log\log \frac{1}{t}$.

For two outer measures μ and ν, μ is said to be *absolutely continuous with respect to* ν, denoted $\mu \ll \nu$, if $\mu(A) = 0$ for every set A for which $\nu(A) = 0$.

▶ Example. For every s, let $\phi_s(t) = t^s \log \frac{1}{t}$. Then $\mathcal{H}^s \ll \mathcal{H}^{\phi_s}$.

▶ Example. For every s, let $\phi_s(t) = \frac{t^s}{\log \frac{1}{t}}$. Then $\mathcal{H}^{\phi_s} \ll \mathcal{H}^s$.

2.2 Algorithmic Information Theory

The *conditional Kolmogorov complexity* of a binary string $\sigma \in \{0,1\}^*$ given binary string $\tau \in \{0,1\}^*$ is

$$K(\sigma|\tau) = \min_{\pi \in \{0,1\}^*} \{\ell(\pi) : U(\pi, \tau) = \sigma\} ,$$

where U is a fixed universal prefix-free Turing machine and $\ell(\pi)$ is the length of π. The *Kolmogorov complexity* of σ is $K(\sigma) = K(\sigma|\lambda)$, where λ is the empty string. An important fact is that the choice of universal machine affects the Kolmogorov complexity by at most an additive constant (which, especially for our purposes, can be safely ignored). See [11, 28, 6] for a more comprehensive overview of Kolmogorov complexity.

We can naturally extend these definitions to Euclidean spaces by introducing "precision" parameters [16, 14]. Let $x \in \mathbb{R}^m$, and $r, s \in \mathbb{N}$. The *Kolmogorov complexity of x at precision r* is

$$K_r(x) = \min \{K(p) \,:\, p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m\} .$$

The *conditional Kolmogorov complexity of x at precision r given $q \in \mathbb{Q}^m$* is

$$\hat{K}_r(x|q) = \min \{K(p \,|\, q) \,:\, p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m\} .$$

The *conditional Kolmogorov complexity of x at precision r given $y \in \mathbb{R}^n$ at precision s* is

$$K_{r,s}(x|y) = \max \left\{\hat{K}_r(x|q) \,:\, q \in B_{2^{-s}}(y) \cap \mathbb{Q}^n\right\} .$$

We typically abbreviate $K_{r,r}(x|y)$ by $K_r(x|y)$.

The *effective Hausdorff dimension* and *effective packing dimension*[4] of a point $x \in \mathbb{R}^n$ are

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r} \quad \text{and} \quad \operatorname{Dim}(x) = \limsup_{r\to\infty} \frac{K_r(x)}{r} .$$

By letting the underlying fixed prefix-free Turing machine U be a universal *oracle* machine, we may *relativize* the definition in this section to an arbitrary oracle set $A \subseteq \mathbb{N}$. The definitions of $K^A_r(x)$, $\dim^A(x)$, $\operatorname{Dim}^A(x)$, etc. are then all identical to their unrelativized versions, except that U is given oracle access to A. Note that taking oracles as subsets of the naturals is quite general. We can, and frequently do, encode a point y into an oracle, and consider the complexity of a point *relative* to y. In these cases, we typically forgo explicitly referring to this encoding, and write e.g. $K^y_r(x)$. We can also *join* two oracles $A, B \subseteq \mathbb{N}$ using any computable bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. We denote the join of A and B by (A, B). We can generalize this procedure to join any countable sequence of oracles.

[4] Although effective Hausdorff was originally defined by J. Lutz [13] using martingales, it was later shown by Mayordomo [26] that the definition used here is equivalent. For more details on the history of connections between Hausdorff dimension and Kolmogorov complexity, see [6, 27].

As mentioned in the introduction, the connection between effective dimensions and the classical Hausdorff and packing dimensions is given by the point-to-set principle introduced by J. Lutz and N. Lutz [14].

▶ **Theorem 2** (Point-to-set principle). *Let $n \in \mathbb{N}$ and $E \subseteq \mathbb{R}^n$. Then*

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x), \text{ and}$$
$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^A(x).$$

An oracle testifying to the the first equality is called a *Hausdorff oracle for E*. Similarly, an oracle testifying to the the second equality is called a *packing oracle for E*.

3 Optimal Hausdorff Oracles

For any set E, there are infinitely many Hausdorff oracles for E. A natural question is whether there is a Hausdorff oracle which minimizes the complexity of every point in E. Unfortunately, it is, in general, not possible for a single oracle to maximally reduce *every* point. We introduce the notion of optimal Hausdorff oracles by weakening the condition to a *single* point.

▶ **Definition 3.** *Let $E \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{N}$. We say that A is Hausdorff optimal for E if the following conditions are satisfied.*

1. *A is a Hausdorff oracle for E.*
2. *For every $B \subseteq \mathbb{N}$ and every $\epsilon > 0$ there is a point $x \in E$ such that $\dim^{A,B}(x) \geq \dim_H(E) - \epsilon$ and for almost every $r \in \mathbb{N}$*

$$K_r^{A,B}(x) \geq K_r^A(x) - \epsilon r.$$

Note that the second condition only guarantees the existence of *one* point whose complexity is unaffected by the addtional information in B. However, we can show that this implies the seemingly stronger condition that "most" points are unaffected. For $B \subseteq \mathbb{N}$, $\epsilon > 0$ define the set

$$N(A, B, \epsilon) = \{x \in E \mid (\forall^\infty r)\, K_r^{A,B}(x) \geq K_r^A(x) - \epsilon r\}.$$

▶ **Proposition 4.** *Let $E \subseteq \mathbb{R}^n$ be a set such that $\dim_H(E) > 0$ and let A be an oracle. Then A is a Hausdorff optimal oracle for E if and only if A is a Hausdorff oracle and $\dim_H(N(A, B, \epsilon)) = \dim_H(E)$ for every $B \subseteq \mathbb{N}$ and $\epsilon > 0$.*

A simple, but useful, result is if B is an oracle obtained by adding additional information to an optimal Hausdorff oracle, then B is also optimal.

▶ **Lemma 5.** *Let $E \subseteq \mathbb{R}^n$. If A is an optimal Hausdorff oracle for E, then the join $C = (A, B)$ is Hausdorff optimal for E for every oracle B.*

We now give some basic closure properties of the class of sets with optimal Hausdorff oracles.

▶ **Observation 6.** *Let $F \subseteq E$. If $\dim_H(F) = \dim_H(E)$ and F has an optimal Hausdorff oracle, then E has an optimal Hausdorff oracle.*

We can also show that having optimal Hausdorff oracles is closed under countable unions.

▶ **Proposition 7.** *Let $E_1, E_2, \ldots$ be a countable sequence of sets and let $E = \cup_n E_n$. If every set E_n has an optimal Hausdorff oracle, then E has an optimal Hausdorff oracle.*

3.1 Outer Measures and Optimal Oracles

In this section we give a sufficient condition for a set to have optimal Hausdorff oracles. Specifically, we prove that if $\dim_H(E) = s$, and there is a metric outer measure, absolutely continuous with respect to $\mathcal{H}^s$, such that $0 < \mu(E) < \infty$, then E has optimal Hausdorff oracles. Although stated in this general form, the main application of this result (in Section 3.2) is for the case $\mu = \mathcal{H}^s$.

For every $r \in \mathbb{N}$, let $\mathcal{Q}^n_r$ be the set of all dyadic cubes at precision r, i.e., cubes of the form

$$Q = [m_1 2^{-r}, (m_1+1)2^{-r}) \times \ldots \times [m_n 2^{-r}, (m_n+1)2^{-r}),$$

where $0 \leq m_1, \ldots, m_n \leq 2^r$. For each r, we refer to the 2^{nr} cubes in $\mathcal{Q}_r$ as $Q_{r,1}, \ldots, Q_{r,2^{nr}}$. We can identify each dyadic cube $Q_{r,i}$ with the unique dyadic rational $d_{r,i}$ at the center of $Q_{r,i}$.

We now associate, to each metric outer measure, a *discrete semimeasure on the dyadic rationals* $\mathbb{D}$. Recall that discrete semimeasure on $\mathbb{D}^n$ is a function $p : \mathbb{D}^n \to [0,1]$ which satisfies $\Sigma_{r,i} p(d_{r,i}) < \infty$.

Let $E \subseteq \mathbb{R}^n$ and μ be a metric outer measure such that $0 < \mu(E) < \infty$. Define the function $p_\mu : \mathbb{D}^n \to [0,1]$ by

$$p_{\mu,E}(d_{r,i}) = \frac{\mu(E \cap Q_{r,i})}{r^2 \mu(E)}.$$

▶ **Observation 8.** *Let μ be a metric outer measure and $E \subseteq \mathbb{R}^n$ such that $0 < \mu(E) < \infty$. Then for every r, every dyadic cube $Q \in \mathcal{Q}_r$, and all $r' > r$,*

$$\mu(E \cap Q) = \sum_{\substack{Q' \subset Q \\ Q' \in \mathcal{Q}_{r'}}} \mu(E \cap Q').$$

▶ **Proposition 9.** *Let $E \subseteq \mathbb{R}^n$ and μ be a metric outer measure such that $0 < \mu(E) < \infty$. Relative to some oracle A, the function $p_{\mu,E}$ is a lower semi-computable discrete semimeasure.*

In order to connect the existence of such an outer measure μ to the existence of optimal oracles, we need to relate the semimeasure p_μ and Kolmogorov complexity. We achieve this using a fundamental result in algorithmic information theory.

Levin's optimal lower semicomputable subprobability measure, relative to an oracle A, on the dyadic rationals $\mathbb{D}$ is defined by

$$\mathbf{m}^A(d) = \sum_{\pi \,:\, U^A(\pi) = d} 2^{-|\pi|}.$$

▶ **Lemma 10.** *Let $E \subseteq \mathbb{R}^n$ and μ be a metric outer measure such that $0 < \mu(E) < \infty$. Let A be an oracle relative to which $p_{\mu,E}$ is lower semi-computable. Then is a constant $\alpha > 0$ such that $\mathbf{m}^A(d) \geq \alpha p_{\mu,E}(d)$, for every $d \in \mathbb{D}^n$.*

Proof. Case and Lutz [3], generalizing Levin's coding theorem [9, 10], showed that there is a constant c such that

$$\mathbf{m}^A(d_{r,i}) \leq 2^{-K^A(d_{r,i}) + K^A(r) + c},$$

for every $r \in \mathbb{N}$ and $d_{r,i} \in \mathbb{D}^n$. The optimality of $\mathbf{m}^A$ ensures that, for every lower semicomputable (relative to A) discrete semimeasure ν on $\mathbb{D}^n$,

$$\mathbf{m}^A(d_{r,i}) \geq \alpha \nu(d_{r,i}).$$ ◀

The results of this section have dealt with the dyadic rationals. However, we ultimately deal with the Kolmogorov complexity of Euclidean points. A result of Case and Lutz [3] relates the Kolmogorov complexity of Euclidean points with the complexity of dyadic rationals.

▶ **Lemma 11** ([3]). *Let $x \in [0,1)^n$, $A \subseteq \mathbb{N}$, and $r \in \mathbb{N}$. Let $Q_{r,i}$ be the (unique) dyadic cube at precision r containing x. Then*

$$K^A_r(x) = K^A(d_{r,i}) - O(\log r).$$

▶ **Lemma 12.** *Let $E \subseteq \mathbb{R}^n$ and μ be a metric outer measure such that $0 < \mu(E) < \infty$. Let A be an oracle relative to which $p_{\mu,E}$ is lower semi-computable. Then, for every oracle $B \subseteq \mathbb{N}$ and every $\epsilon > 0$, the set*

$$N = \{x \in E \mid (\exists^\infty)\ K^{A,B}_r(x) < K^A_r(x) - \epsilon r\}$$

has μ-measure zero.

We now have the machinery in place to prove the main theorem of this section.

▶ **Theorem 13.** *Let $E \subseteq \mathbb{R}^n$ with $\dim_H(E) = s$. Suppose there is a metric outer measure μ such that*

$$0 < \mu(E) < \infty,$$

and either
1. *$\mu \ll \mathcal{H}^{s-\delta}$, for every $\delta > 0$, or*
2. *$\mathcal{H}^s \ll \mu$ and $\mathcal{H}^s(E) > 0$.*

Then E has an optimal Hausdorff oracle A.

Proof. Let $A \subseteq \mathbb{N}$ be a Hausdorff oracle for E such that $p_{\mu,E}$ is computable relative to A. Note that such an oracle exists by the point-to-set principle and routine encoding. We will show that A is optimal for E.

For the sake of contradiction, suppose that there is an oracle B and $\epsilon > 0$ such that, for every $x \in E$ either
1. $\dim^{A,B}(x) < s - \epsilon$, or
2. there are infinitely many r such that $K^{A,B}_r(x) < K^A_r(x) - \epsilon r$.

Let N be the set of all x for which the second item holds. By Lemma 12, $\mu(N) = 0$. We also note that, by the point-to-set principle,

$$\dim_H(E - N) \leq s - \epsilon,$$

and so $\mathcal{H}^s(E - N) = 0$.

To achieve the desired contradiction, we first assume that $\mu \ll \mathcal{H}^{s-\delta}$, for every $\delta > 0$. Since $\mu \ll \mathcal{H}^{s-\delta}$, and $\dim_H(E - N) < s - \epsilon$,

$$\mu(E - N) = 0.$$

Since μ is a metric outer measure,

$$\begin{aligned} 0 &< \mu(E) \\ &\leq \mu(N) + \mu(E - N) \\ &= 0, \end{aligned}$$

a contradiction.

Now suppose that $\mathcal{H}^s \ll \mu$ and $\mathcal{H}^s(E) > 0$. Then, since $\mathcal{H}^s$ is an outer measure, $\mathcal{H}^s(E) > 0$ and $\mathcal{H}^s(E - N) = 0$ we must have $\mathcal{H}^s(N) > 0$. However this implies that $\mu(N) > 0$, and we again have the desired contradiction. Thus A is an optimal Hausdorff oracle for E and the proof is complete. ◀

Recall that $E \subseteq [0,1)^n$ is called an s-set if

$$0 < \mathcal{H}^s(E) < \infty.$$

Since $\mathcal{H}^s$ is a metric outer measure, and trivially absolutely continuous with respect to itself, we have the following corollary.

▶ **Corollary 14.** *Let $E \subseteq [0,1)^n$ be an s-set. Then there is an optimal Hausdorff oracle for E.*

3.2 Sets with optimal Hausdorff oracles

We now show that every analytic set has optimal Hausdorff oracles.

▶ **Lemma 15.** *Every analytic set E has optimal Hausdorff oracles.*

Proof. We begin by assuming that E is compact, and let $s = \dim_H(E)$. Then for every $t < s$, $\mathcal{H}^t(E) > 0$. Thus, by Theorem 1(1), there is a sequence of compact subsets $F_1, F_2, \ldots$ of E such that

$$\dim_H(\bigcup_n F_n) = \dim_H(E),$$

and, for each n,

$$0 < \mathcal{H}^{s_n}(F_n) < \infty,$$

where $s_n = s - 1/n$. Therefore, by Theorem 13, each set F_n has optimal Hausdorff oracles. Hence, by Proposition 7, E has optimal Hausdorff oracles and the conclusion follows.

We now show that every Σ^0_2 set has optimal Hausdorff oracles. Suppose $E = \cup_n F_n$ is Σ^0_1, where each F_n is compact. As we have just seen, each F_n has optimal Hausdorff oracles. Therefore, by Proposition 7, E has optimal Hausdorff oracles and the conclusion follows.

Finally, let E be analytic. By Theorem 1(2), there is a Σ^0_2 subset F of the same Hausdorff dimension as E. We have just seen that F must have an optimal Hausdorff oracle. Since $\dim_H(F) = \dim_H(E)$, by Observation 6 E has optimal Hausdorff oracles, and the proof is complete. ◀

Crone, Fishman and Jackson [4] have recently shown that, assuming the Axiom of Determinacy (AD)[5], *every* subset E has a Borel subset F such that $\dim_H(F) = \dim_H(E)$. This, combined with Lemma 15, yields the following corollary.

▶ **Corollary 16.** *Assuming AD, every set $E \subseteq \mathbb{R}^n$ has optimal Hausdorff oracles.*

▶ **Lemma 17.** *Suppose that $E \subseteq \mathbb{R}^n$ satisfies $\dim_H(E) = \dim_P(E)$. Then E has an optimal Hausdorff oracle. Moreover, the join (A, B) is an optimal Hausdorff oracle, where A and B are Hausdorff and packing oracles, respectively, of E.*

[5] Note that AD is inconsistent with the axiom of choice.

Proof. Let A be a Hausdorff oracle for E and let B be a packing oracle for E. We claim that that the join (A, B) is an optimal Hausdorff oracle for E. By the point-to-set principle, and the fact that extra information cannot increase effective dimension,

$$\begin{aligned}\dim_H(E) &= \sup_{x\in E} \dim^A(x)\\ &\geq \sup_{x\in E} \dim^{A,B}(x)\\ &\geq \dim_H(E).\end{aligned}$$

Therefore

$$\dim_H(E) = \sup_{x\in E} \dim^{A,B}(x),$$

and the first condition of optimal Hausdorff oracles is satisfied.

Let $C \subseteq \mathbb{N}$ be an oracle and $\epsilon > 0$. By the point-to-set principle,

$$\dim_H(E) \leq \sup_{x\in E} \dim^{A,B,C}(x),$$

so there is an $x \in E$ such that

$$\dim_H(E) - \epsilon/4 < \dim^{A,B,C}(x).$$

Let r be sufficiently large. Then, by our choice of B and the fact that additional information cannot increase the complexity of a point,

$$\begin{aligned}K_r^{A,B}(x) &\leq K_r^B(x)\\ &\leq \dim_P(E)r + \epsilon r/4\\ &= \dim_H(E)r + \epsilon r/4\\ &< \dim^{A,B,C}(x)r + \epsilon r/2\\ &\leq K_r^{A,B,C}(x) + \epsilon r.\end{aligned}$$

Since the oracle C and ϵ were arbitrarily, the proof is complete. ◀

3.3 Sets without optimal Hausdorff oracles

In the previous section, we gave general conditions for a set E to have optimal Hausdorff oracles. Indeed, we saw that under the axiom of determinacy, every set has optimal Hausdorff oracles.

However, assuming the axiom of choice (AC) and the continuum hypothesis (CH), we are able to construct sets without optimal Hausdorff oracles.

▶ **Lemma 18.** *Assume AC and CH. Then, for every $s \in (0, 1)$, there is a subset $E \subseteq \mathbb{R}$ with $\dim_H(E) = s$ such that E does not have optimal Hausdorff oracles.*

Let $s \in (0, 1)$. We begin by defining two sequences of natural numbers, $\{a_n\}$ and $\{b_n\}$. Let $a_1 = 2$, and $b_1 = \lfloor 2/s \rfloor$. Inductively define $a_{n+1} = b_n^2$ and $b_{n+1} = \lfloor a_{n+1}/s \rfloor$. Note that

$$\lim_n a_n/b_n = s.$$

Using AC and CH, we order the subsets of the natural numbers such that every subset has countably many predecessors. For every countable ordinal α, let $f_\alpha : \mathbb{N} \to \{\beta \mid \beta < \alpha\}$ be a function such that each ordinal β strictly less than α is mapped to by infinitely many n. Note that such a function exists, since the range is countable assuming CH.

We will define real numbers x_α, y_α via transfinite induction. Let x_1 be a real which is random relative to A_1. Let y_1 be the real whose binary expansion is given by

$$y_1[r] = \begin{cases} 0 & \text{if } a_n < r \leq b_n \text{ for some } n \in \mathbb{N} \\ x_1[r] & \text{otherwise} \end{cases}$$

For the induction step, suppose we have defined our points up to α. Let x_α be a real number which is random relative to the join of $\bigcup_{\beta<\alpha}(A_\beta, x_\beta)$ and A_α. This is possible, as we are assuming that this union is countable. Let y_α be the point whose binary expansion is given by

$$y_\alpha[r] = \begin{cases} x_\beta[r] & \text{if } a_n < r \leq b_n, \text{ where } f_\alpha(n) = \beta \\ x_\alpha[r] & \text{otherwise} \end{cases}$$

Finally, we define our set $E = \{y_\alpha\}$. This set E satisfies $\dim_H(E) = s$, however E does not have an optimal Hausdorff oracle.

3.3.1 Generalization to higher dimension

In this section, we use Lemma 18 to show that there are sets without optimal Hausdorff oracles in $\mathbb{R}^n$ of every possible dimension. We will need the following lemma on giving sufficient conditions for a product set to have optimal Hausdorff oracles. Interestingly, we need the product formula to hold for arbitrary sets, first proven by Lutz [17]. Under the assumption that F is regular, the product formula gives

$$\dim_H(F \times G) = \dim_H(F) + \dim_H(G) = \dim_P(F) + \dim_H(G),$$

for every set G.

▶ **Lemma 19.** *Let $F \subseteq \mathbb{R}^n$ be a set such that $\dim_H(F) = \dim_P(F)$, let $G \subseteq \mathbb{R}^m$ and let $E = F \times G$. Then E has optimal Hausdorff oracles if and only if G has optimal Hausdorff oracles.*

▶ **Theorem 20.** *Assume AC and CH. Then for every $n \in \mathbb{N}$ and $s \in (0, n)$, there is a subset $E \subseteq \mathbb{R}^n$ with $\dim_H(E) = s$ such that E does not have optimal Hausdorff oracles.*

4 Marstrand's Projection Theorem

The following theorem, due to Lutz and Stull [20], gives sufficient conditions for strong lower bounds on the complexity of projected points.

▶ **Theorem 21.** *Let $z \in \mathbb{R}^2$, $\theta \in [0, \pi]$, $C \subseteq \mathbb{N}$, $\eta \in \mathbb{Q} \cap (0,1) \cap (0, \dim(z))$, $\varepsilon > 0$, and $r \in \mathbb{N}$. Assume the following are satisfied.*
1. *For every $s \leq r$, $K_s(\theta) \geq s - \log(s)$.*
2. *$K_r^{C,\theta}(z) \geq K_r(z) - \varepsilon r$.*

Then,

$$K_r^{C,\theta}(p_\theta z) \geq \eta r - \varepsilon r - \frac{4\varepsilon}{1-\eta} r - O(\log r)\,.$$

The second condition of this theorem requires the oracle (C, θ) to give essentially no information about z. The existence of optimal Hausdorff oracles gives a sufficient condition for this to be true, for all sufficiently large precisions. Thus we are able to show that Marstrands projection theorem holds for any set with optimal Hausdorff oracles.

▶ **Theorem 22.** *Suppose $E \subseteq \mathbb{R}^2$ has an optimal Hausdorff oracle. Then for almost every $\theta \in [0, \pi]$,*

$$\dim_H(p_\theta E) = \min\{\dim_H(E), 1\}.$$

This shows that Marstrand's theorem holds for every set E with $\dim_H(E) = s$ satisfying any of the following:

1. E is analytic.
2. $\dim_H(E) = \dim_P(E)$.
3. $\mu \ll \mathcal{H}^{s-\delta}$, for every $\delta > 0$ for some metric outer measure μ such that $0 < \mu(E) < \infty$.
4. $\mathcal{H}^s \ll \mu$ and $\mathcal{H}^s(E) > 0$, for some metric outer measure μ such that $0 < \mu(E) < \infty$.

For example, the existence of exact gauged Hausdorff measures on E guarantees the existence of optimal Hausdorff oracles.

▶ Example. Let E be a set with $\dim_H(E) = s$ and $\mathcal{H}^s(E) = 0$. Suppose that $0 < \mathcal{H}^\phi(E) < \infty$, where $\phi(t) = \frac{t^s}{\log \frac{1}{t}}$. Since $\mathcal{H}^\phi \ll \mathcal{H}^{s-\delta}$ for every $\delta > 0$, Theorem 13 implies that E has optimal Hausdorff oracles, and thus Marstrand's theorem holds for E.

▶ Example. Let E be a set with $\dim_H(E) = s$ and $\mathcal{H}^s(E) = \infty$. Suppose that $0 < \mathcal{H}^\phi(E) < \infty$, where $\phi(t) = t^s \log \frac{1}{t}$. Since $\mathcal{H}^s \ll \mathcal{H}^\phi$, Theorem 13 implies that E has optimal Hausdorff oracles, and thus Marstrand's theorem holds for E.

4.1 Counterexample to Marstrand's theorem

In this section we show that there are sets for which Marstrand's theorem does not hold. While not explicitly mentioning optimal Hausdorff oracles, the construction is very similar to the construction in Section 3.3.

▶ **Theorem 23.** *Assuming AC and CH, for every $s \in (0, 1)$ there is a set E such that $\dim_H(E) = 1 + s$ but*

$$\dim_H(p_\theta E) = s$$

for every $\theta \in (\pi/4, 3\pi/4)$.

This is a modest generalization of Davies' theorem to sets with Hausdorff dimension strictly greater than one. In the next section we give a new proof of Davies' theorem by generalizing this construction to the endpoint $s = 0$.

We will need the following simple observation.

▶ **Observation 24.** *Let $r \in \mathbb{N}$, $s \in (0, 1)$, and $\theta \in (\pi/8, 3\pi/8)$. Then for every dyadic rectangle*

$$R = [d_x - 2^{-r}, d_x + 2^{-r}] \times [d_y - 2^{-sr}, d_y + 2^{-sr}],$$

there is a point $z \in R$ such that $K^\theta_r(p_\theta z) \leq sr + o(r)$.

For every $r \in \mathbb{N}$, $\theta \in (\pi/4, 3\pi/4)$, binary string x of length r and string y of length sr, let $g_\theta(x, y) \mapsto z$ be a function such that

$$K^\theta_r(p_\theta\,(x, z)) \leq sr + o(r).$$

That is, g_θ, given a rectangle

$$R = [d_x - 2^{-r}, d_x + 2^{-r}] \times [d_y - 2^{-sr}, d_y + 2^{-sr}],$$

outputs a value z such that $K_r(p_\theta(x, z))$ is small.

Let $s \in (0,1)$. We begin by defining two sequences of natural numbers, $\{a_n\}$ and $\{b_n\}$. Let $a_1 = 2$, and $b_1 = \lfloor 2/s \rfloor$. Inductively define $a_{n+1} = b_n^2$ and $b_{n+1} = \lfloor a_{n+1}/s \rfloor$. We will also need, for every ordinal α, a function $f_\alpha : \mathbb{N} \to \{\beta \mid \beta < \alpha\}$ such that each ordinal $\beta < \alpha$ is mapped to by infinitely many n. Note that such a function exists, since the range is countable assuming CH.

Using AC and CH, we first order the subsets of the natural numbers and we order the angles $\theta \in (\pi/4, 3\pi/4)$ so that each has at most countably many predecessors.

We will define real numbers x_α, y_α and z_α inductively. Let x_1 be a real which is random relative to A_1. Let y_1 be a real which is random relative to (A_1, x_1). Define z_1 to be the real whose binary expansion is given by

$$z_1[r] = \begin{cases} g_{\theta_1}(x_1, y_1)[r] & \text{if } a_n < r \leq b_n \text{ for some } n \in \mathbb{N} \\ y_1[r] & \text{otherwise} \end{cases}$$

For the induction step, suppose we have defined our points up to ordinal α. Let x_α be a real number which is random relative to the join of $\bigcup_{\beta<\alpha}(A_\beta, x_\beta)$ and A_α. Let y_α be random relative to the join of $\bigcup_{\beta<\alpha}(A_\beta, x_\beta)$, A_α and x_α. This is possible, as we are assuming CH, and so this union is countable. Let z_α be the point whose binary expansion is given by

$$z_\alpha[r] = \begin{cases} g_{\theta_\beta}(x_\alpha, y_\alpha)[r] & \text{if } a_n < r \leq b_n, \text{ for } f_\alpha(n) = \beta \\ y_\alpha[r] & \text{otherwise} \end{cases}$$

Finally, we define our set $E = \{(x_\alpha, z_\alpha)\}$.

4.2 Generalization to the endpoint

▶ **Theorem 25.** *Assuming AC and CH, there is a set E such that $\dim_H(E) = 1$ but*

$$\dim_H(p_\theta E) = 0$$

for every $\theta \in (\pi/4, 3\pi/4)$.

For every $r \in \mathbb{N}$, $\theta \in (\pi/4, 3\pi/4)$, binary string x of length r and string y of length sr, let $g_\theta^s(x, y) \mapsto z$ be a function such that

$$K_r^\theta(p_\theta\,(x, z)) \leq sr + o(r).$$

That is, g_θ^s, given a rectangle

$$R = [d_x - 2^{-r}, d_x + 2^{-r}] \times [d_y - 2^{-sr}, d_y + 2^{-sr}],$$

outputs a value z such that $K_r(p_\theta(x, z))$ is small.

We begin by defining two sequences of natural numbers, $\{a_n\}$ and $\{b_n\}$. Let $a_1 = 2$, and $b_1 = 4$. Inductively define $a_{n+1} = b_n^2$ and $b_{n+1} = (n+1)\lfloor a_{n+1} \rfloor$. We will also need, for every ordinal α, a function $f_\alpha : \mathbb{N} \to \{\beta \mid \beta < \alpha\}$ such that each ordinal $\beta < \alpha$ is mapped to by infinitely many n. Note that such a function exists, since the range is countable assuming CH.

Using AC and CH, we first order the subsets of the natural numbers and we order the angles $\theta \in (\pi/4, 3\pi/4)$ so that each has at most countably many predecessors.

We will define real numbers x_α, y_α and z_α inductively. Let x_1 be a real which is random relative to A_1. Let y_1 be a real which is random relative to (A_1, x_1). Define z_1 to be the real whose binary expansion is given by

$$z_1[r] = \begin{cases} g^1_{\theta_1}(x_1, y_1)[r] & \text{if } a_n < r \le b_n \text{ for some } n \in \mathbb{N} \\ y_1[r] & \text{otherwise} \end{cases}$$

For the induction step, suppose we have defined our points up to ordinal α. Let x_α be a real number which is random relative to the join of $\bigcup_{\beta<\alpha}(A_\beta, x_\beta)$ and A_α. Let y_α be random relative to the join of $\bigcup_{\beta<\alpha}(A_\beta, x_\beta)$, A_α and x_α. This is possible, as we are assuming CH, and so this union is countable. Let z_α be the point whose binary expansion is given by

$$z_\alpha[r] = \begin{cases} g^{1/n}_{\theta_\beta}(x_\alpha, y_\alpha)[r] & \text{if } a_n < r \le b_n, \text{ for } f_\alpha(n) = \beta \\ y_\alpha[r] & \text{otherwise} \end{cases}$$

Finally, we define our set $E = \{(x_\alpha, z_\alpha)\}$.

5 Optimal Packing Oracles

Similarly, we can define optimal *packing* oracles for a set.

▶ **Definition 26.** *Let $E \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{N}$. We say that A is an optimal packing oracle (or packing optimal) for E if the following conditions are satisfied.*

1. *A is a packing oracle for E.*
2. *For every $B \subseteq \mathbb{N}$ and every $\epsilon > 0$ there is a point $x \in E$ such that $\mathrm{Dim}^{A,B}(x) \ge \dim_P(E) - \epsilon$ and for almost every $r \in \mathbb{N}$*
$$K^{A,B}_r(x) \ge K^A_r(x) - \epsilon r.$$

Let $E \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{N}$. For $B \subseteq \mathbb{N}$, $\epsilon > 0$ define the set

$$N(A, B, \epsilon) = \{x \in E \mid (\forall^\infty r)\, K^{A,B}_r(x) \ge K^A_r(x) - \epsilon r\}.$$

▶ **Proposition 27.** *Let $E \subseteq \mathbb{R}^n$ be a set such that $\dim_P(E) > 0$ and let A be an oracle. Then A is packing optimal for E if and only if A is a packing oracle and for every $B \subseteq \mathbb{N}$ and $\epsilon > 0$, $\dim_P(N(A, B, \epsilon)) = \dim_P(E)$.*

▶ **Lemma 28.** *Let $E \subseteq \mathbb{R}^n$. If A is packing optimal for E, then the join $C = (A, B)$ is packing optimal for E for every oracle B.*

We now give some basic closure properties of the class of sets with optimal packing oracles.

▶ **Observation 29.** *Let $F \subseteq E$. If $\dim_P(F) = \dim_P(E)$ and F has an optimal packing oracle, then E has an optimal packing oracle.*

We can also show that having optimal packing oracles is closed under countable unions.

▶ **Lemma 30.** *Let $E_1, E_2, \ldots$ be a countable sequence of sets and let $E = \cup_n E_n$. If every set E_n has an optimal packing oracle, then E has an optimal packing oracle.*

We will need a specific set which has optimal Hausdorff and optimal packing oracles. For every $0 \le \alpha < \beta \le 1$ define the set

$$D_{\alpha,\beta} = \{x \in (0, 1) \mid \dim(x) = \alpha \text{ and } \mathrm{Dim}(x) = \beta\}.$$

▶ **Lemma 31.** *For every $0 \leq \alpha < \beta \leq 1$, $D_{\alpha,\beta}$ has optimal Hausdorff and optimal packing oracles and*

$$\dim_H(D_{\alpha,\beta}) = \alpha$$
$$\dim_P(D_{\alpha,\beta}) = \beta.$$

5.1 Sufficient conditions for optimal packing oracles

▶ **Lemma 32.** *Let $E \subseteq \mathbb{R}^n$ be a set such that $\dim_H(E) = \dim_P(E) = s$. Then E has optimal Hausdorff and optimal packing oracles.*

▶ **Theorem 33.** *Let $E \subseteq \mathbb{R}^n$ with $\dim_P(E) = s$. Suppose there is a metric outer measure μ such that*

$$0 < \mu(E) < \infty,$$

and either
1. *$\mu \ll \mathcal{P}^s$, or*
2. *$\mathcal{P}^s \ll \mu$ and $\mathcal{P}^s(E) > 0$.*

Then E has an optimal packing oracle A.

We now show that every analytic set has optimal packing oracles.

▶ **Lemma 34.** *Every analytic set E has optimal packing oracles.*

5.2 Sets without optimal oracles

In this section, we state results which show that, assuming CH and AC, there are sets without Hausdorff optimal and without packing optimal oracles of arbitrary dimension.

▶ **Theorem 35.** *Assuming CH and AC, for every $0 < s_1 < s_2 \leq 1$ there is a set $E \subseteq \mathbb{R}$ which does not have Hausdorff optimal nor packing optimal oracles such that*

$$\dim_H(E) = s_1 \text{ and } \dim_P(E) = s_2.$$

▶ **Corollary 36.** *Assuming CH and AC, for every $0 < s_1 < s_2 \leq 1$ there is a set $E \subseteq \mathbb{R}$ which has optimal Hausdorff oracles but does not have optimal packing oracles such that*

$$\dim_H(E) = s_1 \text{ and } \dim_P(E) = s_2.$$

▶ **Theorem 37.** *Assuming CH and AC, for every $0 < s_1 < s_2 \leq 1$ there is a set $E \subseteq \mathbb{R}$ which has optimal packing oracles but does not have optimal Hausdorff oracles such that*

$$\dim_H(E) = s_1 \text{ and } \dim_P(E) = s_2.$$

References

1. Krishna B. Athreya, John M. Hitchcock, Jack H. Lutz, and Elvira Mayordomo. Effective strong dimension in algorithmic information and computational complexity. *SIAM J. Comput.*, 37(3):671–705, 2007. doi:10.1137/S0097539703446912.
2. Christopher J. Bishop and Yuval Peres. *Fractals in probability and analysis*, volume 162 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. doi:10.1017/9781316460238.

3 Adam Case and Jack H. Lutz. Mutual dimension. *ACM Transactions on Computation Theory*, 7(3):12, 2015.

4 Logan Crone, Lior Fishman, and Stephen Jackson. Hausdorff dimension regularity properties and games. *arXiv preprint*, 2020. arXiv:2003.11578.

5 Roy O. Davies. Two counterexamples concerning Hausdorff dimensions of projections. *Colloq. Math.*, 42:53–58, 1979.

6 Rod Downey and Denis Hirschfeldt. *Algorithmic Randomness and Complexity*. Springer-Verlag, 2010.

7 Kenneth Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. Wiley, third edition, 2014.

8 Kenneth Falconer, Jonathan Fraser, and Xiong Jin. Sixty years of fractal projections. In *Fractal geometry and stochastics V*, pages 3–25. Springer, 2015.

9 Leonid A. Levin. On the notion of a random sequence. *Soviet Math Dokl.*, 14(5):1413–1416, 1973.

10 Leonid Anatolevich Levin. Laws of information conservation (nongrowth) and aspects of the foundation of probability theory. *Problemy Peredachi Informatsii*, 10(3):30–35, 1974.

11 Ming Li and Paul M.B. Vitányi. *An Introduction to Kolmogorov Complexity and Its Applications*. Springer, third edition, 2008.

12 Jack H. Lutz. Dimension in complexity classes. *SIAM J. Comput.*, 32(5):1236–1259, 2003.

13 Jack H. Lutz. The dimensions of individual strings and sequences. *Inf. Comput.*, 187(1):49–79, 2003.

14 Jack H. Lutz and Neil Lutz. Algorithmic information, plane Kakeya sets, and conditional dimension. *ACM Trans. Comput. Theory*, 10(2):Art. 7, 22, 2018. doi:10.1145/3201783.

15 Jack H Lutz and Neil Lutz. Who asked us? how the theory of computing answers questions about analysis. In *Complexity and Approximation*, pages 48–56. Springer, 2020.

16 Jack H. Lutz and Elvira Mayordomo. Dimensions of points in self-similar fractals. *SIAM J. Comput.*, 38(3):1080–1112, 2008.

17 Neil Lutz. Fractal intersections and products via algorithmic dimension. In *42nd International Symposium on Mathematical Foundations of Computer Science (MFCS 2017)*, 2017.

18 Neil Lutz and D. M. Stull. Bounding the dimension of points on a line. In *Theory and applications of models of computation*, volume 10185 of *Lecture Notes in Comput. Sci.*, pages 425–439. Springer, Cham, 2017.

19 Neil Lutz and D. M. Stull. Dimension spectra of lines. In *Unveiling dynamics and complexity*, volume 10307 of *Lecture Notes in Comput. Sci.*, pages 304–314. Springer, Cham, 2017.

20 Neil Lutz and D. M. Stull. Projection theorems using effective dimension. In *43rd International Symposium on Mathematical Foundations of Computer Science (MFCS 2018)*, 2018.

21 J. M. Marstrand. Some fundamental geometrical properties of plane sets of fractional dimensions. *Proc. London Math. Soc. (3)*, 4:257–302, 1954. doi:10.1112/plms/s3-4.1.257.

22 Pertti Mattila. Hausdorff dimension, orthogonal projections and intersections with planes. *Ann. Acad. Sci. Fenn. Ser. AI Math*, 1(2):227–244, 1975.

23 Pertti Mattila. *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*. Cambridge University Press, 1999.

24 Pertti Mattila. Hausdorff dimension, projections, and the fourier transform. *Publicacions matematiques*, pages 3–48, 2004.

25 Pertti Mattila. Hausdorff dimension, projections, intersections, and besicovitch sets. In *New Trends in Applied Harmonic Analysis, Volume 2*, pages 129–157. Springer, 2019.

26 Elvira Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Inf. Process. Lett.*, 84(1):1–3, 2002.

27 Elvira Mayordomo. Effective fractal dimension in algorithmic information theory. In S. Barry Cooper, Benedikt Löwe, and Andrea Sorbi, editors, *New Computational Paradigms: Changing Conceptions of What is Computable*, pages 259–285. Springer New York, 2008.

28 Andre Nies. *Computability and Randomness*. Oxford University Press, Inc., New York, NY, USA, 2009.

29 Tuomas Orponen. Combinatorial proofs of two theorems of Lutz and Stull. *arXiv preprint*, 2020. `arXiv:2002.01743`.

30 D. M. Stull. Results on the dimension spectra of planar lines. In *43rd International Symposium on Mathematical Foundations of Computer Science*, volume 117 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 79, 15. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018.