Quantum-Inspired Combinatorial Games: Algorithms and Complexity

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Abstract

Recently, quantum concepts inspired a new framework in combinatorial game theory. This transformation uses *discrete* superpositions to yield beautiful new rulesets with succinct representations that require sophisticated strategies. In this paper, we address the following fundamental questions:

- **Complexity Leap**: *Can this framework transform polynomial-time solvable games into intractable games?*
- **Complexity Collapse**: *Can this framework transform* PSPACE-*complete games into ones with complexity in the lower levels of the polynomial-time hierarchy?*

We focus our study on how it affects two extensively studied polynomial-time-solvable games: Nim and Undirected Geography. We prove that both Nim and Undirected Geography make a *complexity leap* over NP, when starting with superpositions: The former becomes Σ_2^p-hard and the latter becomes PSPACE-complete. We further give an algorithm to prove that from *any* classical starting position, quantumized Undirected Geography remains polynomial-time solvable. Together they provide a nearly-complete characterization for Undirected Geography. Both our algorithm and its correctness proof require strategic moves and graph contraction to extend the matching-based theory for classical Undirected Geography.

Our constructive proofs for both games highlight the intricacy of this framework. The polynomial time robustness of Undirected Geography in this quantum-inspired setting provides a striking contrast to the recent result that the disjunctive sum of two Undirected Geography games is PSPACE-complete. We give a Σ_2^p-hardness analysis of quantumized Nim, even if there are no pile sizes of more than 1.

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Supplementary Material We have implemented several games discussed in this paper as web games:
Quantum Nim: `https://turing.plymouth.edu/~kgb1013/DB/combGames/quantumNim.html`
Demi Version: `https://turing.plymouth.edu/~kgb1013/DB/combGames/demiQuantumNim.html`
TransverseWave: `https://turing.plymouth.edu/~kgb1013/DB/combGames/transverseWave.html`

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1 Introduction

In 2006, Allan Goff – a nuclear engineer – introduced a game, called Quantum Tic-Tac-Toe, as an educational tool for learning quantum mechanics [21]. He incorporated elements inspired by the quantum concepts of superpositions and entanglement into Tic-Tac-Toe, a popular two-player game traced back to ancient Egypt and Roman Empire. This variant's popularity made it subject to further study, including exploring the entire game tree [24]. Since then, quantum-inspired rules have been derived for other combinatorial games[1], e.g., Chess [1] and Cops and Robbers [20]. In 2017, Dorbec and Mhalla [13] took a step further and presented a general *discrete* framework for turning combinatorial games into *quantum-inspired* variants, extending Goff's formulation (of superposition of moves and entanglement among Tic-Tac-Toe positions).

In this paper, we study several fundamental questions addressing the computational impacts of this framework to combinatorial games. Superpositions of moves and entanglement among game positions introduce complex *nondeterminism* into the game space, providing a rich structure for algorithm design and complexity characterization.

1.1 Combinatorial Games and a Quantum-Inspired Transformation

Combinatorial games are mathematical games between two players with perfect information and no random elements. Traditionally, each game is defined by a *succinct ruleset*, specifying the domain of game *positions* that map to other positions one can move to (*options*). [4, 23, 12, 2, 32]. In order to obtain a meaningful semblance of entanglement, one needs to include another facet, *moves*: the natural descriptions for transitioning from positions to their options. Each move (e.g., "take two from the third pile" or "move the token to v") could apply to many positions, resulting in a different option for each. In the *normal-play* setting, two players take turns selecting their next move, and the player who is forced to a *terminal position* – a position with no feasible options – loses the game. A ruleset is *impartial* if both players have the same options at every position, and *partisan* otherwise.

The rapid expansion of strategic spaces makes games, e.g., GO, Chess, and Checkers, both favorite intellectual pastimes in the real world and decision problems for study [28, 35, 17]. Many games, such as Nim, Tic-Tac-Toe, and Mancala, have long historical roots and are taught to primary school students. Others are studied in mathematics and computer science, including theorized practical games and abstract formulations based on topology, logic, and graph theory, (e.g., Hex, Poset, Atropos, QSAT, Avoid True, Geography[2], Node-Kayles, Sprouts and Hackenbush [30, 19, 31, 4, 33, 8, 3]). In 1901, Bouton [5] developed a complete mathematical theory for Nim, an impartial game traced back to an ancient Chinese game known as Jian Shi Zi (捡石子 - picking stones). Each Nim position is a collection of piles of (small) stones. At their turn, players take at least one stone from one of the piles. Under normal play, the player taking the last stone wins. Three decades later, Bouton's solution was instrumental in Sprague-Grundy Theory on impartial games [34, 22], which laid the foundation for Combinatorial Game Theory (CGT) [4].

CGT is now a research area drawing interest in both mathematics and computer science [4, 23, 12, 2, 32]. Beyond analyzing the structure and complexity of individual rulesets, CGT studies the system of games as a whole. This holistic approach provides a systematic

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[1] Although different in details, these developments are parallel to other fields. For example, in economic game theory, researchers have explored what happens when agents are allowed to make quantum decisions and how quantum decisions impact game and economic equilibria [14].

[2] We use Geography to mean Generalized Geography (and Directed Geography).

framework for combining games and studying their relationships. The most basic way to combine games is the *disjunctive sum*: For any two games G and H, $G + H$ is a game in which the next player chooses to make a move in exactly one of G and H, leaving the other unchanged. Using this algebraic structure over games, the Sprague-Grundy theorem establishes that every impartial game is equivalent to a single NIM pile. With this equivalence, *a.k.a* the *Grundy value or nimber*, Bouton's solution for NIM – based on bit-wise exclusive OR – provides a complete mathematical theory for the disjunctive sum of impartial games.

The disjunctive sum also acts as a transformation for defining new rulesets from existing ones. For example, let NIM[(1)] denotes a single-pile one-stone NIM. For any position Z, $\text{OnePass}(Z) := Z + \text{NIM}[(1)]$ transforms Z into a new game of playing Z endowed with a single one-time "pass move" shared by the two players. Mathematically, the quantum-inspired framework [21, 13] can be viewed as a new *transformation* of combinatorial games. Before outlining this framework, we remark that by *quantum combinatorial games*, Goff, Dorbec and Mhalla don't mean games defined on *continuous* qubits and unitary transformations as in quantum computing. Rather, quantum combinatorial games draw on quantum concepts of superposition and entanglement to enhance the classical combinatorial games with a "quantum-inspired" *discrete* framework of moves and positions. We think that *"quantum-inspired combinatorial games"* is the more precise and literal name for these games.

This generalization transforms each game Z into a new one $Z^{\mathcal{Q}}$ in which, at their turn, players can either make a classical move, or a superposition of classical moves.[3] Following Goff, Dorbec and Mhalla, the latter type of moves will be referred to as *quantum moves*. Likewise, a *quantum game state* – a quantum position – is a superposition of multiple classical game states, each referred to as a *realization*. Mathematically, quantum moves introduce *nondeterminism* into game states, represented by quantum positions. We will use $\mathbb{M} := \langle m_1 \mid m_2 \mid \ldots \mid m_w \rangle$ to denote a *w-wide superposition* of classical moves, and $\mathbb{G} = \langle g_1 \mid g_2 \mid \ldots \mid g_s \rangle$ to denote an *s-wide quantum position*. With this notation, classical moves and positions correspond to $w = 1$ and $s = 1$, respectively.

In $Z^{\mathcal{Q}}$, $\mathbb{M}$ is said to be *feasible* for $\mathbb{G}$ if $\forall i \in [1 : w]$, m_i is feasible for a non-empty subset of realizations in $\mathbb{G}$, according to Z. $\mathbb{M}$ takes $\mathbb{G}$ to a new quantum position with "nondeterministic" game states resulting from feasible transitions in the "Cartesian product" of $\{m_1, \ldots, m_w\}$ and $\{g_1, \ldots, g_s\}$, according to Z: Realizations in $\mathbb{G}$, according to [21, 13], are "entangled" in that each move m_i applies to the entire nondeterministic composition. If m_i is not feasible for a realization, then the realization can no longer be factored in for making future moves, so we say it *collapses*. As a quantum game progresses, the number of realizations in the quantum state of the board can go up and down. When all possible moves of a player would cause a realization to collapse, we call it a *terminal* realization for that player. Under normal-play[4], the game ends and the current player loses on a quantum game position when all its realizations are terminal for that player.

For positive integer w, let $Z^{\mathcal{Q}(w)}$ denote the quantumized Z in which all quantum moves have width at most w. For partizan games, there are also some subtleties in formalizing the outcome of the quantum extension. We will provide more detailed discussion in Section 4.

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[3] In [13], Dorbec and Mhalla postulated five quantum variants for extending classical rulesets, addressing possible restrictions on the interaction between classical moves and quantum positions. In this shorter version for FUN, we focus on the most natural variant, in which classical moves can be viewed as special cases of superposition moves. More on other variants can be found in the full version of this paper.

[4] We do not apply quantumness to Misère play or Scoring games, the usual alternatives to Normal Play. These seem like excellent areas for future work.

1.2 An Illustration of “Quantum Impact” via Quantum Nim

We use Nim to provide a concrete example. In the classical setting, a well-known losing position for the current player is the configuration consisting of a pair of two-stone piles, which we denote by the 2D vector $(2, 2)$, The current player cannot win at this position because regardless of whether they pick one or two from a pile, the other player can simply do the same to the other pile to win.

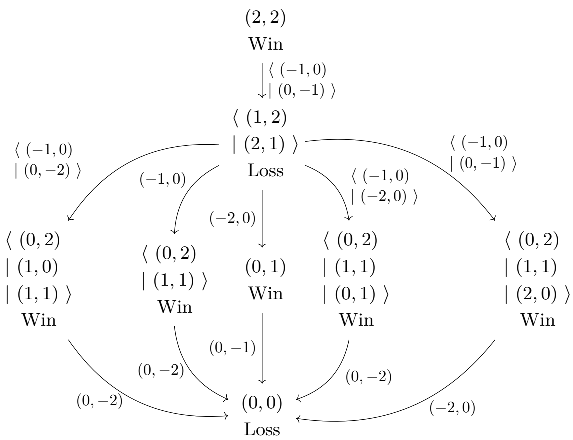

Figure 1 Winning strategy for current player in Quantum Nim $(2, 2)$, showing that quantum moves impact the game’s outcome. (There are four additional move options from $\langle(1, 2) \mid (2, 1)\rangle$ that are not shown because they are symmetric to moves given.)

A move is specified by the index of the pile and the amount to take. Here, we use $(-c, 0)$ and $(0, -c)$ to denote the moves of taking c stones from the first and second pile, respectively. From position $(2, 2)$, in addition to the classical moves of picking one or two stones from a pile, the current player can consider a quantum move, $\langle(-1, 0) \mid (0, -1)\rangle$, formed by the *superposition* of taking one stone from either pile. This *quantum move* yields a *quantum position*, $\langle(1, 2) \mid (2, 1)\rangle$, consisting of a superposition of two Nim positions. Subsequent quantum moves may further increase the number of realizations in the game’s superposition state. For example, if the other player makes the quantum move, $\langle(-1, 0) \mid (0, -2)\rangle$, then the next game position will be a superposition of three classical positions: $\langle(0, 2) \mid (1, 0) \mid (1, 1)\rangle$. (The realization $(2, -1)$ is not included because it’s not a legal Nim position.)

Remarkably, the quantum move $\langle(-1, 0) \mid (0, -1)\rangle$ is a winning move on Quantum Nim at $(2, 2)$. Figure 1 gives the complete game tree after that move from $(2, 2)$. Thus, quantum moves not only enrich players’ strategical domains, but also can alter the winnability. The sophisticated interactions (between superpositions of moves and superpositions of realizations) and the potential explosiveness (in the complexity of quantum configurations) make quantum games fascinating for computational complexity studies.

1.3 Highlights of Our Results

Combinatorial games are fun to play and it is intellectually stimulating to search for optimal moves. The *deep alternation* of elegant rules defines *game trees*, which can *exponentially* expand the space of game strategies, introducing a challenge to the basic decision problem of whether or not the current player has a winning move. Thus, it is fundamental to determine the complexity of a game. If it is *intractable*, this indicates that the game doesn’t have a simple schema players can employ, so they must use complex strategies instead. *Elegant* combinatorial games with *simple rules* and *intractable complexity* are the gold standard for combinatorial game design [15]. In the digital age powered by AI, the lack of an efficient algorithm for optimally playing a combinatorial game gives human players a fighting chance to compete against computer programs. The deep challenge also gives computer programs a reason to learn and improve [8, 9].[5] If it is *tractable*, then, in addition to being a pedagogical illustration, the efficient method can be a source of ideas for more general solution concepts and methodology. An exemplary case is Bouton's polynomial-time Nim solution and its subsequent generalization in Sprague-Grundy theory. Polynomial-time-solvable games are also useful in capturing some practical phenomena, for example in modeling network dynamics [16, 10] or deriving other graph algorithms [25]. Furthermore, understanding the causes behind the transitions between tractable and intractable, through transformation of games, allows one to both turn "simple games" into "sophisticated" games [7] and transform intractable abstract games into real world solutions [11, 29].

Our research program here started with and expanded upon the following question:

- **Complexity Leap**: *Can the quatum-inspired framework transform a polynomial-time solvable combinatorial game into an intractable game?*

In our pursuit of this possible leap in the *quantumized complexity* of game, we have also obtained an affirmative answer to the following basic question:

- **Complexity Collapse**: *Are there* PSPACE*-complete combinatorial games whose quantum extensions fall to the lower levels of polynomial hierarchy?*

We focus on the effect on two extensively studied polynomial-time-solvable games: Nim and Undirected Geography, the undirected version of Geography. In the 1970s, recognizing the graph-theoretical background of a real-world "Word Chain" game called Geography, Richard Karp recommended the game to his then Ph.D. student Tom Schaefer for complexity study [31]. In Geography, a position is defined a directed graph and a starting node (for the token). The two players alternates turns moving the token to an adjacent node and then deleting the node it came from. In the normal-play setting, the player who cannot make a move loses the game. Schaefer proved that deciding the winnability of Geography is PSPACE-complete [31]. Lichtenstein and Sipser [28] then simplified his gadgets to prove that GO is PSPACE-hard. Indeed, both Undirected Geography and Directed Acyclic Geography – i.e., Geography on DAGs – are tractable [18].

In this paper, we prove that superpositions not only enrich the structure, but also impact the complexity of combinatorial games: When starting at a quantum position, both Quantum Nim and Quantum Undirected Geography make a *complexity leap* over NP. The former becomes Σ_2^p-hard and the latter becomes PSPACE-complete.

Complexity Characterization of *Quantum Undirected Geography*. We show that this is PSPACE-complete, even in a position resulting from polynomially-many (quantum) moves from a classical position. We complement this with an algorithmic result, highlighting the fragility of this game. In Section 2.1, we present a polynomial-time algorithm for solving Quantum Undirected Geography when starting at any classical position. The proof of correctness requires carefully-designed strategic moves supported by graph contraction

[5] The rapid exponential growth in decision time makes searching for winning strategies challenging even for game boards with moderate-sizes, such as 19 by 19 for Go or 14 by 14 for Nash's "optimal" Hex size [30].

to extend the matching theory for classical UNDIRECTED GEOGRAPHY [18]. In addition to highlighting the fundamental difference between classical and quantum starts, this algorithmic result provides a sharp contrast to the recent result that the disjunctive sum of two UNDIRECTED GEOGRAPHY games is PSPACE-complete [7].

Complexity Leap in *Quantum Nim* and Beyond. Our Σ_2^p-hardness proof of QUANTUM NIM takes several turns, each of which provides new insight. It involves two logic games of Schaefer [31]: AVOID TRUE and PARTITION-FREE QBF.[6] We establish the following: (1) QUANTUM AVOID TRUE and QUANTUM NIM with Boolean starting quantum position (i.e., each pile in a realization has one or zero stones) are isomorphic to each other. (2) QUANTUM PARTITION-FREE QBF is polynomial-time reducible to QUANTUM AVOID TRUE. (3) QUANTUM PARTITION-FREE QBF is Σ_2^p-complete. These technical steps have several basic implications: First, because AVOID TRUE is PSPACE-complete [31], this isomorphism represents a significant use of superpositions of NIM positions – individually polynomial-time solvable – to encode the intractable game. The encoding shows superposition can be more expressive than disjunctive sum. Second, the isomorphism aforementioned suggests an in-between transformation of rulesets: A *Demi-Quantum* combinatorial game lives in the classical world with an initial quantum endowment: It starts in a quantum position, but during the game, only classical moves are allowed. Our isomorphism result proves that DEMI-QUANTUM NIM is PSPACE-hard. Third, our isomorphism is not between QUANTUM BOOLEAN NIM and AVOID TRUE, but with QUANTUM AVOID TRUE.

The need to prove the latter's intractability led us to a family of fundamental questions, centered around an intuitive conjecture: The quantum generalization of any combinatorial game is always as hard (computationally) to play as the game itself. Particularly, *does a* PSPACE-*hard combinatorial game remain* PSPACE-*hard in the quantum setting?*

We present a complexity-theoretical refutation to this question. Furthermore, motivated by Ko's separation of the polynomial-time hierarchy [26],[7] we prove that for any integer $k > 0$, there exists a PSPACE-complete game whose quantum extension is complete exactly for level k in the polynomial-time hierarchy (in the full version of the paper). Our Σ_2^p-hard proof naturally extends to the quantum generalization of SUBTRACTION games, a family of arithmetic strategy games widely used to teach young children about mathematical thinking. In Section 3.3, we formulate general "Robust Binary-Nim Encoding" properties sufficient for supporting a reduction from QUANTUM NIM, implying that several natural games including BRUSSELS SPROUTS, GO and DOMINEERING are Σ_2^p-hard in the quantum setting.

Practical Board Games. Our studies have inspired the design of a practical board game, called TRANSVERSE WAVE, based on the isomorphism between QUANTUM BOOLEAN NIM and AVOID TRUE. In our web-based version of the game (given in Supplementary Material), one can play either against another human player (sitting at the same computer) or some of AI programs. Being a PSPACE-complete impartial game with simple rules played on a colorful 2D-grid board, this board game has several desirable properties outlined by Eppstein [15]. This board game also enabled the implementation of QUANTUM NIM and DEMI-QUANTUM NIM.

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[6] See Section 3.1 for the rules of these two logic games.
[7] Ko proved the following: For any integer $k > 0$, there exists an oracle O_k such that *relative to* O_k, the polynomial-time hierarchy collapses to exactly to level k.

2 Exploring Quantum Undirected Geography

We first focus on Quantum Undirected Geography. For this ruleset, each move is indicated by a vertex; a classic move is feasible if the indicated vertex is adjacent to the current vertex. Our two complementing algorithmic/complexity characterizations reveal a majestic computational landscape with PSPACE-hard peaks over quantum positions, dipping into Polynomial-Time valleys around classical instances. Furthermore, some PSPACE-hard positions are reachable from classical positions via polynomial-length quantum paths, making them realizable in Quantum Undirected Geography from classical starts.

2.1 Strategic Graph Contraction: Tractability of Classical Starts

▶ **Theorem 1** (Efficient Solution for Classical Starts)**.** *For any Undirected Geography position Z, $Z^{\mathcal{Q}(2)}$ with the same starting position are solvable in polynomial time.*

We will prove the theorem by establishing that for any undirected graph $G = (V, E)$ and starting vertex $s \in V$, position (G, s) is a winning position (of the current player) in the quantum setting with 2-wide quantum moves if and only if (G, s) is a winning position in the classical Undirected Geography. The classical case can be solved by the Fraenkel-Scheinerman-Ullman algorithm guided by an elegant matching theory: (G, s) is a winning position if and only if s is in every maximum matching of G [18]. Our graph-contraction-based algorithm extends this matching theory to the quantum setting.

In order to prove this, we will first present the algorithm that the winner in the classical game (*hero*) will use to win under quantum play. The hero will always make classical moves, but we need to show how they respond to quantum moves by their opponent, the *villain*. If the villain makes a quantum move, the hero will try to make a winning collapsing move (described further below). If they cannot, they can still win by keeping track of the quantum superposition, treating the two quantumly-chosen vertices as one combined (*contracted*) vertex. The hero keeps track of an overlaid graph $G' = (V', E')$: (A) Initially, $V' = \{\{v\} \mid v \in V\}$. We refer to $c(v)$ (the contraction with v) as the element of V' that contains v. (B) E' will be updated so that $(X, Y) \in E' \Leftrightarrow X, Y \in V'$ and $\forall x \in X, y \in Y : (x, y) \in E$. (C) Whenever a player makes a classical move from $a \to b$ (meaning the current player makes a classical move after the previous player makes a classical move) the hero will remove $c(a)$ from V' and all incident edges from E'. (D) Whenever the villain makes a quantum move $a \to \langle v_1 \mid v_2 \rangle$, the hero will again remove $c(a)$ from V' and all incident edges from E'. (E) Whenever the hero follows a quantum move with a classical move, $\langle v_1 \mid v_2 \rangle \to b$, the hero will update G' based on whether they make a collapsing move: (E.1) If the hero collapses that quantum move, then they can remove the remaining v_i as though it had been a classical move by the villain. (E.2) If the hero does not collapse, but $c(v_1) = c(v_2)$, then all of those vertices have all been visited in all realizations. The hero can remove $c(v_1)$ from V' and all incident edges from E'. (E.3) If the hero does not collapse and $c(v_1) \neq c(v_2)$, then the hero removes both $c(v_1)$ and $c(v_2)$ from V' and replaces them with $c(v_1) \cup c(v_2)$. Then the hero will reset E' to match the definition given above.

Next, we describe how the hero chooses their move, using maximum matchings on G'. (In our notation, we consider a matching, M, as both a set of pairs and a function. So, $(a, b) \in M \Leftrightarrow M(a) = b \Leftrightarrow M(b) = a$.). In the classical winning position, the current vertex is in all maximum matchings on the graph [18], meaning after the classical winner's turn, the loser must start from a vertex not contained in some maximum matching. Our hero will maintain a similar invariant on G': there is a maximum matching on G' such that the villain will be starting their turn on a vertex not in that matching. Our algorithm is as follows:

- If the villain makes a classical move to v, the hero considers any maximum matching, M on G', and then moves to $x \in M(c(v))$ such that $(v, x) \in E$. This leaves us with a maximum matching, $M \setminus \{(c(v), c(x))\}$ on the remaining graph that does not include $c(x)$, thus upholding the invariant. The hero removes $c(v)$ and $c(x)$ from V'.
- If the villain makes a quantum move to $\langle\ a \mid b\ \rangle$, and $c(a) = c(b)$, then the hero acts as though the villain moved classically to only a, finds a maximum matching, M on G', then moves to $x \in M(c(a))$ such that $(a, x) \in E$. Subtracting $(c(a), c(x))$ from M results in a maximum matching without $c(x)$, upholding the invariant. The hero removes $c(a) = c(b)$ and $c(x)$ from the partition V' they are keeping track of.
- If the villain moves to $\langle\ a \mid b\ \rangle$, and $c(a) \neq c(b)$, then the hero has additional work to do. Notably, they find a matching, M, such that $\exists x \in M(c(a))$ where $(a, x) \in E$, but $(b, x) \notin E$; or $\exists x \in M(c(b))$ where $(b, x) \in E$, but $(a, x) \notin E$, if one exists. There are three cases:
  1. If $\exists M$ with $x \in M(c(a))$ where $(a, x) \in E$, but $(b, x) \notin E$, then the hero moves to x. Since $(b, x) \notin E$, the realization where the villain moved to b collapses out. Subtracting $(c(a), c(x))$ from M yields a maximum matching without $c(x)$, upholding the invariant. The hero removes $c(a)$ and $c(x)$ from V'.
  2. If $\exists M$ with $x \in M(c(b))$ where $(b, x) \in E$, but $(a, x) \notin E$, then the hero moves to x as in the previous case, with a and b swapping roles, then removes $c(b)$ and $c(x)$ from V'.
  3. If no maximum matching exists with those requirements, then the hero can just use any maximum matching, M, to make their move. For any maximum matching M:

     $$\forall x \in M(c(a)) : (b, x) \in E \quad \text{and} \quad \forall y \in M(c(b)) : (a, y) \in E.$$

     The hero can now move to any x and update V' by removing $c(x)$ and contracting $c(a)$ and $c(b)$ into one element $c(a) \cup c(b)$. In order to continue safely, the hero needs this new contracted vertex to be adjacent to $c(y)$, for any $y \in M(c(b))$. Thus, the hero needs both that $(c(a), c(y)) \in E'$ and $(c(b), c(y)) \in E'$. The latter is already true because $M(c(b)) = c(y)$. For the former, by Lemma 2 below, (a, x) and (b, y) are in a maximum matching on G. Thus, there is another maximum matching with the swapped edges (a, y) and (b, x). Then, by Lemma 3, $(c(a), c(y)) \in E'$.

▶ **Lemma 2.** *If $(c(a), c(b))$ is in a maximum matching of G', then in any realization where neither a nor b has been used, (a, b) is part of a maximum matching on the unvisited graph.*

Proof. Let M be the maximum matching on G' containing $(c(a), c(b))$. Also let H be the remaining subgraph of G that hasn't been visited in the given realization. Then, for each $X \in V'$, there is exactly one vertex in H remaining. Due to the definition of E', that vertex must be adjacent to the vertex of H inside the contraction $M(X)$. Thus, each matched pair in M corresponds to exactly one unique pair of neighbors in H, which creates a maximum matching on that graph. Thus, a and b must be neighbors and (a, b) is in a maximum matching on H. ◀

▶ **Lemma 3.** *If (a, b) is in a maximum matching on G and $c(a) \neq c(b)$, then at any point in the game where a and b are still included in contractions, $(c(a), c(b)) \in E'$.*

Proof. We prove this by contradiction. Assume $(c(a), c(b)) \notin E'$. Thus, $\exists\ (a', b') \notin E$, where $a' \in c(a)$ and $b' \in c(b)$. Without losing generality, we assume that the most recent contraction breaks the statement of the lemma; all prior contraction-graphs G' contained all edges from all maximum matchings in G. Assume that the villain's last quantum move was to $\langle x \mid y\rangle$ and the hero had to respond to a non-collapsing move at vertex z. Thus: (1) In the prior contraction, c', $c'(x) \neq c'(y)$. (2) $c'(x) \cup c'(y) = c(a)$. (3) $\exists\ M$, a matching on G' that the hero used to choose z. (4) WLOG, $a' \in c'(x)$ and $b' \in M(c'(y))$ ◀

Proof of Theorem 1. The invariant the hero maintains is: at the end of the hero's turn, after having moved to x, there is a maximum matching on G' that does not contain the contracted vertex $c(x)$. Thus, either (A) There are no more edges leaving x and the villain loses immediately, or (B) If there is a move and the villain moves from x to y, then y must be contained in one of the other matched pairs, meaning that the hero will be able to move to y's match. (Lemma 2 requires that y is part of one of those matches, because if it wasn't, then (x, y) would be part of a maximum matching on G and that edge will be represented as an edge in a maximum matching in G', which won't work with the invariant.) When there are no moves left in G', there are no moves left in G, since if the edge (x, y) has $c(x) = c(y)$ then only one vertex is remaining in each realization, and if $c(x) \neq c(y)$, then $(c(x), c(y))$ must be in G'. The invariant will be maintained because each turn the hero will start on a vertex in all maximum matchings of G' and will traverse the edge from one of them. Since the hero will always have a move to make on their turn, they will never lose the game. ◀

2.2 Intractability of Quantum Geography Positions

Figure 2 Gadgets for reducing from a GEOGRAPHY edge (a, b). The left is the gadget as it will appear in the *main-board* and all other boards except from (a, b)-*board*. The right is the edge as it appears in (a, b)-*board*. All other parts of the two boards will be the same.

▶ **Theorem 4** (Intractability of Quantum Starts). *QUANTUM UNDIRECTED GEOGRAPHY, when one begins with a superposition of polynomial width ("polynomially wide") is* PSPACE-*complete.*

Proof. The game tree has height $\leq n = |V|$. As seen in the full version of this paper, the quantumization with poly-wide quantum start is in PSPACE. For hardness, we reduce from Quantum GEOGRAPHY, the original version which includes directed edges (and which we prove to be PSPACE-hard in Theorem 21. Consider a GEOGRAPHY instance where the underlying graph has n nodes and m edges. We will create an (undirected) superposition that consists of $m + 1$ entangled realizations. (1) One realization, *main-board*, has a very similar structure, but each arc (a, b) is replaced by a three-part path with two new vertices, ab_1 and ab_2: (a, ab_1), (ab_1, ab_2), and (ab_2, b). (2) For each arc (a, b), we include (a, b)-*board*, which is exactly the same as *main-board* except there is an extra vertex, STOP, and the edge (ab_1, ab_2) is replaced with (STOP, ab_2).

We show the two relevant edge gadgets in Figure 2. The *main-board* can never collapse out unless a player moves to STOP, since all other edges are also in other realizations. If a player does move to STOP, the game immediately ends, as all other realizations with STOP collapse. Thus, the game either ends when there are no moves left in the main game board, or STOP is entered. Additionally, a player can move to STOP iff their opponent reached the ab_2 vertex by moving from b, corresponding to going backwards on arc (a, b) in the Quantum GEOGRAPHY game. This cannot happen when moving from ab_1 since traversing the (ab_1, ab_2) edge collapses out the realization with an edge to STOP. Thus, traversing an edge backwards is always a losing move. Thus, winning strategies for QUANTUM UNDIRECTED GEOGRAPHY, correspond exactly to those for QUANTUM GEOGRAPHY. ◀

▶ **Theorem 5** (Intractability: Reachable Quantum Positions). *Quantum Undirected Geography is* PSPACE*-complete for positions reachable with polynomial number of moves after a classical start.*

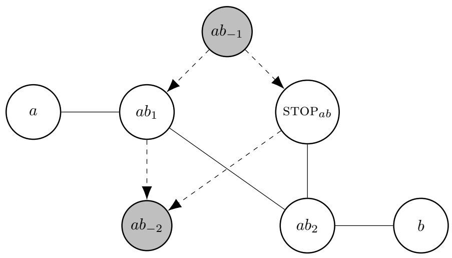

Figure 3 Gadget for a Geography edge (a, b) in Undirected Geography. Prior to the current position, the quantum moves $ab_{-1} \to \langle ab_1 \mid \text{STOP}_{ab}\rangle \to ab_{-2}$ were made.

Proof. To prove this, we use the reduction from Theorem 4, which contains what we refer to as the *core realizations*. Our new usage differs in that we have separate vertices STOP_{ab} instead of a common STOP. We call the non-core (exponential number of) realizations *redundant realizations*, because the players can ignore them. This is because, in each redundant realization, R: (1) There is a core realization such that if it collapses, so does R, and (2) At any position in the game, there is a core realization that contains all available moves in R. To complete the proof, we show that we can reach a superposition consisting only of (1) all core boards and (2) redundant boards. For our new starting position, we take all vertices and edges in our main board from the Theorem 4 reduction, and add vertices and all new edges. As shown in Figure 3, for each arc (a, b) in Quantum Geography we can insert vertices ab_1, ab_2, and STOP_{ab}, along with edges (a, ab_1), (ab_1, ab_2), (STOP_{ab}, ab_2), and (ab_2, b).

In addition, we include two vertices, ab_{-1} and ab_{-2}, which will have already been previously visited in all realizations by the time we reach the start. These vertices are connected by edges (ab_{-1}, ab_1), $(ab_{-1}, \text{STOP}_{ab})$, (ab_1, ab_{-2}), and $(\text{STOP}_{ab}, ab_{-2})$. For some other arc, (c, d), we can connect the gadgets by setting $ab_{-2} = cd_{-1}$. Now we prescribe a series of prior moves across all edges $\{(a_i, b_i) \mid i \in \{1, \cdots, m\}\}$: $(a_1b_1)_{-1} \to \langle (a_1b_1)_1 \mid \text{STOP}_{a_1b_1}\rangle \to (a_1b_1)_{-2} = (a_2b_2)_{-1} \to \cdots \to (a_{m-1}b_{m-1})_{-2} = (a_mb_m)_{-1} \to \langle (a_mb_m)_1 \mid \text{STOP}_{a_mb_m}\rangle (a_mb_m)_{-2} \to x$ were made, where x is the starting vertex of Quantum Geography. By construction, no realizations were ever collapsed in these prior moves. Also, there is a realization for each of the core realizations, by just following the branch that moved to STOP for each edge gives us the main-board. Following the branch that moved to STOP for each edge gadget other than some edge $e = (a, b)$ gives us the (a, b)-board. We now only need to show that the rest of the realizations are redundant. All realizations have only the edges in either the main-board or an (a, b)-board fulfilling the first property for redundancy. For the second property, we note that each of the non-core realizations must contain at least two STOP vertices; its current vertex is either adjacent to one stop (all its moves are in the core realization with the stop) or not (all its moves are in the main board). ◀

3 Complexity Leap in Quantum Nim: Logic Connection

In this section and the next section, we prove the following theorem:

▶ **Theorem 6** (Quantum Leap Over NP). *Quantum Nim with quantum starts is* Σ_2^p*-hard.*

3.1 Proof Outline: The Logic Connection of Quantum Nim

Our proof of Theorem 6 is involved. We first give its high-level steps, connecting Quantum Nim to the quantumized Avoid True. Avoid True is an impartial logic game introduced in Schaefer's landmark paper [31]. A position is given by a *positive* CNF formula, with all variables set to `false`. Two players take turns flipping a variable from `false` to `true`. A move is *feasible* if setting its variable `true` will not make the whole CNF evaluate to `true`.

▶ **Theorem 7.** *Let Quantum Boolean Nim denote the game whose position is a superposition of Boolean Nim. in which each pile has either zero or one stone. Then, Quantum Boolean Nim and Quantum Avoid True are isomorphic rulesets, i.e. they have isomorphic game trees.*

In Section 4, we will demonstrate the complexity of Quantum Avoid True through another logic game, Partition-Free QBF, which is a *partisan* game also played on a formula. Player True wants to make the formula `true` while False wants to make it `false`. Boolean variables of the formula are *partitioned* into two sets, one for each player. At their turn, players can "freely" set any of their unassigned variables. Using a delicate argument to counter an "unwelcome" quantum impact, in Section 4.2 we extend Schaefer's reduction from Partition-Free QBF to Avoid True in the quantum setting as well.

▶ **Theorem 8** (Partisan-Impartial Reduction in Quantum Setting)**.** *There exists a polynomial-time reduction from Quantum Partition-Free QBF to Quantum Avoid True.*

In Section 4.1, we study the quantumized complexity of the family of PSPACE-complete QBF games as a whole, addressing the subtlety of transforming QBF-games into normal-play games in the quantum setting. In particularly, we establish the following:

▶ **Theorem 9** (Complexity Collapes: Σ_2^p-Hardness)**.** *Quantum Partition-Free QBF with a classical start is Σ_2^p-complete for even alternation and Π_2^p-complete for odd alternation.*

Motivated by celebrated complexity characterizations of Lander's NP-intermediate problems [27] and Ko's intricate, meticulous separation of levels of the polynomial-time hierarchy [26], in the full version, we refine the last theorem to prove the following:

▶ **Theorem 10** (Complexity Collapses into Polynomial-Time Hierarchy)**.** *For any integer $k > 0$, for complexity class Σ_k, Π_k, or both Σ_k and Π_k, there exists a classically* PSPACE*-complete combinatorial game whose quantum generalization is complete in that class.*

3.2 Isomorphism Between Two Natural Quantum-Inspired Games

Proof of Theorem 7. Crucial to our complexity analysis, this isomorphism between Quantum Boolean Nim and Quantum Avoid True is polynomial-time computable.

First note that Quantum Avoid True has the following interesting self-isomorphic property. Quantum Avoid True with quantum starts is isomorphic with Quantum Avoid True with classical starts. This is because that the superposition of any two classical Avoid True positions is "equivalent" to the classical position defined by the `and` for their CNFs. Thus, focusing on the logic-to-nim direction, we consider a classical Avoid True position (F, V, T), where $V = \{x_1, \ldots, x_n\}$, F is the formula with m clauses, $C_1, \ldots, C_m$, and $T \subseteq V$ denotes the subset of variables set to `true`. (T is empty at the start.)

We now reduce this position to a Boolean Nim superposition $\mathbb{B}_{(F,V,T)}$ with m realizations – one for each clause – and n piles (encoding Boolean variables). In the realization for C_i, we set piles corresponding to variables in C_i to zero to set up the mapping between collapsing the realization with making the clause `true`. We also set all piles associated with variables in T to zero, to set up the mapping between collapsing the realization with selecting a selected variable. We use these two mappings to inductively establish that the game tree for Quantum Boolean Nim at $\mathbb{B}_{(F,V,T)}$ is isomorphic to the game tree for Quantum Avoid True at (F, V, T). We first demonstrate this reduction on the following example:

$$\underbrace{(x_1 \vee x_2 \vee x_3 \vee x_4)}_{A} \wedge \underbrace{(x_1 \vee x_5 \vee x_6 \vee x_7)}_{B} \wedge \underbrace{(x_1 \vee x_3 \vee x_6)}_{C} \wedge \underbrace{(x_2 \vee x_5 \vee x_8)}_{D}$$

and already-chosen variables, $T = \{x_8\}$. The reduction gives three Nim realizations: $A = (0, 0, 0, 0, 1, 1, 1, 0)$, $B = (0, 1, 1, 1, 0, 0, 0, 0)$, and $C = (0, 1, 0, 1, 1, 0, 1, 0)$. The clause corresponding to D was already true, so the corresponding realization has already collapsed.

To prove the correctness of the reduction, we prove that the current player has a winning strategy at (F, V, T) in Quantum Avoid True iff the next player has a winning strategy at $\mathbb{B}_{(F,V,T)}$ in Quantum Nim. We will prove inductively that the following invariant holds if we play our Nim encoding and its original Quantum Avoid True position in tandem: For every active clause in each realization of Quantum Avoid True there is exactly one realization in the Quantum Nim encoding with the corresponding realization having piles of 0 for each variable in the clause and variables that are no longer `false` for that position.

As the basis of the induction, the stated invariant is true at the start. There is one realization of Quantum Avoid True. In its Nim encoding constructed above, each realization has piles of 0 for the variables in the corresponding clause, and all variables are `false`, so the rest of the piles are at 1. We show that the invariant still holds after any move in Quantum Nim by examining each effect of its "coupled move" on any possible clause.

- Any classical or quantum move targeting a pile of size 1 in the Quantum Nim position does not collapse the Nim realization, consistent with the fact that selecting the corresponding variable results in the corresponding clause remaining active in the Quantum Avoid True instance. After the move, the pile will now be at 0 in the Quantum Nim realization, consistent with the fact that after selecting the corresponding variable in the realization of Quantum Avoid True, the variable is no longer `false`.
- Any classical or quantum move targeting a pile of 0 in Quantum Nim, that is not in the corresponding Quantum Avoid True clause Since this variable must have already been flipped, that means that it is 0 in all realizations and thus cannot be taken.
- If a classical or quantum move is made in the Quantum Nim position on a pile of 0 that corresponds to a variable in the associated Quantum Avoid True clause, then the clause is satisfied, and the corresponding realization collapses. This matches the Quantum Avoid True case where the clause is no longer active.

And it is through establishing this invariant that we get the desired structural morphism between playing Quantum Avoid True and playing its encoding Quantum Nim.

The inverse encoding from Quantum Boolean Nim to Avoid True is the following: Given a position $\mathbb{B}$ in Quantum Boolean Nim, we create a clause from each realization in $\mathbb{B}$. Suppose $\mathbb{B}$ has m realizations and n piles. We use n Boolean variables, $V = \{x_1, \ldots, x_n\}$. For each realization in $\mathbb{B}$, the clause consists of all variables corresponding to piles with zero pebbles. The reduced CNF $F_{\mathbb{B}}$ is the `and` of all these clauses. Taking a stone from a pile collapses a realization for which the pile has no stone is mapped to selecting the corresponding Boolean variable making the clause associated with the realization `true`. Thus, playing Quantum Boolean Nim at position $\mathbb{B}$ is isomorphic to playing Avoid True starting at position $(F_{\mathbb{B}}, V, \emptyset)$. Note that the reduction can be set up in polynomial time. ◀

3.3 Nim Encoding and Structural Witness of Σ_2^p-Hardness

In our reduction from Σ_2^p-hard Quantum Avoid True, our Quantum Nim game uses a poly-wide superposition of perhaps the simplest Nim positions: all Nim positions that we used in our encoding are from the family of Boolean Nim. So, our Σ_2^p-hard intractability proof of Quantum Nim holds for Quantum Boolean Nim and can be extended broadly to the quantum extension of Subtraction games.In this section, we present a more systematic theory to extend these hardness results. Particularly, we can use the Σ_2^p-hard result get results for several other games in the quantum setting. We show that if a game is able to classically properly embed a binary game with properties that we will describe, then its quantumized complexity at a poly-wide superposition is at least Σ_2^p-hard.

▶ **Definition 11** (Robust Binary-Nim Encoding). *A ruleset $\mathcal{R}$ has a* robust binary-Nim encoding *if $\mathcal{R}$ has a position b with the following properties for any n. (1) b has a set M of $n = |M|$ distinct feasible moves. (2) For each $\sigma \in M$, selecting σ moves the game from b to a losing position b_σ, such that position b_σ has $(n-1)$ feasible moves given by $M \setminus \{\sigma\}$. In addition, b_σ recursively induces a robust binary-Nim encoding for $(n-1)$. (3) When $n = 0$, the game will have no available moves.*

▶ **Lemma 12.** *If a game $\mathcal{R}$ has a robust binary-Nim encoding, then it is Σ_2^p-hard to determine the winnability of poly-wide quantum positions in its quantum setting.*

Proof. We reduce from Quantum Boolean Nim at a poly-wide superposition. Suppose there are n piles and a superposition with m realizations $\langle b_1 \mid \cdots \mid b_m \rangle$. Now we focus on the position b in $\mathcal{R}$ that induces the robust binary-Nim encoding for n, with set of moves $M = \{\sigma_1, \ldots, \sigma_n\}$. For each pile i, we associate it with move σ_i. Each realization, $b \in \{b_1, \ldots, b_m\}$, in Quantum Boolean Nim defines a set $S \subseteq [n]$, corresponding to piles with value 1. In the reduced quantum position for $\mathcal{R}$, we make a realization with the position whose feasible moves are $\{\sigma_i | s \in S\}$. This is a direct encoding, where we are just relabeling move i to σ_i, setting up the desired morphism between playing Quantum Boolean Nim and playing its encoding in the quantum setting of $\mathcal{R}$. ◀

We can easily embed Boolean Nim in several games, even games with fixed outcomes like Brussels Sprouts. For "interesting" games, such as Go and Domineering, this process is often simple, as one needs only to be able to make a board state that is a sum of combinatorial game value $*$ with only a single move in each of them.

4 On the Complexity of Quantum Avoid True

It is well known that the Quantified Boolean formula problem (QBF) – determining whether a quantified Boolean formula is true or false – can be viewed as a classical combinatorial game. Technically, combinatorial games must also fulfill the "normal play" requirement, meaning that a player loses *if and only if* they are unable to make a move. There are several equivalent ways to do this (transforming logical QBF decisions into combinatorial games) and are all classically polynomial-time reducible to each other. Complexity-theoretically, QBF is the canonical complete problem for PSPACE, and thus all these QBF variants are PSPACE combinatorial games; they form the bedrock of PSPACE reductions. However, because quantum games are more subtly dependent on the explicit move definitions in the ruleset, these known reductions from the "classical world" don't necessarily continue to hold. So, we need to define a variant that we will use to prove the intractability of Avoid True.

4.1 Quantum Collapse

In this subsection, we consider one natural "end-of-QBF" transformation that contributes to the complexity collapse in the quantum setting. This is critical for our Avoid True proof, an thus allows us to get the hardness for Nim's quantumized complexity. In the literature, the Quantified Boolean formula problem is also known as the Quantified Satisfiability Problem (QSAT). However, in this paper, we will – for clarity of presentation – use the following ruleset naming convention for QBF and QSAT, in order to denote two different ruleset families of combinatorial games rooted in the Quantified Boolean formula problem.

The QBF family: We will use QBF to denote the family of games that *textually* implements the Quantified Boolean formula problem: An instance of QBF is given by a CNF f, whose clauses may contain both positive and negative literals, over two ordered lists of Boolean variables. We have two cases: if there are even variables, then it has the form $(T_1, \ldots, T_n)$ and $(F_1, \ldots, F_n)$. Otherwise, it has the form $(T_1, \ldots, T_n, T_{n+1})$ and $(F_1, \ldots, F_n)$. So, the quantified boolean formula is, respectively:

$$\textbf{Case Even:} \quad \exists T_1 \forall F_1 \cdots \exists T_n \forall F_n \quad f(T_1, \ldots, T_n, F_1, \ldots, F_n)$$
$$\textbf{Case Odd:} \quad \exists T_1 \forall F_1 \cdots \exists T_n \forall F_n \exists T_{n+1} \quad f(T_1, \ldots, T_n, T_{n+1}, F_1, \ldots, F_n)$$

In this game, one player – *Player True* – aims to satisfy the CNF formula, while the other player – *Player False* – wants the formulae to be unsatisfied. Starting with True, the two players alternate turns setting their next variables to `True` or `False`. In other words, in their respective i^{th} turn, Player True sets variable T_i, then Player False sets variable F_i. The variants in this family, as we shall define later, differ in the details of the termination condition in transforming logic QBF into a combinatorial game.

The QSAT Family: We will use QSAT to denote the family of *partition free assignment* QBF games, with relaxations on the variables the players can choose to set at each turn. As in QBF, each instance of QSAT includes a CNF formulae f, and two players – Player True and Player False – who take turns to set their own variables; Player True can only set variables in $\{\cdots T_i \cdots\}$ and Player False can only set variables in $\{\cdots F_i \cdots\}$. Again, we have two cases – Case Even and Case Odd – depending on which player has one more variable. For the game, Player True aims to satisfy the CNF formulae, while Player False wants refute them. Unlike in QBF, the players can "freely" set any of their unassigned variables to `True` or `False`, as opposed to setting variables according to the prescribed order. In the variant we will call Phantom-Move QBF, after all variables have been assigned, the next player has a feasible (*phantom*) move available iff they have a winning assignment.

▶ **Theorem 13** (Quantum Collapses of QBF: Classical Starts). *Quantum Phantom-Move QBF with a classical start is Σ_2^p-complete in Case Even, and is Π_2^p-complete in Case Odd.*

Proof. Notice that the player who makes the final move (the "phantom move") will be allowed to choose to collapse any quantum moves so that the variables are assigned in a way that they will win (if it is possible). Since quantum moves cannot collapse before the end of the game, they can choose the assignments for any variables made during assignments by either player. As such, the optimal move for this collapsing player will be to make exclusively quantum moves, and the optimal move for other player will be to never make a quantum move. So, one player makes a variable assignment of half of the variables, and then only wins if every other assignment of the other variables are winning positions for them. Otherwise, the phantom move player wins. If the phantom move player is False, then this is exactly the Σ_2^p SAT problem. Otherwise, it is exactly the Π_2^p SAT problem.

For formal completeness, we go through the trivial two-way reduction. In Case Even, we can create a Σ_2^p SAT instance, giving True's variables as the "exists" variables, and False's variables as the "for all" variables. In Case Odd, we can use a Π_2^p SAT instance by similarly giving False's variables as the "for all" variables, and True's variables as the "exists" variables. As previously described, both of these will output which player wins correctly.

For the other direction, we can do the same with both, only in the other direction. In other words, make all "exist" variables into True's variables and all "for all" variables into False's variables. Then, we have a corresponding Quantum Phantom Move QBF instance. ◀

We now turn our attention to Partition-Free QBF and its quantumized complexity. In contrast to standard QBF games, the partition-free version relaxes the order requirement on setting the variables. As we discussed before, we will refer to this family of QBF-based games as the QSAT family. Classically, all variants of QSAT are PSPACE-complete games.

This variant of the game is implicitly invoked in several classic reductions, including Schaefer's Avoid True reduction [31], and we will use this fact to prove the quantum version.

▶ **Theorem 14** (Quantum Collapses of Partition-Free QBF). *Phantom Move QSAT with a classical start is Σ_2-complete in Case Even and Π_2-complete in Case Odd.*

Proof. We will call the player that makes the phantom move the PM Player and the other player as the NM Player, for short. Then, the theorem follows from the following observations:

▶ **Observation 15.** *If the PM Player has a winning strategy, then arbitrarily assigning each variable to $\langle$`true` $|$ `false`$\rangle$ is also a winning strategy.*

Proof. Suppose for the sake of contradiction that the PM player has a winning strategy S_1 but the only quantum strategy S_2 isn't a winning strategy. Then, that means that NM Player has a sequence of moves such that in no realizations, the PM player has fulfilled their winning condition, as otherwise they could move into the phantom move. But then, that exact sequence of moves is a winning sequence of moves against S_1, and any possible deviations of it. Therefore S_1 isn't a winning strategy. ◀

▶ **Observation 16.** *If the NM player has a winning strategy, then they have a winning strategy of selecting classic moves independent the PM player's choices.*

Proof. If the NM player has a winning strategy against PM player, then they must also have one against the PM Player's strategy of only selecting quantum assignments as mentioned in Observation 15. Regardless of what this strategy is, the fact that it wins means that it must win against all realizations of this strategy, since otherwise the PM Player could collapse to that realization on the phantom move, then win. The NM Player can do this with only classic moves, as any realization of a quantum strategy must also not have any winning realizations for the NM Player either. ◀

From here, we can form a trivial reduction to and from Σ_2^p-SAT and Π_2^p-SAT for instances where False gets the phantom move and True gets the phantom move, respectively. ◀

4.2 Quantum Lift of Schaefer's Partisan-Impartial Reduction

We now prove Theorem 6, showing that Quantum Boolean Nim is intractable, with complexity between NP and PSPACE. We will reduce to it – via Quantum Avoid True – from the quantum generalization of Phantom-Move QSAT. In Phantom-Move QSAT, one can equivalently define the "phantom move" to be included with the final variable selection, such that the player that would assign the final variable may only do so if they will have reached their win condition upon playing that variable. One can reduce from the other Phantom move variant by introducing a variable v_{n+1} which only appears in a clause as $(v_{n+1} \vee \neg v_{n+1})$ given to the phantom move player. So the hardness remains the same. We will be reducing from this variant of the game. Classically, this game, as proved by Schaefer [31], is PSPACE-complete. Below, we will use this fact to establish the following theorem.

▶ **Theorem 17** (Complexity of QUANTUM AVOID TRUE). *QUANTUM AVOID TRUE is Σ_2^p-hard.*

Proof. Consider a PHANTOM-MOVE QSAT instance Z with CNF. In the proof, we will focus on the case where both players have the same number of variables, which is an even number. This is acceptable since the hardness results remain even when fixed to any arbitrary parity. Recall that the player that goes first will be called the *True player* and the player that goes second the *False player*. Below in both Z and its quantum generalization $Z^{\mathcal{Q}}$, we will denote the True Player's variables by $\{T_1, \ldots, T_m\}$ and the False Player's variables by $\{F_1, \ldots, F_m\}$.

In our proof, we will show that Schaefer's reduction from PHANTOM-MOVE QSAT to AVOID TRUE in the classical setting can be quantum lifted into a QUANTUM AVOID TRUE instance to encode $Z^{\mathcal{Q}}$: Schaefer's reduction (in our notation) is the following: (1) For each True Player's variable T_i, we introduce two new variables T_{i_1} and T_{i_2}, and create a positive clause $(T_{i_1} \vee T_{i_2})$. We will collectively call these clauses *TV clauses.* T_{i_1} will represent assigning a true literal, and T_{i_2} will represent assigning a false literal. (2) For each False Player's variable F_i, we introduce three new variables F_{i_1}, F_{i_2}, and F_{G_i}, and create a clause $(F_{i_1} \vee F_{i_2} \vee F_{iG})$. We will collectively call these clauses *FV clauses.* As before, F_{i_1} will represent assigning a true literal, and F_{i_2} will represent assigning a false literal. The F_{iG} variable is simply an arbitrary variable for the purposes of ensuring a certain clause parity (which we will give the motivation for later). It will function as an alternate truth literal assignment, as it appears in every clause F_{i1} does. (3) For each of the clauses in Z, we replace each instance of positive literal T_i and F_i with T_{i_1} and F_{i_1}, respectively; we replace each instance of negated variable $\neg T_i$ and $\neg F_i$ with T_{i_2} and F_{i_2}, respectively. We will collectively call these clauses *QBF clauses.* (4) For each instance of a variable T_{i_1}, F_{i_1}, T_{i_2}, and F_{i_2} in the QBF clauses, we add, to same clause, variables T'_{i_1}, F_{iG} T'_{i_2}, or F'_{i_2}, respectively. We will collectively call these new variables *duplicate variables.* These serve both as a way to ensure the QBF clauses all have even parity, and for player strategy, as we will cover later.

Through our proof, we will let X denote the AVOID TRUE instance obtained from Schaefer's reduction of PHANTOM-MOVE QSAT instance Z. We will call the first player in X Player True and the second player Player False. Schaefer proved that True can win (the impartial) X if and only if True can win (the partizan) Z. Below, we will extend Schaefer's proof to show that their quantum generalization $Z^{\mathcal{Q}}$ and $X^{\mathcal{Q}}$ has the same winner (when optimally played). Before going on with our proof, we first recall one of the key properties, formulated by Schaefer, and extend it to the quantum setting.

▶ **Lemma 18** (Avoid-True Destiny). *For any classical position created by Schaefer's reduction, if all unsatisfied clauses have an even number of variables, then Player True will win, under arbitrary play by both players for the remainder of the game. If instead all unsatisfied clauses have an odd number of variables, then Player False will win, under arbitrary play by both players for the remainder of the game.*

Lemma 18 captures the "parity" design used in Schaefer's reduction. It also defines a scenario satisfying the condition of "quantumness doesn't matter."

▶ **Lemma 19** (When Quantum Doesn't Matter). *If all realizations of a game classically have only one possible winner, no matter what sequence of moves either player makes, then quantumness doesn't matter.*

Proof. In order for a game to classically have only a single possible winner, all paths to a leaf node in a game tree must be of the same parity. For a game to end, all realizations must have no moves remaining. Otherwise, the rule sets would allow for some kind of move to be made. As such, all games end with all remaining realizations at leaf nodes. Since the number of moves to get here is the same in all realizations, all paths to the game's end must have the same parity as the classical game's terminal nodes. Then, that same player wins no matter what moves either player makes. Therefore, quantumness doesn't matter. ◀

Combining Lemma 18 and Lemma Lemma 19, we get:

▶ **Corollary 20** (Quantum-Avoid True Destiny). *For any position reached in Schaefer's reduction, if in all realizations, there are only TV clauses and/or QBF clauses unsatisfied, then False wins; if in all realizations, there are only FV clauses unsatisfied, than True wins.*

To prove the correctness of this reduction in the quantum setting, we simply need to prove that if True has a winning strategy in the instance of Z^Q that we are reducing from, then True has a winning strategy in X^Q, and that if False has a winning strategy in Z^Q, then False has a winning strategy in X^Q. In the proof below, we will crucially use the following fact that we will establish in Observation 16: True has a winning strategy in Quantum Phantom-Move QSAT if and only if True has a single assignment (of only classical moves) that can always win, regardless of what moves False makes. So, if True is the winner in Z^Q, we perscribe the following strategy for True in X^Q: (1) Assign variables in the TV clauses according the winning strategy for Z^Q (2) Assign all of the F'_{i2} variables. Our proof is based on the following key observation is: If True is able to follow this strategy to completion, then all TV and QBF clauses must be satisfied, resulting in a True win by Corollary 20.

To proceed, we just need to show that True's strategy is always a legal set of moves. First, True must be able to satisfy all m TV clauses, because False can't assign all m FV clauses before True, thus whatever TV clause true assigns last can't be the last. Then, because that variable assignment in Z must have satisfied all clauses regardless of how the FV variables were assigned, that means in no realization is it legal to make a move that would satisfy all FV clauses, as all QBF clauses must be satisfied at the same time. Since all realizations still have the FV clauses unsatisfied, True can then assign all F'_{i2} variables, since none of them appear in an FV clause. So, that strategy is always achievable.

Now, for the second case. Suppose that False wins the instance Z^Q. Then, False applies the following strategy: (1) For False's move $i \leq m-1$, if F_{i1} and F_{i2} are not yet classically assigned, make a quantum move of $[F_{i_1} \mid F_{i_2}]$. Otherwise, play arbitrarily on the FV clauses not yet assigned in all realizations. (2) For the final move, assign the variable in final FV to either `true` or `false`, whichever is a legal move. If both are, choose arbitrarily.

Note that if False is able to carry out this strategy, then by Corollary 20, they will win the game, as they will have satisfied all FV clauses leaving only TV and/or QBF clauses remaining. For the proof of the correctness of this strategy, first note that, as we will prove in Observation 15, if False has a winning strategy in Quantum Phantom-Move QSAT, then repeatedly choosing, for an arbitrary variable, a quantum assignment of $\langle$`true` | `false`$\rangle$ is also a winning strategy. If True has only played consistently, then False will be able to make a move on the final unsatisfied FV clause, as there exists a realization where not all QBF clauses are satisfied. If True ever makes a move with at least one realization that isn't on a variable in a TV clause, then when False moves to the final FV clause, there exists a realization where True hasn't assigned all of the TV clauses, so False can make their final move arbitrarily on the clause. ◀

5 Quantum Graph Games

In this section, we consider quantum transformation of several well-studied PSPACE-complete combinatorial games. In addition to GEOGRAPHY, we will also analyze the following games:

- NODE KAYLES: An impartial game where players alternate turns placing tokens on vertices of a given graph. A player is only able to place a token on vertex if that vertex does not already contain a token and is not adjacent to any vertex with a token. As such, a player is unable to move when the tokens form an *maximal* independent set.
- BIGRAPH NODE KAYLES: A variant of NODE KAYLES in which nodes are partitioned into red nodes and blue nodes, where the blue player can only play on blue vertices and the red player can only play on red vertices.
- SNORT: A game where one player is a blue player, and the other is a red player. Players alternate placing tokens of their color onto vertices of a given graph. Players can't place tokens on vertices adjacent to vertices with a token of the opponent's color.

▶ **Theorem 21.** *QUANTUM GEOGRAPHY with a classical start is* PSPACE-*complete.*

Proof. We have a very simple reduction from classic GEOGRAPHY. We replace each edge in the GEOGRAPHY graph as shown in Figure 4 with a path through two new vertices.

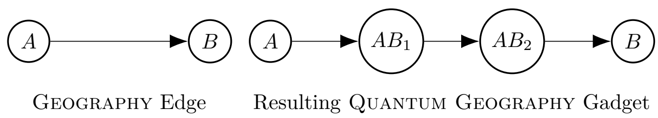

Figure 4 PSPACE Reduction to QUANTUM GEOGRAPHY is a simple transformation on the edges.

Now, if a player ever makes a quantum move from a classical move, e.g. from A to the super position of AB_1 and AC_1, then the opponent can immediately collapse to either AB_2 or AC_2, effectively choosing which of B and C will be moved to. Thus, making a quantum move only gives the next player the power to choose your move and will never give a classically-losing player a winning quantum strategy. ◀

In the full version, we prove the following theorem.

▶ **Theorem 22.** *QUANTUM NODE KAYLES and QUANTUM BIGRAPH NODE KAYLES are* PSPACE-*complete.*

▶ **Theorem 23.** *QUANTUM SNORT is* PSPACE-*complete.*

Proof. We reduce from BIGRAPH NODE KAYLES. We simply create an extra vertex connected to all red vertices and place a red token on it, and then create a vertex with a blue token on it connected to all blue vertices. See Figure 5. ◀

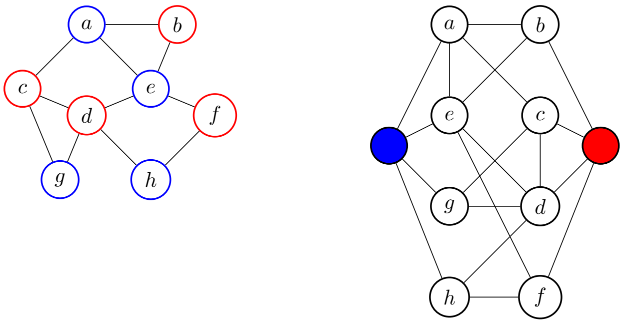

Figure 5 Graph for Quantum BiGraphNodeKayles, followed by the result of the reduction to Quantum Snort on that graph.

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