

Mirror Games Against an Open Book Player*

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Abstract

Mirror games were invented by Garg and Schnieder (ITCS 2019). Alice and Bob take turns (with Alice playing first) in declaring numbers from the set $\{1, 2, \dots, 2n\}$. If a player picks a number that was previously played, that player loses and the other player wins. If all numbers are declared without repetition, the result is a draw. Bob has a simple mirror strategy that assures he won't lose and requires no memory. On the other hand, Garg and Schenier showed that every deterministic Alice needs memory of size linear in n in order to secure a draw.

Regarding probabilistic strategies, previous work showed that a model where Alice has access to a secret random perfect matching over $\{1, 2, \dots, 2n\}$ allows her to achieve a draw in the game w.p. a least $1 - \frac{1}{n}$ and using only polylog bits of memory.

We show that the requirement for secret bits is crucial: for an “open book” Alice with no secrets (Bob knows her memory but not future coin flips) and memory of at most $n/4c$ bits for any $c \geq 2$, there is a Bob that wins w.p. close to $1 - 2^{-c/2}$.

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1 Introduction

In the mirror game, Alice (the first player) and Bob (the second player) take turns picking numbers belonging to the set $\{1, 2, \dots, 2n\}$. A player loses if they repeat a number that has already been picked. After $2n$ rounds, when no more numbers are left to say, then the result of the game is a draw. With perfect memory, both players can keep track of all the numbers that have been announced and avoid losing.

Bob, who plays second, has a simple deterministic, low memory strategy (which can actually be performed even by humans): Bob fixes any perfect matching on the elements $\{1, 2, \dots, 2n\}$; for example $(1, 2), (3, 4), \dots, (2n - 1, n)$. For every number picked by Alice, Bob responds with the matched number. Unless Alice repeats a number, this allows Bob to pick in every turn a number that has not appeared so far. Note that Bob needs to remember just the last number Alice played, i.e. $O(\log n)$ bits, in order to execute this strategy.

This game was suggested by Garg and Schneider [4] as a very simple example of a mirror strategy (hence the name). Given the limited computational resources they require, the question is when can they be implemented. In particular whether Alice has a lower memory strategy as well and if not what is it that breaks the symmetry between the two players.

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What Garg and Schneider showed is that every deterministic “winning” (drawing) strategy for Alice requires space that is linear in n . I.e. there is no mirror like strategy for her.

What about probabilistic strategies? Here we have to consider the model carefully. Suppose that Alice has a secret memory of some bounded size (we call this the “closed book” case) as well as a supply of random bits (that Bob cannot access). Garg and Schneider showed a randomized strategy that manages to draw against any strategy of Bob with probability at least $1 - \frac{1}{n}$ and requires $O(\sqrt{n} \log^2 n)$ bits of memory while relying on access to a *secret random perfect matching oracle*. Feige [2] showed an improved randomized strategy, under the same settings, that draws with the same probability and needs only $O(\log^3 n)$ bits of memory. Menuhin and Naor [6] showed that such strategies can be implemented with either $O(n \log n)$ bits of long lasting randomness (in addition to the $O(\log^3 n)$ bits of memory), or to replace the assumptions with cryptographic ones (assuming that Bob cannot break a one-way function) and obtain a low memory strategy for Alice.

1.1 An Open Book Alice

In contrast to the above model, we consider the case where Alice has *no secret memory*. Alice has $m < 2n$ memory bits, but their content is known to Bob at any point. She has an “unlimited” supply of randomness: in each turn Alice can ask for any number of additional random bits, and then she has access to those bits throughout the rest of the game. However, since Alice has no secrets, Bob knows her random bits *after she asks for them*¹. The question we consider is whether Bob has a strategy that promises to win (forces Alice to lose) with high probability when Alice has such limited memory?

As usual, we assume that Bob is aware of Alice’s strategy and creates his own strategy accordingly. The question is whether for any Alice that has sublinear memory there is a Bob that makes her lose with certain probability. As we shall see, this is indeed the case. We will first see a strategy that makes Alice lose with probability about 1/2 (Theorem 12) and then show how to amplify this to any constant (Theorem 15): any Alice that has memory of size smaller than n/c loses except with probability at most $2^{-\gamma c}$ for some constant $\gamma > 0$.

Note that if Alice chooses all her random bits in advance, then a simple argument shows that she will lose with probability 1: once the random bits are chosen, then her strategy is deterministic and from the work of Garg and Schneider there is the best choice of Bob’s strategy that makes her lose. So the whole difficulty revolves around the issue of Alice choosing the random bits on the fly.

2 Combinatorial Preliminaries

2.1 Oddtown-Eventown

Garg and Schneider [4], in showing their linear lower bound on the space complexity of any deterministic winning or drawing strategy of Alice, used the famous Eventown-Oddtown Theorem of Berlekamp from extremal combinatorics. This theorem puts a bound on the maximum number of even sets where every pair has an odd intersection:

► **Definition 1.** An “Oddtown” is a collection \mathcal{F} of subsets of $\{1, 2, \dots, N\}$ where every subset $s \in \mathcal{F}$ is of even cardinality, but the intersection of every pair of distinct subsets $s_i, s_j \in \mathcal{F}$ has an odd cardinality.

¹ This is similar to the “full information model” in distributed computing.

► **Theorem 2.** *Every Oddtown contains at most N subsets.*

A proof can be found, for instance, in [3]. We will use the following corollary of the theorem:

► **Corollary 3.** *Let \mathcal{F} be a collection of subsets of $\{1, 2, \dots, N\}$ where every subset $s \in \mathcal{F}$ has an even cardinality. For every integer j , if $|\mathcal{F}| > N + j$ then \mathcal{F} contains at least $\lceil \frac{j}{2} \rceil$ disjoint pairs of subsets from \mathcal{F} , where for each pair (s_i, s_j) the union $s_i \cup s_j$ is of even cardinality. We call this a matching P of pairs.*

Proof. As long as \mathcal{F} contains more than N elements, by Theorem 2 it contains pair of distinct subsets that have an intersection of even cardinality. Since every subset in \mathcal{F} has even cardinality, the union of the pair has even cardinality as well. Put the pair in the matching P , delete the pair from \mathcal{F} and continue the process. ◀

▷ **Claim 4.** Let $\mathcal{A}_1, \dots, \mathcal{A}_m$ be events whose union is the entire sample space. Let $H_b = \{\mathcal{A}_i \mid \Pr(\mathcal{A}_i) > b, i \in \{1, \dots, m\}\}$ be the set of events with relative “high” probability. Let \mathcal{A} be the event that one of the events from H_b happens. If $b < \frac{1}{m}$, then $\Pr[\mathcal{A}] \geq 1 - (m - 1) \cdot b$.

► **Definition 5.** *A function $f: \mathbb{N} \rightarrow \mathbb{R}$ is negligible if for every positive integer c there exists an integer N_c such that for all $x > N_c$, $|f(x)| < \frac{1}{x^c}$. We use the term **negl** to denote some negligible function.*

3 Introduction to the Mirror Game

3.1 Strategies and memory

Alice has $m < n$ bits of memory, and we refer to her state as $x \in \{0, 1\}^m$. In addition, at the end of turn $2i$ (after Bob picks a number) Alice chooses random bits $r_i \in \{0, 1\}^\ell$, when this r_i is added to the set of random strings she can access. I.e. at the i th round Alice has at her disposal the memory state $x \in \{0, 1\}^m$ and $R_A^i = (r_1, r_2, \dots, r_i)$. The size of ℓ can be arbitrary large. If the value of i is clear from the context, we will simply use R_A .

Formally, the description of Alice’s strategy consists of two functions:

1. **Next move function** $next: \{0, 1\}^m \times \{0, 1\}^{i \cdot \ell} \rightarrow \{1, 2, \dots, 2n\}$. receiving the current memory state, the random bits chosen in steps r_1, \dots, r_i and outputting a number in $\{1, 2, \dots, 2n\}$.
2. **State transition function** $upd: \{1, \dots, 2n\} \times \{0, 1\}^m \times \{1, \dots, 2n\} \times \{0, 1\}^{i \cdot \ell} \rightarrow \{0, 1\}^m$.

Receiving the last move of Bob, the current memory state, turn number $2i + 1$, the random bits chosen in steps r_1, \dots, r_i and assigning a new memory state.

Note that Bob knows these functions. In the context of the random bits of Alice, when Alice receives a new sequence of random bits r_i , Bob has access to r_i as well (but not to the future bits, before they are selected).

Let R_A and R_B be the random variables that represent the random bits of Alice and Bob respectively. Note that in the beginning of turn $2i + 1$, $R_A^i = (r_1, r_2, \dots, r_i)$ and at that point Bob knows the value of R_A^i . Let S^i be the random variable that represents the subset of numbers that have been picked by either player until turn i . Similarly, X_i is the random variable that represents the memory state of Alice in the i th turn. If the value of i is clear, we will simply use X and S .

Given $x \in \{0, 1\}^m$, turn $i \in \{1, 2, \dots, 2n\}$ and R_A , for a subset $s \subseteq \{1, 2, \dots, 2n\}$ of size i we will consider the following notation:

- $\Pr[S = s] := \Pr_{R_B}[\text{set of used numbers is } s \mid R_A = r_A]$

20:4 Mirror Games Against an Open Book Player

■ $\Pr[X_i = x] := \Pr_{R_B}[X_i = x | R_A = r_A]$, and if i is clear we will use $\Pr[X = x]$.

Now we will consider the conditional probability that s was the subset of chosen numbers that lead to memory state x :

$$\Pr[S = s | x] := \Pr_{R_B}[\text{The set of used numbers is } s | \text{memory state is } x, R_A^i = r_A].$$

► **Definition 6.** For turn $1 \leq i \leq 2n$, memory state $x \in \{0, 1\}^m$ and random strings r_A , denote with $\mathcal{F}_x^{r_A}$ the collection of sets $s \subseteq \{1, 2, \dots, 2n\}$ of size i s.t. it is possible that the state of memory of Alice at turn i is x when the set of numbers played is equal to s and Alice's random string is r_A . Call those sets the feasible sets.

For ease of notation we sometime use \mathcal{F}_x , instead of $\mathcal{F}_x^{r_A}$.

4 Bob's Strategy Against an Open Book Alice

We present a strategy for Bob against any Alice with $o(n)$ memory and without secrets. Namely, Bob has access to Alice's memory state, random bits as well as her strategy, namely the state transition function upd and next move function $next$, but not to any future random bits. We will first see in Section 4.1 a strategy that makes Alice lose with probability about $1/2$. We will then see in Section 4.2 how to amplify this strategy and reduce Alice's probability of winning to any $1/2^c$ constant at the price of limiting her memory to be at most n/ac for some constant a .

The rough outline of the strategy is for Bob to start by a sequence of random moves that do not include any of the numbers selected. At some midpoint Bob changes course. He selects a pair of subsets, so that one of them is the correct one played up to this point and the other one is a "decoy". The pair is selected so that from Alice's point of view these two sets are indistinguishable. Bob continues by avoiding elements that belong to this pair. Alice has to be the first one to venture into this territory and make a guess who is the real one and who is the decoy. She loses with probability $1/2$. The strategy is not computationally efficient, even if Alice has an efficient strategy, since Bob needs to come up with the pairing.

4.1 Bob Can Win With Probability Close to Half

We present a strategy for Bob that makes him win with probability close to half. Let $1 \leq k \leq [n]$ be a value that will be determined later (we will see that $k = 0.4n$). Until turn $2k$ Bob chooses a number uniformly at random from those that have not been picked so far.

Let s be the actual set of used numbers and let $x \in \{0, 1\}^m$ be the memory state of Alice after turn $2k$. We will show that if Alice has low memory, then with high probability Bob can find a partition of the feasible sets $\mathcal{F}_x^{r_A}$ (Definition 6) into pairs. For every pair (s_1, s_2) in the partition, (i) the two subsets s_1 and s_2 have similar conditional probabilities that they are the sets actually played given memory state $x \in \{0, 1\}^m$ and randomness r_A , and (ii) the cardinality of the union of the pair is even. Namely,

1. $\Pr_{R_B}[S = s_1 | X = x, R_A = r_A] \approx \Pr_{R_B}[S = s_2 | X = x, R_A = r_A]$.
2. $|s_1 \cup s_2|$ is even.

Note that this partition does not depend on s , just on R_A and x . But as we show w.h.p s belongs to some pair in the partition, meaning that there exists (s_1, s_2) in the partition s.t. $s \in \{s_1, s_2\}$. Bob's strategy after turn $2k$ is just to pick some number that has not been used and that is not in $s_1 \cup s_2$.

We claim that since $|s_1 \cup s_2|$ is even, Alice must be the one that chooses the first value in $s_1 \cup s_2$ after turn $2k$. Since s_1 and s_2 have similar probabilities, even if a “little birdy” tells Alice about s_1 and s_2 , she will not be able to differentiate between them in real-time. Namely, she does not know whether $s = s_1$ or $s = s_2$. Therefore she loses with probability close to half.

Bob’s strategy for making Alice lose with probability 1/2

1. From turn 2 until turn $2k$: Bob chooses uniformly at random a number that has not been picked so far.
2. Following turn $2k$ Bob finds two sets s_1 and s_2 of size $2k$ s.t. the actual set of number picked so far s is equal to one of them, and $|s_1 \cup s_2|$ is even.
3. From turn $2k$ till the end, Bob picks some number that has not been used and that is not in $s_1 \cup s_2$.

Note that after turn $2k$ Bob’s strategy can be deterministic. For example pick the smallest number that has not been used and that is not in $s_1 \cup s_2$.

▷ **Claim 7.** For every subset $s \subseteq \{1, 2, \dots, 2n\}$ and for every assignment r_A to the random bits of Alice, if $|s| = 2k$ then

$$\Pr_{R_B}[S = s | R_A = r_A] \leq \left(\frac{2k}{2n}\right)^k$$

Proof. For every turn $i \in \{2, 4, \dots, 2k\}$, if all of the numbers picked so far belong to s , then the cardinality of the set of numbers that have not been used and belongs to s is exactly $2k - i + 1$. Thus the probability that Bob chooses a number from s is exactly $\frac{2k-i+1}{2n-i+1}$ independent from r_A . Hence, let $Z = \{2, 4, \dots, 2k\}$ and we get:

$$\Pr_{R_B}[S = s | R_A = r_A] \leq \prod_{i \in Z} \frac{2k - i + 1}{2n - i + 1} \leq \prod_{i \in Z} \frac{2k - 1}{2n - 1} = \left(\frac{2k - 1}{2n - 1}\right)^k \leq \left(\frac{2k}{2n}\right)^k \quad \triangleleft$$

► **Corollary 8.** For every subset $s \subseteq \{1, 2, \dots, 2n\}$ and for every assignment r_A to the random bits of Alice, if $k = 0.4n$ and $|s| = 2k$ then

$$\Pr_{R_B}[S = s | R_A = r_A] \leq 0.4^{0.4n} = 0.4^{0.2 \cdot 2n} \leq 0.84^{2n}$$

These numbers were chosen for convenience, where the goal is to get an upper bound on $\Pr_{R_B}[S = s]$ of the form α^{2n} for some constant $0 < \alpha < 1$.

We call a memory state x *useful* for r_A if $\Pr_{R_B}[X_{2k} = x | R_A = r_A] > 0.9^{2n}$. Once again, this number was chosen for convenience. We want to get a lower bound on $\Pr[X = x | R_A = r_A]$ of the form β^{2n} for some constant $0 < \beta < 1$ s.t. $\alpha < \beta$. Let $U_{r_A} \subseteq \{0, 1\}^m$ be the set of useful memory for r_A . We argue that w.h.p. Alice’s memory state in turn $2k$ is useful.

▷ **Claim 9.** Let X be the random variable representing the memory state after turn $2k$. For any assignment to Alice’s coin flips r_A , if $m < 0.2n$, then

$$\Pr_{R_B}[X \in U_{r_A} | R_A = r_A] \geq 1 - \text{negl}(n).$$

Proof. Let A_x be the event that $X = x$. Note that for every $x \in \{0, 1\}^m$ we get that $\Pr[A_x] \leq 0.9^{2n} \leq \frac{1}{2^m}$. Then by Claim 4:

$$\Pr_{R_B}[x \in U_{r_A}] \geq 1 - (2^m - 1) \cdot 0.9^{2n} > 1 - 0.97^{2n} + 0.9^{2n} = 1 - \text{negl}(n).$$

where the last inequality is true since $2^m \cdot 0.9^{2n} \leq 2^{0.2n} \cdot 0.9^{2n} < 0.97^{2n}$. ◁

20:6 Mirror Games Against an Open Book Player

▷ **Claim 10.** For any assignment r_A , if x is useful, then Bob can find a subset $W_x^{r_A} \subseteq \mathcal{F}_x^{r_A}$ and a partition $P_x^{r_A}$ of $W_x^{r_A}$ into pairs s.t.:

1. $\Pr_{R_B}[S \in W_x^{r_A} | x \in U_{r_A}, R_A = r_A] > 1 - \text{negl}(n)$.
2. For every pair $(s_1, s_2) \in P_x^{r_A}$ we have that $|s_1 \cup s_2|$ is even.
3. For every pair $(s_1, s_2) \in P_x^{r_A}$ the ratio of their conditional probabilities is close to 1:

$$\frac{2n}{2n+1} \leq \frac{\Pr_{R_B}[S = s_1 | x, R_A = r_A]}{\Pr_{R_B}[S = s_2 | x, R_A = r_A]} \leq \frac{2n+1}{2n}$$

Proof. The idea is to partition $\mathcal{F}_x^{r_A}$ into layers of the possible subsets that lead to memory state x according to the values of their conditional probability. The pair $(s_1, s_2) \in P_x^{r_A}$, only if $\Pr[S = s_1 | x]$ and $\Pr[S = s_2 | x]$ belong to the same layer and $|s_1 \cup s_2|$ is even. We consider $(2n)^3$ different layers. For $j \in \{1, \dots, (2n)^3\}$ let the j th interval of probabilities be

$$I_j = \left[\left(\frac{2n}{2n+1} \right)^j, \left(\frac{2n}{2n+1} \right)^{j-1} \right].$$

Every layer L_j represents the collection of subsets whose conditional probability given memory state x and r_A is in the interval I_j . Thus, $L_j = \{s \in \mathcal{F}_x^{r_A} | \Pr_{r_B}[S = s | X = x] \in I_j\}$. Note that if s_1 and s_2 in L_j , then

$$\frac{2n}{2n+1} \leq \frac{\Pr_{r_B}[S = s_1 | x]}{\Pr_{r_B}[S = s_2 | x]} \leq \frac{2n+1}{2n}.$$

Note that every subset in L_j has even cardinality. Thus, if layer L_j contains more than $2n$ subsets, then by Claim 3 it contains at least $\frac{|L_j| - N}{2}$ pairs with an even intersection. Call this set of pairs P_j . Let $P_x^{r_A}$ be the collection of all pairs from $P_1, P_2, \dots, P_{(2n)^3}$. Denote the collection of all subsets that belong to some pair in $P_x^{r_A}$ as $W_x^{r_A}$. Namely,

$$P_x^{r_A} := \{(s_1, s_2) \in P_j | j \in [(2n)^3]\} W_x^{r_A} := \{s \subseteq \{1, 2, \dots, 2n\} | \exists s', (s, s') \in P_x^{r_A}\}.$$

We will see that $P_x^{r_A}$ and $W_x^{r_A}$ meet all the three conditions of the claim:

1. Since x is useful and by Corollary 8 we get that for every $s \in \mathcal{F}_x^{r_A}$:

$$\Pr_{R_B}[S = s | X = x, R_A = r_A] \leq \frac{\Pr[S = s | R_A = r_A]}{\Pr[X = x | R_A = r_A]} \leq \frac{0.84^{2n}}{0.9^{2n}} < 0.934^{2n}$$

Therefore for every $F \subseteq \mathcal{F}_x^{r_A}$, if F contains at most a polynomial number of subsets, then $\Pr[S \in F | X = x]$ is negligible. There are $(2n)^3$ layers, and in every layer i , there are at most $2n$ subsets that do not belong to P_i . Thus we get at most $(2n)^4$ subsets that not belong to $P_x^{r_A}$. In addition define:

$$L_x = \{s \in \mathcal{F}_x^{r_A} | \Pr[S = s | x] < \frac{1}{2^{4n}}\}.$$

Since the cardinality of L_x is at most 2^{2n} , we get that

$$\Pr[S \in L_x | X = x] < 2^{2n} \cdot \frac{1}{2^{4n}} = 2^{-2n}.$$

We choose $(2n)^3$ layers, since $\left(\frac{2n}{2n+1}\right)^{(2n)^3-1} \leq \frac{1}{2^{4n}}$ for large enough n . Overall $W_x^{r_A}$ contains all of the subsets from $\mathcal{F}_x^{r_A}$, except some of the subsets in L_x and at most $(2n)^4$ other subsets, then the claim is follows.

2. Correct by the construction of $P_x^{r_A}$.
3. By construction for every pair $(s_1, s_2) \in P_x^{r_A}$ there exists j s.t $(s_1, s_2) \in L_j$. ◁

► **Observation 11.** *Since $|s_1 \cup s_2|$ is even, then Alice is the first to pick an element of $s_1 \cup s_2$ after turn $2k$.*

Now we will show that even if from turn $2k$ and on Alice has an unlimited size memory and randomness, and even if Alice knows $W_x^{r_A}$ for every memory state x , she still loses with probability close to half.

► **Theorem 12.** *If Alice has no secrets, and memory of size at most $0.2n$ bits, then there is a strategy for Bob where Alice loses with probability at least $\frac{1}{2} \cdot \frac{2n}{2n+1} - \text{negl}(n)$*

Proof. Let r_A be assignment to the random bits of Alice. If a memory state X is useful for r_A and $S \in W_x^{r_A}$, then there exists $(s_1, s_2) \in P_x^{r_A}$ s.t. $S \in (s_1, s_2)$, $|s_1 \cup s_2|$ is even and

$$\frac{2n}{2n+1} \leq \frac{\Pr[S = s_1|x]}{\Pr[S = s_2|x]} \leq \frac{2n+1}{2n}$$

From turn $2k$, Bob chooses a number that has not been used and not in $s_1 \cup s_2$. Then by Observation 11 the first one after turn $2k$ to pick a number from $s_1 \cup s_2$ is Alice. Intuitively, since s_1 and s_2 have similar conditional probability given X and r_A , Alice cannot determine whether $S = s_1$ or $S = s_2$. Then the best choice Alice can make the first time she picks a number from $s_1 \cup s_2$, is to choose a number from the set with the lowest conditional probability. But even then, she loses w.p. similar to half.

Formally, let a be the first number from $s_1 \cup s_2$ that Alice picks after turn $2k$. We want to analyze:

$$\Pr_{R_B}[a \in S | X \text{ is useful}, S \in W_X^{r_A}, R_A = r_A]$$

Note that if $a \in S$, then Alice loses. Indeed, for any randomness r_A and useful memory state $x \in U_{r_A}$:

$$\begin{aligned} & \Pr_{r_B}[a \in S | X = x, S \in \{s_1, s_2\}, (s_1, s_2) \in P_x^{r_A}, R_A = r_A] \\ & \geq \min_{s' \in \{s_1, s_2\}} \Pr_{r_B}[S = s' | X = x, S \in \{s_1, s_2\}, (s_1, s_2) \in P_x^{r_A}, R_A = r_A] \end{aligned} \quad (1)$$

$$\begin{aligned} & = \min_{s' \in \{s_1, s_2\}} \frac{\Pr_{r_B}[S = s' | X = x, R_A = r_A]}{\Pr_{r_B}[S \in \{s_1, s_2\} | X = x, R_A = r_A]} \\ & \geq \min_{s' \in \{s_1, s_2\}} \frac{\Pr_{r_B}[S = s' | X = x, R_A = r_A]}{\Pr_{r_B}[S = s_1 | X = x, R_A = r_A] + \Pr_{r_B}[S = s_2 | X = x, R_A = r_A]} \\ & \geq \frac{\left(\frac{2n}{2n+1}\right)^j}{2 \left(\frac{2n}{2n+1}\right)^{j-1}} \end{aligned} \quad (2)$$

$$= \frac{2n}{2n+1} \cdot \frac{1}{2} \quad (3)$$

Where all the probabilities are over Bob's coins R_B . Inequality 1 is true since given $S \in (s_1, s_2)$, without knowing S , decide if $S = s_1$ or $S = s_2$ is hard in terms of information at least as choosing number from S . Inequality 2 is true by property number 3 of Claim 10.

By Claim 9 the memory state x in turn $2k$ is useful with probability at least $1 - \text{negl}(n)$. By Claim 10 (first property) $S \in W_x^{r_A}$ with probability of at least $1 - \text{negl}(n)$. Finally, by Equation 3 we get that Alice loses w.p. at least $\frac{2n}{2n+1} \cdot \frac{1}{2} - \text{negl}(n)$. ◀

4.2 Amplifying the Probability that Bob Wins

For simplicity, we assume that n is even. Let $c \geq 2$ s.t. n/c is even as well. We will show a strategy for Bob where he wins with probability about $1 - (\frac{1}{2})^{c/2}$ against any open book Alice with bounded memory. The idea is to use similar arguments to the previous section, but instead of finding only one pair of subsets with similar conditional probability and an even union, we would like to find a collection of such pairs. We define $c/2$ breakpoints in time where each breakpoint represents some even numbered turn. Each breakpoint and memory state x , define a new partition to pairs of subsets (where the subsets are possible one that could have been used since the last breakpoint). The subsets are of size n/c and the matched subsets should have similar conditional probabilities.

For $1 \leq t \leq c/2$ and a set s of the numbers used between breakpoint $t-1$ and breakpoint t , we will show that w.h.p. s belongs to some pair of subsets (s_1^t, s_2^t) in the partition of feasible subsets for the given memory and Alice's coins; as before, s_1^t and s_2^t have similar conditional probability and an even union. Then Bob adds this pair to the collection of pairs that he already has from previous breakpoints. He uses this collection of pairs in order to define his winning strategy against Alice. Let us formalize the foregoing discussion.

Breakpoint number t for $t \in \{0, 1, \dots, c/2\}$ is chosen at turn number $k_t := t \cdot \frac{n}{c}$. Since n/c is even then k_t is even.

The span of turns between two successive breakpoints is called an **epoch**. In particular, for $t \in \{1, \dots, c/2\}$, the span of n/c turns between turns $k_{t-1} + 1$ and k_t is called a t -**epoch** (note that the epoch ends at the t th breakpoint.)

At the end of each breakpoint $t \in \{1, \dots, c/2\}$, Bob comes up with new pair (s_1^t, s_2^t) . We will show later how Bob finds these pairs. Denote with $C_t = (s_1^1, s_2^1), \dots, (s_1^t, s_2^t)$ the collection of pairs that Bob comes up until breakpoint number t . Let $C_0 := \emptyset$ and $C_t := C_{t-1} \cup \{(s_1^t, s_2^t)\}$.

Bob's amplified strategy:

Phase 1

- **From turn 2 Until turn $n/2$:** during the t th epoch Bob chooses *uniformly at random* among the elements that do not belong to any subset s that is part of a pair in C_{t-1} . Namely, from

$$\{1, \dots, 2n\} \setminus \bigcup \{s \mid s \in \{s_1^j, s_2^j\} \text{ and } (s_1^j, s_2^j) \in C_{t-1}\}$$

- Let $C = C_{c/2} = \{(s_1^1, s_2^1), \dots, (s_1^{c/2}, s_2^{c/2})\}$ be the collection of pairs of subsets from the first $n/2$ turns. For any turn $n/2 < i \leq 2n$, let C_i^* be all pairs $(s_1^i, s_2^i) \in C$ such that no member of $s_1^i \cup s_2^i$ was played from breakpoint t until turn i .

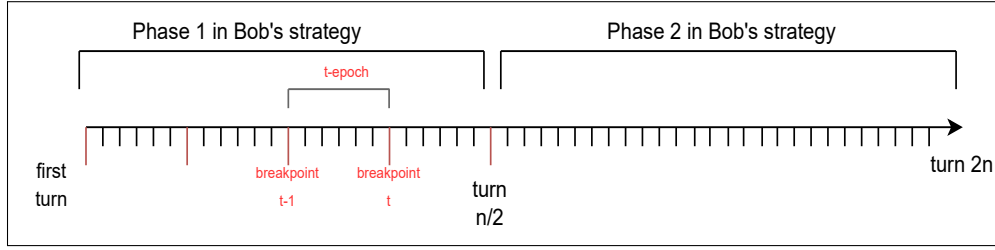
Phase 2

- **For every turn $i \in \{n/2 + 2, n/2 + 4, \dots, 2n\}$,** Bob picks some number that has not been used and not in

$$\bigcup \{s \mid s \in \{s_1^j, s_2^j\} \text{ and } (s_1^j, s_2^j) \in C_i^*\}$$

We still need to specify how Bob sets the pairs (s_1^t, s_2^t) in each breakpoint t , and what is the meaning of similar conditional probability in this case.

Let X^t be the random variable that represents the memory state of Alice at breakpoint number t , let S^t be the r.v. that represents the set of numbers that have been used during the t th epoch, let R_A^t be the r.v. that represents the randomness of Alice until breakpoint



■ **Figure 1** The red lines represent the breakpoints, the black lines represent the turns throughout the game. Not to scale.

number t and let H^t be the r.v. that represent all the history of the game until breakpoint number t . Namely,

$$H^t := (S^1, \dots, S^t, X^1, \dots, X^t, C^t)$$

Note that H_t represents the set of numbers that were played, the memory states of Alice in each breakpoint and the pairs that Bob chooses, until break point t .

Given history h^{t-1} and randomness r_A . Denote with $\mathcal{F}_x^{h^{t-1}, r_A}$ the collection of feasible sets $s \subseteq \{1, \dots, 2n\}$ of size n/c s.t. it is possible that the set of numbers picked during the t th epoch was s given all the history h^{t-1} until breakpoint $t - 1$ and Alice's random string in breakpoint number t is r_A .

For simplicity, we use the notation \mathcal{F}_x^t , instead of $\mathcal{F}_x^{h^{t-1}, r_A}$. We want to find a partition into pairs P_t of \mathcal{F}_x^t that includes most of the subsets in \mathcal{F}_x^t . First we show an upper bound for any set s on the probability that the set played in the $(t) - \text{epoch}$ is s . The number of values from $\{1, \dots, 2n\}$ that are not in C_{t-1} is at least n . Therefore, in each turn during epoch t , Bob has at least n numbers from which he chooses one at random one in Phase 1 of his strategy. If size of s is n/c , then, similarly to Claim 7, we get

$$\Pr_{R_B}[S^t = s | R_A = r_A, H^{t-1} = h^{t-1}] \leq \left(\frac{n/c}{n}\right)^{n/2c} = \left(\frac{1}{c}\right)^{2n/4c}$$

Let $\alpha = \left(\frac{1}{c}\right)^{1/4c}$. Note that $\alpha < 1$. Let $\alpha < \beta < 1$. We call a memory state x **useful** if $\Pr_{R_B}[X^t = x | H^{t-1} = h^{t-1}, R_A = r_A] > \beta^{2n}$. We want to show that w.h.p. Alice's memory state in each breakpoint is useful. Choose $\delta \in (0, 1)$ s.t. $2^\delta \cdot \beta < 1$. Recall that the motivation to define a useful memory state is to have an upper bound on the conditional probability of any set of numbers to be the one used in the $t - \text{epoch}$ over Bob's coins.

Let $U_t^{r_A} \subseteq \{0, 1\}^m$ be the set of useful memory states.

▷ **Claim 13.** Let x be the memory state after turn k_t (breakpoint t). For every randomness r_A and history h^{t-1} , if $m < \delta \cdot 2n$, then $\Pr_{R_B}[X \in U_t^{r_A} | R_A = r_A, H^{t-1} = h^{t-1}] \geq 1 - \text{negl}(n)$.

Proof. By Claim 4:

$$\Pr[x \in U_t^{r_A} | R_A = r_A, H^{t-1} = h^{t-1}] \geq 1 - (2^m - 1) \cdot \beta^{2n} > 1 - (2^\delta \beta)^{2n} + \beta^{2n}$$

which is $1 - \text{negl}(n)$ ◁

20:10 Mirror Games Against an Open Book Player

Recall that the motivation to define a useful memory state is to have an upper bound on the conditional probability of set of used numbers during (t) – epoch over Bob’s coins. In our case, $\Pr[S^t = s | X^t = x, H^{t-1} = h^{t-1}, R^A = r_A, x \in U_t^{r_A}]$ is bounded from above by a^{2n} , for some constant a that is smaller than one. If x is useful we get,

$$\begin{aligned} & \Pr[S^t = s | X^t = x, H^{t-1} = h^{t-1}, R_A = r_A] \\ & \leq \frac{\Pr[S^t = s | R_A = r_A, H^{t-1} = h^{t-1}]}{\Pr[X^t = x | R_A = r_A, H^{t-1} = h^{t-1}]} \\ & \leq \left(\frac{\alpha}{\beta}\right)^{2n} \end{aligned}$$

Thus Claim 10 holds in this case as well. Namely,

▷ **Claim 14.** For any assignment r_A . If x is useful, then Bob can find a subset $W_x^t \subseteq \mathcal{F}_x^t$ and a partition P_x^t of W_x^t into pairs s.t:

1. $\Pr[S^t \in W_x^t | X^t = x, x \in U_t^{r_A}, H^{t-1} = h^{t-1}] > 1 - \text{negl}(n)$
2. For every pair $(s_1, s_2) \in P_x^t$, $|s_1 \cup s_2|$ is even.
3. For every pair $(s_1, s_2) \in P_x^t$:

$$\frac{2n}{2n+1} \leq \frac{\Pr[S^t = s_1 | X^t = x, x \in U_t^{r_A}, H^{t-1} = h^{t-1}]}{\Pr[S^t = s_2 | X^t = x, x \in U_t^{r_A}, H^{t-1} = h^{t-1}]} \leq \frac{2n+1}{2n}$$

Claim 14 gives us the appropriate definition of similar probability for two subsets. Therefore, in every breakpoint t and for a given set s of values picked during t – epoch, Bob choose the pair (s_1^t, s_2^t) s.t. $s \in (s_1^t, s_2^t)$ and $(s_1^t, s_2^t) \in P_x^t$. This pair is then added to C_{t-1} to get C_t .

► **Theorem 15.** *If Alice has no secrets, and memory of size of at most $\delta \cdot 2n$ bits, then there exists a strategy for Bob such that against it Alice loses with probability of at least*

$$1 - \left(\frac{n+1}{2n+1}\right)^{c/2} - \text{negl}(n)$$

Proof. If memory state x^t is useful and $S^t \in W_x^t$, then Bob can find a pair $(s_1^t, s_2^t) \in P_x^t$ s.t. $S^t \in (s_1^t, s_2^t)$, $|s_1^t \cup s_2^t|$ is even and

$$\frac{2n}{2n+1} \leq \frac{\Pr[s_1^t | x, H^{t-1} = h^{t-1}]}{\Pr[s_2^t | x, H^{t-1} = h^{t-1}]} \leq \frac{2n+1}{2n}.$$

From breakpoint t and on, Bob chooses a number that has not been used and not in C_t . Moreover, at every round in epoch $1 \leq t' \leq c/2$ there are at least n numbers that have not been used and are not in $C_{t'}$. Thus Bob has probability of picking a specific number is at most $1/n$.

As in Observation 11, for every $1 \leq t \leq c/2$, the first one to pick a number from $s_1^t \cup s_2^t$ after breakpoint t is Alice, since Bob avoids picking from these sets until Alice does so (and by parity she will be the first to do so). Call the element chosen a_t and the number of turn q_t (note that the q_t ’s are not necessarily increasing in t ; furthermore, q_t might be smaller than $n/2$). Therefore, as in Theorem 12, Eq. 3 and applying Claim 14 we get that,

$$\begin{aligned} & \Pr[a_t \in S^t | X^t \in U_t, S^t \in \{s_1^t, s_2^t\}, (s_1^t, s_2^t) \in P_x^t, H^{t-1} = h^{t-1}, R_A = r_A] \\ & \geq \min_{s' \in \{s_1^t, s_2^t\}} \Pr[S^t = s' | X^t \in U_t, S^t \in \{s_1, s_2\}, (s_1, s_2) \in P_x^t, H^{t-1} = h^{t-1}, R_A = r_A] \\ & \geq \frac{2n}{2n+1} \cdot \frac{1}{2} \end{aligned} \tag{5}$$

In turn q_t Alice needs to determine whether $S^t = s_1^t$ or $S^t = s_2^t$. We claim that even if Alice already guessed (before turn q_t) the values of $S_{j_1}, \dots, S_{j_\ell}$ for breakpoints j_1, \dots, j_ℓ different from t , she still has the same probability of guessing S^t correctly as in Eq. 5. The reason is that s_1^t and s_2^t have similar conditional probability given all of the history until breakpoint $t - 1$. Therefore knowledge of those guesses does not give her any advantage at turn q_t .

By Claim 13 the memory state x^t in breakpoint t is useful with a probability of at least $1 - \text{negl}(n)$. By Claim 14 $S^t \in W_x^t$ with probability of at least $1 - \text{negl}(n)$. Overall we get that for every turn q_t Alice picks a number that has already been played with probability at least

$$\frac{2n}{2n+1} \cdot \frac{1}{2} - \text{negl}(n)$$

In order not to repeat the same number twice and lose the game, Alice needs to guess the right set for S^t for every $t = 1, 2, \dots, \frac{c}{2}$ (in some order). But whenever she makes such a guess, her probability of succeeding given that she has succeeded so far is roughly $(n+1)/(2n)$. Therefore the probability she succeeds in all of the cases is at most

$$\left(\frac{n+1}{2n+1}\right)^{c/2} + \text{negl}(n) \quad \blacktriangleleft$$

► **Corollary 16.** *If Alice has no secrets, and memory of size at most $\frac{n}{4c}$ bits (for $c \geq 2$), then there exists a strategy for Bob that for every $\epsilon > 0$ and large enough n , Alice loses with probability at least*

$$1 - \left(\frac{1}{2} + \epsilon\right)^{c/2}.$$

Proof. The only requirement on δ in Theorem 15 is that $2^\delta \beta < 1$. Therefore we can choose δ s.t.:

$$\delta \in \left(-\frac{3}{4} \log \beta, -\log \beta\right).$$

The only requirement on $\beta \in (0, 1)$ is that $\beta > \alpha = (\frac{1}{c})^{1/4c}$. By choosing β s.t.: $\log \beta \in (\log \alpha, \frac{2}{3} \log \alpha)$ we get that $\delta > -\frac{3}{4} \log \beta > -\frac{3}{4} \cdot \frac{2}{3} \log \alpha = -\frac{1}{2} \log \alpha$. Therefore

$$\delta > -\frac{1}{2} \log \left(\left(\frac{1}{c}\right)^{1/4c}\right) = \frac{1}{8c} \log c > \frac{1}{8c}.$$

By Theorem 15 the corollary follows. ◀

5 Further Work

Note that the strategy for Bob requires exponential computation. The work suggests the question of whether Bob's strategy can be made to be computationally efficient. In particular, a possible direction for getting an efficient strategy for Bob is to simulate Alice on various future selections, with the hope that this reveals which set and matching she is actually storing in her memory and therefore yield a move to play that will make her lose.

Another question is whether it is possible to have a closed book Alice with little memory and only a few bits of long term randomness where even an all powerful Bob will not be able to beat her. Recall that under computational limitations on Bob this is possible, as pointed out by Menuhin and Naor [6].

Is there an extension of Berlekamp's Theorem that will tell us that given enough subsets not only are there two whose union is of even cardinality but there are c sets such that the union of any c' of them is of even cardinality for all $c' \leq c$. This type of result would have made the amplification much simpler.

What happens if both Alice and Bob have limited memory? Is there some sort of stable solution where neither party wants to deviate from a scripted solution (assuming they prefer winning over drawing and that over losing)?

Finally, what is the precise bound on the the memory Alice actually needs in order to survive this ordeal. We can think of a strategy that requires around n bits. Is this tight? Also is there a better amplification where you get that a $o(n)$ memory Alice has a probability of winning that is exponentially small in n .

Adversarial streams and sampling. Recently there has been a lot of exciting research on streaming in adversarial environments. A streaming algorithm is called adversarially robust if its performance holds even when the elements in the stream are chosen adaptively and in an adversarial manner after seeing the current approximation or sample. The question is whether tasks that can be done with low memory against a stream that does not change as a function of the seeing the current state can be done with low memory even if the status is public. For instance, on the positive side, Ben-Eliezer, Jayaram, Woodruff and Yogev [1] showed general transformations for a family of tasks, for turning a streaming algorithm to be adversarially robust (with some overhead). On the other hand, Kaplan, Mansour, Nissim and Stemmer [5] showed a problem that requires polylogarithmic amount of memory in the static case but any adversarially robust algorithm for it requires a much larger memory.

The general question we ask is under what circumstance can we argue that a game where a player has no deterministic low memory strategy does not have an open book randomized one as well.

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