Twisted Ways to Find Plane Structures in Simple Drawings of Complete Graphs

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Abstract

Simple drawings are drawings of graphs in which the edges are Jordan arcs and each pair of edges share at most one point (a proper crossing or a common endpoint). We introduce a special kind of simple drawings that we call generalized twisted drawings. A simple drawing is generalized twisted if there is a point O such that every ray emanating from O crosses every edge of the drawing at most once and there is a ray emanating from O which crosses every edge exactly once.

Via this new class of simple drawings, we show that every simple drawing of the complete graph with n vertices contains $\Omega(n^{\frac{1}{2}})$ pairwise disjoint edges and a plane path of length $\Omega(\frac{\log n}{\log \log n})$. Both results improve over previously known best lower bounds. On the way we show several structural results about and properties of generalized twisted drawings. We further present different characterizations of generalized twisted drawings, which might be of independent interest.

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1 Introduction

Simple drawings are drawings of graphs in the plane such that vertices are distinct points in the plane, edges are Jordan arcs connecting their endpoints, and edges intersect at most once either in a proper crossing or in a shared endpoint. The edges and vertices of a drawing partition the plane (or, more exactly, the plane minus the drawing) into regions, which are called the *cells* of the drawing. If a simple drawing is plane (that is, crossing-free), then its cells are classically called *faces*.

In the past decades, there has been significant interest in simple drawings. Questions about plane subdrawings of simple drawings of the complete graph on n vertices, K_n, have attracted particularly close attention.

Rafla [18] conjectured that every simple drawing of K_n contains a plane Hamiltonian cycle. The conjecture has been shown to hold for $n \leq 9$ [1], as well as for several special classes of simple drawings, like straight-line, monotone, and cylindrical drawings, but remains open in general. If Rafla's conjecture is true, then this would immediately imply that every simple drawing of the complete graph contains a plane perfect matching. However, to-date even the existence of such a matchging is still unknown.

Ruiz-Vargas [20] showed in 2017 that every simple drawing of K_n contains $\Omega(n^{\frac{1}{2}-\varepsilon})$ pairwise disjoint edges for any $\varepsilon > 0$, which improved over a series of previous results: $\Omega((\log n)^{\frac{1}{6}})$ in 2003 [15], $\Omega(\frac{\log n}{\log \log n})$ in 2005 [16], $\Omega((\log n)^{1+\varepsilon})$ in 2009 [9], and $\Omega(n^{\frac{1}{3}})$ in 2013 and 2014 [10, 12, 21].

Pach, Solymosi, and Tóth [15] showed that every simple drawing of K_n contains a subdrawing of $K_{c \log^{\frac{1}{8}} n}$, for some constant c, that is either *convex* or *twisted*[1]. They further showed that every simple drawing of K_n contains a plane subdrawing isomorphic to any fixed tree with up to $c \log^{\frac{1}{6}} n$ vertices, for some constant c. This implies that every simple drawing of K_n contains a plane path of length $\Omega((\log n)^{\frac{1}{6}})$, which has been the best lower bound known prior to this paper.

Concerning general plane substructures, it follows from a result of Ruiz-Vargas [20] that every simple drawing of K_n contains a plane subdrawing with at least $2n-3$ edges. Further, García, Pilz, and Tejel [13] showed that every maximal plane subdrawing of a simple drawing of K_n is biconnected. Note that, in contrast to straight-line drawings, simple drawings of K_n in general do not contain triangulations, that is, plane subdrawings where all faces (except at most one) are 3-cycles.

In this paper, we introduce a new family of simple drawings, which we call *generalized twisted* drawings. The name stems from the fact that one can show that any twisted drawing is weakly isomorphic to a generalized twisted drawing (but not every generalized twisted drawing is weakly isomorphic to a twisted drawing). It follows, that for any n there exists a generalized twisted drawing. Two drawings D and D' are *weakly isomorphic* if there is a bijection between the vertices and edges of D and D' such that a pair of edges in D crosses exactly when the corresponding pair of edges in D' crosses.

▶ **Definition 1.** *A simple drawing D is* ***c-monotone*** *(short for circularly monotone) if there is a point O such that any ray emanating from O intersects any edge of D at most once.*

A simple drawing D of K_n is ***generalized twisted*** *if there is a point O such that D is c-monotone with respect to O and there exists a ray r emanating from O that intersects every edge of D.*

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[1] In their definition for simple drawings, *convex* means that there is a labeling of the vertices to $v_1, v_2, \ldots, v_n$ such that (v_i, v_j) $(i < j)$ crosses (v_k, v_l) $(k < l)$ if and only if $i < k < j < l$ or $k < i < l < j$, and *twisted* means that there is a labeling of the vertices to $v_1, v_2, \ldots, v_n$ such that (v_i, v_j) $(i < j)$ crosses (v_k, v_l) $(k < l)$ if and only if $i < k < l < j$ or $k < i < j < l$.

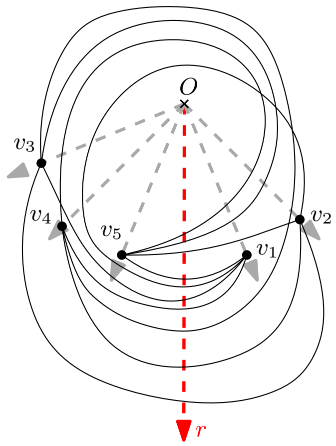


Figure 1 A generalized twisted drawing of K_5. All edges cross the (red) ray r.

We label the vertices of c-monotone drawings $v_1, \ldots, v_n$ in counterclockwise order around O. In generalized twisted drawings, they are labeled such that the ray r emerges from O between the ray to v_1 and the one to v_n. Figure 1 shows an example of a generalized twisted drawing of K_5.

Generalized twisted drawings turn out to have quite surprising structural properties. We show some crossing properties of generalized twisted drawings in Section 2 and with that also prove that they always contain plane Hamiltonian paths (Theorem 3). This result is an essential ingredient for showing that any simple drawing of K_n contains $\Omega(\sqrt{n})$ pairwise disjoint edges (Theorem 9 in Section 3), as well as a plane path of length $\Omega(\frac{\log n}{\log \log n})$ (Theorem 10 in Section 4). In Section 5, we present different characterizations of generalized twisted drawings that are of independent interest. We conclude with an outlook on further work and open problems in Section 6. *Full proofs are available at arXiv:2203.06143.*

2 Twisted preliminaries

In this section, we show some properties of generalized twisted drawings, which will be used in the following sections.

▶ **Lemma 2.** *Let D be a generalized twisted drawing of K_4, with vertices $\{v_1, v_2, v_3, v_4\}$ labeled counterclockwise around O. Then the edges v_1v_3 and v_2v_4 do not cross.*

Proof Sketch. Assume, for a contradiction, that the edge v_1v_3 crosses the edge v_2v_4. There are (up to strong isomorphism) two possibilities to draw the crossing edges v_1v_3 and v_2v_4, depending on whether v_1v_3 crosses the (straight-line) segment from O to v_4 or not; cf. Figure 2. In both cases, there is only one way to draw v_1v_2 such that the drawing stays generalized twisted, yielding two regions bounded by all drawn edges (v_1v_3, v_2v_4, v_1v_2). The vertices v_3 and v_4 lie in the same region. It is well-known that every simple drawing of K_4 has at most one crossing. Thus, the edge v_3v_4 cannot leave this region. However, it is impossible to draw v_3v_4 without leaving the region such that it is c-monotone and crosses the ray r (see the dotted arrows in Figure 2 for necessary emanating directions of v_3v_4). ◀

Using the crossing property of Lemma 2, it follows directly that generalized twisted drawings always contain plane Hamiltonian paths.

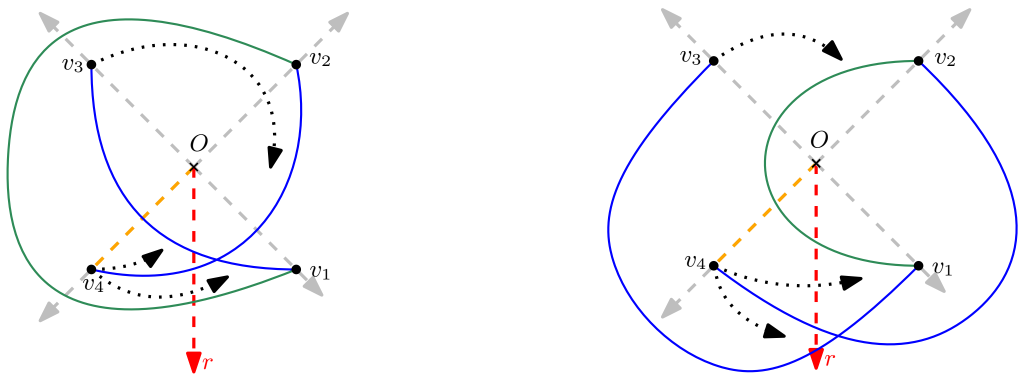


Figure 2 The two possibilities to draw v_1v_3 and v_2v_4 crossing and generalized twisted.

▶ **Theorem 3.** *Every generalized twisted drawing of K_n contains a plane Hamiltonian path.*

Proof of Theorem 3. Let D be a generalized twisted drawing of K_n. Consider the Hamiltonian path $v_1, v_{\lceil \frac{n}{2} \rceil+1}, v_2, v_{\lceil \frac{n}{2} \rceil+2}, v_3, \ldots, v_{\lceil \frac{n}{2} \rceil-1}, v_n, v_{\lceil \frac{n}{2} \rceil}$ if n is odd or the Hamiltonian path $v_1, v_{\lceil \frac{n}{2} \rceil+1}, v_2, v_{\lceil \frac{n}{2} \rceil+2}, v_3, \ldots, v_{n-1}, v_{\lceil \frac{n}{2} \rceil}, v_n$ if n is even. See for example the Hamiltonian path v_1, v_4, v_2, v_5, v_3 in Figure 1. Take any pair of edges (v_i, v_j) and (v_k, v_l) of the path, where we can assume without loss of generality that $i < j$ and $k < l$. If the two edges share an endpoint, they are adjacent and do not cross. Otherwise, if they do not share an endpoint, either $i < k < j < l$ or $k < i < l < j$ by definition of the path. In any of the two cases, (v_i, v_j) and (v_k, v_l) cannot cross by Lemma 2. Therefore, no pair of edges cross, and the Hamiltonian path is plane. ◀

Analogous to the proof of Theorem 3, one can argue that in every generalized twisted drawing of K_n with n odd, the Hamiltonian cycle $v_1, v_{\lceil \frac{n}{2} \rceil+1}, v_2, v_{\lceil \frac{n}{2} \rceil+2}, \ldots, v_{\lceil \frac{n}{2} \rceil-1}, v_n, v_{\lceil \frac{n}{2} \rceil}, v_1$ is plane. We strongly conjecture that every generalized twisted drawing of K_n contains a plane Hamiltonian cycle, but its structure for even n is still an open problem.

Theorem 3 will be used heavily in the next two sections. Further, the following statement, which has been implicitly shown in [10] and [12], will be used in all remaining sections.

▶ **Lemma 4.** *Let D be a simple drawing of a complete graph containing a subdrawing D', which is a plane drawing of $K_{2,n}$. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2\}$ be the sides of the bipartition of D'. Let D_A be the subdrawing of D induced by the vertices of A. Then D_A is weakly isomorphic to a c-monotone drawing. Moreover, if all edges in D_A cross the edge b_1b_2, then D_A is weakly isomorphic to a generalized twisted drawing.*

3 Disjoint edges in simple drawings

In this section, we show that every simple drawing of K_n contains at least $\left\lfloor \sqrt{\frac{n}{48}} \right\rfloor$ pairwise disjoint edges, improving the previously known best bound of $\Omega(n^{\frac{1}{2}-\varepsilon})$, for any $\varepsilon > 0$, by Ruiz-Vargas [20]. In addition to the properties of generalized twisted drawings from Section 2, we use the following theorems and observations to prove this new lower bound.

▶ **Theorem 5** ([13]). *For $n \geq 3$, every maximal plane subdrawing of any simple drawing of K_n is biconnected.*

The following theorem is a direct consequence of Corollary 5 in [19].

▶ **Theorem 6.** *Let D be a simple drawing of K_n with $n \geq 3$. Let H be a connected plane subdrawing of D containing at least two vertices, and let v be a vertex in $D \setminus H$. Then D contains two edges incident to v that connect v with H and do not cross any edges of H.*

▶ **Observation 7.** *For any $n \geq 3$, the number of edges in a planar graph with n vertices is at most $3n - 6$.*

A drawing is *outerplane* if it is plane, and all vertices lie on the unbounded face of the drawing. A graph is *outerplanar* if it can be drawn outerplane. Outerplanar graphs have a smaller upper bound on their number of edges than planar graphs.

▶ **Observation 8.** *For any $n \geq 3$, the number of edges in an outerplanar graph with n vertices is at most $2n - 3$.*

▶ **Theorem 9.** *Every simple drawing of K_n contains at least $\left\lfloor \sqrt{\frac{n}{48}} \right\rfloor$ pairwise disjoint edges.*

Proof. Let D be a simple drawing of K_n, and let M be a maximal plane matching of D. If $m := |M| \geq \sqrt{\frac{n}{48}}$, then Theorem 9 holds. So assume that $|M| < \sqrt{\frac{n}{48}}$. We will show how to find another plane matching, whose size is at least $\lfloor \sqrt{\frac{n}{48}} \rfloor$.

The overall idea is the following: Let H be a maximal plane subdrawing of D whose vertex set is exactly the vertices matched in M and that contains M. We will find a face f in H that contains much more unmatched vertices inside than matched vertices on its boundary. Then we will show that there exists a subset of the vertices inside that face, which induces a subdrawing of D that is weakly isomorphic to a generalized twisted drawing and contains enough vertices to guarantee the desired size of the plane matching.

We start towards finding the face f. By Theorem 5, H is biconnected. Thus, H partitions the plane into faces, where the boundary of each face is a simple cycle. Note that the vertices of H are exactly the vertices that are matched in M, and the vertices inside faces are the vertices that are unmatched in M. Let U be the set of vertices of D that are not matched by any edge of M. We denote the set of vertices of U inside a face f_i by $U(f_i)$, the number of vertices in $U(f_i)$ by $u(f_i)$, and the number of vertices on the boundary of the face f_i by $|f_i|$.

We next show that there exists a face f of H such that $u(f) \geq \frac{\sqrt{48n}}{12}|f|$. Assume for a contradiction that for every face f_i it holds that

$$u(f_i) < \frac{\sqrt{48n}}{12}|f_i|.$$

There are exactly $n - 2m$ unmatched vertices. As every unmatched vertex is in the interior of a face of H (that might be the unbounded face), we can count the unmatched vertices by summing over the number of vertices in each face (including the unbounded face). Thus,

$$n - 2m \leq \sum_{f_i} u(f_i) < \frac{\sqrt{48n}}{12} \sum_{f_i} |f_i|. \tag{1}$$

The number of edges in H is $\frac{1}{2}\sum_{f_i} |f_i|$. Since H is plane, we can use Observation 7 to bound the number of edges of H by $3n' - 6$, where n' is the number of vertices in H. As the vertices of H are exactly the matched vertices, their number is $n' = 2m$. Hence,

$$\sum_{f_i} |f_i| \leq 6 \cdot 2m - 12.$$

From $m < \sqrt{\frac{n}{48}}$ it follows that

$$\sum_{f_i} |f_i| < 12\sqrt{\frac{n}{48}} - 12 \tag{2}$$

and

$$n - 2\sqrt{\frac{n}{48}} < n - 2m. \tag{3}$$

Putting equations (1) to (3) together we obtain that

$$n - 2\sqrt{\frac{n}{48}} < \frac{\sqrt{48n}}{12}(12\sqrt{\frac{n}{48}} - 12) = n - \sqrt{48n}.$$

However, this inequality cannot be fulfilled by any $n \geq 0$. Thus, there exists at least one face f_i with $u(f_i) \geq \frac{\sqrt{48n}}{12}|f_i|$. We call that face f. (If there are several such faces, we take an arbitrary one of them and call it f.)

As a next step, we will find two vertices on the boundary of f to which many vertices inside f are connected via edges that do not cross each other or H. From f and the set $U(f)$, we construct a plane subdrawing H' as follows; cf. Figure 3 (left). We add the vertices and edges on the boundary of f. Then we iteratively add all the vertices in $U(f)$, where for each added vertex v we also add two edges of D incident to v such that the resulting drawing stays plane. Two such edges exist by Theorem 6. Since the matching M is maximal, any edges between two unmatched vertices must cross at least one edge of M and thus must cross the boundary of f. Hence, no edge in H' can connect two vertices of $U(f)$ (as they are unmatched). Consequently, every vertex in $U(f)$ is connected in H' to exactly two vertices that both lie on the boundary of f.

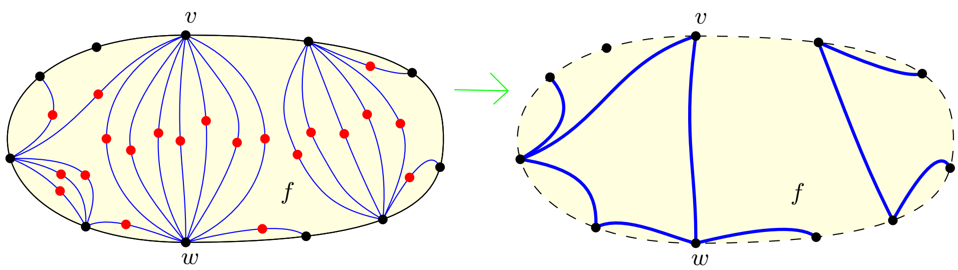


Figure 3 Left: The face f in H containing the plane drawing H' (blue lines) inside. Right: We can obtain an outerplane drawing from H' by interpreting bundles of edge pairs incident to the same black vertices as plane edges.

We consider the edges in H' that connect a vertex in $U(f)$ as a pair of edges. Every edge in such a pair is contained in exactly one pair, since it is incident to exactly one unmatched vertex. Thus, we can see every such pair of edges as one *long edge* incident to two vertices on the boundary of f. If several of those long edges have the same endpoints, we call them a bundle of edges; see Figure 3 (right).

From the long edges, we can define a graph G' as follows. The vertices of G' are the vertices of D that lie on the boundary of f. Two vertices u and v are connected in G' if there is at least one long edge in H' that connects them. By the definition of long edges, G' is outerplanar (as can be observed in Figure 3 (right)). Note that every unmatched vertex in

$U(f)$ defines a long edge, so the number of long edges is $u(f) \geq \frac{\sqrt{48n}}{12}|f|$. From Observation 8, it follows that G' has at most $2|f|-3$ edges. As a consequence, there is a pair of vertices on the boundary of f such that the number of long edges in its bundle is at least

$$\frac{1}{(2|f|-3)}\frac{\sqrt{48n}}{12}|f| > \frac{\sqrt{48n}}{24}.$$

This implies that there are two vertices, say v and w, to which more than $\frac{\sqrt{48n}}{24}$ vertices inside f have two plane incident edges. We call the set of vertices in $U(f)$ that have plane edges to both vertices v and w the set U_{vw}. This set is marked in Figure 4 (left). We denote the subdrawing of D induced by U_{vw} by D_{vw}; see Figure 4 (right).

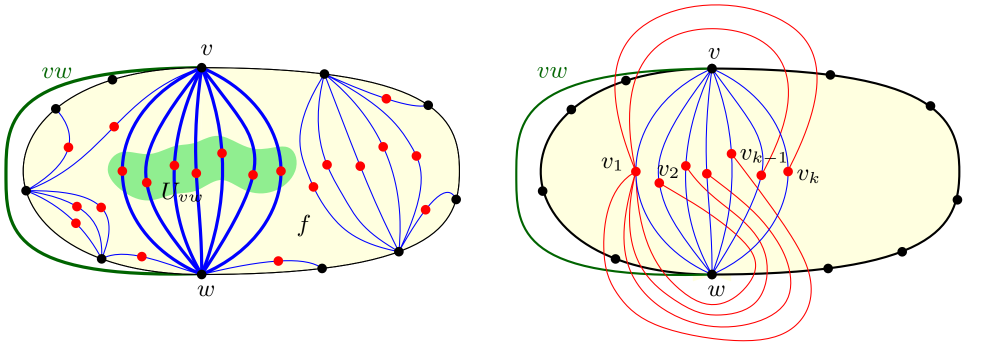


Figure 4 The subdrawing D' induced by U_{vw} and the edges in D_{vw}. Left: The set U_{vw}. Right: The edges adjacent to the leftmost vertex, v_1, are drawn (in red).

We show that all edges between vertices in U_{vw} cross the edge vw. Let x and y be two vertices of D_{vw}. Let R_1 be the region bounded by the edges xv, vy, yw, and wx that lies inside the face f; see Figure 5. We show that xy and vw lie completely outside R_1. The edge xy has to lie either completely inside or completely outside R_1, because it is adjacent to all edges on the boundary of R_1. As M is maximal and the edge xy connects two unmatched vertices, it has to cross at least one matching edge. Thus, xy has to lie completely outside R_1. (There can be no matching edges in R_1, as R_1 is contained inside the face f.) As H is a maximal plane subdrawing, vw cannot lie inside the face f and thus has to be outside R_1. Since both edges vw and xy lie completely outside R_1 and the vertices along the boundary of R_1 are sorted $vxwy$, the two edges have to cross. Thus, all edges of D_{vw} cross the edge vw.

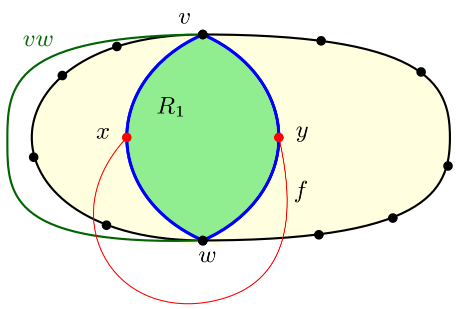


Figure 5 The edge xy has to cross the edge vw.

Since the edges from vertices in U_{vw} to v and w are plane, it follows from Lemma 4 that D_{vw} is weakly isomorphic to a generalized twisted drawing. Thus, D_{vw} contains at least $\lfloor \frac{1}{2}\frac{\sqrt{48n}}{24} \rfloor$ pairwise disjoint edges by Theorem 3. Hence, D contains at least $\lfloor \sqrt{\frac{n}{48}} \rfloor$ pairwise disjoint edges. ◀

4 Plane paths in simple drawings

In the previous section, we used generalized twisted drawings to improve the lower bound on the number of disjoint edges in simple drawings of K_n. In this section, we show that generalized twisted drawings are also helpful to improve the lower bound on the length of the longest path in such drawings, where the length of a path is the number of its edges, to $\Omega(\frac{\log n}{\log \log n})$. This improves the previously known best bound of $\Omega((\log n)^{\frac{1}{6}})$, which follows from a result of Pach, Solymosi, and Tóth [15].

▶ **Theorem 10.** *Every simple drawing D of K_n contains a plane path of length $\Omega(\frac{\log n}{\log \log n})$.*

To prove the new lower bound, we first show that all c-monotone drawings on n vertices contain either a generalized twisted drawing on $\sqrt{n}$ vertices or a drawing weakly isomorphic to an x-monotone drawing on $\sqrt{n}$ vertices. We know that drawings weakly isomorphic to generalized twisted drawings or x-monotone drawings contain plane Hamiltonian paths (by Theorem 3 and Observation 11 below). We conclude that c-monotone drawings contain plane paths of the desired size. We then show that every simple drawing of the complete graph contains either a c-monotone drawing or a plane d-ary tree. With easy observations about the length of the longest path in d-ary trees and by putting all results together, we obtain that every simple drawing D of K_n contains a plane path of length $\Omega(\frac{\log n}{\log \log n})$.

4.1 Plane paths in c-monotone drawings

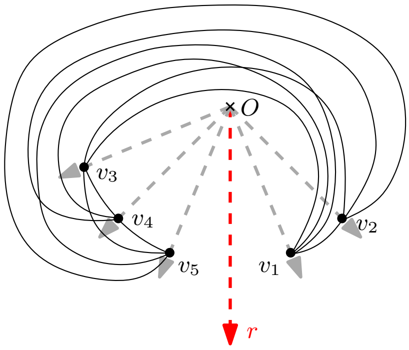
A simple drawing is *x-monotone* if any vertical line intersects any edge of the drawing at most once (see Figure 6b). This family of drawings has been studied extensively in the literature (see for example [2, 5, 7, 11, 17]). By definition, c-monotone drawings in which there exists a ray emanating from O, which crosses all edges of the drawing, are generalized twisted. In contrast, consider a c-monotone drawing D such that there exists a ray r emanating from O that crosses no edge of D. Then it is easy to see that D is strongly isomorphic to an x-monotone drawing. (A c-monotone drawing on the sphere can be cut along the ray r and the result drawn on the plane such that all rays are vertical lines and the ray r is to the very left of the drawing.) Figure 6a shows a c-monotone drawing D of K_5 where no edge crosses the ray r, and Figure 6b shows an x-monotone drawing of K_5 strongly isomorphic to D. We will call simple drawings that are strongly isomorphic to x-monotone drawings *monotone* drawings. In particular, any c-monotone drawing for which there exists a ray emanating from O that crosses no edge of the drawing is monotone.

It is well-known that any x-monotone drawing of K_n contains a plane Hamiltonian path. For instance, assuming that the vertices are ordered by increasing x-coordinates, the set of edges $v_1v_2, v_2v_3 \ldots, v_{n-1}v_n$ form a plane Hamiltonian path.

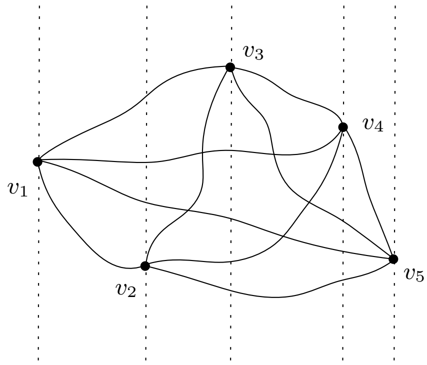
▶ **Observation 11.** *Every monotone drawing of K_n contains a plane Hamiltonian path.*

We will show that c-monotone drawings contain plane paths of size $\sqrt{n}$, by showing that any c-monotone drawing of K_n contains a subdrawing of $K_{\sqrt{n}}$ that is either generalized twisted or monotone. To do so, we will use Dilworth's Theorem on chains and anti-chains in partially ordered sets. A *chain* is a subset of a partially ordered set such that any two distinct elements are comparable. An *anti-chain* is a subset of a partially ordered set such that any two distinct elements are incomparable.

▶ **Theorem 12** (Dilworth's Theorem, [8])**.** *Let P be a partially ordered set of at least $(s-1)(t-1)+1$ elements. Then P contains a chain of size s or an antichain of size t.*



(a) A c-monotone drawing D of K_5 such that the ray r crosses no edge of D.



(b) An x-monotone drawing of K_5 strongly isomorphic to D of Figure 6a.

Figure 6 Two strongly isomorphic monotone drawings of K_5.

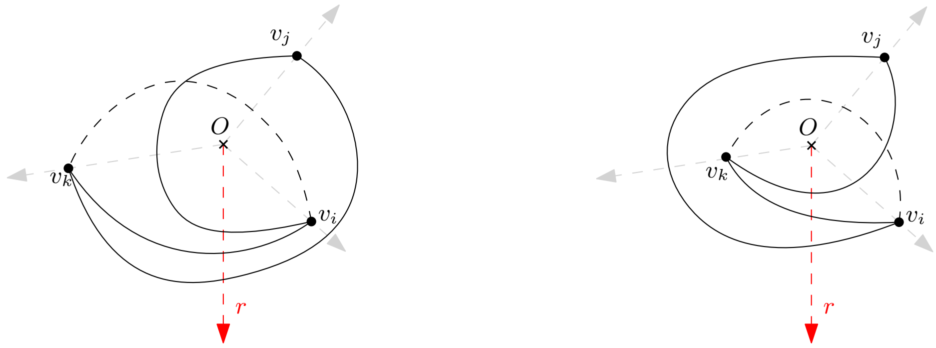


Figure 7 If edges v_iv_j and v_jv_k cross r in a c-monotone drawing, then v_iv_k must also cross r.

▶ **Theorem 13.** *Let s, t be two integers, $1 \leq s, t \leq n$, such that $(s-1)(t-1)+1 \leq n$. Let D be a c-monotone drawing of K_n. Then D contains either a generalized twisted drawing of K_s or a monotone drawing of K_t as subdrawing. In particular, if $s = t = \lceil \sqrt{n} \rceil$, D contains a complete subgraph K_s whose induced drawing is either generalized twisted or monotone.*

Proof Sketch. Without loss of generality we may assume that the vertices of D appear counterclockwise around O in the order $v_1, v_2, \ldots, v_n$. Let r be a ray emanating from O, keeping v_1 and v_n on different sides. We define an order, $\preceq$, in this set of vertices as follows: $v_i \preceq v_j$ if and only if either $i = j$ or $i < j$ and the edge (v_i, v_j) crosses r.

We show that $\preceq$ is a partial order. The relation is clearly reflexive and antisymmetric. Besides, if $v_i \preceq v_j$ and $v_j \preceq v_k$, then $v_i \preceq v_k$, because $i < j$ and $j < k$ imply $i < k$, and if v_iv_j and v_jv_k cross r, then v_iv_k also crosses r (see Figure 7). Hence, the relation is transitive.

In this partial order $\preceq$, a chain consists of a subset $v_{i_1}, \ldots, v_{i_{s-1}}$ of pairwise comparable vertices, that is, a subset of vertices such that their induced subdrawing is generalized twisted (all edges cross r). An antichain, $v_{j_1}, \ldots, v_{j_{t-1}}$, consists of a subset of pairwise incomparable vertices, that is, a subset of vertices such that their induced subdrawing is monotone (no edge crosses r). Therefore, the first part of the theorem follows from applying Theorem 12 to the set of vertices of D and the partial order $\preceq$.

Finally, observe that if $s = t \leq \lceil \sqrt{n} \rceil$, then $(s-1)(t-1)+1 \leq n$. Thus, D contains a complete subgraph $K_{\lceil \sqrt{n} \rceil}$ whose induced subdrawing is either generalized twisted or monotone. ◀

Combining Theorems 3 and 13 with Observation 11, we obtain the following theorem.

▶ **Theorem 14.** *Every c-monotone drawing of K_n contains a plane path of length $\Omega(\sqrt{n})$.*

4.2 Plane paths in simple drawings

To show that any simple drawing of K_n contains a plane path of length $\Omega(\frac{\log n}{\log\log n})$, we will use d-ary trees. A d-ary tree is a rooted tree in which no vertex has more than d children. It is well-known that the height of a d-ary tree on n vertices is $\Omega(\frac{\log n}{\log d})$.

Proof of Theorem 10. Let v be a vertex of D and let $S(v)$ be the star centered at v, that is, the set of edges of D incident to v. $S(v)$ can be extended to a maximal plane subdrawing H that must be biconnected by Theorem 5. See Figure 8 for a depiction of $S(v)$ and H.

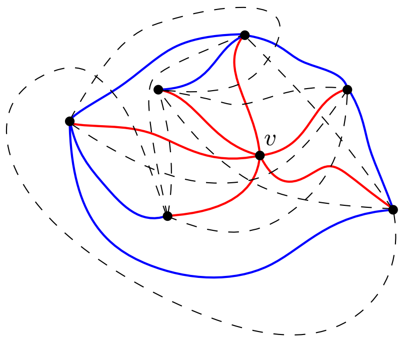


Figure 8 A simple drawing of K_7. The red edges show the star $S(v)$, the red and blue edges together form a maximal plane subdrawing H. Dashed edges are edges of K_7 that are not in H.

Assume first that there is a vertex w in $H \setminus v$ that has degree at least $(\log n)^2$ in H. Let U_{vw} be the set of vertices neighboured in H to both, v and w. Note that $|U_{vw}| \geq (\log n)^2$. The subdrawing H' of H consisting of the vertices in U_{vw}, the vertices v, and w, and the edges from v to vertices in U_{vw}, and from w to vertices in U_{vw} is a plane drawing of $K_{2,|U_{vw}|}$. From Lemma 4, it follows that the subdrawing of D induced by U_{vw} is weakly isomorphic to a c-monotone drawing. Therefore, by Theorem 14, the subdrawing induced by U_{vw} contains a plane path of length $\Omega(\sqrt{|U_{vw}|}) = \Omega(\log n)$.

Assume now that the maximum degree in $H \setminus v$ is less than $(\log n)^2$. Since H is biconnected, $H \setminus v$ contains a plane tree T of order $n-1$ whose maximum degree is at most $(\log n)^2$. Thus, considering that T is rooted, the diameter of T is at least $\Omega(\frac{\log n}{\log\log n})$. Therefore, since T is plane, it contains a plane path of length at least $\Omega(\frac{\log n}{\log\log n})$ and the theorem follows. ◀

5 Characterizing generalized twisted drawings

In previous sections, we have seen how generalized twisted drawings were used to make progress on open problems of simple drawings. In addition to this, generalized twisted drawings are also interesting in their own right and have some quite surprising structural properties. Despite the fact that research on generalized twisted drawings is rather recent and still ongoing, there are already several interesting characteristics and structural results. Some of them will be presented in this section.

One characterization involves curves crossing every edge once. From the definition of generalized twisted drawing (see Figure 1), there always exists a simple curve that crosses all edges of the drawing exactly once (for instance, a curve that starts at O and follows r until it reaches a point Z on r in the unbounded cell). In Theorem 15, we show that the converse is also true. That is, every simple drawing D of K_n in which we can add a simple curve that crosses every edge of D exactly once is weakly isomorphic to a generalized twisted drawing.

Another characterization is based on what we call *antipodal vi-cells*. For any three vertices in a simple drawing D of K_n, the three edges connecting them form a simple cycle which we call a *triangle*. Every such triangle partitions the plane (or sphere) into two disjoint regions which are the *sides* of the triangle (in the plane a bounded and an unbounded one). Two cells of D are called *antipodal* if for each triangle of D, they lie on different sides. Further, we call a cell with a vertex on its boundary a vertex-incident-cell or, for short, a *vi-cell*.

By definition, every generalized twisted drawing D contains two antipodal cells, namely, the cell containing the starting point of the ray r and the unbounded cell. This follows from the fact that the ray r crosses every edge exactly once. Hence, r crosses the boundary of any triangle exactly three times, so the cells containing the "endpoints" of r must be on different sides of the triangle.

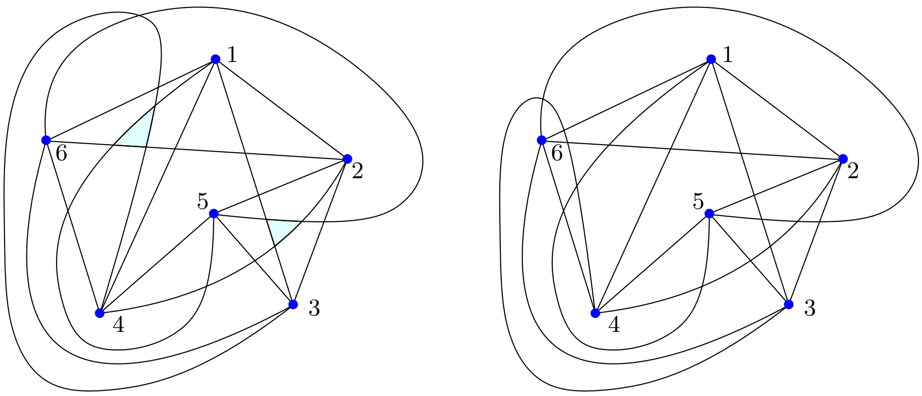


Figure 9 Two weakly isomorphic drawings of K_6 that are not weakly isomorphic to any generalized twisted drawing. Antipodal cells are marked in blue.

It turns out that the converse (existence of two antipodal cells implies weakly isomorphic to generalized twisted) is not true. Figure 9 (left) shows a drawing of K_6 that contains two antipodal cells, but no antipodal vi-cells. From Theorem 15 bellow it will follow that such drawings cannot be weakly isomorphic to a generalized twisted drawing. However, we observed that for all generalized twisted drawings of K_n with $n \leq 6$, both, the cell containing the startpoint of the ray r and the unbounded cell, are vi-cells. Figure 10 shows all (up to strong isomorphism) simple drawings of K_6 that are weakly isomorphic to generalized twisted drawings. We show that this is true in general. More than that, we show in Theorem 16 that every drawing of K_n that is weakly isomorphic to a generalized twisted drawing contains a pair of antipodal vi-cells. In the other direction, we show in Theorem 15 that every simple drawing containing a pair of antipodal vi-cells is weakly isomorphic to a generalized twisted drawing.

The final characterization is based on the extension of a given drawing of the complete graph to a drawing containing a spanning, plane bipartite graph that has all vertices of the original drawing on one side of the bipartition. From the definition of generalized twisted drawings, it follows that any genereralized twisted drawing D of K_n can be extended to a simple drawing D' of K_{n+2} including new vertices O and Z such that D' contains a plane drawing of a spanning bipartite graph. One side of the bipartition consists of all vertices in D and the other side of the bipartition consists of the new vertices O and Z. Moreover,

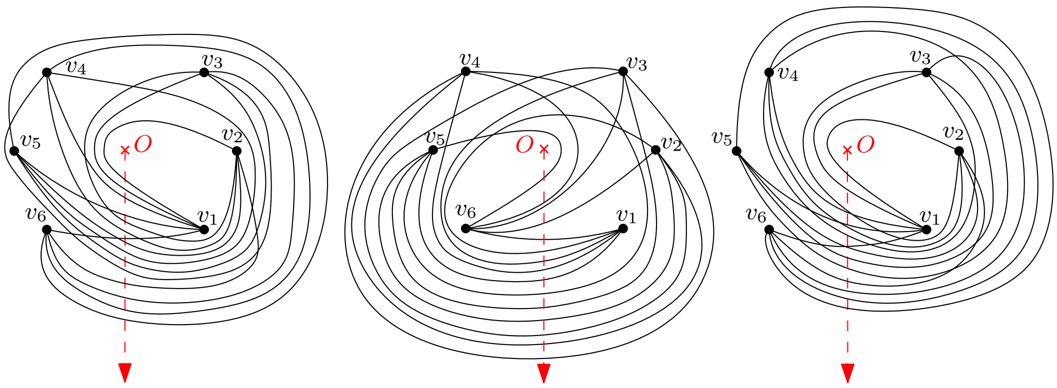


Figure 10 All different generalized twisted drawings of K_6 (up to weak isomorphism). The rightmost drawing is twisted.

the edge OZ crosses all edges of D. One way to add the new vertices and edges incident to them is to draw (1) the vertex O at point O, (2) the vertex Z in the unbounded cell on the ray r, (3) the edge OZ straight-line (along the ray r), (4) edges from O to the vertices of D straight-line (along the inner segment of the rays crossing through the vertices), and (5) edges from Z to the vertices of D first far away in a curve and the final part straight-line (along the outer segment of the rays crossing through the vertices). The converse, that every drawing that can be extended like this is weakly isomorphic to a generalized twisted drawing, has already been shown in Lemma 4.

We show the following characterizations.

▶ **Theorem 15** (Characterizations of generalized twisted drawings). *Let D be a simple drawing of K_n. Then, the following properties are equivalent.*

Property 1 *D is weakly isomorphic to a generalized twisted drawing.*

Property 2 *D contains two antipodal vi-cells.*

Property 3 *D can be extended by a simple curve c such that c crosses every edge of D exactly once.*

Property 4 *D can be extended by two vertices, O and Z, and edges incident to the new vertices such that D together with the new vertices and edges is a simple drawing of K_{n+2}, the edge OZ crosses every edge of D, and no edge incident to O crosses any edge incident to Z.*

To prove Theorem 15, we will first show that Property 1 implies Property 2 (Theorem 16). We next show that Property 2 implies Property 3 (Theorem 17). Then, we show that Property 3 implies Property 4 (Theorem 18). By Lemma 4, Property 4 implies Property 1. Thus, all properties are equivalent. In a full version of this work, we will extend the theorem to show that also strong isomorphism to a generalized twisted drawing is equivalent to the properties of Theorem 15. We show this by proving that any simple drawing of K_n fulfilling Property 4 is strongly isomorphic to a generalized twisted drawing. However, the reasoning for strong isomorphism is quite lengthy and would exceed the space constraints of this submission.

▶ **Theorem 16.** *Every simple drawing of K_n which is weakly isomorphic to a generalized twisted drawing of K_n, with $n \geq 3$, contains a pair of antipodal vi-cells. In generalized twisted drawings the cell containing O and the unbounded cell form such a pair.*

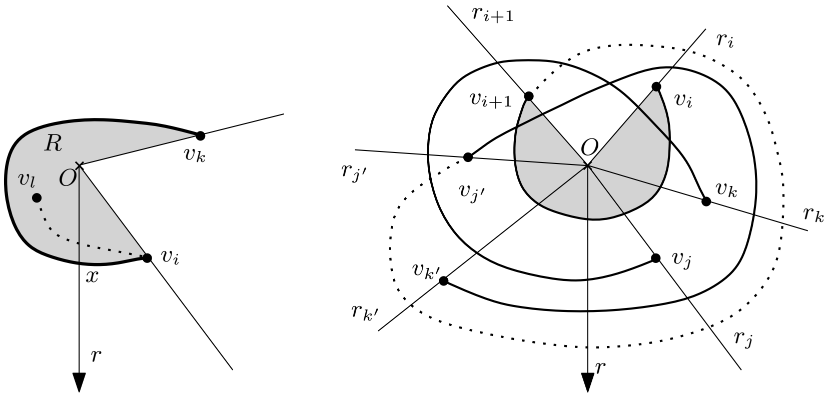


Figure 11 Left: If there is a vertex v_l in R, it cannot be connected to v_i without crossing r before x. Right: If the edge v_jv_k crosses the segment $\overline{Ov_i}$ and the edge $v_{j'}v_{k'}$ crosses the segment $\overline{Ov_{i+1}}$, then there is no way of connecting v_{i+1} and $v_{j'}$.

Proof sketch. We first show that every generalized twisted drawing D of K_n, with $n \geq 3$, contains a pair of antipodal vi-cells, where O lies in a cell of that pair. Let c be the segment OZ, where Z is a point on r in the unbounded cell. By definition of generalized twisted, c crosses every edge of D once, so O and Z are in two antipodal cells C_1 and C_2, respectively.

To prove that C_1 is a vi-cell, we use the following properties. First, if we take the first edge v_iv_k that crosses c (as seen from O) at point x, then we can prove that $k = i + 1$ and the bounded region R defined by the edge v_iv_{i+1} and the segments $\overline{Ov_i}$ and $\overline{Ov_{i+1}}$ is empty (see Figure 11, left). Second, using this empty region we can prove that D cannot contain simultaneously an edge v_jv_k crossing $\overline{Ov_i}$ and another edge $v_{j'}v_{k'}$ crossing $\overline{Ov_{i+1}}$ (see Figure 11, right). Therefore, at least one of the segments $\overline{Ov_i}$ and $\overline{Ov_{i+1}}$ is uncrossed, and O necessarily lies in a vi-cell (with either v_i or v_{i+1} on the boundary). Finally, arguing on the last edge crossing c and the unbounded cell, we can show that Z also lies in a vi-cell.

To show that also every drawing which is weakly isomorphic to a generalized twisted drawing contains a pair of antipodal vi-cells, we use Gioan's Theorem [6, 14]. By Gioan's Theorem, any two weakly isomorphic drawings of K_n can be transformed into each other with a sequence of triangle-flips and at most one reflection of the drawing. A *triangle-flip* is an operation which transforms a triangular cell $\triangle$ that has no vertex on its boundary by moving one of its edges across the intersection of the two other edges of $\triangle$. We show that if a drawing D_1 contains two antipodal vi-cells, then after performing a triangle flip on D_1, the resulting drawing D_2 still has two antipodal vi-cells. The main argument is that triangle-flips are only applied to cells without vertices on their boundary, and thus the antipodality of the vi-cells cannot change. ◀

▶ **Theorem 17.** *In any simple drawing D of K_n that contains a pair of antipodal vi-cells, it is possible to draw a curve c that crosses every edge of D exactly once.*

Proof sketch. Let (C_1, C_2) be a pair of antipodal vi-cells of D. Let v_1 be a vertex on the boundary of C_1 and v_2 a vertex on the boundary of C_2. We construct the curve as follows: First, we draw a simple curve c from C_1 to C_2 such that (1) it emanates from v_1 in C_1 and ends in C_2 very close to v_2, (2) does not cross any edge incident to v_1, (3) only intersects edges of D in proper crossings, and (4) has the minimum number of crossings with edges of D among all curves that fulfill (1), (2) and (3). This curve c always exists since $S(v_1)$ is a plane drawing that has only a face in which both v_1 and v_2 lie (see Figure 12, left).

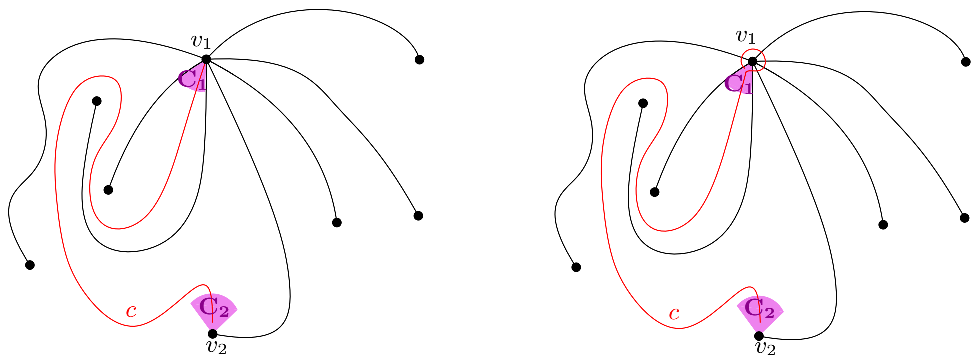


Figure 12 Building a curve such that it crosses every edge of D once and its endpoints do not lie on any edges or vertices of D.

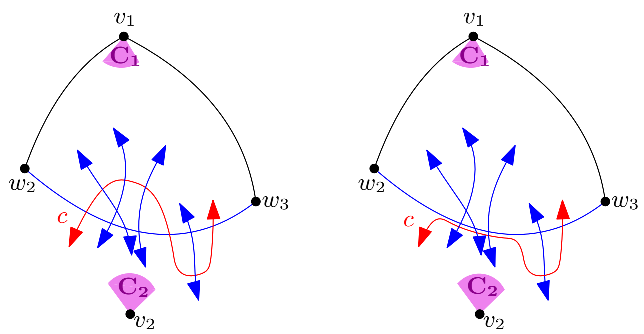


Figure 13 Decreasing the number of crossings between c and the edge w_2w_3.

Then, we prove that c crosses every edge w_2w_3 in D that is not incident to v_1 exactly once. On the one hand, since c connects two antipodal cells, the endpoints of c have to be on two different sides of the triangle T formed by v_1, w_2 and w_3. Thus, c has to cross w_2w_3 an odd number of times because it does not cross $S(v_1)$ and must cross the boundary of T an odd number of times. On the other hand, if c crosses w_2w_3 at least three times, then we can prove that c can be redrawn as shown in Figure 13, decreasing the number of crossings, which contradicts (4). Therefore, c crosses every edge w_2w_3 at most twice and, consequently, only once.

Finally, we change the end of c from v_1 to a point in C_1 in the following way (see Figure 12, right). From some point of c sufficiently close to v_1 and inside C_1, we reroute c by going around v_1 such that only the edges incident to v_1 are crossed, and end at a point in C_1. ◀

▶ **Theorem 18.** *Let D be a simple drawing of K_n in which it is possible to draw a simple curve c that crosses every edge of D exactly once. Then, D can be extended by two vertices O and Z (at the position of the endpoints of the curve), and edges incident to those vertices such that the obtained drawing is a simple drawing of K_{n+2}, no edge incident to O crosses any edge incident to Z, and all edges in D cross the edge OZ.*

Proof sketch. Let $c = OZ$ be the curve crossing every edge of D once, oriented from O to Z. Let wu be an edge of D, oriented from w to u, crossing OZ at a point x. We say that wu is a *top* (respectively *bottom*) edge if the clockwise order of w, Z, u and O around x is w, Z, u, O (respectively w, O, u, Z). See Figure 14. With these definitions, we can prove that there is a vertex w_1 in D such that all the oriented edges emanating from w_1 are top in relation to c.

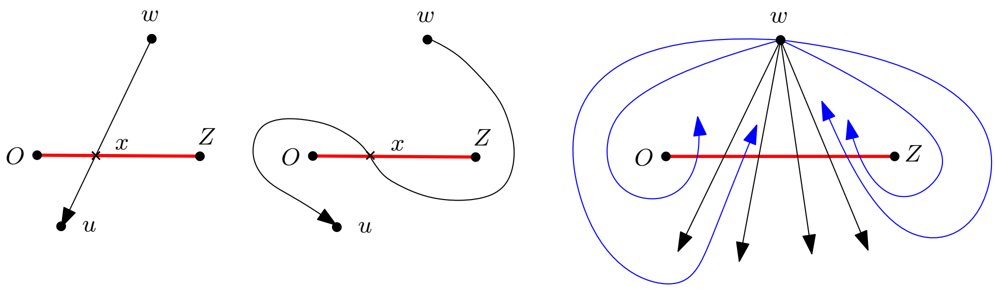


Figure 14 Top and bottom edges. For simplicity, the curve OZ is drawn as a horizontal line. Left: A top edge wu. Centre: A bottom edge wu. Right: The (black) top and (blue) bottom edges of $S(w)$.

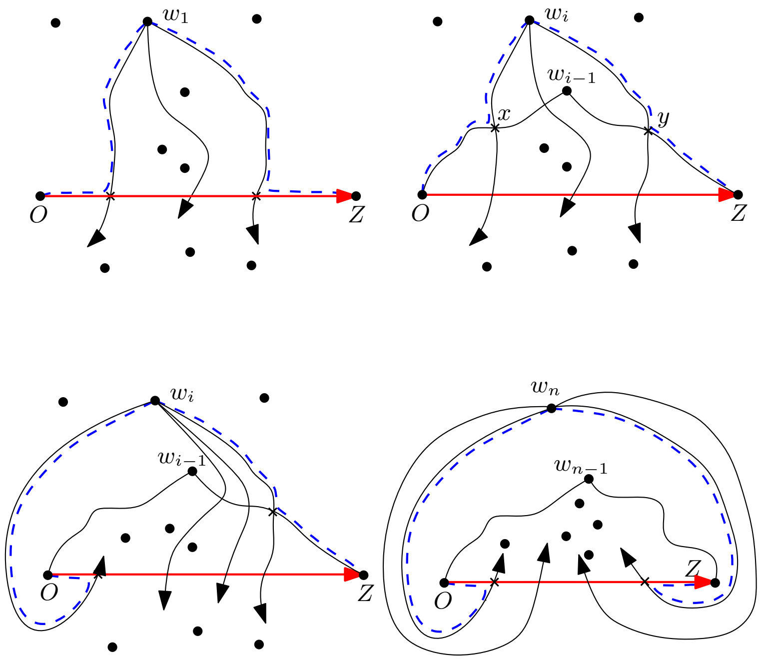


Figure 15 Building the (dashed) edges w_iO and w_iZ.

Thus, by removing w_1 and all its incident edges from D, there is a vertex w_2 in the new drawing such that all its incident edges are top, and so on. As a consequence, there is a *natural* order $w_1, w_2, \ldots, w_n$ of the vertices of D such that for any vertex w_i, the edges w_iw_j with $j > i$ are top, and the edges w_iw_j with $j < i$ are bottom.

Given the natural order $w_1, w_2, \ldots, w_n$, our construction of the extended drawing is as follows. Let D'_0 be the simple drawing formed by the vertices and edges of D, O and Z as new vertices, and c as the edge connecting O and Z. From D'_0, we build new drawings $D'_1, D'_2, \ldots, D'_n$, by adding in step i the edges w_iO and w_iZ. These two edges are added very close to some edges in D'_{i-1}. Figure 15 illustrates how these two edges are added in each step.

In the first step, the edge Ow_1 follows the curve OZ until the crossing point between OZ and the first top edge w_1u emanating from w_1, and then it follows this top edge until reaching w_1. The edge Zw_1 is built in an analogous way, taking the last top edge emanating from w_1. See Figure 15 top-left. For $i = 2, \ldots, n-1$, in step i we do different constructions depending on whether the first and last top edges of $S(w_i)$ cross the edges $w_{i-1}O$ and $w_{i-1}Z$. If the first top edge w_iu_1 crosses $w_{i-1}O$ at a point x and the last top edge w_iu_k crosses $w_{i-1}Z$ at a point y (see Figure 15 top-right), then Ow_i follows Ow_{i-1} until x, and then it follows u_1w_i until w_i. The edge Zw_i is built following Zw_{i-1} until y and then following u_kw_i. On the contrary, if the first and the last top edges of $S(w_i)$ only cross one of $w_{i-1}O$ and $w_{i-1}Z$, say $w_{i-1}Z$ (see Figure 15 bottom-left), then Ow_i follows OZ until the crossing point between OZ and the last bottom edge of $S(w_i)$, and then it follows this bottom edge until w_i. The edge Zw_i is built as in the first step, using the last top edge of $S(w_i)$. In the last step, we build Ow_n and Zw_n as in the first step, but using the first and the last bottom edges of $S(w_n)$ instead of the first and last top edges. See Figure 15 bottom-right.

By a detailed analysis of cases, we can prove for $i = 1, \ldots, n$ that D'_i is a simple drawing such that no edge incident to O crosses any edge incident to Z. Therefore, D'_n is the drawing of K_{n+2} satisfying the required properties. ◀

6 Conclusion and outlook

Generalized twisted drawings have a suprisingly rich structure and many useful properties. We showed several of those properties in Section 2 and different characterizations of generalized twisted drawings in Section 5. We have proven in Section 2 that every generalized twisted drawing on an odd number of vertices contains a plane Hamiltonian cycle, and therefore one especially interesting open question is the following.

▶ **Conjecture 19.** *Every generalized twisted drawing of K_n contains a plane Hamiltonian cycle.*

Using properties of generalized twisted drawings has turned out to be helpful for investigating simple drawings in general. We first improved the lower bound on the number of disjoint edges in simple drawings of K_n to $\Omega(\sqrt{n})$ (Section 3). Then generalized twisted drawings played the central role to improve the lower bound on the length of plane paths contained in every simple drawing of K_n to $\Omega(\frac{\log n}{\log \log n})$ (Section 4).

On the other hand, from Theorem 17 it immediately follows that no drawing that is weakly isomorphic to a generalized twisted drawing can contain three interior-disjoint triangles (since the endpoints of the curve crossing every edge once must be on opposite sides of every triangle, the maximum number of interior-disjoint triangles is two). Up to strong isomorphism, there are only two simple drawings of K_4. The plane drawing contains three interior-disjoint triangles. Thus, (up to strong isomorphism) the only drawing of K_4 that is weakly isomorphic to a generalized twisted drawing, is the drawing with a crossing. Hence, in every generalized twisted drawing all subdrawings induced by 4 vertices contain a crossing and thus every generalized twisted drawing is crossing maximal. Up to strong isomorphism, there are two crossing maximal drawings of K_5: the convex drawing of K_5 and the twisted drawing of K_5. Since the convex drawing contains three interior-disjoint triangles, the only (up to strong isomorphism) drawing of K_5 that is weakly isomorphic to a generalized twisted drawing is the twisted drawing of K_5 (that is drawn generalized twisted in Figure 1).

It is part of our ongoing work to show that for $n \geq 7$, a drawing is weakly isomorphic to a generalized twisted drawing if and only if all subdrawings induced by five vertices are weakly isomorphic to the twisted K_5. Interestingly, the $n \geq 7$ is necessary as there is a drawing with 6 vertices that contains only twisted drawings of K_5 but is not weakly isomorphic to a generalized twisted drawing (see the drawings in Figure 9). There are (up to strong isomorphism) three more simple drawings of K_6 that consist of only twisted drawings of K_5 and they are all weakly isomorphic to generalized twisted drawings (see Figure 10).

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