Hop-Spanners for Geometric Intersection Graphs

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Abstract

A t-spanner of a graph $G = (V, E)$ is a subgraph $H = (V, E')$ that contains a uv-path of length at most t for every $uv \in E$. It is known that every n-vertex graph admits a $(2k-1)$-spanner with $O(n^{1+1/k})$ edges for $k \geq 1$. This bound is the best possible for $1 \leq k \leq 9$ and is conjectured to be optimal due to Erdős' girth conjecture.

We study t-spanners for $t \in \{2, 3\}$ for geometric intersection graphs in the plane. These spanners are also known as *t-hop spanners* to emphasize the use of graph-theoretic distances (as opposed to Euclidean distances between the geometric objects or their centers). We obtain the following results: (1) Every n-vertex unit disk graph (UDG) admits a 2-hop spanner with $O(n)$ edges; improving upon the previous bound of $O(n \log n)$. (2) The intersection graph of n axis-aligned fat rectangles admits a 2-hop spanner with $O(n \log n)$ edges, and this bound is the best possible. (3) The intersection graph of n fat convex bodies in the plane admits a 3-hop spanner with $O(n \log n)$ edges. (4) The intersection graph of n axis-aligned rectangles admits a 3-hop spanner with $O(n \log^2 n)$ edges.

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1 Introduction

Graph spanners were introduced by Awerbuch [7] and by Peleg and Schäffer [54]. A spanner of a graph G is a spanning subgraph H with bounded distortion between graph distances in G and H. For an edge-weighted graph $G = (V, E)$, a spanning subgraph H is a t-spanner if $d_H(u, v) \leq t \cdot d_G(u, v)$ for all $u, v \in V$, where d_H and d_G are the shortest-path distances in H and G, respectively. The parameter $t \geq 1$ is the *stretch factor* of the spanner. A long line of research is devoted to finding spanners with desirable features, which minimize the number of edges, the weight, or the diameter; refer to a recent survey by Ahmed et al. [2].

In abstract graphs, all edges have unit weight. In a graph G of girth g, any proper subgraph H has stretch at least $g - 1$. In particular, a complete bipartite graph does not have any subquadratic size t-spanner for $t < 3$. The celebrated greedy spanner by Althöfer et al. [3] finds, for every n-vertex graph and parameter $t = 2k - 1$, a t-spanner with $O(n^{1+\frac{1}{k}})$ edges; and this bound matches the lower bound from the Erdős girth conjecture [31].

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Geometric Setting: Euclidean and Metric Spanners. Given a set P of n points in a metric space (M, δ), consider the complete graph G on P where the weight of an edge uv is the distance $\delta(u, v)$. If M has doubling dimension d (e.g., Euclidean spaces of constant dimension) the greedy algorithm by Althöfer et al. [3] constructs an $(1+\varepsilon)$-spanner with $\varepsilon^{-O(d)}n$ edges [45]. Specifically, every set of n points in $\mathbb{R}^d$ admits a $(1+\varepsilon)$-spanner with $O(\varepsilon^{-d}n)$ edges, and this bound is the best possible [45].

Gao and Zhang [38] considered data structures for approximating the *weighted* distances in *unit disk graphs* (*UDG*), which are intersection graphs of unit disks in $\mathbb{R}^2$. Importantly, the *weight* of an edge is the Euclidean distance between the centers. They designed a well-speared pair-decomposition (WSPD) of size $O(n \log n)$ for an n-vertex UDG. For the unit ball graphs in doubling dimensions, Eppstein and Khodabandeh [30] construct $(1+\varepsilon)$-spanners which also have bounded degree and total weight $O(w(MST))$, generalizing earlier work in $\mathbb{R}^d$ by Damian et al. [23]; see also [46]. Fürer and Kasiviswanathan [37] construct a $(1+\varepsilon)$-spanner with $O(\varepsilon^{-2}n)$ edges for the intersection graph of n disks of arbitrary radii in $\mathbb{R}^2$.

Hop-Spanners for Geometric Intersection Graphs. Unit disk graphs (UDG) were the first geometric intersection graphs for which the hop distance was studied (i.e., the unweighted version), motivated by applications in wireless communication. Spanners in this setting are often called *hop-spanners* to emphasize the use of graph-theoretic distance (i.e., hop distance), as opposed to the Euclidean distance between centers.

For an n-vertex UDG G, Yan et al. [57] constructed a subgraph H with $O(n \log n)$ edges and $d_H(u, v) \leq 3d_G(u, v) + 12$, which is a 15-hop spanner. Catusse et al. [19] showed that every n-vertex UDG admits a 5-hop spanner with at most $10n$ edges (as well as a noncrossing $O(1)$-spanner with $O(n)$ edges). Biniaz [9] improved this bound to $9n$. Dumitrescu et al. [28] recently showed that every n-vertex UDG admits a 5-hop spanner with at most $5.5n$ edges, a 3-hop spanner with at most $11n$ edges, and a 2-hop spanner with $O(n \log n)$ edges. In this paper, we improve the bound on the size of 2-hop spanners to $O(n)$, and initiate the study of minimum 2-hop spanners of other classes of geometric intersection graphs.

Our Contributions.

1. Every unit disk graph on n vertices admits a 2-hop spanner with $O(n)$ edges (Theorem 2 in Section 2). This bound is the best possible; and it generalizes to intersection graphs of translates of a convex body in the plane (shown in the full version of the paper).
2. The intersection graph of n axis-aligned fat rectangles in $\mathbb{R}^2$ admits a 2-hop spanner with $O(n \log n)$ edges (Theorem 15 in Section 3). This bound is the best possible: We establish a lower bound of $\Omega(n \log n)$ for the size of 2-hop spanners in the intersection graph of n homothets of any convex body in the plane (Theorem 19 in Section 4).
3. The intersection graph of n fat convex bodies in $\mathbb{R}^2$ admits a 3-hop spanner with $O(n \log n)$ edges (shown in the full version of the paper).

Related Previous Work. While our upper bounds are constructive, we do not attempt to minimize the number of edges in a k-spanner for a given graph. The *minimum k-spanner* problem is to find a k-spanner H of a given graph G with the minimum number of edges. This problem is NP-hard [16, 54] for all $2 \leq k \leq o(\log n)$; already for planar graphs [10, 41]. It is also hard to approximate up to a factor of $2^{(\log^{1-\varepsilon} n)/k}$, for $3 \leq \log^{1-2\varepsilon} n$ and $\varepsilon > 0$, assuming $NP \not\subseteq BPTIME(2^{\text{polylog}(n)})$ [25]; see also [27, 29, 42]. On the positive side, Peleg and Krtsarz [43] gave an $O(\log(m/n))$-approximation for the minimum 2-spanner problem for graphs G with n vertices and m edges; see also [20]. There is an $O(n)$-time algorithm for the minimum 2-spanner problem over graphs of maximum degree at most four [17].

Classical graph optimization problems (which are often hard and hard to approximate) typically admit better approximation ratios or are fixed-parameter tractable (FPT) for geometric intersection graphs. Three main strategies have been developed to take advantage of geometry: (i) Divide-and-conquer strategies using separators and dynamic programming [4, 8, 24, 18, 34, 35, 36, 47]; (ii) Local search algorithms [14, 21, 40, 51]; and (iii) Bounded VC-dimension and the ε-net theory [1, 6, 13, 53, 50, 52]. It is unclear whether separators and local search help find small k-hop spanners. Small hitting sets and ε-nets help finding large cliques in geometric intersection graphs, and this is a tool we use, as well.

Relation to Edge Clique and Biclique Covers. A 2-hop spanner H of a graph $G = (V, E)$ is union of stars $\mathcal{S}$ such that every edge in E is induced by a star in $\mathcal{S}$. Thus the minimum 2-spanner problem is equivalent to minimizing the sum of sizes of stars in $\mathcal{S}$. As such, the 2-spanner problem is similar to the *minimum dominating set* and *minimum edge-clique cover* problems [32, 49]. In particular, the size of a 2-hop spanner is bounded above by the minimum *weighted* edge clique cover, where the weight of a clique K_t is $t-1$ (i.e., the size of a spanning star). Recently, de Berg et al. [24] proposed a divide-and-conquer framework for optimization problems on geometric intersection graphs. Their main technical tool is a weighted separator theorem, where the weight of a separator is $W = \sum_i w(t_i)$ for a decomposition of the subgraph induced by the separator into cliques K_{t_i}, and sublinear weights $w(t) = o(t)$. For 2-hop spanners, however, each clique K_t requires a star with $t-1$ edges, so the weight function would be linear $w(t) = t-1$.

Every biclique (i.e., complete bipartie graph) $K_{s,t}$ admits a 3-hop spanner with $s+t-1$ edges (as a union of two stars). Hence an *edge biclique cover*, with total weight W and weight function $w(K_{s,t}) = s+t$, yields a 3-hop spanners with at most W edges. Every n-vertex graph has an edge biclique cover of weight $O(n^2/\log n)$, and this bound is tight [33, 56]. (In contrast, every n-vertex graph has a 3-hop spanner with $O(n^{3/2})$ edges [3].) Better bounds are known for semi-algebraic graphs, where the edges are defined in terms of semi-algebraic relations of bounded degree. For instance, an incidence graph between n points and m hyperplanes in $\mathbb{R}^d$ admits an edge biclique cover of weight $O((mn)^{1-1/d} + m + n)$ [5, 11, 55]. Recently, Do [26] proved that a semi-algebraic bipartite graph on $m+n$ vertices, where the vertices are points in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, resp., has an edge biclique cover of weight $O_\varepsilon(m^{\frac{d_1 d_2 - d_2}{d_1 d_2 - 1}+\varepsilon} n^{\frac{d_1 d_2 - d_1}{d_1 d_2 - 1}+\varepsilon} + m^{1+\varepsilon} + n^{1+\varepsilon})$ for any $\varepsilon > 0$. For $d_1 + d_2 \leq 4$, this result yields nontrivial 3-hop spanners. For a UDG with $m = n$ unit disks, $d_1 = d_2 = 2$ gives a 3-hop spanner with $W \leq O_\varepsilon(n^{4/3+\varepsilon})$ edges. But for the intersection graph of arbitrary disks in $\mathbb{R}^2$, $d_1 = d_2 = 3$ gives $O_\varepsilon(n^{3/2+\varepsilon})$, which is worse than the default $O(n^{3/2})$ guaranteed by the greedy algorithm [3].

Representation. Our algorithms assume a geometric representation of a given intersection graphs (it is NP-hard to recognize UDGs [12], disk graphs [39, 48], or box graphs [44]). Given a set of geometric objects of bounded description complexity, the intersection graph and the hop distances can easily be computed in polynomial time. Chan and Skrepetos [22] designed near-quadratic time algorithms to compute all pairwise hop-distances in the intersection graph of n geometric objects (e.g., balls or hyperrectangles in $\mathbb{R}^d$). In a UDG, the hop-distance between a given pair of disks can be computed in optimal $O(n \log n)$ time [15].

2 Two-Hop Spanners for Unit Disk Graphs

In this section, we prove that every n-vertex UDG has a 2-hop spanner with $O(n)$ edges. The proof hinges on a key lemma, Lemma 1, in a bipartite setting. A unit disk is a closed disk of unit diameter in $\mathbb{R}^2$; two unit disks intersect if and only if their centers are at distance at most 1 apart. For finite sets $A, B \subset \mathbb{R}^2$, let $U(A, B)$ denote the unit disk graph on $A \cup B$, and let $G(A, B)$ denote the bipartite subgraph of $U(A, B)$ of all edges between A and B.

▶ **Lemma 1.** *Let $P = A \cup B$ be a set of n points in the plane such that* $\mathrm{diam}(A) \leq 1$, $\mathrm{diam}(B) \leq 1$, *and A (resp., B) is above (resp., below) the x-axis. Then there is a subgraph H of $U(A, B)$ with at most $5n$ edges such that for every edge ab of $G(A, B)$, H contains a path of length at most 2 between a and b.*

We construct the graph H in Lemma 1 incrementally: In each step, we find a subset $W \subset A \cup B$, together with a subgraph $H(W)$ of at most $5|W|$ edges that contains a uv-path of length at most 2 for every edge uv between $u \in W$ and $v \in N(W)$ (cf. Lemma 5); and then recurse on $P \setminus W$. We show that $\bigcup_W H(W)$ is a 2-hop spanner for $U(A, B)$.

Section 2.1 establishes a technical lemma about the interaction pattern of disks in the bipartite setting. One step of the recursion is presented in Section 2.2. The proof of Lemma 1 is in Section 2.3. Lemma 1, combined with previous work [9, 19, 28] that reduced the problem to a bipartite setting, implies the main result of this section.

▶ **Theorem 2.** *Every n-vertex unit disk graph has a 2-hop spanner with $O(n)$ edges.*

Proof. Let P be a set of centers of n unit disks in the plane, and let $G = (P, E)$ be the UDG on P. Consider a tiling of the plane with regular hexagons of diameter 1, where each point in P lies in the interior of a tile. A tile τ is *nonempty* if $\tau \cap P \neq \emptyset$. Clearly $\mathrm{diam}(P \cap \tau) \leq \mathrm{diam}(\tau) = 1$. For each nonempty tile τ, let S_τ be a spanning star on $P \cap \tau$.

For each pair of tiles, σ and τ, at distance at most 1 apart, Lemma 1 yields a graph $H_{\sigma,\tau} := G(A, B) \subset G$ for $A = P \cap \sigma$ and $B = P \cap \tau$ with $5(|P \cap \sigma| + |P \cap \tau|)$ edges. Let H be the union of all stars S_τ and all graphs $H_{\sigma,\tau}$. It is easily checked that H is a 2-hop spanner of G: Indeed, let $uv \in E$. If u and v are in the same tile τ, then S_τ contains uv or a uv-path of length 2. Otherwise u and v are in different tiles, say σ and τ, at distance at most 1, and $H_{\sigma,\tau}$ contains uv or a uv-path of length 2.

It remains to bound the number of edges in H. The union of all stars S_τ is a spanning forest on P, which has at most $n-1$ edges. Every tile σ is within unit distance from 18 other tiles [9]. The total number of edges in $H_{\sigma,\tau}$ over all pairs of tiles is $\sum_{\sigma,\tau} 5(|P\cap\sigma|+|P\cap\tau|) \leq 18 \sum_\sigma 5(|P \cap \sigma|) = 90n$. Overall, H has less than $91n$ edges, as required. ◀

2.1 Properties of Unit-Disk Hulls

Let $A \subset \mathbb{R}^2$ be a finite set of points above the x-axis. Let $\mathcal{D}$ be the set of all unit disks with centers on or below the x-axis. Let $M(A)$ be the union of all unit disks $D \in \mathcal{D}$ such that $A \cap \mathrm{int}(D) = \emptyset$, and let $\mathrm{hull}(A) = \mathbb{R}^2 \setminus \mathrm{int}(M(A))$; see Fig. 1.

For every $p \in \mathbb{R}^2$ above the x-axis, let $X(p)$ denote its vertical projection onto $\partial\mathrm{hull}(A)$; this is well defined by Lemma 3(1) below. Let $L(p)$ and $R(p)$ denote the points in $A \cap \partial\mathrm{hull}(A)$ immediately to the left and right of $X(p)$ if such a point exists; that is, $L(p)$ (resp., $R(p)$) is the point in $A \cap \partial\mathrm{hull}(A)$ with the largest (resp., smallest) x-coordinate that still satisfies $L(p)_x \leq X(p)_x$ (resp., $R(p)_x \geq X(p)_x$).

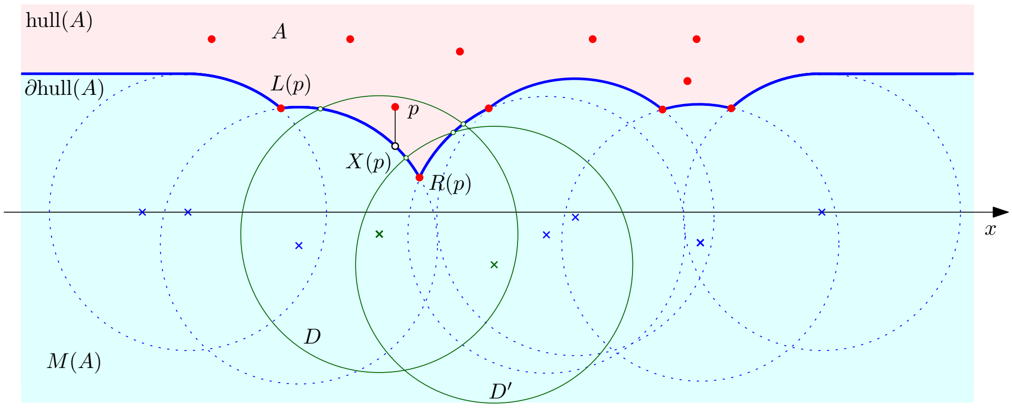

Figure 1 A point set A (red), region $M(A)$ (light blue), and hull(A) (pink). A point $p \in A$ in a disk $D \in \mathcal{D}$, its vertical projection $X(p) \in \partial\text{hull}(A)$, and the two adjacent points $L(p), R(p) \in A$.

▶ **Lemma 3.** *For every finite set $A \subset \mathbb{R}^2$ above the x-axis, the following holds:*

1. *$\partial\text{hull}(A)$ is an x-monotone curve.*
2. *For every $D \in \mathcal{D}$, the intersection $D \cap \partial\text{hull}(A)$ is connected (possibly empty).*
3. *For every $D \in \mathcal{D}$ and every $p \in A$, if $p \in D$, then D contains $X(p)$. Further, $L(p)$ or $R(p)$ exists, and D contains $L(p)$ or $R(p)$ (possibly both).*
4. *Let $D, D' \in \mathcal{D}$. Suppose that ∂D intersects $\partial\text{hull}(A)$ at points with x-coordinates x_1 and x_2, and $\partial D'$ intersects $\partial\text{hull}(A)$ at points with x-coordinates x_1' and x_2'. If $x_1 \leq x_1' \leq x_2' \leq x_2$, then $D' \cap \text{hull}(A) \subset D \cap \text{hull}(A)$.*

The proof (in the full version of the paper) is a straightforward extension of previous results [28, Lemma 4].

2.2 One Incremental Step

Let A and B be finite point sets above and below the x-axis, respectively, and let $P = A \cup B$. For every point $p \in \mathbb{R}^2$, let $N(p) \subset P$ denote the points in P on the opposite side of the x-axis within unit distance from p; refer to Fig. 2. For a point set $S \subset \mathbb{R}^2$, let $N(S) = \bigcup_{p \in S} N(p)$. Suppose that a unit circle centered at $p \in A$ intersects $\partial\text{hull}(B)$ at points $p_1, p_2 \in \mathbb{R}^2$; or a unit circle centered at $p \in B$ intersects $\partial\text{hull}(A)$ at points $p_1, p_2 \in \mathbb{R}^2$. Define $I(p) = N(N(p)) \setminus (N(p_1) \cup N(p_2))$; see Fig. 2 for an example.

▶ **Lemma 4.** *Let $P = A \cup B$ be a finite set of points in the plane such that A (resp., B) is above (resp., below) the x-axis. For every $p \in P$, $N(I(p)) \subset N(p)$.*

Proof. We may assume w.l.o.g. that $p \in A$. Let $v \in I(p)$, and let D_v (resp., D_p) denote the unit disk centered on v (resp., p). As $v \in N(N(p))$, D_v contains some point $u \in N(p)$. Clearly, D_p contains u. By Lemma 3(3), D_v and D_p contain $X(u)$. As $D_p \cap \text{hull}(B)$ has endpoints p_1 and p_2, Lemma 3(1)–(2) implies that $X(u)$ has x-coordinate between p_1 and p_2. By definition of $I(P)$, $D_v \cap \text{hull}(B)$ does not contain either p_1 or p_2, so it only contains points between p_1 and p_2. By Lemma 3(4), $N(v) \subset N(p)$. ◀

We construct a spanner by repeatedly applying the following lemma:

▶ **Lemma 5.** *Let $P = A \cup B$ be a set of n points in the plane such that $\text{diam}(A) \leq 1$, $\text{diam}(B) \leq 1$, and A (resp., B) is above (resp., below) the x-axis. Then there exists a nonempty subset $W \subset P$ and a graph $H(W)$ with the following properties:*

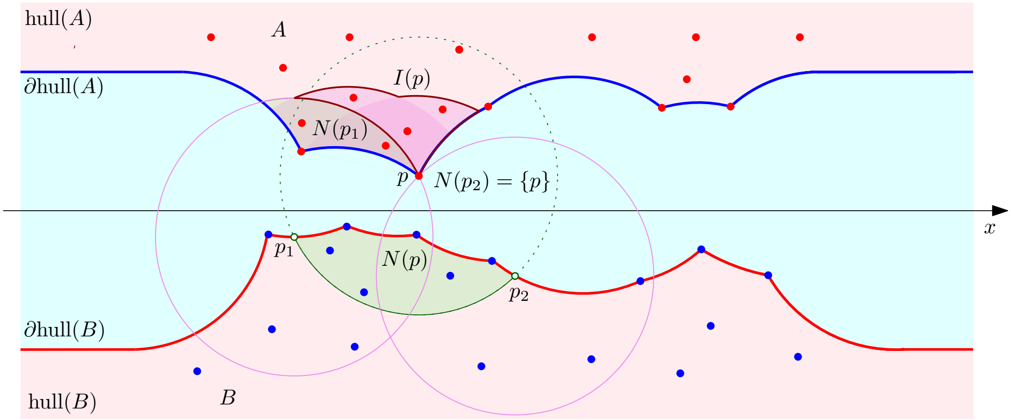

Figure 2 A point $p \in A$ and its neighbors $N(p) \subset B$. The unit circle centered at p intersecting $\partial\text{hull}(B)$ at p_1 and p_2. The sets $N(p_1)$, $N(p_2)$, and $I(p)$.

1. *$H(W)$ is a subgraph of $U(A, B)$;*
2. *$H(W)$ contains at most $5|W|$ edges;*
3. *for every edge ab in the neighborhood of W in $G(A, B)$, $H(W)$ contains an ab-path of length at most 2.*

Proof. Let $m \in \mathbb{R}^2$ be the point that maximizes $|N(m)|$ (breaking ties arbitrarily) and let $k = |N(m)|$. Notice that m might not be in P. By Lemma 3(3), every point in $N(m)$ is within unit distance of $L(m)$ or $R(m)$; and $L(m), R(m) \in P$. Thus there exists a point $v \in P$ such that $|N(v)| \geq k/2$.

Now let $p \in P$ be the point that maximizes $|N(p)|$; and note that $|N(p)| \geq k/2$. Let $W = N(p) \cup I(p) \cup \{p\}$. Let $H(W)$ be the spanning star centered at p connected to all points in $N(N(p))$ and to all points in $N(p)$. We verify that $H(W)$ has the required properties:

1. Every point in $N(p)$ is within unit distance of p. As $p \in A$ and $N(N(p)) \subset A$, every point in $N(N(p))$ is within unit distance of p. Thus $H(W)$ is a subgraph of $U(A, B)$.
2. By definition of k, $|N(p_1)| \leq k$ and $|N(p_2)| \leq k$. Thus, $|N(N(p))| \leq 2k + |I(p)|$. Further, $|W| = |N(p)| + |I(p)| \geq k/2 + |I(p)|$. Thus $|N(N(p))| \leq 4|W|$. The spanning star $H(W)$ has $|N(N(p))| + |N(p)| - 1$ edges, so it has at most $5|W|$ edges.
3. For every $v \in N(p)$, all neighbors of v are in $N(N(p))$ by the definition of $N(.)$, so the spanning star contains a path of length at most 2 to each neighbor. For every $v \in I(p) \cup \{p\}$, all neighbors of v are in $N(p)$ by Lemma 4, so the spanning star contains a path of length at most 2 to each neighbor. ◀

2.3 Proof of Lemma 1

We can now construct a sparse 2-hop spanner in the bipartite setting. We restate Lemma 1.

▶ **Lemma 1.** *Let $P = A \cup B$ be a set of n points in the plane such that* $\text{diam}(A) \leq 1$, $\text{diam}(B) \leq 1$, *and A (resp., B) is above (resp., below) the x-axis. Then there is a subgraph H of $U(A, B)$ with at most $5n$ edges such that for every edge ab of $G(A, B)$, H contains a path of length at most 2 between a and b.*

Proof. Apply Lemma 5 to find a subset $W \subset P$ and a subgraph $H(W)$. Let H be the union of $H(W)$ and the spanner constructed by recursing on $P \setminus W$. Since H is the union of subgraphs of $U(A, B)$, it is itself a subgraph of $U(A, B)$.

Stretch analysis. Suppose $a \in A$ and $b \in B$ are neighbors in $G(A, B)$. We assume w.l.o.g. that a was removed before or at the same time as b during the construction of H as part of some subset W. Then H includes a subgraph $H(W)$ that, by construction, connects a to all neighbors that have not yet been removed (including b) by paths of length at most 2.

Sparsity analysis. Each subgraph $H(W)$ in H is responsible for removing some set of points W and has at most $5|W|$ edges. Charge 5 edges to each of the $|W|$ points removed. As each point is removed exactly once, H contains at most $5n$ edges. ◀

3 Two-Hop Spanners for Axis-Aligned Squares

For intersection graphs of n unit disks, we found 2-hop spanners with $O(n)$ edges in Section 2. This bound does not generalize to intersection graphs of disks of arbitrary radii, as we establish a lower bound of $\Omega(n \log n)$ in Section 4. Here, we construct 2-hop spanners with $O(n \log n)$ edges for such graphs under the L_∞ norm (where unit disks are really unit squares). The result also holds for axis-aligned fat rectangles.

We prove a linear upper bound for the 1-dimensional version of the problem (Section 3.1), and then address axis-aligned fat rectangles in the plane (Section 3.2). The *fatness* of a set $s \subset \mathbb{R}^2$ is the ratio $\varrho_{\text{out}}/\varrho_{\text{in}}$ between the radii of a minimum enclosing disk and a maximum inscribed disk of s. A collection S of geometric objects is α-fat if the fatness of every $s \in S$ is at most α; and it is *fat*, for short, if it is α-fat for some $\alpha \in O(1)$.

3.1 Two-Hop Spanners for Interval Graphs

Let $G(S)$ be the intersection graph of a set S of n closed segments in $\mathbb{R}$. Assume w.l.o.g. that $G(S)$ is connected: otherwise, we can apply this construction to each connected component.

We partition $\bigcup S$ into a collection of disjoint intervals $\mathcal{I} = \{I_1, \ldots, I_m\}$ as follows. Let $I_0 = \{p_0\}$ be the interval containing only the leftmost point in $\bigcup S$, and let $k := 1$. While p_{k-1} lies to the left of the rightmost point in $\bigcup S$, let p_k be the rightmost point of any segment in S that intersects p_{k-1}; let $I_k = (p_{k-1}, p_k]$; and set $k := k + 1$. As $G(S)$ is connected, this process terminates. For every $k \in \{1, \ldots, m\}$, define the *covering segment* c_k to be some segment that intersects p_{k-1} and has right endpoint p_k; see Fig. 3. Notice that by construction of I_k, c_k is guaranteed to exist, and $I_k \subset c_k$.

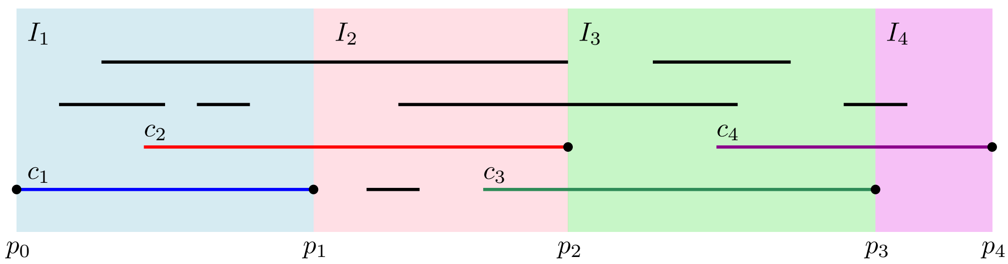

Figure 3 A set of segments S, with $\bigcup S$ partitioned into intervals $\mathcal{I} = \{I_1, \ldots, I_4\}$. Each $I_k \in I$ is contained in some covering segment $c_k \in S$.

▶ **Lemma 6.** *The set of intervals $\mathcal{I}$ defined above has the following properties:*

1. *$\mathcal{I}$ is a partition of $\bigcup S$;*
2. *every segment $s \in S$ intersects at most 2 intervals in I;*
3. *if two segments $a, b \subset \bigcup S$ intersect (with a, b not necessarily elements of S), then there is some interval in I that intersects both segments.*

The proof is straightforward; see the full version of the paper.

▶ **Theorem 7.** *Every n-vertex interval graph admits a 2-hop spanner with at most $2n$ edges.*

Proof. We construct the 2-hop spanner H as the union of stars. For every interval $I_k \in I$, construct a star H_k centered on the covering segment c_k with an edge to every segment that intersects I_k. As $I_k \subset c_k$, every segment that intersects I_k also intersects c_k, so there is an edge between the two segments in $G(S)$. Define $H = \bigcup_{k=1}^{m} H_k$.

Stretch analysis. Suppose $s_1, s_2 \in S$ intersect. By Lemma 6(3), $s_1 \cap s_2$ intersects some interval I_k. Thus, the star $H_k \subset H$ connects s_1 and s_2 by a path of length at most 2.

Sparsity analysis. Suppose the star $H_k \subset H$ has j edges. The corresponding interval $I_k \in \mathcal{I}$ intersects $j+1$ segments in S. Charge 1 edge to each of the segments intersecting I_k. By Lemma 6(2), each of the n segments in S is charged at most twice. ◀

▶ **Corollary 8.** *The intersection graph of a set of n axis-aligned rectangles in $\mathbb{R}^2$ that all intersect a fixed horizontal or vertical line admits a 2-hop spanner with at most $2n$ edges.*

3.2 Two-Hop Spanners for Axis-Aligned Fat Rectangles

Let $G(S)$ be the intersection graph of a set S of n axis-aligned α-fat closed rectangles in the plane. For every pair of intersecting rectangles $a, b \in S$, select some representative point in $a \cap b$. Let $C(S)$ denote the set comprising the representatives for all intersections.

Setup for a Divide & Conquer Strategy. We recursively partition the plane into slabs by splitting along horizontal lines. The recursion tree $\mathcal{P}$ is a binary tree, where each node $P \in \mathcal{P}$ stores a slab, denoted slab(P), that is bounded by horizontal lines b_P and t_P on the bottom and top, respectively. The node P also stores a subset $S(P) \subset S$ of (not necessarily all) rectangles in S that intersect slab(P).

Let the *inside set* $\text{In}(P) \subset S(P)$ be the set of rectangles contained in int(slab(P)). Let the *bottom set* $B(P) \subset S(P)$ be rectangles that intersect the line b_P, the *top set* $T(P) \subset S(P)$ be the rectangles that intersect t_P, and the *across set* $A(P) = B(P) \cap T(P)$; see Fig. 4.

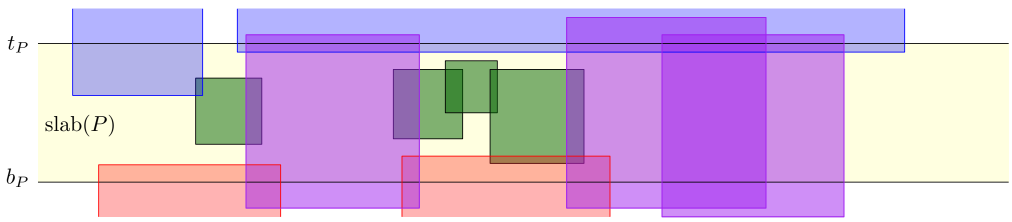

Figure 4 A horizontal slab(P) is bounded by b_P and t_P. Rectangles in the inside set In(P) (green), bottom set $B(P)$ (red), top set $T(P)$ (blue), and across set $A(P) = B(P) \cap T(P)$ (purple). Some red and blue fat rectangles are shown only partially, in a small neighborhood of slab(P).

We define the root node P_r to have a slab large enough to contain all rectangles in S, and define $S(P_r) = S$. We define the rest of the space partition tree recursively. Let $P \in \mathcal{P}$. Define $C(P) \subset C(S)$ to be the set $C(S) \cap \text{int}(\text{slab}(P))$. If $C(P) = \emptyset$, then P is a leaf and has no children. Otherwise, P has two children P_1 and P_2. Let c_P be a horizontal line with at most half the points in $C(P)$ on either side. Let slab(P_1) (resp., slab(P_2)) be the slab bounded by b_P and c_P (resp., c_P and t_P); and let $S(P_1) \subset S(P) \setminus A(P)$ (resp.,

$S(P_2) \subset S(P) \setminus A(P))$ be the set of rectangles that intersect this slab, excluding rectangles in $A(P)$. Notice that no rectangles in $A(P)$ appear in the children of P, whereas rectangles in the sets $\mathrm{In}(P)$, $B(P) \setminus A(P)$, and $T(P) \setminus A(P)$ appear in one or both of the children.

Spanner Construction. We construct a spanner $H(S)$ for $G(S)$ as the union of subgraphs $H(P)$ for each node P in the space partition tree.

We construct $H(P)$ such that there is a path of length at most 2 between every rectangle $s \in A(P)$ and every rectangle in $S(P)$ that s intersects. Every edge in $G(S(P))$ requiring such a path involves a rectangle in $B(P)$, a rectangle in $T(P)$, or a rectangle in $\mathrm{In}(P)$. We construct three subgraphs to deal with these three categories of edges.

By Corollary 8, we can construct a subgraph $H_B(P)$ of $G(B(P))$ with at most $2\,|B(P)|$ edges that is a 2-hop spanner for $B(P)$. Similarly, we can construct a 2-hop spanner $H_T(P)$ for $G(T(P))$ with $2\,|T(P)|$ edges. As all rectangles in $A(S)$ intersect b_P, we can apply Corollary 8 to construct a 2-hop spanner $H'_{\mathrm{In}}(P)$ with at most $2\,|A(P)|$ edges for $G(A(P))$. To construct $H_{\mathrm{In}}(P)$, we partition $\bigcup\big(A(P) \cap \mathrm{slab}(P)\big)$ analogously to the 1-dimensional case.

Recall that by Lemma 6(1), the line segment $\bigcup\big(A(P) \cap b_P\big)$ can be partitioned into intervals I_k, each of which is contained in some covering segment $c_k \in A(P)$. As every $s \in A(P)$ is an axis-aligned rectangle that spans two horizontal lines b_P and t_P, the segments in I_k can be extended upward to form axis-aligned rectangles $\widehat{I}_k$, each with an associated *covering rectangle* $\widehat{c}_k \in S$ corresponding to the covering segment c_k in the 1-dimensional case. Let $\widehat{\mathcal{I}}$ denote the set of all 2-dimensional intervals $\widehat{I}_k$.

We construct $H_{\mathrm{In}}(P)$ from $H'_{\mathrm{In}}(P)$ using these intervals. For every $s \in \mathrm{In}(P)$, if s intersects some $\widehat{I}_k \in \widehat{\mathcal{I}}$, add an edge between s and $\widehat{c}_k$ to $H'_{\mathrm{In}}(P)$. Let $H(P) = H_B(P) \cup H_T(P) \cup H_{\mathrm{In}}(P)$.

Stretch and Weight Analysis. We start with a technical lemma (Lemma 9), which is used in the stretch and weight analysis for the graph $H(P)$ of a single node $P \in \mathcal{P}$ (Lemma 10). Notice that the intervals in $\widehat{\mathcal{I}}$ act similarly to the 1-dimensional intervals in $\mathcal{I}$: in particular, Lemma 6 carries over, with I_k replaced by $\widehat{I}_k$, and with the line segment $\bigcup S$ replaced by the region $\bigcup\big(A(P) \cap \mathrm{slab}(P)\big)$.

▶ **Lemma 9.** *Let w denote the smallest width of any rectangle in $S(P)$, where the width of a rectangle $s \in A(P)$ is the length of $s \cap b_P$. Then for any $k \in \mathbb{N}$, the union of any $2k$ contiguous intervals in $\widehat{I}$ has width at least kw.*

Proof. By construction, every covering rectangle $\widehat{c}_k$ intersects $\widehat{I}_k$ and $\widehat{I}_{k-1}$. By Lemma 6(2), $\widehat{c}_k$ does not intersect any other intervals in $\widehat{I}$. Thus, $\widehat{c}_l \subset \widehat{I}_{k-1} \cup \widehat{I}_k$. This means that every pair of intervals has width at least w. As there are k disjoint pairs of intervals in a set containing $2k$ contiguous intervals, such a set must have width at least kw. ◀

▶ **Lemma 10.** *The subgraph $H(P)$ has the following properties:*
1. *for every edge $ab \in G(P)$ with $a \in A(P)$, $H(P)$ contains an ab-path of length at most 2;*
2. *$H(P)$ contains $O(\alpha^2\,|S(P)|)$ edges.*

Proof.
1. Every $b \in S(P)$ is in $B(P)$, $T(P)$, or $\mathrm{In}(P)$. If $b \in B(P)$, then the claim follows from the definition of $H_B(P)$ and the fact that $H_B(P)$ is a subgraph of $H(P)$. Similarly, the claim holds when $b \in T(P)$.
   Suppose $b \in \mathrm{In}(P)$. By Lemma 6(3), if there is an edge ab in $G(P)$ then both a and b intersect some interval $\widehat{I}_k$. By construction of $H_{\mathrm{In}}(P)$, there is an edge between a and c_k and between b and c_k (or else either a or b is equal to c_k) and so there is a path of length at most 2 between a and b in $H_{\mathrm{In}}(P)$. As $H_{\mathrm{In}}(P)$ is a subgraph of $H(P)$, this proves the claim.

2. By construction, $H_B(P)$ contains $2\,|B(P)|$ edges, $H_B(P)$ contains $2\,|T(P)|$ edges, and $H'_{\text{In}}(P)$ contains $2\,|A(P)|$ edges.
   We now bound the number of edges that are added to $H'_{\text{In}}(P)$ to produce $H_{\text{In}}(P)$. Let h be the distance between b_P and t_P. Every rectangle $a \in A(P)$ has width at least $\Omega(\frac{h}{\alpha})$, as a is α-fat and has height at least h. Further, notice that every rectangle $b \in \text{In}(P)$ has width less than αh, as otherwise it would cross b_P or t_P.
   Let $\widehat{I}_l, \widehat{I}_r \in \widehat{I}$, resp., be the leftmost and rightmost intervals that b intersects. As these intervals are interior-disjoint, the intervals between $\widehat{I}_l$ and $\widehat{I}_r$ (if any exist) must have a total length less than αh; otherwise, b could not intersect both. By Lemma 9, any consecutive $2\alpha^2$ intervals (all of length at least h/α) have width at least αh. Thus, b can intersect at most $2\alpha^2 - 1$ intervals other than $\widehat{I}_l$ and $\widehat{I}_r$.
   By construction, this implies that $b \in \text{In}(P)$ adds at most $O(\alpha^2)$ edges to H'_{In} during the construction of H_{In}. Thus, H_{In} has at most $2|A(P)| + \alpha^2|\text{In}(P)|$ edges. As $\text{In}(P)$, $B(P)$, $T(P)$, and $A(P)$ are all subsets of $S(P)$, $H(P)$ has at most $O(\alpha^2|S(P)|)$ edges. ◀

We prove that $H(S) = \bigcup_{P\in\mathcal{P}} H(P)$ has $O(\alpha^2 n \log n)$ edges and that it is a 2-hop spanner. We begin by considering the size. While some $H(P)$ may contain many edges, we bound the total size of $H(S)$ by showing that every rectangle in S is involved in $O(\log n)$ subproblems.

▶ **Lemma 11.** *For every rectangle $s \in S$, the following hold:*
1. *there are $O(\log n)$ nodes $P \in \mathcal{P}$ where $s \in \text{In}(P)$;*
2. *there are $O(\log n)$ nodes $P \in \mathcal{P}$ where $s \in B(P) \setminus A(P)$; symmetrically, there are $O(\log n)$ nodes $P \in \mathcal{P}$ where $s \in T(P) \setminus A(P)$;*
3. *there are $O(\log n)$ nodes $P \in \mathcal{P}$ where $s \in A(P)$.*

Proof. Notice that for any k, the slabs of nodes at level k in the space partition tree have pairwise disjoint interiors. Since S contains n rectangles, there are at most $\binom{n}{2}$ intersections in $G(S)$. Thus, $|C(S)| \le \binom{n}{2}$, and so the tree has $O(\log n)$ levels.

1. For every level $k \in \mathbb{N}$ in the space partition tree, there is only one node P where $s \in \text{In}(P)$. Suppose for the sake of contradiction that $s \in \text{In}(P_1)$ and $s \in \text{In}(P_2)$ with P_1 and P_2 in the same level and $P_1 \neq P_2$. By the definition of $\text{In}(.)$, s is contained in $\text{slab}(P_1)$ and in $\text{slab}(P_2)$. As these slabs are disjoint, this is impossible. Summation over $O(\log n)$ levels of the recursion tree completes the proof.
2. For every level $k \in \mathbb{N}$ in the tree, consider the node P with the highest slab such that $\text{slab}(P) \cap s \neq \emptyset$. Notice that $s \in B(P)$ and $s \notin T(P)$, so $s \in B(P) \setminus A(P)$. Any other node P' in this level that s intersects lies strictly below P (as nodes within a level have pairwise disjoint slab interiors) and s is connected, so $s \in B(P')$ only if $s \in T(P')$. Thus, P is the only node in level k where $s \in B(P) \setminus A(P)$. A symmetric argument proves that there is only one P per level where $s \in T(P) \setminus A(P)$.
3. For every level $k \in \mathbb{N}$ in the tree, there are at most two nodes P such that $s \in A(P)$. Suppose for the sake of contradiction that there exist distinct P_1, P_2, and P_3 at level k such that $s \in A(P_1) \cap A(P_2) \cap A(P_3)$. The interiors of the corresponding slabs are disjoint, so we may assume w.l.o.g. that P_1 lies below P_2, which lies below P_3. As s is connected, it intersects b_P and t_P for every node P between P_1 and P_3. In particular, s must be in $A(P)$ for the sibling P of P_2. Then s is also in $A(P')$ for the parent P' of P_2. This is a contradiction – if s were in the A set of the parent of P_2, it would not have been added to the set $S(P_2) \subset S(P') \setminus A(P')$ of rectangles for the child. ◀

▶ **Corollary 12.** *For every $s \in S$, there are $O(\log n)$ nodes $P \in \mathcal{P}$ where $s \in S(P)$.*

Proof. This follows from the fact that for every node P, $S(P)$ is the union of the four sets mentioned in Lemma 11: $S(P) = \mathrm{In}(P) \cup (B(P) \setminus A(P)) \cup (T(P) \setminus A(P)) \cup A(P)$. ◀

▶ **Lemma 13.** *$H(S)$ has $O(\alpha^2 n \log n)$ edges.*

Proof. For every node P, $H(P)$ has $O(\alpha^2 |S(P)|)$ edges by Lemma 10. Charge $O(\alpha^2)$ edges to each rectangle in $S(P)$. By Corollary 12, each rectangle is charged at most $O(\log n)$ times, and so $H(S)$ has at most $O(\alpha^2 n \log n)$ edges. ◀

▶ **Lemma 14.** *$H(S)$ is a 2-hop spanner for $G(S)$.*

Proof. Let ab be an edge in $G(S)$. As the rectangles a and b intersect, there is some point $p \in C(S)$ that lies in $a \cap b$. Since p is not in the interior of any slab at the leaf level, a horizontal line of the space partition contains p. Assume w.l.o.g. that this line is b_P for some node P. If both a and b are present in $S(P)$, then $H_B(P)$ contains an ab-path of length at most 2. Otherwise, there is some node P' for which both a and b are in $S(P')$ but either a or b is *not* in the set for either child of P'. Assume w.l.o.g. that a was removed. By construction, a rectangle is removed exactly when it is in $A(P')$. By Lemma 10, $H(P')$ contains an ab-path of length at most 2. As $H(S) = \bigcup_{P \in \mathcal{P}} H(P)$, this proves that $H(S)$ contains such a path. ◀

The previous two lemmata prove the following theorem.

▶ **Theorem 15.** *The intersection graph of every set of n axis-aligned α-fat rectangles in the plane admits a 2-hop spanner with $O(\alpha^2 n \log n)$ edges.*

4 Lower Bound Constructions

In this section, we define a class of graphs for which any 2-hop spanner has at least $\Omega(n \log n)$ edges, then show that these graphs can be realized as the intersection graph of n homothets of any convex body in the plane.

Construction of $F(h)$. For every $h \in \mathbb{N}$, we construct a graph $F(h)$, which contains $2^h(h+1)$ vertices. The vertex set is $V = \{0, \ldots, 2^h - 1\} \times \{0, \ldots, h\}$. For each vertex $v = (x, i)$, we call i the *level* of v. For each level $i \in \{0, \ldots, h\}$, partition the vertices with level less than or equal to i into 2^i groups of $2^{h-i}(i+1)$ consecutive vertices based on their x-coordinates. In particular, for every level $i \in \{0, \ldots, h\}$, let $\{0, \ldots, 2^h - 1\} = \bigcup_{k=1}^{2^i - 1} X_{k,i}$, where $X_{k,i} = \{2^{h-i}k, 2^{h-i}k + 1, \ldots, 2^{h-i}(k+1) - 1\}$. This defines groups $V_{k,i} = X_{k,i} \times \{0, \ldots, i\}$ for $k \in \{0, \ldots, 2^i - 1\}$. Notice that $(x, \ell) \in V_{k,i}$ for $k = \lfloor x/2^i \rfloor$ and $i \geq \ell$. Finally, add edges to the graph $F(h)$ such that every group $V_{k,i}$ is a clique; see Fig. 5.

We show that any 2-hop spanner for $F(h)$ with $n = 2^h(h+1)$ vertices has $\Omega(2^h h^2) = \Omega(n \log n)$ edges. We do this by first showing that a 2-hop spanner contains $\Omega(2^{h-i}h)$ edges in each clique induced by a group $V_{k,i}$, and these edges are distinct from the edges required by any other group. This result follows from the following lemma:

▶ **Lemma 16.** *Suppose that the vertex set of the complete graph K_{2n} is partitioned into two sets A and B each of size n, and call edges between A and B bichromatic. Then every 2-hop spanner of K_{2n} contains n bichromatic edges.*

Proof. Let S be a 2-hop spanner for K_{2n}. If every vertex in A is incident to a bichromatic edge in S, then clearly S contains at least $|A| = n$ bichromatic edges. Otherwise, there is some $a \in A$ that has no direct edges to B in S. For every $b \in B$, S contains a 2-hop path between a and b, that is, a path (a, a_b, b) for some $a_b \in A$. The edges $a_b b$ are bichromatic and distinct for all $b \in B$, so S contains at least $|B| = n$ bichromatic edges. ◀

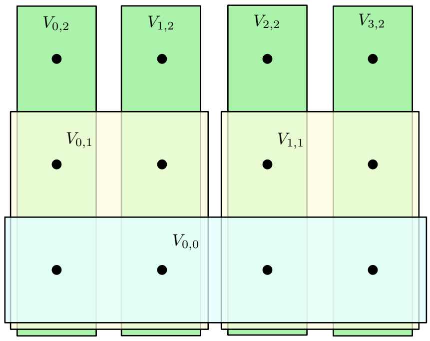

Figure 5 Vertices of $F(2)$ grouped by cliques $V_{k,i}$.

▶ **Lemma 17.** *For all $h \in \mathbb{N}$, $F(h)$ has $n = 2^h(h+1)$ vertices and $\Omega(n \log n)$ edges.*

Proof. Notice that every $X_{k,i}$, for $i < h$, can be written as $X_{2k,i+1} \cup X_{2k+1,i+1}$. Accordingly, we can partition $V_{k,i}$ into two sets of equal size:

$$V_{k,i} = \Big(X_{2k,i+1} \times \{0, \dots, i\}\Big) \cup \Big(X_{2k+1,i+1} \times \{0, \dots, i\}\Big).$$

Call edges that cross between these two sets $V_{k,i}$-bichromatic.

We claim that the set of $V_{k,i}$-bichromatic edges and $V_{k',i'}$-bichromatic edges are disjoint unless $k' = k$ and $i' = i$. If $i = i'$, then the claim follows from the fact that $V_{k,i}$ and $V_{k',i}$ are disjoint. Otherwise, assume w.l.o.g. that $i < i'$. Notice that either $X_{k',i'}$ is contained within $X_{2k,i+1}$ or $X_{2k+1,i+1}$, or it is disjoint from both. The $V_{k,i}$-bichromatic edges cross from $X_{2k,i+1}$ to $X_{2k+1,i+1}$ while $V_{k',i'}$-bichromatic edges stay within $X_{k',i'}$, so the edge sets must be disjoint.

Let S be a 2-hop spanner of $F(h)$. Each vertex set $V_{k,i}$ contains $2^{h-i}(i+1)$ vertices, so the partition described above involves two sets of size $2^{h-i-1}(i+1)$. As $V_{k,i}$ is a clique, Lemma 16 implies that S contains at least $2^{h-i-1}(i+1)$ $V_{k,i}$-bichromatic edges. Every level i contains 2^i groups $V_{k,i}$, so S contains at least $2^{h-1}(i+1)$ bichromatic edges in each level. Summation over all h levels (excluding the level where $i = h$) yields at least $\sum_{i=0}^{h-1} 2^{h-1}(i+1) = \Omega(2^h h^2) = \Omega(n \log n)$ edges. ◀

Geometric Realization of $F(h)$. We realize $F(h)$ as the intersection graph of a set $S(h)$ of homothets of any convex body for all $h \in \mathbb{N}$. The construction is recursive. To construct $S(h+1)$, we form two copies of $S(h)$ to realize vertices in the first h levels, then add homothets to realize the vertices in level $h+1$.

▶ **Lemma 18.** *For every convex body $C \subset \mathbb{R}^2$ and every $h \in \mathbb{N}$, the n-vertex graph $F(h)$ can be realized as the intersection graph of a set $S(h)$ of n homothets of C.*

Proof. Let C be a convex body (i.e., a compact convex set with nonempty interior) in the plane. Let $o \in \partial C$ be an extremal point of C. Then there exists a (tangent) line L such that $C \cap L = \{o\}$. Assume w.l.o.g. that o is the origin, L is the x-axis, and C lies in the upper halfplane. We construct $S(h)$ recursively from $S(h-1)$. Let $s(a,i) \in S(h)$ denote the homothet that represents the vertex $(a,i) \in F(h)$. We maintain two invariants: (I1) for every $a \in \{0, \dots, 2^h - 1\}$, there is some point p_a on the x-axis such that every $s(a,i) \in S(h)$ is tangent to the x-axis and intersects the x-axis exactly at p_a; and (I2) whenever $s_1, s_2 \in S(h)$ intersect, $s_1 \cap s_2$ has nonempty interior.

Construction. $F(0)$ has a single vertex $(0,0)$ and no edges, so it can be represented as the single convex body C with the extremal point o on the x-axis.

We now construct $S(h)$ from $S(h-1)$; see Fig. 6. By invariant (I2), there is some $\varepsilon > 0$ such that for every $s \in S(h-1)$, translating s by ε in any direction does not change the intersection graph. Duplicate $S(h-1)$ to form the sets $S_1(h-1)$ and $S_2(h-1)$, and translate every homothet in $S_2(h-1)$ by ε in the positive x direction. Let $S'(h) = S_1(h-1) \cup S_2(h-1)$. Notice that for every clique $V_{k,i}$ in $F(h-1)$, there is a corresponding clique in the intersection graph of $S'(h)$ that contains both the vertices in the clique $V_{k,i}$ realized by $S_1(h-1)$ *and* the vertices in the clique $V_{k,i}$ realized by $S_2(h-1)$.

The x-axis is still tangent to all $s \in S'(h)$, and there are 2^h distinct points on the x-axis that intersect some $s \in S'(h)$. Each point p_a has a neighborhood that intersects only the homothets in $S'(h)$ that contain p_a, since every convex body that does not contain p_a has a positive distance from p_a by compactness. For each p_a, add a homothetic copy C_a of C completely contained within that neighborhood, tangent to the x-axis and containing p_a. Let $S(h)$ be the union of $S'(h)$ and these C_a.

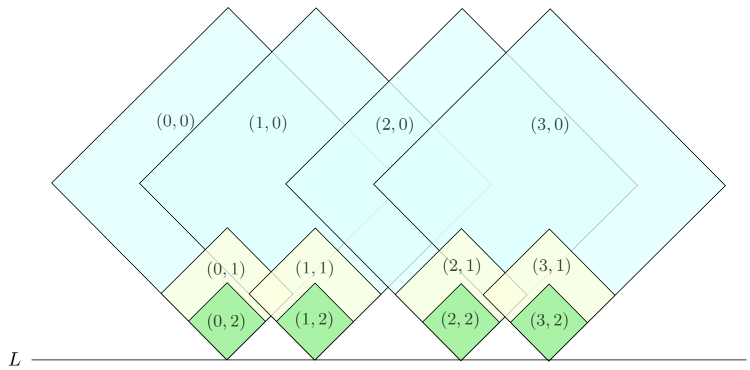

Figure 6 Realization of $F(2)$ with homothets of squares, all tangent to L. Each homothet is labeled with the vertex of $F(2)$ that is represents.

Correctness. For $0 \le i < h$, let the homothet $s_1(a,i) \in S_1(h-1)$ represent $(2a,i)$ in $F(h)$, and let the homothet $s_2(a,i) \in S_2(h-1)$ represent $(2a+1,i)$ in $F(h)$. Let the homothets C_a represent $(a,h) \in F(h)$.

This correspondence implies that, for all $0 \le i < h$, vertices in the clique $V_{k,i}$ in $F(h)$ have been realized by homothets corresponding to a clique $V_{\lfloor \frac{k}{2} \rfloor, i}$ in $S_1(h-1)$ or $S_2(h-1)$. By construction of $S(h)$, any two such homothets intersect. Similar reasoning applies in the opposite direction: any intersection between two homothets in S' corresponds to an edge in some clique in $F(h)$. When $i = h$, notice that every C_a intersects exactly the homothets in $S'(h)$ that intersect p_a, which by assumption were the homothets representing points with the same x-coordinate. Thus, any clique $V_{k,h}$ in $F(h)$ is represented in $S(h)$, and there are no edges involving C_a that do not correspond to such a clique in $F(h)$. ◀

The previous two lemmata imply the following theorem.

▶ **Theorem 19.** *For every convex body $C \subset \mathbb{R}^2$, there exists a set S of n homothets of C such that every 2-hop spanner for the intersection graph of S has $\Omega(n \log n)$ edges.*

5 Outlook

We have shown that every n-vertex UDG admits a 2-hop spanner with $O(n)$ edges; and this bound generalizes to the intersection graphs of translates of any convex body in the plane (see the full paper). The proof crucially relies on new results on the α-hull of a planar point set. It remains an open problem whether these results generalize to higher dimensions, and whether unit ball graphs admit 2-hop spanners with $O_d(n)$ edges in $\mathbb{R}^d$ for any $d \geq 3$.

We proved that the intersection graph of n axis-aligned squares in $\mathbb{R}^2$ admits a 2-hop spanner with $O(n \log n)$ edges, and this bound is the best possible. However, it is unclear whether the upper bound generalizes to Euclidean disks of arbitrary radii (or to fat convex bodies) in the plane. For fat convex bodies and for axis-aligned rectangles, we obtained 3-hop spanners with $O(n \log n)$ and $O(n \log^2 n)$ edges, respectively. However, it is unclear whether the logarithmic factors are necessary. Do these intersection graphs admit weighted edge biclique covers of weight $O(n)$? In general, we do not even know whether a linear bound can be established for any constant stretch: Is there a constant $t \in \mathbb{N}$ for which every intersection graph of n disks or rectangles admits t-hop spanner with $O(n)$ edges?

Finally, it would be interesting to see other classes of intersection graphs (e.g., for strings or convex sets in $\mathbb{R}^2$, set systems with bounded VC-dimension or semi-algebraic sets in $\mathbb{R}^d$) for which the general bound of $O(n^{1+1/\lceil t/2 \rceil})$ edges for t-hop spanners can be improved.

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