

# Parameterised Partially-Predrawn Crossing Number

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## Abstract

Inspired by the increasingly popular research on extending partial graph drawings, we propose a new perspective on the traditional and arguably most important geometric graph parameter, the *crossing number*. Specifically, we define the *partially predrawn crossing number* to be the smallest number of crossings in any drawing of a graph, part of which is prescribed on the input (not counting the prescribed crossings). Our main result – an FPT-algorithm to compute the partially predrawn crossing number – combines advanced ideas from research on the classical crossing number and so called *partial planarity* in a very natural but intricate way. Not only do our techniques generalise the known FPT-algorithm by Grohe for computing the standard crossing number, they also allow us to substantially improve a number of recent parameterised results for various drawing extension problems.

**2012 ACM Subject Classification** Theory of computation → Fixed parameter tractability; Theory of computation → Computational geometry

**Keywords and phrases** Crossing Number, Drawing Extension, Partial Planarity, Parameterised Complexity

**Digital Object Identifier** 10.4230/LIPIcs.SoCG.2022.46

**Related Version** *Full Version*: <https://arxiv.org/abs/2202.13635>

**Funding** *Thekla Hamm*: Supported by the Austrian Science Fund (projects P31336, Y1329, and W1255-N23).

*Petr Hliněný*: Supported by the Czech Science Foundation, project no. 20-04567S.

## 1 Introduction

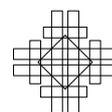
Determining the crossing number, i.e. the smallest possible number of pairwise transverse intersections (called *crossings*) of edges in any drawing, of a graph is among the most important problems in discrete computational geometry. As such its general computational complexity is well-researched: Probably most famously, it is known that graphs with crossing number 0, i.e. planar graphs, can be recognised in polynomial time [27, 20, 28]. Generally, computing the crossing number of a graph is NP-hard, even in very restricted settings [16, 19, 25, 4], and also APX-hard [3]. However there is a fixed-parameter algorithm for the problem, and even one that can compute a drawing of a graph with at most  $k$  crossings in time in  $\mathcal{O}(f(k)n)$  or decide that its crossing number is larger than  $k$  [17, 22].

More recently, so called *graph drawing extension* problems have received increased attention. Instead of being given an entirely abstract graph as an input, here the input is a *partially drawn graph*  $\mathcal{P} = (G, \mathcal{H})$ , meaning that a subgraph  $H$  of the input graph  $G$  is given with a fixed drawing  $\mathcal{H}$  which must not be changed in the solution. This is motivated by immediate applications in network visualisation [23], as well as a more general line of research in which important computational problems are extended to the setting in which parts of the solution are prescribed which can lead to useful insights for dynamic or divide-and-conquer type algorithms and heuristics [5, 14]. In this context it is natural to define the *partially predrawn crossing number* as the smallest number of pairwise crossings of edges in any



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38th International Symposium on Computational Geometry (SoCG 2022).  
Editors: Xavier Goaoc and Michael Kerber; Article No. 46; pp. 46:1–46:15  
Leibniz International Proceedings in Informatics

**LIPICs** Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



drawing which coincides with (i.e., extends) the given fixed drawing of the pre drawn skeleton, minus the number of “unavoidable” crossings already contained in the fixed drawing of the skeleton. We name this problem PARTIALLY PRE DRAWN CROSSING NUMBER.

Of course, the problem of computing the partially pre drawn crossing number is more general than the one of computing the classical crossing number (which is captured by the former by simply letting the pre drawn skeleton be empty), and thus the known hardness results for computing the classical crossing number carry over. To the best of our knowledge, the partially pre drawn crossing number problem has so far not been explicitly studied in literature, although, there are papers which study *partially embedded planarity*, i.e. the property of having partially pre drawn crossing number 0, and variants thereof. In particular, similarly to ordinary planarity, partially drawn graphs extendable to planar drawings can be recognised in polynomial time [1], and in analogy to the Kuratowski theorem, there is also a neat list of forbidden “partially drawn minors” (Figure 3) which characterise partially drawn graphs extendable to planar drawings [21].

If one allows a non-zero number of crossings, the only algorithmic results on extending partially drawn graphs with constrained crossings we are aware of are those for scenarios with a few edges or vertices outside of the pre drawn skeleton or/and with a small number of crossings for each edge. We give a brief list of these algorithmic results:

- An algorithm to determine the exact partially pre drawn crossing number of a partially drawn graph in FPT time parameterised by the number of edges which are not fixed by the pre drawn skeleton [6] (the “rigid” case in the paper).
- An algorithm to determine whether there is a 1-planar drawing (or more generally a drawing in which each edge outside of the pre drawn skeleton has at most  $c$  crossings) which coincides with the given partial drawing in FPT time parameterised by ( $c$  and) the number of edges which are not fixed by the pre drawn skeleton [13, 15].
- An algorithm to determine whether there is a 1-planar drawing which coincides with the given partial drawing in XP time parameterised by the vertex cover size of the edges which are not fixed by the pre drawn skeleton [12].
- An algorithm to determine whether there is a simple drawing in which each edge outside of the pre drawn skeleton has at most  $c$  crossings which coincides with the given partial drawing in FPT time parameterised by  $c$  and the number the edges which are not fixed by the pre drawn skeleton [15].

We remark that all these parameterised algorithms require the given pre drawn skeleton to be connected, and the last three algorithms are easily adapted to output drawings minimising the number of crossings under the requirement of the respective properties.

## Contributions

The foundation of our main contribution is a fixed-parameter algorithm for an exact computation of the partially pre drawn crossing number  $k$  of a given partially drawn graph.

► **Theorem 1.1.** *PARTIALLY PRE DRAWN CROSSING NUMBER is in FPT when parameterised by the solution value (i.e., by the number of crossings which are not pre drawn).*

We employ a technique similar to the approach showing fixed-parameter tractability of classical crossing number devised by Grohe [17]. This means we proceed in two phases:

- I. We iteratively reduce the input partially drawn graph  $\mathcal{P}$  until we cannot find a large flat grid in it, and so we bound its treewidth by a function of  $k$ , or decide that the partially pre drawn crossing number of  $\mathcal{P}$  is larger than  $k$ . Importantly, each reduction step is guaranteed to preserve the solution value (unless it is  $> k$ ).

- II. We devise an  $\text{MSO}_2$ -encoding for the property that any partially drawn graph has the partially predrawn crossing number at most  $k$ . The key idea is to encode the predrawn skeleton of the input in a 3-connected planar “frame” which is added to the input partially drawn graph. Using the bounded treewidth of the involved graph with the frame, we then apply Courcelle’s theorem [7] in order to decide this property.

Note that the second step is an interesting result in its own right:

► **Lemma 1.2.** *For every  $k \geq 0$  there is an  $\text{MSO}_2$ -formula  $\psi_k$  such that the following holds. Given a partially drawn graph  $\mathcal{P}$ , one can in polynomial time construct a graph  $G'$  such that  $\psi_k$  is true on  $G'$  if and only if the partially predrawn crossing number of  $\mathcal{P}$  is at most  $k$ . This claim holds also if some edges of  $\mathcal{P}$  are marked as “uncrossable” and we compute the crossing number over such drawings extending  $\mathcal{P}$  that do not have crossings on the “uncrossable” edges.*

While our high-level approach is similar to Grohe’s [17], in each phase we are faced with some caveats, on which we elaborate in the respective sections, due to the fact that we must respect the given predrawn skeleton and that we have to observe also the treewidth of the derived graph which encodes the predrawn skeleton, i.e. of  $G'$  from Lemma 1.2.

In this regard, we also give a concrete example (see Proposition 5.1) of a fundamentally different behaviour of the partially predrawn crossing number compared to the classical one (which can partly explain the difficulties we face in Theorem 1.1, compared to [17]). In a nutshell, we show that for fixed  $k$  a partially drawn graph can have arbitrarily many nested cycles which are “critical” for having crossing number  $> k$ .

Based on the proof of Theorem 1.1 we are also able to give an improved algorithm to determine whether there is a drawing in which each edge outside of the predrawn skeleton has at most  $c$  crossings which coincides with the given partial drawing. Specifically we can show the following theorem, where the *partially predrawn  $c$ -planar crossing number* of a partially drawn graph  $\mathcal{P}$  is as the partially predrawn crossing number above while restricted to only drawings of  $\mathcal{P}$  in which each edge outside of the predrawn skeleton has at most  $c$  crossings.

► **Theorem 1.3.** *PARTIALLY PREDRAWN  $c$ -PLANAR CROSSING NUMBER is in FPT when parameterised by the solution value (i.e., by the number of crossings which are not predrawn).*

Compared to the algorithm given in [13], Theorem 1.3 presents an additional improvement in two important aspects. Not only can our algorithm solve the  $c$ -planar drawing extension problem parameterised by the number of new crossings (a less restrictive parameter than the combination of  $c$  and  $|E(G) \setminus E(H)|$ ), but we can also handle disconnected initial drawings.

We also can combine our techniques with structural insights from [15] to drop the connectivity requirement on the input in the setting that we want to determine the partially predrawn  $c$ -planar crossing number restricted to simple drawings:

► **Theorem 1.4.** *Given a partially drawn graph, one can in FPT time parameterised by  $c$  and the number of edges not contained in the predrawn skeleton, decide the minimum number of crossings in a simple drawing which coincides with the given simple partial drawing and in which each edge outside of the predrawn skeleton has at most  $c$  crossings.*

Full proofs of the \*-marked statements are left for [arXiv:2202.13635](https://arxiv.org/abs/2202.13635).



■ **Figure 1** Two drawings of the same graph (solid lines) with the same rotation scheme. However both drawings are not equivalent. All dashed curves need to be mapped without crossing each other or any solid line by any homeomorphism from the left to the right the drawing. This is not possible.

## 2 Preliminaries

We use standard terminology for undirected simple graphs [9] and assume basic understanding of *parameterised complexity* [8, 10], and of *Courcelle’s theorem together with MSO logic* [2, 7] and treewidth. We refer also to the full preprint paper for additional background on these notions. Regarding embeddings and drawings of graphs we mostly follow [24].

For  $r \in \mathbb{N}$ , we write  $[r]$  as shorthand for the set  $\{1, \dots, r\}$ .

### 2.1 Partial graph drawings

A *drawing*  $\mathcal{G}$  of a graph  $G$  in the Euclidean plane  $\mathbb{R}^2$  is a function that maps each vertex  $v \in V(G)$  to a distinct point  $\mathcal{G}(v) \in \mathbb{R}^2$  and each edge  $e = uv \in E(G)$  to a simple open curve  $\mathcal{G}(e) \subset \mathbb{R}^2$  with the ends  $\mathcal{G}(u)$  and  $\mathcal{G}(v)$ . We require that  $\mathcal{G}(e)$  is disjoint from  $\mathcal{G}(w)$  for all  $w \in V(G) \setminus \{u, v\}$ . In a slight abuse of notation we often identify a vertex  $v$  with its image  $\mathcal{G}(v)$  and an edge  $e$  with  $\mathcal{G}(e)$ . Throughout the paper we will moreover assume that: there are finitely many points which are in an intersection of two edges, no more than two edges intersect in any single point other than a vertex, and whenever two edges intersect in a point, they do so transversally (i.e., not tangentially).

The intersection (a point) of two edges is called a *crossing* of these edges. A drawing  $\mathcal{G}$  is *planar* (or a *plane graph*) if  $\mathcal{G}$  has no crossings, and a graph is *planar* if it has a planar drawing. The number of crossings in a drawing  $\mathcal{G}$  is denoted by  $\text{cr}(\mathcal{G})$ . A drawing  $\mathcal{G}$  is *c-planar* (or a *c-plane graph*) if every edge in  $\mathcal{G}$  contains at most  $c$  crossings, and a graph is *c-planar* if it has a  $c$ -planar drawing. The *planarisation*  $\mathcal{G}^\times$  of a drawing  $\mathcal{G}$  of  $G$  is the plane graph obtained from  $\mathcal{G}$  by making each crossing point a new degree-4 vertex of  $\mathcal{G}^\times$ . The inclusion-maximal connected subsets of the set-complement  $\mathbb{R}^2 \setminus \mathcal{G}$  are called the *faces* of  $\mathcal{G}$ . For any drawing exactly one of these faces is infinite and referred to as the *outer face*.

A *partial drawing* of a graph  $G$  is a drawing of an arbitrary subgraph  $H$  of  $G$ . A *partially drawn graph*  $\mathcal{P} = (G, \mathcal{H})$ , with an implicit reference to  $H$ , is a graph  $G$  together with a partial drawing  $\mathcal{H}$  of  $H \subseteq G$ , and then  $\mathcal{H}$  is called the *predrawn skeleton* of  $(G, \mathcal{H})$ . We say that two drawings  $\mathcal{G}_1$  and  $\mathcal{G}_2$  of the same graph  $G$  are *equivalent* if there is a homeomorphism of  $\mathbb{R}^2$  onto itself taking  $\mathcal{G}_1^\times$  onto  $\mathcal{G}_2^\times$  [24]. For connected  $\mathcal{G}_1^\times$  and  $\mathcal{G}_2^\times$ , this is the same as requiring equal rotation systems and the same outer face. However, for disconnected drawings, [21] in addition to equal rotation systems and outer face it is necessary to specify which faces of each connected component of  $\mathcal{G}_1^\times$  contain which other connected components and in which orientation, and match this specification with  $\mathcal{G}_2^\times$  (see also Figure 1).

In this setup, we also say that two *partially drawn graphs* are *isomorphic* if there exists an isomorphism which gives an equivalence of their predrawn skeletons.

## 2.2 Problem definitions

The PARTIALLY PREDRAWN CROSSING NUMBER problem takes as an input a partially drawn graph  $(G, \mathcal{H})$  and an integer  $q$ . The task is to decide whether there is a drawing  $\mathcal{G}$  of  $G$ , the restriction of which to the predrawn skeleton  $H$  is equivalent to  $\mathcal{H}$  (we can shortly say that  $\mathcal{G}$  *extends*  $\mathcal{H}$ ), such that  $\mathcal{G}$  has at most  $q + \text{cr}(\mathcal{H})$  crossings. The smallest value of the parameter  $q$  for which  $(G, \mathcal{H})$  is a **yes**-instance of PARTIALLY PREDRAWN CROSSING NUMBER is called the *partially predrawn crossing number* of  $(G, \mathcal{H})$ , denoted by  $\text{pd-cr}(G, \mathcal{H})$ . Note that  $\text{pd-cr}(G, \emptyset)$  is the (called classical for distinction) *crossing number*  $\text{cr}(G)$  of  $G$ .

Likewise, the PARTIALLY PREDRAWN  $c$ -PLANAR CROSSING NUMBER problem takes as an input a partially drawn graph  $(G, \mathcal{H})$  and an integer  $q$ . The task is to decide whether there is a drawing  $\mathcal{G}$  of  $G$  in which every edge in  $E(G) \setminus E(H)$  has at most  $c$  crossings and the restriction of which to  $H$  is equivalent to  $\mathcal{H}$ , such that  $\mathcal{G}$  has altogether at most  $q + \text{cr}(\mathcal{H})$  crossings. The smallest  $q$  (which may not be defined in general; a trivial example for which  $q$  is not defined is given by  $c = 1$  and  $G$  not 1-planar) for which  $(G, \mathcal{H})$  is a **yes**-instance of PARTIALLY PREDRAWN  $c$ -PLANAR CROSSING NUMBER is called the *partially predrawn  $c$ -planar crossing number* of  $(G, \mathcal{H})$ .

## 2.3 A parameterised algorithm for classical crossing number

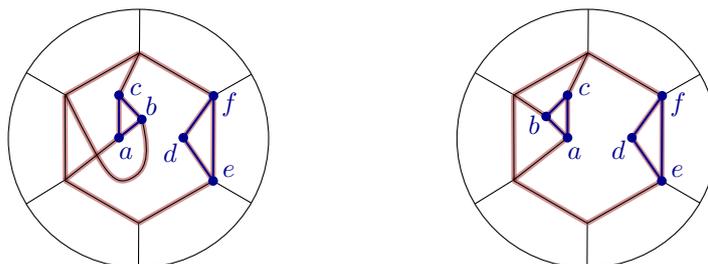
We outline the high-level idea of Grohe’s algorithm [17] to decide the classical crossing number of a graph in FPT time and note some obstacles that we need to overcome. Due to lack of space in the main paper, we leave the complete formal recapitulation together with some supplementary definitions for the full preprint paper.

The algorithm proceeds in two phases.

### Phase I – Bounding Treewidth

Consider a graph  $G$  in which some edges are marked as “uncrossable”, and the question of whether there is a drawing of  $G$  with at most  $k$  crossings in which no “uncrossable” edge is crossed for a fixed parameter  $k$ . To improve readability, we shortly say that a drawing is *conforming* if no edge marked “uncrossable” is crossed in it. Grohe [17] showed that in polynomial time one can (i) confirm that the answer to this question is no, (ii) find a tree decomposition of  $G$  with width bounded in  $k$ , or (iii) find a connected planar subgraph  $I \subseteq G$  where  $|V(I)| \geq 6$  together with a cycle  $C$  that is disjoint from  $V(I)$  and contains  $N(I)$  such that the following holds. If  $G'$  arises from  $G$  by contracting  $I$  to a vertex  $v_I$  and additionally marking all edges incident to  $v_I$  and all edges of  $C$  as “uncrossable”, then any crossing-minimum conforming drawing of  $G$  arises from a crossing-minimum conforming drawing of  $G'$  by replacing  $v_I$  with a planar drawing of  $G[V(I) \cup V(C)]$  where the drawing of  $C$  is distorted to match that in the drawing of  $G'$  and  $I$  is drawn in an  $\varepsilon$ -neighbourhood of  $v_I$ . Conversely, every crossing-minimum conforming drawing of  $G'$  arises from a crossing-minimum conforming drawing of  $G$  by contracting  $I$  and placing the resulting vertex on the drawing of some vertex in  $I$ .

In the partially drawn setting we can however not simply contract a subgraph  $I$  without losing information about its parts that are potentially fixed by the partial drawing of the instance. In particular, reinserting some unrestricted planar drawing of  $I$  can violate the partial drawing (see Figure 2).



■ **Figure 2** Example where predrawn parts (blue) make it impossible to simply insert a planar drawing of  $I$  (brown underlay). If the partial drawing is as on the left,  $I$  can be drawn planarly as depicted on the right but not while preserving equivalence of the partial drawing (cf. Figure 1).

### Phase II – MSO Encoding

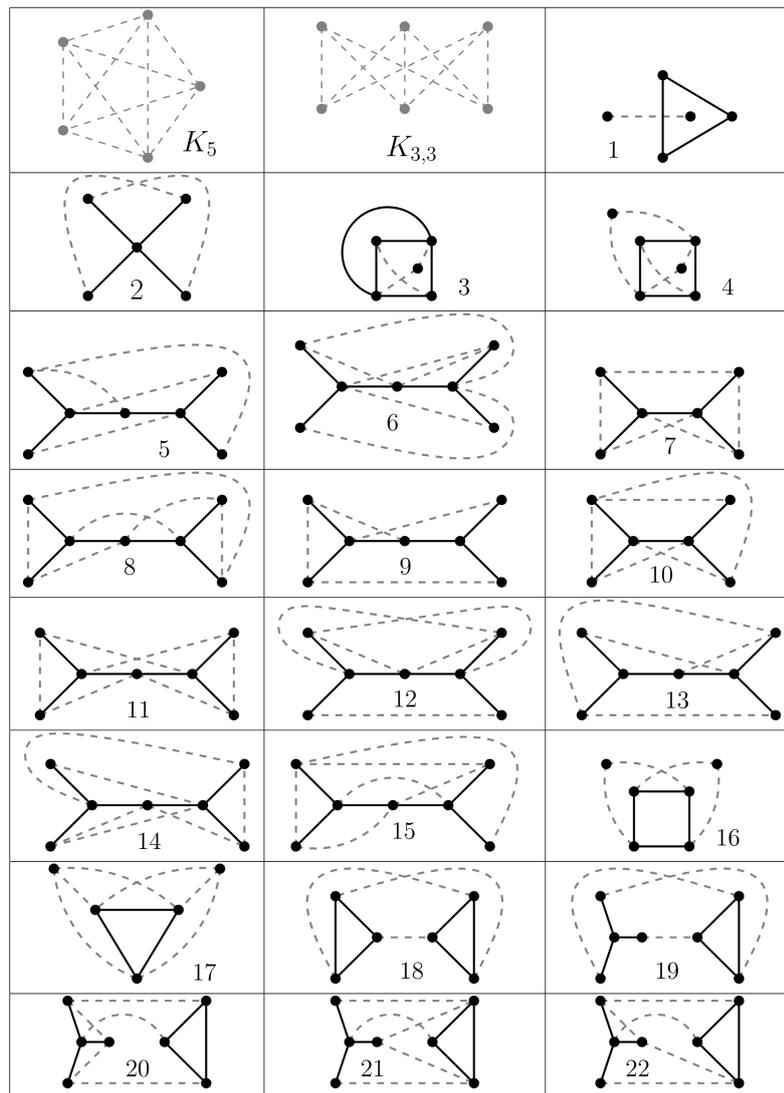
After having reduced  $G$  to a graph of treewidth bounded in the desired crossing number, one can apply Courcelle’s theorem to decide whether  $\text{cr}(G) \leq k$  for any fixed  $k$ . For that it is sufficient to encode in  $\text{MSO}_2$  logic the existence of at most  $k$  pairs of edges such that, after planarising a hypothetical crossing between the two edges of each pair, the resulting graph is planar. To express planarity, one simply excludes the existence of subdivisions of the two Kuratowski obstructions  $K_5$  and  $K_{3,3}$ . The task of interpreting the planarisation of hypothetical crossings, “guessed” by existential quantifiers, is a more subtle one. In order to avoid heavy tools of finite model theory here, we can apply the following trick: instead of  $G$ , use the graph  $G^{(k)}$  which subdivides  $k$ -times every edge of  $G$ , and “guess”  $k$  pairs of the subdivision vertices which are pairwise identified to make the planarisation.

This of course does not carry over easily to the partially drawn setting as the Kuratowski obstructions do not capture the predrawn skeleton shape, i.e., there could be partially drawn graphs with high crossing number and not containing any  $K_5$  or  $K_{3,3}$  subdivisions. Here, instead, we will use the corresponding planarity obstructions for partially drawn graphs from [21], described next in Section 2.4. This brings two new complications to be resolved; namely that the list of obstructions is not finite, and that we have to encode the input drawing of the given partially drawn graph in an abstract way which can be “read” by an  $\text{MSO}_2$ -formula.

## 2.4 Characterising partially predrawn planarity

We use the mentioned result of Jelínek, Kratochvíl and Rutter [21] characterising partially predrawn planarity, that is, the question of whether a given partially drawn graph  $(G, \mathcal{H})$  admits a planar drawing which extends  $\mathcal{H}$ , by means of forbidding so-called PEG-minors. In this context we assume  $\text{cr}(\mathcal{H}) = 0$ . The forbidden obstructions are formed by one “easy” infinite family described separately (the *alternating chains*) and a list of 24 specific partially drawn graphs shown in Figure 3. However, since PEG-minors are not suitable for our application, we relax the characterisation of [21] to make a larger finite obstruction set and a simpler-to-handle containment relation (essentially a “partially drawn topological minor”).

A *subdivision* of an edge in a partially drawn graph  $(G, \mathcal{H})$  is the same subdivision in the graph  $G$ , which is correspondingly applied to  $\mathcal{H}$  if the subdivided edge is from  $\mathcal{H}$ . A partially drawn graph  $(G_1, \mathcal{H}_1)$  is a (*partially drawn*) *subgraph* of  $(G, \mathcal{H})$  if  $G_1 \subseteq G$ ,  $\mathcal{H}_1 \subseteq \mathcal{H}$  and the drawing  $\mathcal{H}_1$  is equivalent to the restriction of  $\mathcal{H}$  to  $\mathcal{H}_1$ . Note that in general one may have an edge of  $G_1$  which is predrawn in  $\mathcal{H}$  but not in  $\mathcal{H}_1$ .

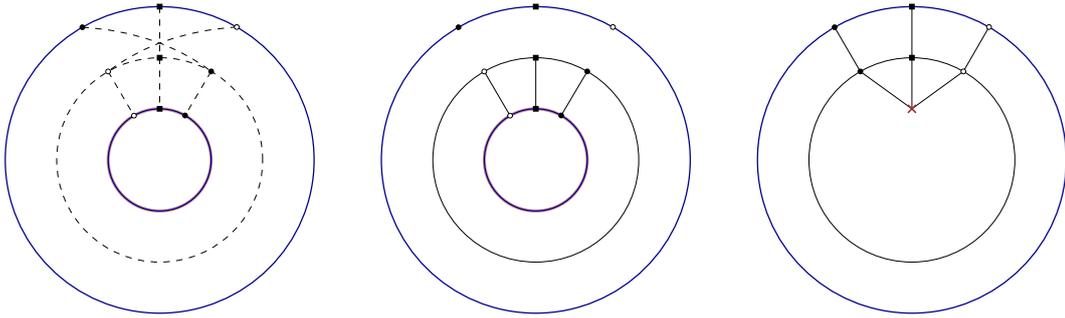


■ **Figure 3** (A picture copied from arXiv:1204.2915v1 with permission of the authors.) The list of 24 partially drawn graphs [21] that are the obstructions (as PEG-minors) for partially drawn graphs which can be extended to planar drawings. The solid black edges and vertices form the predrawn skeleton of the graphs, and dashed edges are the non-fixed ones.

► **Theorem 2.1** (adapted from [21]). *There is a finite family  $\mathcal{K}$  of partially drawn graphs such that the following is true. A partially drawn graph  $\mathcal{P} = (G, \mathcal{H})$  admits a planar drawing which extends  $\mathcal{H}$  if and only if  $\text{cr}(\mathcal{H}) = 0$  and the following hold:*

- i. *there is no alternating chain in  $\mathcal{P}$  (see the preprint version for the full definition), and*
- ii. *no subdivision of a partially drawn graph from  $\mathcal{K}$  is isomorphic to a partially drawn subgraph of  $\mathcal{P}$ .*

Briefly put, the family  $\mathcal{K}$  from Theorem 2.1 is composed of all graphs obtained from the obstructions  $(G, \mathcal{H})$  in Figure 3 [21] by possible iterative splittings (of vertices of degree  $> 3$  in  $G$ ) and possible releasing of certain edges from  $\mathcal{H}$ . The *splitting* of a vertex  $v$  is performed by partitioning the neighbourhood of  $v$  into two disjoint sets  $N_1$  and  $N_2$ , and replacing  $v$



■ **Figure 4** Instance (left) cannot be drawn with 0 crossings (same node styles indicate adjacencies to vertices on dashed cycle), but subinstance (middle) consisting only of  $H$  (blue), and induced graph on  $I$  (brown underlay) and  $C$ , as well as subinstance (right) in which  $I$  is contracted can.

with two new adjacent vertices  $v_1$  and  $v_2$  such that the neighbourhood of  $v_1$  is  $N_1 \cup \{v_2\}$  and the neighbourhood of  $v_2$  is  $N_2 \cup \{v_1\}$ . The *release* of an edge  $f \in E(H)$  from  $\mathcal{H}$  is allowed if  $f$  is a bridge, i.e.  $f$  is not contained in any cycle of  $H$ , and is performed as follows: If one end (resp., both ends) of  $f$  is of degree  $> 2$  in  $H$ , subdivide  $f$  once (twice), and denote by  $f'$  the edge resulting from  $f$  such that both ends of  $f'$  are of degree  $\leq 2$  in  $H$ . Then remove  $f'$  only from  $\mathcal{H}$  (but keep it in  $G$ ). We leave the details for the full preprint paper.

### 3 Algorithm for partially predrawn crossing number

Note that, regarding the input partially drawn graph  $(G, \mathcal{H})$ , we may as well assume that  $\mathcal{H}$  is a plane graph; otherwise, we replace  $\mathcal{H}$  with its planarisation  $\mathcal{H}^\times$  (and accordingly adjust  $G$ , which formally means to move to the partially drawn graph  $((G - E(H)) \cup \mathcal{H}^\times, \mathcal{H}^\times)$ ). This is sound since neither do we care about the number of crossings prescribed by  $\mathcal{H}$ , nor do we have any restrictions on single edges in  $H$ , and hence do not care to identify them. Thus, we will assume planar  $\mathcal{H}$  throughout the rest of the section, unless we explicitly say otherwise.

#### 3.1 Phase I – Treewidth

To show that we can arrive at an input graph with small treewidth, we prove a statement analogous to Grohe’s iterative contraction for the partially predrawn setting. Approaching this, however, it becomes quite clear that contracting a subgraph  $I$  must be treated much more delicately. The role of the cycle  $C$  in that case is that it could be treated as an interface to glue together two drawings – any planar drawing of the contracted part and any drawing of  $G$  after contraction with at most  $k$  crossings in which no “uncrossable” edge is crossed. For actually gluing the parts together, the drawing of  $C$  might need to be “flipped” in either of these two drawings. This can create a problem in terms of being equivalent to  $\mathcal{H}$  on  $H$ . Even if we ensure that each of the two drawings we would potentially like to glue together to a drawing of  $G$  are compatible with  $\mathcal{H}$  or the contraction of  $\mathcal{H}$ , this compatibility is not invariant under flipping  $C$  (see e.g. Figure 4).

For this purpose we consider the notion of  $(\mathcal{H}, \mathcal{I})$ -flippability for  $C$  and  $I$ . Essentially, we say that  $C$  is  $(\mathcal{H}, \mathcal{I})$ -flippable in a graph  $D$ , if the orientation of  $C$  with respect to  $I$  in a planar drawing of  $D$  that is equivalent to  $\mathcal{H}$  on  $H$  is not determined by  $\mathcal{H}$ . Otherwise  $C$  is  $(\mathcal{H}, \mathcal{I})$ -unflippable in  $D$ . A formal definition that makes use of the non-equivalence of drawing two disconnected triangles described in Figure 1 is given in the full preprint paper. Using this formal definition it can be decided in polynomial time whether a cycle is  $(\mathcal{H}, \mathcal{I})$ -flippable in a graph, or not.

To facilitate readability, we say that for a partially drawn graph  $(G, \mathcal{H})$  where some edges of  $G$  are marked as “uncrossable”, the drawings of  $G$  that we want to consider, are *k-crossing conforming*. More formally, a *k-crossing conforming* drawing is a drawing of  $G$  with at most  $k + \text{cr}(\mathcal{H})$  crossings that is equivalent to  $\mathcal{H}$  on the predrawn skeleton  $H$  and in which no “uncrossable” edge is crossed. The following key theorem is fully stated and proved in the preprint paper.

► **Theorem 3.1.** *For all  $k \in \mathbb{N}$  there exists  $w \in \mathbb{N}$ , such that given a partially drawn graph  $(G, \mathcal{H})$  in which some edges are marked “uncrossable”, in FPT-time parameterised by  $k$  we can*

1. *decide that there is no  $k$ -crossing conforming drawing of  $(G, \mathcal{H})$ ; or*
2. *find a tree decomposition of  $G$  of width at most  $w$ ; or*
3. *find an equivalent instance  $(G', \mathcal{H}')$  with the property that  $|V(G')| < |V(G)|$ .*

**Sketch of proof.** We start by applying the result by Grohe [17] for  $k$  with  $G$  as input. If the algorithm of [17] decides that the number of crossings in any drawing of  $G$  in which no “uncrossable” edge is crossed is more than  $k$  times, we can safely return that the same is true for any such drawing that is equivalent to  $\mathcal{H}$  on the predrawn skeleton. Similarly, if the algorithm returns a tree decomposition of width at most  $w$ , we can return that tree decomposition.

In the last case, the algorithm finds a subgraph  $I \subseteq G$  and a cycle  $C$  in  $G$  as described in Subsection 2.3 for bounding treewidth. We distinguish whether there is a 0-crossing conforming drawing of  $(G[V(I) \cup V(C)] \cup H, \mathcal{H})$ , or not. Recall that, as we assume  $\mathcal{H}$  to be planarised, edges marked as “uncrossable” are irrelevant in this context because no edge should be crossed. Hence deciding whether there is a 0-crossing conforming drawing of  $(G[V(I) \cup V(C)] \cup H, \mathcal{H})$  is equivalent to deciding whether  $\text{pd-cr}(G[V(I) \cup V(C)] \cup H, \mathcal{H}) = 0$ . This can be decided in linear time using the result by Angelini et al. [1].

▷ **Case 1.** There is no 0-crossing conforming drawing of  $(G[V(I) \cup V(C)] \cup H, \mathcal{H})$ .

In this case we claim that there is no  $k$ -crossing conforming drawing of  $(G, \mathcal{H})$ . Assume for a contradiction that there is such a drawing  $\mathcal{G}$ . In particular this drawing has at most  $k$  crossings and no “uncrossable” edge is crossed in it. Hence, because of the choice of  $I$  and  $C$ , no edge of  $G[V(I) \cup V(C)]$  is crossed in  $\mathcal{G}$ . But as there are exactly  $\text{cr}(\mathcal{H})$  crossings involving only edges of  $H$  in  $\mathcal{G}$ , this means that the restriction of  $\mathcal{G}$  to  $G[V(I) \cup V(C)]$  is a 0-crossing conforming drawing of  $(G[V(I) \cup V(C)] \cup H, \mathcal{H})$ ; a contradiction.

▷ **Case 2.** There is a 0-crossing conforming drawing of  $(G[V(I) \cup V(C)] \cup H, \mathcal{H})$ .

This is the case in which we attempt to construct an equivalent instance with fewer vertices. Informally speaking, if we find an  $(\mathcal{H}, \mathcal{I})$ -flippable cycle  $C$ , we will essentially be able to flip any planar drawing of the contracted subgraph to appropriately match the interface in a drawing of  $G$  after the contraction. Hence we can simply contract  $I$  in  $G$  and  $\mathcal{H}$ .

If we find a cycle that is  $(\mathcal{H}, \mathcal{I})$ -unflippable and the cycle remains unflippable after the contraction of the subgraph is performed, any planar drawing of the contracted subgraph automatically matches the interface in a drawing of  $G$  after contraction. Hence we can simply contract  $I$  in  $G$  and  $\mathcal{H}$ .

The last case is that the cycle we find is  $(\mathcal{H}, \mathcal{I})$ -unflippable but it seems to be flippable after the contraction of the subgraph is performed. In this case the orientation of the cycle is fixed in any planar drawing of the subgraph  $I$  for contraction, but both orientations of the cycle are possible after the contraction is performed. We must therefore appropriately force the orientation of  $C$  in the drawing after performing the contraction to match the one which is in fact forced before the contraction. We will do this by extending  $\mathcal{H}$  carefully. ◀

We can iteratively apply Theorem 3.1  $\mathcal{O}(|V(G)|)$  times to reduce our instance to a graph of small treewidth. Hence from now on we focus on the case that we are given a partially drawn graph  $(G, \mathcal{H})$  and a tree decomposition of  $G$  whose width  $w$  is bounded in the inquired crossing number.

This is already a crucial step towards the targeted application of Courcelle’s theorem. However we still need to incorporate the information on the partial drawing  $\mathcal{H}$  into a graph structure of small treewidth. For this we will define a *framing* of  $(G, \mathcal{H})$ . Note that even though we assume in this definition  $\mathcal{H}$  to be planar, the definition also applies to the general case in which we first planarise  $\mathcal{H}$  into  $\mathcal{H}^\times$  and correspondingly adjust  $G$ .

► **Definition 3.2.** A framing of a partially drawn graph  $(G, \mathcal{H})$ , where  $\mathcal{H}$  is a plane graph, is an ordinary (abstract) graph  $F$  constructed as follows. See Figure 5. We start with the initial drawing  $\mathcal{D} := \mathcal{H}$  and continue by the following steps in order:

1. While the graph of  $\mathcal{D}$  is not connected, we iteratively add edges from  $G$  to  $\mathcal{D}$  that can be inserted in a planar way and which connect two previously disconnected components. If this is no longer possible while the graph is still disconnected, let  $B$  be a face of  $\mathcal{D}$  incident to more than one connected component. We pick a vertex  $v$  on  $B$  and connect  $v$  to an arbitrary vertex from each component incident to  $B$  which does not contain  $v$ . We will call all edges added in this step the connector edges (of the resulting framing).
2. We replace each edge  $f = uw$  of the drawing  $\mathcal{D}$  from Step 1 (including the connector edges) by three internally disjoint paths of length 3 between  $u$  and  $w$ . We will call these three paths together the framing triplet of  $f$ , and denote by  $\mathcal{D}'$  the resulting drawing.
3. Around each vertex  $v \in V(\mathcal{H}^\times)$  in the drawing  $\mathcal{D}'$  from Step 2, we add a cycle on the neighbours of  $v$  in  $\mathcal{D}'$  in the cyclic order given by  $\mathcal{D}'$ . We will call these cycles the framing cycles, and all edges of the resulting planar drawing  $\mathcal{D}''$  the frame edges.
4. Finally, we set  $F := \mathcal{D}'' \cup G$  where  $\mathcal{D}''$  is the underlying graph of  $\mathcal{D}''$  from Step 3.

We remark that Step 1 of the construction of a framing  $F$  of  $(G, \mathcal{H})$  is not deterministic, and hence a partially drawn graph can admit multiple framings. Note also that possible connector edges introduced in Step 1 are no longer present in resulting  $F$  (only their vertices and derived frame triplets are present). Moreover, the most important aspect of Definition 3.2 is that the frame  $(\mathcal{D}'')$  defined after Step 3 is a 3-connected planar graph which hence combinatorially captures the drawing  $\mathcal{H}$  within the framing  $F$ .

As the last step in preparation for applying Courcelle’s theorem we need to show that the framing construction does not considerably increase the treewidth:

► **Lemma 3.3.\*** Let  $F$  be a framing of a partially drawn graph  $(G, \mathcal{H})$ , and  $G^\circ = (G - E(\mathcal{H})) \cup \mathcal{H}^\times$ . Then  $\text{tw}(F) \in \mathcal{O}(16^{k+1} \text{tw}(G^\circ) / \log(\text{tw}(G^\circ)))$ , where  $k = \text{pd-cr}(G, \mathcal{H})$ .

### 3.2 Phase II – MSO<sub>2</sub>-encoding

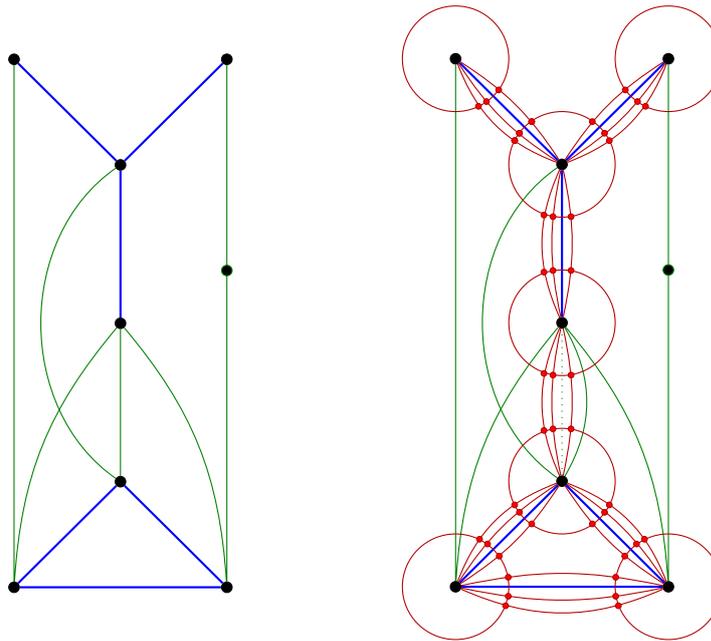
Our aim now is to prove key Lemma 1.2. In closer detail, we are first going to show:

► **Lemma 3.4.\*** Let  $\mathcal{P}_1 = (G_1, \mathcal{H}_1)$  be a partially drawn graph where  $\mathcal{H}_1$  is plane. There exists an MSO<sub>2</sub>-formula  $\sigma$ , depending on  $\mathcal{P}_1$ , such that the following is true:

- For any partially drawn graph  $\mathcal{P}_2 = (G_2, \mathcal{H}_2)$  with plane  $\mathcal{H}_2$  and any framing  $\bar{G}_2$  of  $\mathcal{P}_2$  we have that  $\bar{G}_2 \models \sigma$ , if and only if some subdivision of  $\mathcal{P}_1$  is a partially drawn subgraph of  $\mathcal{P}_2$ .

To combinatorially characterise the partially drawn subgraph containment, we use Definition 3.2 and the following concept of a “framing-aware” minor. Considering framings  $\bar{G}_1$  of  $(G_1, \mathcal{H}_1)$  and  $\bar{G}_2$  of  $(G_2, \mathcal{H}_2)$ , we say that  $\bar{G}_1$  is a *framing topological minor* of  $\bar{G}_2$  if there is a topological-minor embedding of  $\bar{G}_1$  into  $\bar{G}_2$  which additionally satisfies

- every edge of  $G_1$  (resp., of  $H_1$ ) is mapped into a path of  $G_2$  (resp., of  $H_2$ ),



■ **Figure 5** (Definition 3.2) A *framing* of a partially drawn graph  $(G, \mathcal{H})$ : the graph is on the left, such that the predrawn skeleton  $\mathcal{H}$  is drawn with thick blue edges and the remaining edges of  $E(G) \setminus E(H)$  are in green. The framing of  $(G, \mathcal{H})$  on the right has the frame edges drawn in red; for every edge of  $\mathcal{H}$  and for the chosen one connector edge between the two components of  $H$ , we get a framing triplet, and for every vertex of  $\mathcal{H}$  a framing cycle.

- every framing cycle in  $\bar{G}_1$  is mapped into a corresponding framing cycle in  $\bar{G}_2$ ,
- whenever an edge  $f \in E(H_1)$  is mapped into a path  $P_f \subseteq H_2$ , the framing triplet of  $f$  in  $\bar{G}_1$  is embedded (as three internally-disjoint paths) in the union of the framing cycles and triplets of the internal vertices and edges of  $P_f$  in  $\bar{G}_2$ , and
- the analogous condition (as the previous point) applies also to framing triplets of the connector edges of  $\bar{G}_1$ , which are embedded in  $\bar{G}_2$ .

See Figure 6 for a natural illustration of this concept.

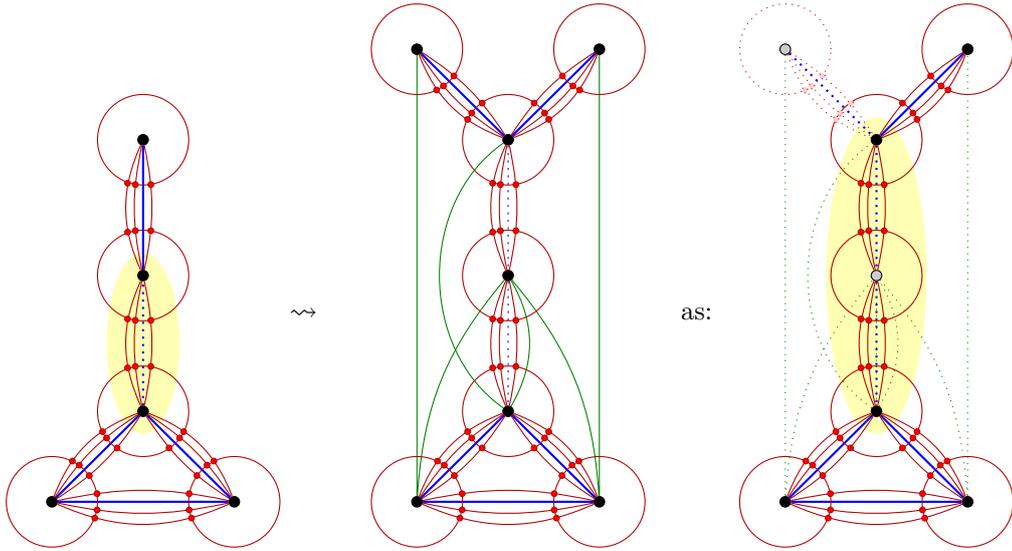
However, to state the desired characterisation we still need to technically generalise Definition 3.2 to an *extended framing* of a partially drawn graph  $(G, \mathcal{H})$  which, informally, allows us to use possible additional connector vertices and arbitrary connector edges between the components of  $\mathcal{H}$ . See the preprint paper for all details.

► **Lemma 3.5.\*** *Let  $\mathcal{P}_1 = (G_1, \mathcal{H}_1)$  and  $\mathcal{P}_2 = (G_2, \mathcal{H}_2)$  be partially drawn graphs where  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are plane. Let  $\bar{G}_2$  be a framing of  $\mathcal{P}_2$ . Then some subdivision of  $\mathcal{P}_1$  is a partially drawn subgraph of  $\mathcal{P}_2$ , if and only if there exists an extended framing  $\bar{G}_1$  of  $\mathcal{P}_1$  such that  $\bar{G}_1$  is a restricted topological minor of  $\bar{G}_2$ .*

We now finish a proof sketch of Lemma 3.4 easily. Let  $\mathcal{F}$  be the finite set of all distinct extended framings of  $\mathcal{P}_1$ . Using Lemma 3.5, we may write the formula  $\sigma \equiv \bigvee_{\bar{G}_1 \in \mathcal{F}} \sigma[\bar{G}_1]$  where  $\bar{G}_2 \models \sigma[\bar{G}_1]$  routinely expresses that  $\bar{G}_1$  is a framing topological minor of  $\bar{G}_2$  (this description uses auxiliary precomputed labels distinguishing the types of edges in  $\bar{G}_2$ ).

We also need to address the other kind of obstruction in Theorem 2.1 with the following:

- **Lemma 3.6.\*** *There exists an MSO<sub>2</sub>-formula  $\tau$  such that the following is true:*
- *For any partially drawn graph  $\mathcal{P}_2 = (G_2, \mathcal{H}_2)$  and any framing  $\bar{G}_2$  of  $\mathcal{P}_2$  we have that  $\bar{G}_2 \models \tau$ , if and only if there exists an alternating chain in  $\mathcal{P}_2$ .*



■ **Figure 6** An illustration of the framing topological minor relation; the framing  $\bar{G}_1$  (of the 5-vertex partially drawn graph  $(G_1, \mathcal{H}_1)$ ) on the left is embedded in the framing  $\bar{G}_2$  (of the 7-vertex graph  $(G_2, \mathcal{H}_2)$ ) in the middle, and this embedding is emphasised as a topological minor in the picture on the right. Notice that the framing triplet in  $\bar{G}_1$  highlighted in the left picture with yellow background is mapped (as three internally disjoint red paths) into a union of two framing triplets plus the intermediate framing cycle in  $\bar{G}_2$ , as highlighted with yellow background in the picture on the right.

Now we can sketch a proof of the key Lemma 1.2 which we reformulate slightly for clarity:

► **Lemma 3.7** (Lemma 1.2). *For every  $k \geq 0$  there is an  $MSO_2$ -formula  $\psi_k$  such that the following holds. Given a partially drawn graph  $\mathcal{P}$ , with some edges of  $\mathcal{P}$  marked as “uncrossable”, one can in polynomial time construct a graph  $G'$  such that  $G' \models \psi_k$  if and only if there exists a  $k$ -crossing conforming drawing of  $\mathcal{P}$ .*

**Sketch of proof.** Recall that we may assume  $\mathcal{H}$  to be a plane graph. We first give a rough outline of what we want to achieve and then sketch the core steps of the proof.

The graph  $G'$  will be based on a framing (as used above). Imagine a conforming drawing  $\mathcal{G}$  of  $G$  (extending  $\mathcal{H}$ ) with  $\text{cr}(\mathcal{G}) = k$  and its planarisation  $\mathcal{G}^\times$ . If we were able to “guess”, within the formula  $\psi_k$ , the additional  $k$  vertices (those of  $\mathcal{G}^\times$ ) making the crossings, then we would finish by checking partially predrawn planarity of the result (i.e., of the guessed  $\mathcal{G}^\times$ ). Using Theorem 2.1, the latter would follow by an application of Lemmas 3.4 and 3.6.

Specifically, for the task of “guessing the crossings”, we subdivide each edge of  $\mathcal{P}$  which is not marked as “uncrossable” by  $k$  new vertices, called *auxiliary vertices* of this partially drawn subdivision  $\mathcal{P}_0 = (G_0, \mathcal{H}_0)$  of  $\mathcal{P}$ . A subdivision clearly does not change the crossing number;  $\text{cr}(\mathcal{P}) = \text{cr}(\mathcal{P}_0)$ . Then we interpret “guessing a crossing” in  $\mathcal{P}_0$  as picking (with existential quantifiers in  $\psi_k$ ) a pair  $r'_1, r''_1 \in V(G_0) \setminus V(G)$  of auxiliary vertices such that not both  $r'_1$  and  $r''_1$  are from edges of  $H$ , and identifying  $r'_1 = r''_1$ . Let  $\mathcal{P}_0[r'_1 = r''_1]$  denote the graph after such an identification. Note that since we do not identify auxiliary pairs from two edges of  $H$ , the following holds – if  $\bar{G}_0$  is a framing of  $\mathcal{P}_0$ , then  $\bar{G}_0[r'_1 = r''_1]$  is a graph isomorphic to the corresponding framing of  $\mathcal{P}_0[r'_1 = r''_1]$ .

We let  $G' = \bar{G}_0$  be a framing of  $\mathcal{P}_0 = (G_0, \mathcal{H}_0)$ . Let  $\mathbf{r}' = (r'_i : i \in [k])$  and  $\mathbf{r}'' = (r''_i : i \in [k])$  be two  $k$ -tuples of vertex variables (which are used to specify the  $k$  identifications of vertex pairs in  $\mathcal{P}_0[\mathbf{r}' = \mathbf{r}'']$ ). We write the desired formula as

$$\psi_k \equiv \exists \mathbf{r}', \mathbf{r}'' \left( \bigwedge_{r, s \in \mathbf{r}' \cup \mathbf{r}''} r \neq s \wedge \bigwedge_{i \in [k]} \chi(r'_i, r''_i) \wedge \psi'_k[\mathbf{r}', \mathbf{r}''] \right),$$

where  $\chi(r'_i, r''_i)$  checks that  $r'_i, r''_i$  are auxiliary vertices and not both coming from edges of  $H$  (using precomputed labels of the auxiliary vertices). The formula  $\psi'_k[r', r'']$  then tests whether the partially drawn graph  $\mathcal{P}_0[r' = r'']$  admits a planar drawing extending  $\mathcal{H}_0$ . This is a technical task based on Lemmas 3.4 and 3.6, and we leave full details for the preprint paper. ◀

Finally, we summarise how **Theorem 1.1** follows from the previous claims. Given a partially drawn graph  $(G, \mathcal{H})$  and an integer  $k > 0$ , we first make  $\mathcal{H}$  planarised. Then, using Theorem 3.1, we either conclude that  $\text{pd-cr}(G, \mathcal{H}) > k$ , or we iteratively reduce the input to an equivalent instance  $(G', \mathcal{H}')$  with the same solution value  $k$ . Moreover, using also Lemma 3.3, we have that the tree-width of any framing  $\tilde{G}'$  of  $(G', \mathcal{H}')$  is bounded in terms of  $k$ . We can hence efficiently decide whether  $\text{pd-cr}(G', \mathcal{H}') \leq k$  using Courcelle's theorem applied with the formula  $\psi_k$  from Lemma 3.7 to a framing  $\tilde{G}'$  of  $(G', \mathcal{H}')$ .

(\*) We can also observe that the FPTruntime of this procedure is  $\mathcal{O}(f(k) \cdot |V(G)|^3)$ .

#### 4 Restricting crossings per edge

Next we outline some nice consequences of our techniques for previously considered drawing extension settings. Firstly, we are able to trivially modify our FPT-algorithm for PARTIALLY PREDRAWN CROSSING NUMBER by additionally encoding the fact that in a solution every edge in  $E(G) \setminus E(H)$  has at most  $c$  crossings by introducing  $k$  auxiliary vertices for each edge in  $E(H)$ , but only  $\min\{c, k\}$  auxiliary vertices for each edge in  $E(G) \setminus E(H)$  in the proof of Lemma 3.7. This immediately gives us Theorem 1.3 restated from above.

► **Theorem 1.3.** *PARTIALLY PREDRAWN  $c$ -PLANAR CROSSING NUMBER is in FPT when parameterised by the solution value (i.e., by the number of crossings which are not predrawn).*

Another closely related problem that has been considered in literature asks for the smallest number of non-predrawn crossings in a *simple* drawing that coincides with the given partially drawn graph, in which each edge in  $E(G) \setminus E(H)$  has at most  $c$  crossings. I.e., compared to PARTIALLY PREDRAWN  $c$ -PLANAR CROSSING NUMBER we only allow drawings in which no pair of edges crosses more than once (crossings between adjacent edges can always be avoided). The difficulty for our approach here is that we need to record the information of which edges in  $\mathcal{H}^\times$  correspond to the same edge in the non-planarised predrawn skeleton  $H$  (this part can be handled by an MSO<sub>2</sub>-formula with help of special edge labels, cf. [15]), and more importantly to keep this information, even during our iterative reduction of  $G$  and  $\mathcal{H}^\times$  described in Section 3.1. The latter seems to be a deep problem, not easy to overcome and a good direction for continuing research.

Nevertheless, using the more restrictive parameterisation by  $|E(G) \setminus E(H)| + c$  (which also naturally bounds the crossing number), we are able to give an improvement on the best known result in [15]: finding the least number of crossings in a simple drawing which coincides with the given partial drawing and in which each edge outside of the predrawn skeleton has at most  $c$  crossings in FPT-time. The known result assumes that the planarised predrawn skeleton is connected, an assumption that we can easily drop using our MSO<sub>2</sub>-encoding in combination with a crucial structural lemma which we adapt from [15] to “stitch” together relevant edges in  $\mathcal{H}^\times$  that correspond to the same edge in  $H$ . This improvement over [15] results in **Theorem 1.4** stated in the Introduction.

## 5 Conclusion

To summarise, we have shown that some algorithmic results for the classical crossing-number can be extended to the partially pre drawn setting, similarly to the respective planarity question [1]. However, what can we say about structural properties of the partially pre drawn crossing number?

For instance, what can we say about the minimal graphs of a certain crossing-number value? We call a partially drawn graph  $\mathcal{P} = (G, \mathcal{H})$  *k-crossing-critical* if the partially pre drawn crossing number of  $\mathcal{P}$  is at least  $k$ , but this crossing number drops down below  $k$  after deleting any edge, pre drawn or not, from  $\mathcal{P}$  (alternatively, one may also include removing any edge from  $H$  while keeping it in  $G$  to the definition). We have recently gotten a complete rough asymptotical characterisation of classical  $k$ -crossing-critical graphs [11], but here we see an important difference in behaviour. For classical  $k$ -crossing-critical graphs, optimal drawings (i.e. those achieving the minimum number of crossings) can never contain a collection of edge-disjoint cycles drawn nested in each other and of size arbitrarily large compared to  $k$  (this is implicit in [18] or [11]). In contrast to that, we provide:

► **Proposition 5.1.\*** *For each  $k \geq 8$  and  $m > 0$ , there exists a partially drawn graph  $\mathcal{P} = (G, \mathcal{H})$  such that  $\mathcal{P}$  is  $k$ -crossing-critical and that an optimal (with minimum crossings) drawing of  $\mathcal{P}$  extending  $\mathcal{H}$  contains at least  $m$  vertex-disjoint nested cycles from  $G - E(H)$ .*

Consequently, even a rough characterisation of partially drawn  $k$ -crossing-critical graphs is a widely open question worth further investigation. Unfortunately, already at the starting point of this track we lack a good analogue of the result [26], saying that a  $k$ -crossing-critical graph has its crossing number bounded in terms of  $k$ , whose proof simply breaks down in the partially pre drawn setting. Having a result like [26] in the pre drawn setting we could, as a first step, adapt the arguments from Section 3 to prove that partially drawn  $k$ -crossing-critical graphs have treewidth bounded in terms of  $k$ .

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