A Methodology for Designing Proof Search Calculi for Non-Classical Logics

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Abstract

In this talk I present a methodology for designing proof search calculi for a wide range of non-classical logics, such as modal and tense logics, bi-intuitionistic (linear) logics and grammar logics. Most of these logics cannot be easily formalised in the traditional Gentzen-style sequent calculus; various structural extensions to sequent calculus seem to be required. One of the more expressive extensions of sequent calculus is Belnap's display calculus, which allows one to formalise a very wide range of logics and which provides a generic cut-elimination method for logics formalised in the calculus. The generality of display calculus derives partly from the pervasive use of structural rules to capture properties of the underlying semantics of the logic of interest, such as various frame conditions in normal modal logics, that are not easily captured by introduction rules alone. Unlike traditional sequent calculi, the subformula property in display calculi does not typically give an immediate bound on the search space (assuming contraction is absent) in proof search, as new structures may be created and their creation may not be driven by any introduction rules for logical connectives. This line of work started out as an attempt to "tame" display calculus, to make it more proof search friendly, by eliminating or restricting the use of structural rules. Two key ideas that make this possible are the adoption of *deep inference*, allowing inference rules to be applied inside a nested structure, and the use of *propagation rules* in place of structural rules. A brief survey of the applications of this methodology to a wide range of logics is presented, along with some directions for future work.

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1 Summary

Non-classical logics, such as modal logics and intermediate logics, have generally been challenging to formalise in the traditional Gentzen's sequent calculi. This has motivated the development of a variety of structural extensions of sequent calculi as alternative proof-theoretic formalisms for these logics. Notable formalisms include display calculi [14], hypersequent calculi [1], tree-hypersequent calculi [19], nested sequent calculi [15, 2], the calculus of structures [12] and labelled sequent calculi [6, 17]. Among these formalisms, display calculi and labelled sequent calculi are perhaps the more general ones, allowing one to design proof systems for a wide range of non-classical logics that satisfy cut admissibility. In display calculus, this is achieved by essentially defining a *structural connective* for each logical connective, and internalising the underlying semantic conditions (e.g., frame conditions in modal logics) into structural rules manipulating the relevant structural connectives. Similarly, in labelled sequent calculi, the labels in a sequent and their relations can be seen as a representation of Kripke frames in the underlying semantics of the logics, and the "structural rules" manipulating these labels and relations are derived directly from the frame conditions characterising the logics.

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However, the generality and the ease in which one represents logics in these calculi come with a price of the loss of some of the more appealing features of Gentzen sequent calculi from the perspective of proof search. The *subformula property* in the traditional sequent calculi provides an immediate way to bound the search space in proof search (assuming contraction is absent), but this is not the case for display calculi or labelled calculi in general. This is a consequence of the use of extended structural rules, which can potentially create structures of an arbitrary size (reading the inference rules bottom up), independently of the (sub)formulas in the end sequent. Another property that arises naturally from a formulation of a logic in the traditional sequent calculus is what I call the *separation property* – given a sequent calculus for a logic, one can extract a sound and complete proof system for any of its sublogics (defined by a selection of connectives) by simply selecting the introduction rules for the connectives defining the sublogic. This property is generally difficult to prove directly in display calculi, as proofs of a formula in a sublogic may require the use of structural connectives that sit outside the sublogic.

In this talk, I present a methodology for designing proof calculi for non-classical logics, for which both the subformula property and the separation property hold. This methodology is based on a refinement process, starting with a "display-like" calculus for a logic, and ending with a nested sequent calculus for the same logic. The syntactic framework for the refinement is that of nested sequent calculus. We generalise the notion of a traditional (one-sided or two-sided) sequent to a tree of sequents, and adopt display-like structural rules that act on the tree of sequents. For example, the familiar *display rules* [14] in our setting becomes essentially a rule that rotates the tree structure of a nested sequent [10].

Our methodology proceeds in three phases. In the first phase, given a logic of interest, we first extend the logic by adding the adjoints of its connectives (if needed). For example, if the logic of interest is an intuitionistic logic, then we will extend it by an exclusion (or subtraction) connective; if it is a modal logic, we extend it to tense logic. We then design a display calculus for the extended logic and produce a *shallow* nested sequent calculus (where introduction rules can be applied only to the root sequent in a nested sequent). A shallow nested calculus is for most part a notational variant of the display calculus. Provided that the shallow nested calculus satisfies Belnap's eight conditions [14], we get cut-elimination for free. The structural rules in the shallow calculus consist of the *internal* structural rules (that change the structures within a sequent, e.g., contraction/weakening) and the *external* structural rules (that change the shape of the tree of a nested sequent, e.g., display postulates and various rules that correspond to frame conditions in modal logic).

In the second phase, we transform the shallow nested sequent calculus into a *deep* nested sequent calculus, where inference rules (including introduction rules) can be applied to any sequent in the nested sequent. A key technical requirement for the deep calculus is that the *only* rules that are allowed change the structure of a nested sequent (i.e., the tree-shape of the nested sequent) are introduction rules. This means in particular that *all* external structural rules are absent in the deep calculus. In place of external structural rules, we introduce *propagation rules* [8, 10] into the deep calculus. These propagation rules determine how formulas in a sequent in the tree of sequents can be propagated to other sequents in the same tree. An important consequence of the absence of external structural rules is that in the proof of a formula, every (logical or structural) connective occurring in the proof also occurs in the formula. This gives us immediately the separation property.

In the last phase, we obtain the deep nested sequent calculus for the logic we started with by simply omitting the introduction rules for the connectives that are not in the logic; by the separation property, this gives us a sound and complete proof system for our logic.

Over the past decade or so, my collaborators and I have applied this methodology to design proof calculi that are amenable to proof search for various non-classical logics. Our early work focused on classical propositional tense logics [8, 10], showing that we can obtain sound and complete cut-free proof systems for all modal logics that can be characterised using *path axioms* [10], which subsume all modal logics in the modal logic cube. This result naturally extends to multi-modal logics, which we have demonstrated by giving cut-free proof systems for a family of grammar logics [21] and their proof search procedures. We applied the same methodology to solve the problem of finding a cut-free proof system for bi-intuitionistic logic [7, 20], which had evaded previous attempts [18]. This was later extended to a version of bi-intuitionistic tense logic [9], which contains an intuitionistic modal logic as its subsystem. Lastly, we have also applied this methodology to design a proof system [4] for full intuitionistic linear logic (FILL) [13] and prove its NP-completeness.

Although our methodology has been successfully applied to design proof calculi for a wide range of logics, it is currently not clear what the limit of its applicability is. We know, for example, modal logics admitting pseudo-transitive axioms of the form $\Box^m p \to \Box^n p$, for $m, n > 1$, do not seem to be expressible in the deep nested sequent calculus without any external structural rules.

There are indications that our deep nested calculi may allow for a more syntax directed proof of the interpolation theorem for a wide range of modal/tense logics and bi-intuitionistic logic [16]. However, proving interpolation for FILL in the deep nested sequent calculus remains a challenge and is a subject of an on-going research.

Our methodology is mainly aimed at bridging display calculi and (deep) nested sequent calculi. It will be interesting to see how this methodology can be generalised to design proof search calculi in a different syntactic framework. For this, we can leverage on existing work on relating different formalisms [5, 11, 3].

References

1. Arnon Avron. Hypersequents, logical consequence and intermediate logics for concurrency. *Ann. Math. Artif. Intell.*, 4:225–248, 1991. doi:10.1007/BF01531058.
2. Kai Brünnler. Deep sequent systems for modal logic. *Arch. Math. Log.*, 48(6):551–577, 2009.
3. Agata Ciabattoni, Tim S. Lyon, Revantha Ramanayake, and Alwen Tiu. Display to labeled proofs and back again for tense logics. *ACM Trans. Comput. Log.*, 22(3):20:1–20:31, 2021. doi:10.1145/3460492.
4. Ranald Clouston, Jeremy E. Dawson, Rajeev Goré, and Alwen Tiu. Annotation-free sequent calculi for full intuitionistic linear logic. In *Computer Science Logic 2013 (CSL 2013), CSL 2013, September 2-5, 2013, Torino, Italy*, volume 23 of *LIPIcs*, pages 197–214. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2013. doi:10.4230/LIPIcs.CSL.2013.197.
5. Melvin Fitting. Nested sequents and prefixed tableaus. In Martin Giese and Roman Kuznets, editors, *TABLEAUX 2011 - Workshops, Tutorials, and Short Papers, Bern, Switzerland, July 4-8, 2011*, volume IAM-11-002 of *Technical Report*, page 69, 2011.
6. Dov M Gabbay. *Labelled deductive systems*, volume 33 of *Oxford Logic guides*. Clarendon Press/Oxford Science Publications, 1996.
7. Rajeev Goré, Linda Postniece, and Alwen Tiu. Cut-elimination and proof-search for bi-intuitionistic logic using nested sequents. In Carlos Areces and Robert Goldblatt, editors, *Advances in Modal Logic 7, papers from the seventh conference on "Advances in Modal Logic," held in Nancy, France, 9-12 September 2008*, pages 43–66. College Publications, 2008. URL: http://www.aiml.net/volumes/volume7/Gore-Postniece-Tiu.pdf.
8. Rajeev Goré, Linda Postniece, and Alwen Tiu. Taming displayed tense logics using nested sequents with deep inference. In *TABLEAUX*, volume 5607 of *Lecture Notes in Computer Science*, pages 189–204. Springer, 2009.

9 Rajeev Goré, Linda Postniece, and Alwen Tiu. Cut-elimination and proof search for bi-intuitionistic tense logic. In Lev D. Beklemishev, Valentin Goranko, and Valentin B. Shehtman, editors, *Advances in Modal Logic 8, papers from the eighth conference on "Advances in Modal Logic," held in Moscow, Russia, 24-27 August 2010*, pages 156–177. College Publications, 2010. URL: `http://www.aiml.net/volumes/volume8/Gore-Postniece-Tiu.pdf`.

10 Rajeev Goré, Linda Postniece, and Alwen Tiu. On the correspondence between display postulates and deep inference in nested sequent calculi for tense logics. *Log. Methods Comput. Sci.*, 7(2), 2011. `doi:10.2168/LMCS-7(2:8)2011`.

11 Rajeev Goré and Revantha Ramanayake. Labelled tree sequents, tree hypersequents and nested (deep) sequents. In Thomas Bolander, Torben Braüner, Silvio Ghilardi, and Lawrence S. Moss, editors, *Advances in Modal Logic 9, papers from the ninth conference on "Advances in Modal Logic," held in Copenhagen, Denmark, 22-25 August 2012*, pages 279–299. College Publications, 2012. URL: `http://www.aiml.net/volumes/volume9/Gore-Ramanayake.pdf`.

12 Alessio Guglielmi. A system of interaction and structure. *ACM Trans. Comput. Log.*, 8(1):1, 2007. `doi:10.1145/1182613.1182614`.

13 Martin Hyland and Valeria de Paiva. Full intuitionistic linear logic (extended abstract). *Ann. Pure Appl. Log.*, 64(3):273–291, 1993. `doi:10.1016/0168-0072(93)90146-5`.

14 Nuel D. Belnap Jr. Display logic. *J. Philos. Log.*, 11(4):375–417, 1982. `doi:10.1007/BF00284976`.

15 Ryo Kashima. Cut-free sequent calculi for some tense logics. *Studia Logica*, 53(1):119–136, 1994. `doi:10.1007/BF01053026`.

16 Tim Lyon, Alwen Tiu, Rajeev Goré, and Ranald Clouston. Syntactic interpolation for tense logics and bi-intuitionistic logic via nested sequents. In Maribel Fernández and Anca Muscholl, editors, *28th EACSL Annual Conference on Computer Science Logic, CSL 2020, January 13-16, 2020, Barcelona, Spain*, volume 152 of *LIPIcs*, pages 28:1–28:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. `doi:10.4230/LIPIcs.CSL.2020.28`.

17 Sara Negri. Proof analysis in modal logic. *J. Philos. Log.*, 34(5-6):507–544, 2005. `doi:10.1007/s10992-005-2267-3`.

18 Luís Pinto and Tarmo Uustalu. Relating sequent calculi for bi-intuitionistic propositional logic. In Steffen van Bakel, Stefano Berardi, and Ulrich Berger, editors, *Proceedings Third International Workshop on Classical Logic and Computation, CL&C 2010, Brno, Czech Republic, 21-22 August 2010*, volume 47 of *EPTCS*, pages 57–72, 2010. `doi:10.4204/EPTCS.47.7`.

19 Francesca Poggiolesi. The method of tree-hypersequents for modal propositional logic. In David Makinson, Jacek Malinowski, and Heinrich Wansing, editors, *Towards Mathematical Philosophy*, volume 28 of *Trends in logic*, pages 31–51. Springer, 2009. `doi:10.1007/978-1-4020-9084-4_3`.

20 Linda Postniece. Deep inference in bi-intuitionistic logic. In Hiroakira Ono, Makoto Kanazawa, and Ruy J. G. B. de Queiroz, editors, *Logic, Language, Information and Computation, 16th International Workshop, WoLLIC 2009, Tokyo, Japan, June 21-24, 2009. Proceedings*, volume 5514 of *Lecture Notes in Computer Science*, pages 320–334. Springer, 2009. `doi:10.1007/978-3-642-02261-6_26`.

21 Alwen Tiu, Egor Ianovski, and Rajeev Goré. Grammar logics in nested sequent calculus: Proof theory and decision procedures. In Thomas Bolander, Torben Braüner, Silvio Ghilardi, and Lawrence S. Moss, editors, *Advances in Modal Logic 9, papers from the ninth conference on "Advances in Modal Logic," held in Copenhagen, Denmark, 22-25 August 2012*, pages 516–537. College Publications, 2012. URL: `http://www.aiml.net/volumes/volume9/Tiu-Ianovski-Gore.pdf`.