

# On Quantitative Algebraic Higher-Order Theories

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## Abstract

We explore the possibility of extending Mardare *et al.*'s quantitative algebras to the structures which naturally emerge from Combinatory Logic and the  $\lambda$ -calculus. First of all, we show that the framework is indeed applicable to those structures, and give soundness and completeness results. Then, we prove some negative results clearly delineating to which extent categories of metric spaces can be models of such theories. We conclude by giving several examples of non-trivial higher-order quantitative algebras.

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## 1 Introduction

One way of seeing program semantics is as the science of program equivalence. Each way of giving semantics to programs implicitly identifies which programs are equivalent. Similarly, a notion of program equivalence can be seen as a way of attributing meaning to programs (namely, the equivalence class to which the program belongs). This point of view makes semantics a powerful source of ideas and techniques for program transformation and program verification, with the remarkable advantage that such techniques can be defined in a compositional and modular way.

However, there are circumstances in which equivalences between programs, being purely dychotomous, are just not informative enough: two programs are either equivalent or not, period. No further quantitative or causal information can be extracted from two programs which are *slightly* different, although not equivalent. Furthermore, as program equivalences are usually congruences, and therefore preserved by any context, programs that only differ in *peculiar* circumstances are also just non-equivalent. For these reasons, methods alternative to program equivalence have to be looked for in all (very common) situations involving transformations that replace a program by one which is only *approximately* equivalent [31], or when the specifications are either not precise or not to be met precisely (e.g. in modern cryptography [27], in which most security properties hold in an approximate sense, namely modulo a negligible probability).



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The considerations above led the scientific community to question the possibility of broadening the scope of program semantics from a science of equivalences to a science of *distances* between programs. By the way, the possibility of interpreting programs in domains having a metric structure has been known since the 1990s [19, 18]. Recently, Mardare, Panangaden, and Plotkin have introduced a notion of quantitative algebra [29] that generalizes usual equational reasoning to a setting in which the compared entities can be at a certain distance. In this way, various notions of quantitative algebra have been shown to be captured through a formal system, *à la* Birkhoff [8].

Still, when the programs at hand are higher-order functional programs, the construction of a metric semantics faces several obstacles. First, it is well-known that the category  $\mathbf{Met}$  of metric spaces and non-expansive maps, providing the standard setting of the approaches just recalled, is *not* a model of the simply typed  $\lambda$ -calculus (more precisely, it is not cartesian closed). Furthermore, finding relevant *sub*-categories of  $\mathbf{Met}$  enjoying enough structure to model higher-order programs can lead to trivial (i.e. discrete) models, and several (mostly negative) results have remained so far in the *folklore* (with a few notable exceptions, e.g. [21]).

In this paper we bite the tail of the dragon: we apply the framework of quantitative equational theories and algebras from Mardare *et al.* to the cases of combinatory logic and the  $\lambda$ -calculus, and we try to highlight features and obstacles in the construction of higher-order quantitative algebras, at the same time showing the existence of several interesting models.

There are various reasons for exploring combinatory algebras, *i.e.* applicative structures where the  $\xi$ -rule fails. The first is that these structures naturally arise in various contexts, most notably in Game Semantics and in particular in the Geometry of Interaction [23], as axiomatized by Abramsky *et al.* [1]. The  $\xi$ -rule can then be enforced only by introducing a rather complex notion of equivalence relation, whose fine structure is usually rather awkward to grasp. The second reason is that combinatory algebras, being indeed algebras, might appear at first sight to be amenable straightforwardly in the first order framework of quantitative algebras of Mardare *et al.* We show that this is illusory, because the impact of the basic assumption that constructors are non-expansive, *i.e.* the Axiom NExp (see Section 4) is very strong, even in a context which could appear to be algebraically well-behaved. Finally, even if it is convenient to assume the  $\xi$ -rule, in reasoning about higher-order programming languages, showing that it holds in implementations is not at all immediate and, when side-effects are present, it needs to be carefully phrased.

The contributions of this paper are threefold:

- Following the framework defined by Mardare *et al.*, we introduce quantitative generalizations of the standard notions of *weak  $\lambda$ -theories* and  *$\lambda$ -theories* [6], and of their algebras. This is in Section 3, Section 4, and Section 5, respectively.
- We study properties and examples of algebras for such theories, as suitable sub-categories of  $\mathbf{Met}$ . Notably, we highlight the relevance of *ultra-metric* and *injective* metric spaces in the construction of non-trivial (i.e. non discrete) algebras. Some examples are discussed through Section 2 and Section 5, further properties and examples are in Section 6.
- Finally, we discuss algebras obtained by relaxing the conditions from Mardare *et al.*: either by replacing metrics by *partial* metrics [9, 34], *i.e.* generalized metrics in which self-distances  $d(x, x)$  need not be zero, or by relaxing the non-expansiveness condition and introducing a class of *approximate* quantitative algebras. This is in Section 7 and Section 8.

## 2 Preliminaries on Metric Spaces

In this section we discuss a few properties of metric spaces and their associated categories, which provide the general setting for quantitative algebras in the sense of Mardare *et al.* In particular, we recall the definition of ultra-metric spaces, as well as *partial ultra-metric spaces* [9, 34]. The latter is a class of generalized metric spaces in which self-distances  $a(x, x)$  are not required to be 0 but only smaller than any distance of the form  $a(x, y)$ .

► **Definition 1.** A pair  $(X, a)$  formed by a set  $X$  and a function  $a : X \times X \rightarrow \mathbb{R}_{\geq 0}^{\infty}$  is called: (i) a pre-metric space if it satisfies, for all  $x, y \in X$ ,  $a(x, x) = 0$  (refl) and  $a(x, y) = a(y, x)$  (symm); (ii) a (pseudo-)metric space if it satisfies (refl), (symm), and, for all  $x, y, z \in X$ ,  $a(x, y) \leq a(x, z) + a(z, y)$  (trans); (iii) an ultra-metric space if it satisfies (refl), (symm) and, for all  $x, y, z \in X$ ,  $a(x, y) \leq \max\{a(x, z), a(z, y)\}$  (trans\*); (iv) a partial ultra-metric space if it satisfies (symm), (trans\*) and, for all  $x, y \in X$ ,  $a(x, y) \geq a(x, x), a(y, y)$  (refl\*).

Since all metrics we consider are “pseudo”, from now on we will omit this prefix. Observe that an ultra-metric space is also a metric space. Moreover, a partial ultra-metric space  $(X, a)$  also yields an ultra-metric space  $(X, a^*)$ , with  $a^*(x, y) = 0$  if  $x = y$  and  $a^*(x, y) = a(x, y)$  otherwise. Usually, partial metric spaces are defined using a stronger version of the triangular law, given by  $a(x, y) \leq a(x, z) + a(z, y) - a(z, z)$ . However, for partial *ultra*-metrics this condition is equivalent to (trans\*) (see e.g. [34]).

The natural morphisms to consider between metric (ultra-metric, partial ultra-metric) spaces  $(X, a)$  and  $(Y, b)$ , hoping to get a continuous currification, are the *non-expansive* functions, i.e. those functions  $f : X \rightarrow Y$  such that for all  $x, y \in X$ ,  $b(f(x), f(y)) \leq a(x, y)$ . We let  $\text{Met}$  (resp.  $\text{UMet}$ ,  $\text{PUMet}$ ) indicate the category of metric spaces (resp. ultra-metric spaces, partial ultra-metric spaces) and non-expansive maps. All categories  $\text{Met}$ ,  $\text{UMet}$  and  $\text{PUMet}$  are cartesian, the product of  $(X, a)$  and  $(Y, b)$  being given by  $(X \times Y, \max\{a, b\})$ . In  $\text{UMet}$  and  $\text{PUMet}$  the cartesian functors  $\{-\} \times X$  have right-adjoints given, respectively, by  $(\text{UMet}(X, \{-\}), \Phi_{a, \{-\}})$  and  $(\text{PUMet}(X, \{-\}), \Phi_{a, \{-\}})$ , where for all metric space  $(Y, b)$ ,  $\Phi_{a, b}(f, g) = \sup\{b(f(x), g(x)) \mid x \in X\}$ . For this reason, both categories are cartesian closed.

By contrast,  $\text{Met}$  is *not* cartesian closed. Indeed, the functor  $(\text{Met}(X, \{-\}), \Phi_{a, \{-\}})$  is right-adjoint in  $\text{Met}$  (and thus also in  $\text{UMet}$ ) to the functor  $(X \times \{-\}, a + \{-\})$ , but for all metric spaces  $(Y, b)$ ,  $(X \times Y, a + b)$  is isomorphic to the cartesian product  $(X \times Y, \max\{a, b\})$  only when  $X$  and  $Y$  are ultra-metrics. Instead, the exponential of  $(X, a)$  and  $(Y, b)$  in  $\text{Met}$ , if it exists, is necessarily of the form  $(\text{Met}(X, Y), \Xi_{a, b})$  (as shown in the long version), where

$$\Xi_{a, b}(f, g) = \inf\{\delta \mid \forall x, y \in X \max\{\delta, a(x, y)\} \geq b(f(x), g(y))\}$$

We use the Greek letter  $\Xi$ , since, as we’ll see, this metric is tightly related to the interpretation of the  $\xi$ -rule of the  $\lambda$ -calculus. Notice that in general  $\Xi_{a, b}$  is only a pre-metric. Indeed, the category of pre-metric spaces and non-expansive functions is cartesian closed, while the exponential of  $(X, a)$  and  $(Y, b)$  exists in  $\text{Met}$  precisely when  $\Xi_{a, b}$  further satisfies (trans).

We will exploit the following useful characterization of exponentiable objects in  $\text{Met}$  (we recall that an object  $A$  in a cartesian category  $\mathbb{C}$  is exponentiable when, for all object  $B$ , the exponential of  $B$  and  $A$  exists in  $\mathbb{C}$ , so  $\mathbb{C}$  is cartesian closed iff all its objects are exponentiable):

► **Theorem 2** ([13]). A metric space  $(X, a)$  is exponentiable in  $\text{Met}$  iff for all  $x_0, x_2 \in X$  and  $\alpha, \beta \in \mathbb{R}_{\geq 0}^{\infty}$  such that  $a(x_0, x_2) = \alpha + \beta$ , the condition below holds:

$$\forall \epsilon > 0 \exists x_1 \in X \text{ s.t. } a(x_0, x_1) < \alpha + \epsilon \text{ and } a(x_1, x_2) < \beta + \epsilon \quad (*)$$

Condition (\*) intuitively requires  $X$  to have “enough points”. For example, the set  $\mathbb{N}$ , as a subspace of  $\mathbb{R}$ , is not exponentiable in  $\text{Met}$  (take  $x_0 = 0, x_1 = 1$  and  $\alpha = \beta = 1/2$ : a point between 0 and 1 is “missing”). Instead, condition (\*) always holds when  $(X, a)$  is *injective* (see [22, 13]): for any collection of points  $\{x_i\}_{i \in I}$  in  $X$  and positive reals  $\{r_i\}_{i \in I}$  such that  $a(x_i, x_j) \leq r_i + r_j$ , there is a point lying in the intersection of all balls  $B(x_i, r_i)$ . This implies that the sub-category  $\text{InjMet}$  of  $\text{Met}$  formed by injective metric spaces is cartesian closed. Since the Euclidean metric is injective, there is a cartesian closed sub-category of  $\text{Met}$  formed by “simple types” over closed real intervals, that we’ll use as working example.

► **Example 3.** Let  $\text{IntST}$  be the set of *simple types over the intervals*, defined by  $[a, b] \in \text{IntST}$ , for all intervals  $[a, b]$  (with  $a, b \in \mathbb{R}_{\geq 0}$  and  $a \leq b$ ) and  $i, j \in \text{IntST} \Rightarrow (i \times j), (i \rightarrow j) \in \text{IntST}$ . For any  $i \in \text{IntST}$ , the metric spaces  $(\mathcal{I}_i, d_i^{\mathcal{I}})$  are defined by  $\mathcal{I}_{[a,b]} := [a, b]$ ,  $\mathcal{I}_{i \times j} := \mathcal{I}_i \times \mathcal{I}_j$ ,  $\mathcal{I}_{i \rightarrow j} := \text{Met}(\mathcal{I}_i, \mathcal{I}_j)$ ,  $d_{[a,b]}^{\mathcal{I}}(x, y) := |x - y|$ ,  $d_{i \times j}^{\mathcal{I}} := \max\{d_i^{\mathcal{I}}, d_j^{\mathcal{I}}\}$  and  $d_{i \rightarrow j}^{\mathcal{I}} := \Xi_{d_i^{\mathcal{I}}, d_j^{\mathcal{I}}}$ .

The “analytic knife” provided by metrics is rather blunt when dealing with isometries in  $\mathbb{R}$ , because these are isolated points in  $\Xi_{a,b}$ . Examples of isometries are the identity and functions which have a right or left inverse. We have:

► **Proposition 4.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an isometry in  $\mathbb{R}$ . Then  $f$  is isolated in the metric  $\Xi_{a,b}$ . Moreover the identity is isolated in all injective spaces.*

### 3 Many-Sorted Quantitative Theories and Algebras

In this section we introduce quantitative theories and algebras in the sense of [29]. In order to cover both the typed and the untyped case, we consider *many-sorted* theories and algebras, hence combining the quantitative (but one-sorted) approach from [29] with the qualitative (but many-sorted) approach from [24].

**Notation.** For any set  $I$ , an *I-sorted set*  $A$  is an  $I$ -indexed family of sets  $A = (A_i)_{i \in I}$  (i.e. an object of  $\text{Set}^I$ ), and an *I-sorted function*  $f : A \rightarrow B$  between  $I$ -sorted sets is an  $I$ -indexed family of functions  $f = (f_i : A_i \rightarrow B_i)_{i \in I}$  (i.e. a morphism in  $\text{Set}^I(A, B)$ ). For a set  $I$ , we denote by  $I^*$  the set of all finite lists of elements of  $I$ , we let  $w$  range over  $I^*$  and use  $*$  for concatenation. For  $A$  an  $I$ -sorted set and  $w = i_1 \dots i_k \in I^*$ , we let  $A_w := \prod_{j=1}^k A_{i_j}$ .  $\text{Var}$  denotes a distinguished  $I$ -sorted containing, such that for all  $i \in I$ ,  $\text{Var}_i$  is a countably infinite set of variables. For any  $I$ -sorted set  $A$  and function  $f : \text{Var} \rightarrow A$ , and pairwise disjoint variables  $x_1, \dots, x_n$ , with  $x_j \in \text{Var}_{i_j}$  and  $a_1, \dots, a_n$  with  $a_j \in A_{i_j}$ , we let  $f_{\bar{x}, \bar{a}} : \text{Var} \rightarrow A$  indicate the  $I$ -sorted function mapping  $x_j$  to  $a_j$  and behaving as  $f$  on all other variables.

► **Definition 5 (Many-Sorted Signature).** *An  $I$ -sorted signature  $\Sigma$  is an  $I^* \times I$ -sorted set  $\{\Sigma_{w,i} \mid w \in I^*, i \in I\}$  (i.e. an object of  $\text{Set}^{I^* \times I}$ ).*

The objects  $\sigma \in \Sigma_{w,i}$  will be called *symbols* of the signature.

► **Definition 6 ( $\Sigma$ -Algebra).** *A  $\Sigma$ -algebra is a pair  $(A, \Omega^A)$  where  $A$  is a  $I$ -sorted family and  $\Omega^A$  associates each symbol  $\sigma \in \Sigma_{w,i}$  with a function  $\sigma_A : A_w \rightarrow A_i$ , where  $A_w = A_{i_1} \times \dots \times A_{i_k}$ , for  $w = i_1 \dots i_k$ . For any object  $A$  of  $\text{Set}^I$ , the free  $\Sigma$ -algebra over  $A$ , noted  $\mathbb{F}_{\Sigma}(A)$ , is the  $I$ -sorted set defined by the following conditions: (i) for all  $x \in A_i$ ,  $x \in \mathbb{F}_{\Sigma}(A)_i$ ; (ii) for all  $\sigma \in \Sigma_{w,i}$  and  $v_1 \in \mathbb{F}_{\Sigma}(A)_{w(1)}, \dots, v_k \in \mathbb{F}_{\Sigma}(A)_{w(k)}$ , then  $\sigma_{\mathbb{F}_{\Sigma}(A)}(v_1, \dots, v_k) := \sigma(v_1, \dots, v_k) \in \mathbb{F}_{\Sigma}(A)_i$ .*

Intuitively,  $\mathbb{F}_{\Sigma}(A)_i$  is the set of “terms of sort  $i$  with parameters in  $A$ ”. Free algebras enjoy the following universal property:

► **Proposition 7.** For any  $\Sigma$ -algebra  $(A, \Omega^A)$  and map  $f \in \text{Set}^I(B, A)$  there exists a unique  $\Sigma$ -homomorphism  $f^\# : \mathbb{F}_\Sigma(B) \rightarrow A$  extending  $f$ , that is, such that  $f = f^\# \circ \eta_B$ , where  $\eta_B : B \rightarrow \mathbb{F}_\Sigma(B)$  is the inclusion map.

Given a function  $f \in \text{Set}^I(B, A)$ , if  $t \in \mathbb{F}_\Sigma(B)_i$  is some term of sort  $i$  with parameters  $b_1, \dots, b_n$  in  $B$ ,  $f^\#t \in \mathbb{F}_\Sigma(A)_i$  is the result of “substituting” each parameter  $b_i$  in  $t$  with  $f(b_i)$ .

Let us now introduce the equational language of quantitative theories.

► **Definition 8.** Let  $\Sigma$  be an  $I$ -sorted signature.

- (i) A quantitative  $\Sigma$ -equation over  $\mathbb{F}_\Sigma(\text{Var})$  is an expression of the form  $t \simeq_\epsilon^i s$ , where  $i \in I$ ,  $t, s \in \mathbb{F}_\Sigma(\text{Var})_i$  and  $\epsilon \in \mathbb{Q}_{\geq 0}$ .
- (ii) For all  $\epsilon \in \mathbb{Q}_{\geq 0}$ , let  $\mathcal{V}(\text{Var})$  be the set of indexed  $\Sigma$ -equations of the form  $x \simeq_\epsilon^i y$ , for some  $i \in I$  and  $x, y \in \text{Var}_i$ , and  $\mathcal{V}(\mathbb{F}_\Sigma(\text{Var}))$  be the set of indexed  $\Sigma$ -equations of the form  $t \simeq_\epsilon^i s$ , where  $i \in I$  and  $t, s \in (\mathbb{F}_\Sigma(\text{Var}))_i$ .

► **Definition 9.** A consequence relation on the free  $\Sigma$ -algebra  $\mathbb{F}_\Sigma(\text{Var})$  is a relation  $\vdash \subseteq \wp(\mathcal{V}(\mathbb{F}_\Sigma(\text{Var}))) \times \mathcal{V}(\mathbb{F}_\Sigma(\text{Var}))$  closed under all instances of the following rules (where  $\epsilon, \delta$  vary over all  $\mathbb{Q}_{\geq 0}$ ):

(Cut) if  $\Gamma \vdash \phi$  for all  $\phi \in \Gamma'$  and  $\Gamma' \vdash \psi$ , then  $\Gamma \vdash \psi$ ;

(Assumpt) if  $\phi \in \Gamma$ , then  $\Gamma \vdash \phi$ ;

(Refl)  $\emptyset \vdash t \simeq_0^i t$ ;

(Symm)  $\{t \simeq_\epsilon^i s\} \vdash s \simeq_\epsilon^i t$ ;

(Triang)  $\{t \simeq_\epsilon^i s, s \simeq_\delta^i u\} \vdash t \simeq_{\epsilon+\delta}^i u$ ;

(Max)  $\{t \simeq_\epsilon^i s\} \vdash t \simeq_{\epsilon+\delta}^i s$ ;

(Arch)  $\{t \simeq_\delta^i s \mid \delta > \epsilon\} \vdash t \simeq_\epsilon^i s$ ;

(NExp)  $\{t_1 \simeq_{\epsilon_1}^{i_1} s_1, \dots, t_k \simeq_{\epsilon_k}^{i_k} s_k\} \vdash \sigma(t_1, \dots, t_k) \simeq_\epsilon^i \sigma(s_1, \dots, s_k)$ , for all  $\sigma \in \Sigma_{i_1 \dots i_k, i}$ ;

(Subst) if  $f : \text{Var} \rightarrow \mathbb{F}_\Sigma(\text{Var})$ , then  $\Gamma \vdash t \simeq_\epsilon^i s$  implies  $f^\#\Gamma \vdash f^\#t \simeq_\epsilon^i f^\#s$ .

Notice that rule (Arch) has infinitely many assumptions.

We let  $\mathcal{E}(\mathbb{F}_\Sigma(\text{Var})) = \wp_{\text{fin}}(\mathcal{V}(\mathbb{F}_\Sigma(\text{Var}))) \times \mathcal{V}(\mathbb{F}_\Sigma(\text{Var}))$  indicate the set of quantitative inferences on  $\mathbb{F}_\Sigma(\text{Var})$  and  $\mathcal{E}(\text{Var}) = \wp_{\text{fin}}(\mathcal{V}(\text{Var})) \times \mathcal{V}(\mathbb{F}_\Sigma(\text{Var}))$  indicate the set of basic quantitative inferences. Axioms for theories will be basic quantitative inferences.

► **Definition 10 (Many-Sorted Quantitative Theory).** Let  $S \subseteq \mathcal{E}(\text{Var})$  be a set of basic quantitative inferences. Let  $\vdash_S$  be the smallest consequence relation including  $S$ . The quantitative equational theory over  $\Sigma$  generated by  $S$  is the set  $\mathcal{U}_S := (\vdash_S) \cap \mathcal{E}(\mathbb{F}_\Sigma(\text{Var}))$ . The elements of  $S$  are the axioms of  $\mathcal{U}_S$ .

To the syntactic notion of quantitative theory there corresponds a semantic notion of quantitative algebra, given by a  $\Sigma$ -algebra endowed with suitable metrics.

► **Definition 11 (Many-Sorted Quantitative Algebra).** Let  $\Sigma$  be an  $I$ -sorted signature. A quantitative  $\Sigma$ -algebra is a tuple  $\mathcal{A} = (A, \Omega^A, d^A)$  where  $(A, \Omega^A)$  is a  $\Sigma$ -algebra and  $d^A$  is an  $I$ -sorted family of metrics  $d_i^A : A_i \times A_i \rightarrow \mathbb{R}_{\geq 0}^\infty$  such that for all  $\sigma \in \Sigma_{w, i}$ ,  $\sigma_A : \text{Met}(A_w, A_i)$ .

Given a quantitative  $\Sigma$ -algebra, we can define a multicategory  $\text{Met}^A$  whose objects are the metric spaces  $(A_i, d_i^A)$ , and where for all  $w = i_1 \dots i_k$ ,  $\text{Met}^A(A_{i_1}, \dots, A_{i_k}; A_i) \subseteq \text{Met}(A_w, A_i)$  contains all functions  $f \in \text{Met}(A_w, A_i)$  such that for some term  $t_f \in \mathbb{F}_\Sigma(A + \{x_1 : w(1), \dots, x_k : w(k)\})$ ,  $f(a_1, \dots, a_k) = f_{\bar{x}, \bar{a}}^\#(t_f)$ . For brevity, we will often abbreviate  $\text{Met}^A(A_{i_1}, \dots, A_{i_k}; A_i)$  as  $\text{Met}^A(A_w; A_i)$ .

► **Definition 12.** Let  $\mathcal{A} = (A, \Omega^A, d^A)$  be a quantitative  $\Sigma$ -algebra. For any  $f : \text{Var} \rightarrow A$ , we say that  $\mathcal{A}$  satisfies a quantitative equation  $\phi = t \simeq_\epsilon^i u$  relative to  $f$  (denoted  $\models_{\mathcal{A}}^f \phi$ ) when  $d_i^A(f^\sharp(t), f^\sharp(u)) \leq \epsilon$ . We say that  $\mathcal{A}$  satisfies a quantitative inference  $\Gamma \vdash \phi$  (denoted  $\Gamma \vDash_{\mathcal{A}} \phi$ ) if for all  $f : \text{Var} \rightarrow A$ , if  $\models_{\mathcal{A}}^f \psi$  holds for all  $\psi \in \Gamma$ , then  $\vDash_{\mathcal{A}}^f \phi$  also holds.

Notice that the interpretation of rule (Nexp) implies that functional terms need to be interpreted as non-expansive morphisms.

► **Remark 13.** All constructions from this section can be adapted to the case of *partial* ultra-metric spaces by replacing, in Definition 9, the rule (Refl) with the following rule:

(PRefl)  $\{t \simeq_\epsilon^i u\} \vdash t \simeq_\epsilon^i t$ ;

and requiring in Def. 11 that the  $d_i^A$  are partial ultra-metrics and  $\sigma^A \in \text{PUMet}(A_w, A_i)$ .

## 4 Quantitative Weak $\lambda$ -Theories and Algebras

As is well-known (see e.g. [5]), a purely algebraic approach to the  $\lambda$ -calculus is provided by *combinatory logic* **CL**. Hence, it is natural to start from this calculus. The equational theory of **CL** captures so-called *weak  $\lambda$ -theories* [5], namely  $\lambda$ -theories where the  $\xi$ -rule (discussed in more detail in Section 5) may fail. In this section we introduce quantitative weak  $\lambda$ -theories and we discuss their algebras, of which **Met** itself is a notable example.

► **Definition 14 (Applicative Signature).** Let  $T$  be a set of sorts (called types) endowed with a binary function  $\rightarrow : T \times T \rightarrow T$ . An applicative signature  $\Sigma$  is a  $T$ -sorted signature which includes symbols  $\cdot_{i,j} \in \Sigma_{(i \rightarrow j) * i, j}$ , for all  $i, j \in T$ .

We will often note  $\cdot_{i,j}(t, u)$  infix, i.e.  $t \cdot_{i,j} u$ , or simply as  $tu$ , when clear from the context. For all  $w = i_1 \dots i_n \in T^*$  and  $j \in T$ , we let  $w \rightarrow j := i_1 \rightarrow \dots \rightarrow i_n \rightarrow j$ . A notable example of applicative signature is the following:

► **Definition 15 (CL-Signature).** Let  $\Sigma^{\text{CL}}$  be the applicative signature which includes symbols  $l_i : i \rightarrow i$ ,  $K_{ij} : i \rightarrow j \rightarrow i$ ,  $S_{ijk} : (i \rightarrow j \rightarrow k) \rightarrow (i \rightarrow j) \rightarrow (i \rightarrow k)$ , for all  $i, j, k \in T$ . The terms of combinatory logic are the elements of the free  $\Sigma^{\text{CL}}$ -algebra,  $\mathbb{F}_{\text{CL}}(\text{Var})$ .

Definition 15 above comprises both the typed and untyped case. In typed Combinatory Logic the set of types  $T$  includes at least a *base type*  $o$ , i.e. a type which is not in the image of  $\rightarrow$  and  $\rightarrow$  is injective, while in the untyped case  $T$  is a singleton set  $\{\star\}$  and hence  $\star \rightarrow \star = \star$ . In the traditional language of “syntax and semantics”, used for instance in [5], when  $f : \text{Var} \rightarrow A$ , the function  $f^\sharp$  of Proposition 7, amounts to the notion of *interpretation* of a term  $t$  in the environment  $f$ , namely  $f^\sharp(t) = \llbracket t \rrbracket_f$ .

We now introduce the natural notion of theory for a **CL**-signature:

► **Definition 16 (CL-Theory).** The quantitative equational theory over  $\mathbb{F}_{\text{CL}}(\text{Var})$ ,  $\mathcal{U}_{\text{CL}}$  is generated by the axioms  $\emptyset \vdash l_i t \simeq_0^i t$ ,  $\emptyset \vdash K_{ij} t u \simeq_0^i t$ , and  $\emptyset \vdash S_{ijk} t u w \simeq_0^k t w (u w)$ . We call (quantitative) weak  $\lambda$ -theory any theory including  $\mathcal{U}_{\text{CL}}$ .

► **Example 17.** The set **IntST** (cf. Example 3) is a particular instance of the set  $T$ . Let  $\mathcal{I}(\Sigma^{\text{CL}})$  be the signature obtained by enriching  $\Sigma^{\text{CL}}$  with 0-ary symbols  $\bar{r} \in \mathcal{I}(\Sigma)_{(), [a, b]}$  for all  $r \in [a, b]$ , and  $k$ -ary symbols  $\bar{f} \in \mathcal{I}(\Sigma)_{[a_1, b_1], \dots, [a_n, b_n], [a, b]}$  for all  $f \in \text{Met}(\prod_i [a_i, b_i], [a, b])$ . Let  $\mathcal{U}_{\text{CL}}^{\mathcal{I}}$  be the theory obtained by extending  $\mathcal{U}_{\text{CL}}$  with all axioms  $\emptyset \vdash \bar{f} \bar{r}_1 \dots \bar{r}_k \simeq_0^{[a, b]} \bar{s}$  whenever  $f(r_1, \dots, r_k) = s$  as well as all axioms  $\emptyset \vdash \bar{r} \simeq_\epsilon^{[a, b]} \bar{s}$  for all rational  $\epsilon \geq |r - s|$ .

A well-known property of Combinatory Logic is *functional completeness*: for any term  $t$  and variable  $x$ , one can construct a term  $\Lambda_x(t)$  so that  $\Lambda_x(t)$  “simulates”  $\lambda$ -abstraction in the sense that one can prove  $\Lambda_x(t)u \simeq t[u/x]$ . This leads to the following definition:

► **Definition 18** (Quantitative Weak  $\lambda$ -Algebra). *An applicative quantitative  $\Sigma$ -algebra  $\mathcal{A} = (A, \Omega^A, d^A)$  is called a quantitative weak  $\lambda$ -algebra if for all  $w \in I^*$ ,  $j \in I$ , and  $f \in \text{Met}(A_w, A_j)$ , the set  $\Lambda(f) = \{g \in A_{w \rightarrow j} \mid \forall (x_1, \dots, x_k) \in A_w \quad g \cdot_A x_1 \cdot_A \dots \cdot_A x_k = f(x_1, \dots, x_k)\}$  is non-empty.*

► **Proposition 19.** *Any quantitative  $\Sigma^{\text{CL}}$ -algebra satisfying  $\mathcal{U}_{\text{CL}}$  is a quantitative weak  $\lambda$ -algebra. Vice versa, any quantitative weak  $\lambda$ -algebra satisfies  $\mathcal{U}_{\text{CL}}$ .*

► **Example 20.** We obtain a quantitative weak  $\lambda$ -algebra by letting  $\mathcal{I} = (\mathcal{I}_i, \Omega^{\mathcal{I}}, d_i^{\mathcal{I}})$ , where  $\bar{r}^{\mathcal{I}} = r$ ,  $\bar{f}^{\mathcal{I}} = f$ , and  $f \cdot^{\mathcal{I}} x = f(x)$ . It is clear that  $\mathcal{I} \models \mathcal{U}_{\text{CL}}^{\mathcal{I}}$  (cf. Example 17).

Following [30], the condition from Definition 18 can be specified in categorical terms: a cartesian multicategory  $\mathbb{C}$  is a model of **CL** precisely when for all objects  $A, B$  of  $\mathbb{C}$  there is an object  $A \overset{\text{vw}}{\rightrightarrows} B$  (called a *very weak exponential of A and B*) together with a surjective natural transformation  $\Phi : \mathbb{C}(\_, A \overset{\text{vw}}{\rightrightarrows} B) \rightarrow \mathbb{C}(\_, A; B)$ . When  $\mathbb{C}$  is the multicategory  $\text{Met}^A$ , the conditions of Definition 18 imply that  $A_{i \rightarrow j}$  is a very weak exponential of  $A_i$  and  $A_j$  in  $\text{Met}^A$ : a family of multiarrows  $\mathbf{Ev}_{w,i,j}^A : \text{Met}^A(A_w; A_{i \rightarrow j}) \Rightarrow \text{Met}^A(A_{w * i}; A_j)$ , natural in  $w$ , is given by  $\mathbf{Ev}_{w,i,j}^A(f)(z, x) = f(z) \cdot^A x$ , and the non-emptiness of the sets  $\Lambda(f)$  corresponds to the surjectivity of this transformation.

Notice that  $\text{Met}$  itself admits very weak exponentials for all of its objects, *i.e.* it is a *very weak CCC* in the sense of [30], provided we endow  $\text{Met}(X, Y)$  with the metric  $\Theta_{a,b}$  for metric spaces  $(X, a)$  and  $(Y, b)$ , where for  $f, g : X \rightarrow Y$   $\Theta_{a,b}(f, g)$  is 0 if  $f = g$ , and otherwise is  $\sup\{b(f(x), g(y)) \mid x, y \in X\}$ . Intuitively, when  $f \neq g$ ,  $\Theta_{a,b}(f, g)$  measures the diameter of the interval spanned by the image of both  $f$  and  $g$ . However, the metric  $\Theta_{a,b}$  is in general rather odd since the identity is an isolated point whenever  $(X, a)$  is infinite and not trivial.

► **Example 21.** The constructions just sketched yields a different weak  $\lambda$ -algebra over the reals  $\mathcal{I}_{\text{weak}} = (\mathcal{I}_i, \Omega^{\mathcal{I}}, d^{\text{weak}})$ , where  $d^{\text{weak}}$  is defined like  $d^{\mathcal{I}}$  but for  $d_{i \rightarrow j}^{\text{weak}} = \Theta_{d_i^{\text{weak}}, d_j^{\text{weak}}}$ . Notice that we still have  $\mathcal{I}_{\text{weak}} \models \mathcal{U}_{\text{CL}}^{\mathcal{I}}$ , since  $\mathcal{I}$  and  $\mathcal{I}_{\text{weak}}$  agree on distances of types  $[a, b]$ .

The result below adapts to the many-sorted case a similar result for one-sorted quantitative equational theories [29]. The proof is similar to that of Theorem 37, so we omit it.

► **Theorem 22** (Soundness and Completeness of Quantitative Weak  $\lambda$ -Theories). *For any quantitative weak  $\lambda$ -theory  $\mathcal{U}$  over  $\Sigma^{\text{CL}}$ ,  $\Gamma \vdash \phi \in \mathcal{U}$  iff  $\Gamma \models_{\mathcal{A}} \phi$  holds for any quantitative weak  $\lambda$ -algebra  $\mathcal{A}$  such that  $\mathcal{A} \models \mathcal{U}$ .*

► **Remark 23.** Following Remark 13, in the case of partial ultra-metric spaces we will talk of *partial weak  $\lambda$ -theories* and *partial weak  $\lambda$ -algebras*.

## 5 Quantitative $\lambda$ -Theories and Algebras

As we recalled, weak  $\lambda$ -theories do not fully capture the equational theory of the  $\lambda$ -calculus, as they fail to capture the so-called  $\xi$ -rule [5]. In our quantitative setting, this rule can be expressed as the inference  $t \underset{\epsilon}{\simeq}^j u \vdash \lambda x.t \underset{\epsilon}{\simeq}^{i \rightarrow j} \lambda x.u$  provided the equation on the left of  $\vdash$  is *locally universally quantified*: the righthand equation holds under the condition that, for all possible value of  $x$ , the lefthand equation holds. This kind of quantitative inferences

differ from those seen so far. The reason for this proviso is that it involves the higher-order operator  $\lambda$ , which “binds” the variable  $x$ . The example below shows that quantitative weak  $\lambda$ -algebras fail to capture this rule.

► **Example 24.** The  $\xi$ -rule fails in the weak  $\lambda$ -algebra  $\mathcal{I}$ : let  $f, g : [0, b] \rightarrow [0, b + \epsilon]$  (where  $b \in \mathbb{R}_{\geq 0}$  and  $\epsilon \in \mathbb{Q}_{\geq 0}$ ) be, respectively, the identity function  $f = \text{id}$  and the function  $g(x) = x + \epsilon$ ; for any  $s \in [a, b]$ , we then have  $|f(s) - g(s)| \leq \epsilon$ , which shows  $\mathcal{I} \models \bar{f}x \stackrel{[a, b + \epsilon]}{\simeq_\epsilon} \bar{g}x$ . However, since  $d_{[a, b] \rightarrow [a, b + \epsilon]}^{\mathcal{I}}(f, g) = b + \epsilon$ , we deduce  $\mathcal{I} \not\models \lambda x. \bar{f}x \stackrel{[a, b] \rightarrow [a, b + \epsilon]}{\simeq_\epsilon} \lambda x. \bar{g}x$ .

In order to define quantitative  $\lambda$ -theories we could follow Curry [5] and “strengthen” the set of axioms, in fact mere equalities, satisfied by a  $\Sigma^{\text{CL}}$ -algebra and essentially do away with the  $\xi$ -rule and all higher order features. The alternative, that we develop in this section, is to take abstraction and the  $\xi$ -rule as first class elements of our theories and algebras. This will require a number of generalizations of the original approach of [29].

At the level of syntax, the first step is to enrich the class of symbols with higher-order operators of the form  $\lambda_i x$ . The occurrence of the variable  $x$  is part of the symbol  $\lambda_i x$  itself.

► **Definition 25 ( $\lambda$ -Signature).** *Given an applicative  $T$ -sorted signature  $\Sigma$ , let  $\Sigma^\lambda$  be the applicative  $T$ -sorted signature further including the symbols  $\lambda_i x \in \Sigma_{j, i \rightarrow j}^\lambda$ , for all  $x \in \text{Var}_i$  and  $i, j \in T$ . The  $\lambda$ -terms are the elements of the free  $\Sigma^\lambda$ -algebra,  $\mathbb{F}_\lambda(\text{Var})$ .*

Terms  $\lambda_i x(t)$  will be denoted by  $\lambda_i x.t$  or simply  $\lambda x.t$ . Free and bound variables, open and closed  $\lambda$ -terms are defined as usual. For a  $\lambda$ -term  $t$ , we denote by  $\text{fv}(t)$ ,  $\text{bd}(t)$ ,  $\text{var}(t)$  the sets of free, bound, and all variables in  $t$ , respectively. In order to simplify the notation we deal with bound variables by implementing directly Barendregt’s “hygiene condition”. For any function  $f : \text{Var} \rightarrow \mathbb{F}_\lambda(\text{Var})$  there exists a function  $f^b : \mathbb{F}_\lambda(\text{Var}) \rightarrow \mathbb{F}_\lambda(\text{Var})$  such that  $f^b(t)$  corresponds to the substitution of  $f(x)$  for  $x$  in  $t$ , for any variable  $x$  occurring free in  $t$ . Given pairwise disjoint variables  $x_1, \dots, x_n$ , with  $x_j \in \text{Var}_j$  and terms  $t_1, \dots, t_n$ , with  $t_j \in \mathbb{F}_\lambda(\text{Var})_j$ , we indicate the “substitution”  $(\text{id}_{\bar{x}, \bar{t}})^b(u)$  simply as  $u[t_j/x_j]$ .

In order to be able to express correctly the  $\xi$ -rule we generalize quantitative equations to expressions of the form  $t \stackrel{X, i}{\simeq_\epsilon} u$ , where  $X$  indicates a finite set of variables which are intended to be “locally quantified” on the left of  $\vdash$ .

► **Definition 26 ( $\Sigma^\lambda$ -equation).** *A quantitative  $\lambda$ -equation is an expression of the form  $t \stackrel{X, i}{\simeq_\epsilon} s$ , where  $i \in I$ ,  $t, s \in \Lambda_i$ ,  $X \subseteq_{\text{fin}} \text{Var}$ ,  $\epsilon \in \mathbb{Q}_{\geq 0}$ . The set  $X$  is the set of locally quantified variables in the equation.*

We let  $\mathcal{V}(\Lambda)$  indicate the set of quantitative  $\lambda$ -equations.

► **Definition 27.** *A consequence relation on  $\Lambda$  is a relation  $\vdash \subseteq \wp(\mathcal{V}(\Lambda)) \times \mathcal{V}(\Lambda)$  closed under the rules (Cut)-(Nexp) from Def. 9 (with  $t \stackrel{i}{\simeq_\epsilon} u$  everywhere replaced by  $t \stackrel{X, i}{\simeq_\epsilon} u$ ), together with the following rules:*

(Subst) *if  $\Gamma \vdash t \stackrel{X, i}{\simeq_\epsilon} s$  and let  $f$  be the identity on  $X$  and, for all  $x \in \text{Var} \setminus X$ ,  $\text{fv}(f(x)) \cap \text{bd}(t, s, \Gamma) = \emptyset$ , then  $\Gamma \vdash t \stackrel{X, i}{\simeq_\epsilon} s$  implies  $f^b(\Gamma) \vdash f^b(t) \stackrel{X, i}{\simeq_\epsilon} f^b(s)$ ;*

(Abstraction) *if  $X \subseteq X'$  and  $\text{fv}(t, s) \cap X' = \emptyset$ , then  $\{t \stackrel{X, i}{\simeq_\epsilon} s\} \vdash t \stackrel{X', i}{\simeq_\epsilon} s$ ;*

(Concretion) *if  $X' \subseteq X$  then  $\{t \stackrel{X, i}{\simeq_\epsilon} s\} \vdash t \stackrel{X', i}{\simeq_\epsilon} s$ ;*

We call  $\mathcal{U}_\lambda$  the quantitative theory generated by the axioms below apart from  $(\eta)$  and we denote by  $\vdash^\lambda$  the corresponding consequence relation, and  $\mathcal{U}_{\lambda\eta}$  the quantitative theory generated by all the axioms below, including  $(\eta)$ , with consequence relation  $\vdash^{\lambda\eta}$ :



- ( $\alpha$ ) if  $x, y \in \text{Var}_i$  and  $y \notin \text{var}(\lambda_i x.t)$ , then  $\emptyset \vdash \lambda x.t \simeq_0^{X,i} \lambda_i y.t[y/x]$ .  
 ( $\xi$ ) if  $x \in X$ , then  $t \simeq_\epsilon^{X,j} u \vdash \lambda_i x.t \simeq_\epsilon^{X,i \rightarrow j} \lambda_i x.u$ ;  
 ( $\beta$ ) if  $(\lambda_i x.t)u \in \Lambda_j$ ,  $\text{fv}(u) \cap \text{bd}(t) = \emptyset$ , then  $\emptyset \vdash (\lambda_i x.t)u \simeq_0^{X,j} t[u/x]$ .  
 ( $\eta$ ) if  $\lambda_i x.(tx) \in \Lambda_{i \rightarrow j}$ ,  $x \notin \text{fv}(t)$ , then  $\vdash t \simeq_0^{X,i \rightarrow j} \lambda_i x.(tx)$ .

Any theory including  $\mathcal{U}_\lambda$  ( $\mathcal{U}_{\lambda\eta}$ ) is called a quantitative (extensional)  $\lambda$ -theory.

► **Example 28.** Consider the  $\lambda$ -signature  $\mathcal{I}(\Sigma)^\lambda$  (cf. Example 17). Let  $\mathcal{U}_{\lambda\eta}^{\mathbb{R}}$  be the extensional  $\lambda$ -theory obtained by enriching  $\mathcal{U}_{\lambda\eta}$  with all real-valued axioms as in Example 17.

We now introduce a class of applicative algebras suitable to account for abstraction operators. This is done by requiring the existence of suitable “closing maps” that send a closed  $\lambda$ -term of the form  $\lambda_{i_1} x_1 \dots \lambda_{i_n} x_n.t$  onto some point of  $A_{i_1 \rightarrow \dots \rightarrow i_n \rightarrow j}$ .

Given any  $T$ -index set  $A$ , extend the definition of  $\Sigma^\lambda$  to  $\Sigma^{\lambda,A}$  so as to contain as 0-ary constructors all elements in  $A$  and correspondingly the notion of  $\mathbb{F}_{\lambda,A}(\text{Var})$ .

► **Definition 29.** A quantitative applicative  $\lambda$ -algebra is a structure  $\mathcal{A} = (A, \Omega^A, \Lambda^A, d^A)$ , where  $(A, \Omega^A, d^A)$  is a quantitative applicative algebra and  $\Lambda_{w,j}^A : (\mathbb{F}_{\lambda,A}(\text{Var}))_{w \rightarrow j}^0 \rightarrow A_{i \rightarrow j}$ . We call applicative  $\lambda$ -algebra the structure  $\mathcal{A} = (A, \Omega^A, \Lambda^A)$  without the metric.

The functions  $\Lambda_{i,j}^A : (\mathbb{F}_{\lambda,A}(\text{Var}))_{i \rightarrow j}^0 \rightarrow A_{i \rightarrow j}$  are intended to define a choice in the set  $\Lambda$  of Definition 18. This will be apparent in view of Definitions 30, 31, 32 below, which will enforce that, in suitable structures, the interpretations of the terms  $\mathbb{F}_{\lambda,A}(\text{Var})_{i \rightarrow j}^0$  become essentially the domain of  $\Lambda$  in Definition 18. We point out that a slight modification of these definitions would permit to recover precisely the categorically weaker notion of Quantitative Weak  $\lambda$ -algebra of Definition 18.

► **Proposition 30 (Interpretation).** Let  $\mathcal{A}$  be a quantitative applicative  $\lambda$ -algebra, and  $\rho : \text{Var} \rightarrow A$ . Then there exists a function  $\rho^\natural : \mathbb{F}_\lambda(\text{Var}) \rightarrow A$ , where  $\rho^\natural(t)$  is defined by cases as  $\rho^\natural(x) = \rho(x)$ ,  $\rho^\natural(t_1 \cdot t_2) = \rho^\natural(t_1) \cdot_A \rho^\natural(t_2)$ , and  $\rho^\natural(\lambda_i x.t) = \Lambda_{w^*,i,j}^A(\lambda \vec{y} \lambda_i x.t) \cdot_A \rho^\natural(y)$ , where  $\lambda \vec{y} \lambda_i x.t$  is the closure of the term  $\lambda_i x.t$  w.r.t. its free variables  $\vec{y}$  of types  $w$ .

To define higher-order structure for a quantitative applicative  $\lambda$ -algebras  $\mathcal{A}$  it is useful to define the multicategory generated by  $\mathcal{A}$ :

► **Definition 31 (Representable Functions).** For any quantitative applicative  $\lambda$ -algebra  $\mathcal{A}$ ,  $\text{Met}^A$  is the multicategory with objects the metric spaces  $(A_i, d_i^A)$ , and where, for  $w = i_1 \dots i_k$ ,  $\text{Met}^A(A_{i_1}, \dots, A_{i_k}; A_i)$  (abbreviated as  $\text{Met}^A(A_w; A_i)$ ) is the set of  $f \in \text{Met}(A_w, A_i)$  such that for some  $t_f \in \mathbb{F}_{\lambda,A}(\text{Var})_{w \rightarrow i}^0$ ,  $f(a_1, \dots, a_n) = \Lambda_{w^*,i}^A(t_f) \cdot^A \vec{a}$ .

Notice that the function  $\Lambda^A$  yields a family of maps  $\Lambda_{w^*,i,j}^A : \text{Met}^A(A_{w^*}; A_j) \rightarrow \text{Met}^A(A_w; A_{i \rightarrow j})$ , given by  $\Lambda_{w^*,i,k}(h)(a)(b) = \Lambda_{w^*,i,j}^A(t_h) \cdot^A \langle a, b \rangle$ .

While cartesian closed (multi)categories are the algebras for extensional  $\lambda$ -theories, an algebra for a  $\lambda$ -theory is a cartesian multicategory in which for all objects  $A, B$  there is an object  $A \overset{w}{\rightrightarrows} B$  (called a *weak exponential*, [30]) together with a natural *retraction*  $\mathbb{C}(\_, A; B) \Rightarrow \mathbb{C}(\_, A \overset{w}{\rightrightarrows} B)$ . To account for the quantitative  $\xi$ -rule, this picture must be slightly adapted, by requiring the maps forming the retraction to be also *non-expansive*. This leads to the following definition:

► **Definition 32 (Quantitative  $\lambda$ -Algebra).** For any quantitative applicative  $\lambda$ -algebra  $\mathcal{A} = (A, \Omega^A, \Lambda^A, d^A)$ ,  $\mathcal{A}$  is a quantitative (extensional)  $\lambda$ -algebra if the maps  $\Lambda_{w^*,i,j}^A, \mathbf{Ev}_{w^*,i,j}^A$  form a family of retractions (resp. isomorphisms) natural in  $w$  and non-expansive (with respect to the pre-metrics  $\Xi_{d_w^*, d_j^A}$  over  $\text{Met}^A(A_w; A_j)$ ).

## 4:10 On Quantitative Algebraic Higher-Order Theories

The definition above can be expressed in more abstract terms using the language of *enriched* categories: the multicategory  $\text{Met}^A$  is enriched over the cartesian closed category of pre-metric spaces and non-expansive functions, where  $\text{Met}^A(A_w; A_i)$  is endowed with the pre-metric  $\Xi_{d_w^A, d_i^A}$ . Then  $\mathcal{A}$  is a quantitative (resp. extensional)  $\lambda$ -algebra when the maps  $\Lambda_{w*i, j}^A, \text{Ev}_{w, i, j}^A$  form an enriched natural retraction (resp. isomorphism) from  $\text{Met}^A(A_{w*i}; A_j)$  to  $\text{Met}^A(A_w; A_{i \rightarrow j})$ . Notice that this condition implies that the pre-metrics  $\Xi_{d_w^A, d_i^A}$  are indeed metrics.

► **Example 33.**  $\mathcal{I}$  becomes a quantitative  $\lambda$ -algebra by defining  $\Lambda^{\mathcal{I}}$  inductively on  $\mathbb{F}_{\lambda, \mathcal{I}}(\text{Var})^0$ , exploiting the cartesian closed structure of the subcategory of  $\text{Met}$  formed by the spaces  $\mathcal{I}_i$ .

Let us now show how quantitative  $\lambda$ -algebras are captured by quantitative  $\lambda$ -theories.

► **Definition 34.** Let  $\mathcal{A} = (A, \Omega^A, \Lambda^A, d^A)$  be a quantitative applicative  $\lambda$ -algebra. For any  $\rho : \text{Var} \rightarrow A$ , we say that  $\mathcal{A}$  satisfies a quantitative equation  $\phi = t \stackrel{X, i}{\simeq_\epsilon} u$  relative to  $\rho$ , (denoted  $\vDash_{\mathcal{A}}^\rho \phi$ ), where  $X = \{x_1, \dots, x_n\}$ , with  $x_1 \in A_{i_1}, \dots, x_n \in A_{i_n}$ , when for all  $a_1, b_1 \in A_{i_1}, \dots, a_n, b_n \in A_{i_n}$ , the following condition holds:

$$d_i^A(\rho_{\vec{x}, \vec{a}}^\natural(t), \rho_{\vec{x}, \vec{b}}^\natural(u)) \leq \max\{\epsilon, d_{i_1}^A(a_1, b_1), \dots, d_{i_n}^A(a_n, b_n)\}$$

We say that  $\mathcal{A}$  satisfies a quantitative  $\lambda$ -inference  $\Gamma \vdash \phi$  (denoted  $\Gamma \vDash_{\mathcal{A}} \phi$ ) if for all  $\rho : \text{Var} \rightarrow A$ , if  $\vDash_{\mathcal{A}}^\rho \psi$  holds for all  $\psi \in \Gamma$ , then  $\vDash_{\mathcal{A}}^\rho \phi$  also holds.  $\mathcal{A}$  satisfies a quantitative  $\lambda$ -theory  $\mathcal{U}$  (denoted  $\mathcal{A} \vDash \mathcal{U}$ ) if it satisfies all the inferences in  $\mathcal{U}$ .

Definition 34 of satisfiability is admittedly more complex than Definition 12. Yet, this is the price one has to pay in order to be able to express the quantitative  $\xi$ -rule. Indeed, the definition of  $\vDash_{\mathcal{A}}^\rho \phi$  treats “locally quantified” variables by applying a condition reminiscent of the metrics  $\Xi$  from Section 2: for all locally quantified variables  $\vec{x}$  in  $\phi = t \stackrel{X, i}{\simeq_\epsilon} u$ , when the  $\vec{x}$  are replaced in  $t$  and  $u$  by *different* points  $\vec{a}, \vec{b}$ , the distance between the resulting terms must be bounded by either  $\epsilon$  or any of the  $d_{i_j}^A(a_j, b_j)$ . This ensures that, whenever  $\vDash_{\mathcal{A}} t \stackrel{\{x\}, j}{\simeq_\epsilon} u$  is satisfied, we can conclude  $\Xi(\lambda x.t, \lambda x.u) \leq \epsilon$ , as the  $\xi$ -rule requires.

► **Example 35.** Def. 34 solves the problem from Example 24: with  $r = 0$  and  $s = \epsilon$ , from the fact that  $|f(s) - g(r)| = 2\epsilon > \max\{|r - s|, \epsilon\}$ , it follows that  $\mathcal{I} \not\vDash \bar{f}x \stackrel{X, j}{\simeq_\epsilon} \bar{g}x$ , hence blocking the counter-example to the  $\xi$ -rule. Rather, it holds that  $\mathcal{I} \vDash \mathcal{U}_{\lambda\eta}^{\mathcal{I}}$  (cf. Example 28).

► **Proposition 36.** A quantitative applicative  $\lambda$ -algebra is a quantitative  $\lambda$ -algebra (resp. a quantitative extensional  $\lambda$ -algebra) iff it satisfies  $U_\lambda$  (resp.  $U_{\lambda\eta}$ ).

From the argument of the proposition above one can also deduce that a quantitative applicative  $\lambda$ -algebra is a weak  $\lambda$ -algebra iff it satisfies  $(\alpha)$  and  $(\beta)$ , and is an (extensional)  $\lambda$ -algebra iff it furthermore satisfies  $(\xi)$  (and  $(\eta)$ ).

We conclude this section by showing soundness and completeness of quantitative (extensional)  $\lambda$ -theories. The proof is based on the construction of a “quantitative term model”.

► **Theorem 37** (Soundness and Completeness of Quantitative  $\lambda$ -theories). Let  $\mathcal{U}$  be a quantitative  $\lambda$ -theory (resp. a quantitative extensional  $\lambda$ -theory) over  $\Sigma^\lambda$ . Then  $\Gamma \vdash^\lambda \phi \in \mathcal{U}$  (resp.  $\Gamma \vdash^{\lambda\eta} \phi \in \mathcal{U}$ ) iff  $\Gamma \vDash_{\mathcal{A}} \phi$  holds for any quantitative  $\Sigma^\lambda$ -algebra (resp. quantitative extensional  $\Sigma^\lambda$ -algebra)  $\mathcal{A}$  such that  $\mathcal{A} \vDash \mathcal{U}$ .

► **Remark 38.** Also in this case the whole construction scales to the case of partial ultra-metric spaces. Following Remark 23, we will speak of *partial  $\lambda$ -theories* and *partial  $\lambda$ -algebras*.

## 6 Metric Constraints

In this section we take a closer look at the several obstacles one might face when looking for higher-order quantitative algebras. First, as seen in Section 2, in higher-order types the *unique* distance,  $\Xi$ , making both application and abstraction non-expansive operations might not be a metric. Moreover, even if such a metric exists, several conditions might lead higher-order distances to be trivial (i.e. discrete), or have plenty of isolated points. But discrete metrics and isolated points convey no more information than equivalences, while one of the main reasons to look for semantics of program distances is to be able to compare informatively programs which are not equivalent. Despite what look like strong limitations, we conclude this section by presenting a few examples of non-discrete quantitative  $\lambda$ -algebras.

**Existence of Exponential Objects.** Given metric spaces  $(X, a)$  and  $(Y, b)$ , if  $(Y, b)$  is ultra-metric, then  $\Xi_{a,b} = \Phi_{a,b}$  is always a metric, which means that  $\text{Met}(X, Y)$  is their exponential object in  $\text{Met}$ . When  $(Y, b)$  is not ultra-metric, condition  $(*)$  from Theorem 2 provides a useful *sufficient* criterion to check if  $\Xi_{a,b}$  is a metric (and thus, if some candidate quantitative applicative  $\lambda$ -algebra  $\mathcal{A}$  is a quantitative  $\lambda$ -algebra). We will now show that, under very mild hypotheses, the validity of  $(*)$  is also *necessary* for  $\mathcal{A}$  to be a quantitative  $\lambda$ -algebra.

Let a quantitative applicative  $\lambda$ -algebra  $\mathcal{A}$  be *observationally complete* when it contains the metric space  $(\mathbb{R}_{\geq 0}^{\infty}, |\cdot - \cdot|)$  and for all sort  $i$ ,  $\text{Met}^{\mathcal{A}}(A_i; \mathbb{R}_{\geq 0}^{\infty}) \simeq \text{Met}(A_i, \mathbb{R}_{\geq 0}^{\infty})$ . In other words,  $\mathcal{A}$  contains *all* observations on  $A_i$  with target  $\mathbb{R}_{\geq 0}^{\infty}$ . Moreover, let a *quantitative  $\lambda$ -pre-algebra* be as a quantitative  $\lambda$ -algebra  $\mathcal{A}$ , but where the  $d_i^{\mathcal{A}}$  need only be pre-metrics. Given a quantitative  $\lambda$ -pre-algebra  $\mathcal{A}$ , let  $\mathcal{A}^*$  indicate the restriction of  $\mathcal{A}$  to those sorts  $i$  for which  $d_i^{\mathcal{A}}$  is a metric (i.e. it also satisfies (trans)).

► **Proposition 39.** *Let  $\mathcal{A}$  be an observationally complete quantitative extensional  $\lambda$ -pre-algebra. For any  $A$  in  $\mathcal{A}^*$ ,  $A$  is exponentiable in  $\text{Met}^{\mathcal{A}^*}$  iff for all  $\alpha, \beta \in \text{Im}(d^{\mathcal{A}})$  and  $x_0, x_2 \in X$  with  $a(x_0, x_2) = \alpha + \beta$ , condition  $(*)$  holds.*

Proposition 39 has a positive side: it provides a sufficient condition for exponentiability which is slightly weaker than Theorem 2, as  $(*)$  needs only hold for distances  $\alpha, \beta$  in the *image* of the distance functions  $d^{\mathcal{A}}$  of the pre-algebra. Notice that, if the  $d^{\mathcal{A}}$  are discrete, condition  $(*)$  trivially holds. On the other hand, Proposition 39 has a negative side: if condition  $(*)$  fails (i.e. some space  $A$  does not contain “enough points”), then  $\mathcal{A}$  fails to be a quantitative  $\lambda$ -algebra. For instance, no algebra containing  $\mathbb{N}$ , with the metric inherited from  $\mathbb{R}$ , as one of its objects, can be a  $\lambda$ -algebra.

**Existence of Compact Algebras.** We have the following negative result.

► **Proposition 40.** *There are no non-trivial one-sorted weak quantitative  $\lambda$ -algebras in  $\text{Met}$  which are compact.*

By contrast, in the multi-sorted case, compact  $\lambda$ -algebras do exist, e.g. take the restriction of the quantitative  $\lambda$ -algebra  $\mathcal{I}$  to compact intervals  $[a, b]$ , i.e. with  $a, b < \infty$ , or simply the full type structure on a finite base set.

**Distances and Observational Equivalence.** The next two results relate distances in quantitative  $\lambda$ -theories with observational equivalence for the associated  $\lambda$ -theory, clearly indicating the (limited) extent to which a metric can deviate from being discrete on *pure* closed  $\lambda$ -terms. We recall that *pure* means that no constants appear in the syntax, or categorically, that the  $\Sigma^{\lambda}$ -signature has only the  $\cdot$  symbols.

► **Proposition 41.** *In a quantitative  $\lambda$ -algebra  $A$ , i.e. a model of the simply typed  $\lambda$ -calculus, terms which are not equated in the maximal theory are all at the same distance from one another. Moreover each  $A_i$  is a bounded pseudo-metric space.*

The maximal non-trivial theory of the *pure* simply typed  $\lambda$ -calculus is the theory *FTS* of the *full type structure* over a two-element base set [7]. Proposition 41 implies then that any quantitative  $\lambda$ -algebra for *FTS* is discrete. We recall that “pure” means that no constants appear in the syntax. Next, we consider the untyped  $\lambda$ -calculus:

► **Proposition 42.** *In a non-trivial weak quantitative  $\lambda$ -algebra  $A$ , the maximal distance between any two points is bounded by  $d(\llbracket K \rrbracket, \llbracket K(\text{SKK}) \rrbracket)$ . Hence all pairs of terms which can be applied, by a given term, on  $\llbracket K \rrbracket$  and  $\llbracket K(\text{SKK}) \rrbracket$  respectively, are that distance apart. Moreover, if  $A$  is a non-trivial quantitative  $\lambda$ -algebra then for any two solvable terms,  $t$  and  $s$ , which are not equated in the maximal theory  $\mathcal{H}^*$  (see [5]) and  $Y$ , fixed-point combinator we have  $d(\llbracket t \rrbracket, YK) = d(\llbracket s \rrbracket, YK)$ . If the distance is ultra-metric we have also  $d(\llbracket t \rrbracket, YK) = d(\llbracket t \rrbracket, \llbracket s \rrbracket)$ . In any case, if the theory equates all unsolvable terms then  $d(\llbracket t \rrbracket, YK) \leq d(\llbracket t \rrbracket, \llbracket s \rrbracket)$ .*

As a consequence of Böhm Theorem (see [5]), Proposition 42 implies that any quantitative  $\lambda$ -algebra for the *pure* untyped  $\lambda$ -calculus is discrete over  $\beta\eta$ -normal forms.

**Positive Examples.** The above limiting results apply only to terms which are not equated in the maximal theories of the  $\lambda$ -calculus, either typed or untyped ([5, 7]). Clearly these terms are significant computationally, and this is the bad news, but these terms are rather special and hence Propositions 41 and 42 have only a limited negative impact, and this is the good news. For instance, in the maximal theory of the simply typed  $\lambda$ -calculus Church, numerals are equated up to parity, so Proposition 41 does not have any bearing on the mutual distance of two different even, or two different odd, numerals. Indeed, rather intriguing distances in quantitative  $\lambda$ -algebras do exist, even in the category of complete (not necessarily ultra-) metric spaces and non-expansive functions, as the following examples show.

Any complete partial order model of Combinatory Logic, and hence in particular of  $\lambda$ -calculus (e.g. any Scott’s inverse limit  $D_\infty$  model, [5]), can be endowed with the metric

$$d(d_1, d_2) = \begin{cases} 0 & \text{if } d_1 = d_2 \\ 1/2 & \text{if } d_1 \text{ and } d_2 \text{ have an upper bound} \\ 1 & \text{otherwise .} \end{cases}$$

One can check that application is non-expansive, and that the space is complete; moreover the space of representable functions (*i.e.* functions determined by the elements of the model), endowed with the supremum metric, is isometrically embedded in the space. Alternatively, one can consider the term model of the simply typed  $\lambda$ -calculus with a base constant  $\perp$ . By strong normalization, it consists of the  $\beta\eta$ -normal forms. Let  $\sqsubseteq$  be the order relation defined on normal forms of the same type by  $\lambda\vec{x}. \perp \sqsubseteq \lambda\vec{x}.t$  and  $\lambda\vec{x}.x_i t_1 \dots t_k \sqsubseteq \lambda\vec{x}.x_i u_1 \dots u_k$ , if  $t_i \sqsubseteq u_i$  for all  $i = 1, \dots, k$  (corresponding to the natural order relation on Böhm trees, see [5]). The set of  $\beta\eta$ -normal forms can be endowed with a notion of distance by putting, for all type  $\sigma \in T$  and  $t, u$  terms of type  $\sigma$ ,  $d_\sigma(t, u)$  be 0 if  $t = u$ , 1/2 if  $t$  and  $u$  have an upper bound, and 1 otherwise.

Yet other distances can be given on the term model of the simply typed  $\lambda$ -calculus by putting  $d_\sigma(t, u)$  be 0 if  $t =_{\beta\eta} u$  and otherwise  $1/N$ , where  $N = \max\{n \mid \llbracket t \rrbracket = \llbracket u \rrbracket \text{ in the full type hierarchy over } n \text{ points}\}$ .

## 7 Partial Quantitative $\lambda$ -Algebras

In this section we discuss *partial metrics*, and the natural generalization of quantitative  $\lambda$ -algebras to *partial quantitative  $\lambda$ -algebras*. In particular we define two non-trivial such algebras for the simply typed  $\lambda$ -calculus. The first  $\lambda$ -algebra that we consider is defined on the term model of  $\beta\eta$ -normal forms of the simply typed  $\lambda$ -calculus with a constant  $\perp$  of base type. The latter is defined within a  $D_\infty$   $\lambda$ -model *à la* Scott. In both cases we define an ultra-metric distance using a suitable notion of *term approximants*.

**The Partial  $\lambda$ -Algebra of the Term Model.** Let  $T$  be the set of simple types built over the base type  $o$ , and let  $\sigma, \tau$  range over  $T$ . The  $\beta\eta$ -normal forms of the simply typed  $\lambda$ -calculus with constant  $\perp$  of type  $o$  can be endowed with a structure of applicative  $\lambda$ -algebra:

► **Proposition 43.** *Let  $\mathcal{NF} = (NF, \Omega^{NF}, \Lambda^{NF})$  be the structure where:*

- *$NF$  is the  $T$ -indexed set of typed  $\beta\eta$ -normal forms with constant  $\perp$  of type  $o$ ,*
  - *for all  $\sigma, \tau, \cdot_{\sigma, \tau} : NF_{\sigma \rightarrow \tau} \times NF_\sigma \rightarrow NF_\tau$  is defined by  $t \cdot_{\sigma, \tau} s = [ts]_{\beta\eta}$ ,*
  - *$\Lambda_{\sigma, \tau}^{NF} : \Lambda_{\sigma \rightarrow \tau}^0 \rightarrow NF_{\sigma \rightarrow \tau}$  is defined by:  $\Lambda_{\sigma, \tau}^{NF}(\lambda x.t) = [\lambda x.t]_{\beta\eta}$ ,*
- where  $[t]_{\beta\eta}$  denotes the  $\beta\eta$ -normal form of  $t$ . Then  $\mathcal{NF}$  is an applicative  $\lambda$ -algebra.

The signature of this algebra can be enriched with *projection operators* providing the approximants of a given normal form. Intuitively, the  $n^{\text{th}}$  approximant of a normal form is the term whose Böhm tree [5] is obtained by cutting all branches at depth  $n$  and by labelling leaves at level  $n$  of type  $\sigma_1 \rightarrow \dots \rightarrow \sigma_m \rightarrow o$  by the term  $\lambda x_1 \dots x_m. \perp$ . More precisely:

► **Definition 44.** *For all  $\sigma \in T$  and for all  $n \in \mathbb{N}$ ,  $\pi_\sigma^n : NF_\sigma \rightarrow NF_\sigma$  is defined by induction on  $n$  as follows: for all  $\lambda x_1 \dots x_m. x_i t_1 \dots t_k \in NF_\sigma$ ,  $\pi_0^\sigma(t) = \lambda x_1 \dots x_m. \perp$  and  $\pi_{n+1}^\sigma(t) = \lambda x_1 \dots x_m. x_i (t_1)_n \dots (t_k)_n$ .*

In the sequel, we will denote the approximant  $\pi_\sigma^n(t)$  simply by  $t_n$ .

► **Definition 45 (Distance on Normal Forms).** *We define a family of functions  $d^{NF} = \{d_\sigma^{NF}\}_\sigma$ , where  $d_\sigma^{NF} : NF_\sigma \times NF_\sigma \rightarrow \mathbb{R}_{\geq 0}$  is defined inductively by*

$$d_o^{NF}(t, s) = \begin{cases} 0 & \text{if } t = s \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad d_{\sigma \rightarrow \tau}^{NF}(t, t') = \frac{1}{2^m}$$

where  $m$  is the largest  $n \in \mathbb{N}$ , if it exists, such that

- (1)  $t_n = t'_n$ ,
  - (2)  $\forall s, s' \in NF_\sigma. (d_\sigma^{NF}(s, s') \leq \frac{1}{2^n} \implies d_\tau^{NF}(ts, ts') \leq \frac{1}{2^n})$ ;
- if such a maximal  $n$  does not exist, then  $d_{\sigma \rightarrow \tau}^{NF}(t, t')$  is set to 0.

► **Lemma 46.** *For all  $\sigma, \tau$ ,  $(NF_\sigma, d_\sigma^{NF})$  is a partial ultra-metric space and  $\cdot_{\sigma, \tau}$  is non-expansive.*

The fact that  $\cdot_{\sigma, \tau}$  is non-expansive follows immediately from the definition of  $d_\sigma^{NF}$ .

► **Remark 47.** Notice that  $d^{NF}$  is not reflexive. *E.g.*, let  $u = \lambda x.xI$  of appropriate type  $\sigma_1 \rightarrow \sigma_2$ ,  $t = \lambda x.x(xt')$  and  $s = \lambda x.x(xs')$  of type  $\sigma_1$ , and  $t', s'$  of appropriate type  $\tau$  such that  $d_\tau^{NF}(t', s') = 1$ . Then  $d_{\sigma_1}^{NF}(t, s) = \frac{1}{2}$ , but  $d_{\sigma_2}(ut, us) = 1$ , *i.e.*  $d_{\sigma_1 \rightarrow \sigma_2}(u, u) = 1$ .

From the definition of  $d^{NF}$ , it immediately follows that application on normal forms is a non-expansive operator. Hence we have:

► **Proposition 48.**  $\mathcal{NF} = (NF, \Omega^{NF}, \Lambda^{NF}, d^{NF})$  is a partial quantitative extensional  $\lambda$ -algebra.

► **Remark 49.** If we drop condition 2 in Definition 45, then we get an ultra-metric (reflexivity holds), however application is expansive. Namely, let  $t = \lambda x_1 x_2. x_1(x_2 t')$  and  $s = \lambda x_1 x_2. x_1(x_2 s')$  of appropriate types  $\sigma_1$  such that  $d_{\sigma'}^{NF}(t', s') = 1$ . Then  $d_{\sigma_1}^{NF}(t, s) = 1/4$ , but for  $u = \lambda x. xII$  of type  $\sigma_1 \rightarrow \sigma_2$ , we get  $d_{\sigma_2}^{NF}(ut, us) = 1$ . Notice that the above terms can be taken to be affine (similar counterexamples can be built also in the purely linear case).

**The Partial  $\lambda$ -Algebra of  $D_\infty$ .** Any inverse limit domain model *à la* Scott of  $\lambda$ -calculus yields an applicative  $\lambda$ -algebra. On such models a notion of approximant naturally arises, by considering for any given element of the domain its projections on the domains of the inverse limit construction. This leads to the following definition:

► **Definition 50.**

- (i) Let  $D_\infty = \bigsqcup_n D_n$  be an inverse limit domain model *à la* Scott. For all  $a \in D_\infty$  we define the  $n^{\text{th}}$  approximant of  $a$ ,  $a_n$ , as the projection of  $a$  into  $D_n$ .
- (ii) Let  $D$  be the  $T$ -indexed set  $\{D_\sigma\}_\sigma$ , where, for all  $\sigma$ ,  $D_\sigma = D_\infty$ .
- (iii) Let  $d_\sigma^D : D_\sigma \times D_\sigma \rightarrow \mathbb{R}_{\geq 0}$  be the distance function defined by induction on types by

$$d_\sigma^D(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad d_{\sigma \rightarrow \tau}^D(a, b) = \frac{1}{2^m}$$

where  $m$  is the maximal  $n \in \mathbb{N}$ , if it exists, such that

- (1)  $a_n = b_n$ ,
  - (2)  $\forall c, d \in D_\sigma. (d_\sigma^D(c, d) \leq \frac{1}{2^n} \implies d_\tau^D(ac, bd) \leq \frac{1}{2^n})$ ;
- if such a maximal  $n$  does not exist, then  $d_{\sigma \rightarrow \tau}^D(a, b) = 0$ .

► **Lemma 51.** For all  $\sigma, \tau$ ,  $(D_\sigma, d_\sigma^D)$  is a partial ultra-metric space and  $\cdot_{\sigma, \tau}$  is non-expansive.

► **Proposition 52.** Let  $\mathcal{D} = (D, \Omega^D, \Lambda^D, d^D)$  be the structure where the functions  $\Lambda_{\sigma, \tau}^D$  are the interpretations of closed typed  $\lambda$ -terms on  $D_\infty$ . Then  $\mathcal{D}$  is a partial quantitative extensional  $\lambda$ -algebra.

**Partial  $\lambda$ -Algebras with Approximants.** The two examples above of partial quantitative  $\lambda$ -algebras can be viewed as special cases of a general construction, which can be carried out on any applicative algebra which includes projection operators. Namely, using the system of approximants given by the projection operators, one can endow the algebra with an ultra-metric, getting a quantitative applicative algebra. If moreover the algebra satisfies the  $(\beta)$ -rule, then it is a quantitative (weak)  $\lambda$ -algebra.

► **Definition 53** (Applicative Algebra with Approximants). An applicative algebra  $\mathcal{A} = (A, \Omega^A)$  has approximants if it includes projection operators  $\pi_\sigma^n : A_\sigma \rightarrow A_\sigma$ , for all  $\sigma \in T$  and  $n \in \mathbb{N}$ , satisfying the following property: for all  $n \in \mathbb{N}$ , for all  $a, b \in A_\sigma$ ,  $\pi_\sigma^{n+1}(a) = \pi_\sigma^{n+1}(b)$  implies  $\pi_\sigma^n(a) = \pi_\sigma^n(b)$ .

► **Proposition 54.** Let  $\mathcal{A} = (A, \Omega^A)$  be an applicative  $\lambda$ -algebra with approximants. Let  $d_\sigma^A : A_\sigma \times A_\sigma \rightarrow \mathbb{R}_{\geq 0}$  be the family of functions defined by induction on types as follows:

$$d_\sigma^A(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad d_{\sigma \rightarrow \tau}^A(a, b) = \frac{1}{2^m}$$

where  $m$  is the maximal  $n \in \mathbb{N}$ , if it exists, such that

- (1)  $\pi_{\sigma \rightarrow \tau}^n(a) = \pi_{\sigma \rightarrow \tau}^n(b)$ ,  
 (2)  $\forall c, d \in A_\sigma. (d_\sigma^A(c, d) \leq \frac{1}{2^n} \implies d_\tau^A(ac, bd) \leq \frac{1}{2^n})$ ;  
 if the maximal  $n$  does not exist, then  $d_{\sigma \rightarrow \tau}^A(a, b) = 0$ . Then  $(A, \Omega^A, d^A)$  is a quantitative applicative algebra.

## 8 Approximate Quantitative Algebras

As we have seen, finding non-discrete quantitative (weak)  $\lambda$ -algebras is difficult. One difficulty arises from the non-expansiveness requirement on application. In Section 7 we have shown how to define non-trivial ultra-metric quantitative  $\lambda$ -algebras, still maintaining the non-expansiveness requirement for application, but at the price of the partiality of the metric. Here we present a different approach: we relax rule (NExp) for application, so as to get quantitative  $\lambda$ -algebras with full pseudo-metric distances. Namely, we introduce the notion of *approximate applicative algebra*: this amounts to an applicative algebra with approximants (see Definition 53 above), and operators  $\{\cdot^n\}_{n \in \mathbf{N}}$  approximating application. Projection operators immediately induce an ultra-metric on the algebra, by just considering condition 1 in Proposition 54 (and dropping condition 2). In general, application is expansive w.r.t. this metric (see Remark 49). However, the milder *uniform non-expansiveness* condition for approximant operators is satisfied in many cases, including the term algebra of normal forms and the  $D_\infty$  model of Section 7. This approach is quite general, since it works both for the typed and the untyped  $\lambda$ -calculus.

► **Lemma 55.** *Let  $\mathcal{A} = (A, \Omega^A)$  be an applicative algebra with approximants. Let  $e_\sigma^A : A_\sigma \times A_\sigma \rightarrow \mathbb{R}_{\geq 0}$  be the family of functions defined by:  $e_\sigma^A(a, b) = \frac{1}{2^m}$ , where  $m$  is the maximal  $n \in \mathbf{N}$ , if it exists, such that  $a_n = b_n$ , otherwise we put  $e_\sigma^A(a, b) = 0$ . Then for all  $\sigma$   $(A, e_\sigma^A)$  is an ultra-metric space.*

► **Definition 56** (Approximate Quantitative Algebra).

- (i) An approximate algebra  $\mathcal{A} = (A, \Omega^A)$  is an applicative algebra with approximants whose signature includes also a family of operators  $\cdot_{\sigma, \tau}^n : A_{\sigma \rightarrow \tau} \times A_\sigma \rightarrow A_\tau$ , for all  $\sigma, \tau \in T$  and  $n \in \mathbf{N}$  (the operators  $\cdot_{\sigma, \tau}^n$  will be simply denoted by  $\cdot^n$ ).
- (ii) An approximate quantitative algebra  $\mathcal{A} = (A, \Omega^A, e^A)$  is an approximate algebra where the operators  $\cdot_{\sigma, \tau}^n$  satisfy the following conditions:
- (1) for all  $a \in A_{\sigma \rightarrow \tau}$ ,  $b \in A_\sigma$ ,  $n \in \mathbf{N}$ ,  $e_\tau^A(a \cdot^{n+1} b, a \cdot b) \leq e_\tau^A(a \cdot^n b, a \cdot b)$ ;
  - (2) for all  $a \in A_{\sigma \rightarrow \tau}$ ,  $b \in A_\sigma$ , for all  $\epsilon > 0$  there exists  $n \in \mathbf{N}$  s.t.  $e_\sigma^A(a \cdot^n b, a \cdot b) \leq \epsilon$ ;
  - (3) (uniform non-expansiveness)  $\forall n > 0 \exists \epsilon_n > 0$  s.t.  $\forall \epsilon \leq \epsilon_n, \forall a, b \in A_{\sigma \rightarrow \tau}, \forall c, d \in A_\sigma$ ,  $e_{\sigma \rightarrow \tau}^A(a, b) \leq \epsilon$  and  $e_\sigma^A(c, d) \leq \epsilon$  implies  $e_\sigma^A(a \cdot^n c, b \cdot^n d) \leq \epsilon$ .

Conditions 1 and 2 above express the fact that the operators  $\cdot^n$  approximate the behaviour of application; condition 3 replaces rule (NExp) for application.

**The Approximate  $\lambda$ -Algebra of the Term Model.** The  $\lambda$ -algebra  $\mathcal{NF}$  can be extended to an approximate quantitative  $\lambda$ -algebra by defining operators  $\cdot_{\sigma, \tau}^n$  as follows: for all  $t \in NF_{\sigma \rightarrow \tau}, s \in NF_\sigma$ , for all  $n \in \mathbf{N}$ ,  $t \cdot_{\sigma, \tau}^n s = t_n \cdot_{\sigma, \tau} s_n$ . One can check that the approximant operators satisfy all conditions of Definition 56.

**The Approximate  $\lambda$ -Algebra of  $D_\infty$ .** The  $\lambda$ -algebra  $\mathcal{D}$  can be extended to an approximate quantitative  $\lambda$ -algebra by defining operators  $\cdot_{\sigma,\tau}^n$  as follows: for all  $a \in D_{\sigma \rightarrow \tau}, b \in D_\sigma$ , for all  $n \in \mathbb{N}$ ,  $a \cdot_{\sigma,\tau}^n b = a_{n+1} \cdot_{\sigma,\tau} b$ . One can check that the approximant operators satisfy all conditions of Definition 56. Notice that the approximate algebra of  $D_\infty$  yields a  $\lambda$ -algebra for the untyped  $\lambda$ -calculus.

Finally, notice that in dealing with partial and approximate algebras we have considered applicative algebras over an extended signature. For lack of space, we have not developed corresponding approximate theories including extra operators and the suitable rules on them. In particular, rule (NExp) has to be replaced by a rule expressing uniform non-expansiveness of approximant operators. We leave this as future work; here we just observe that the appropriate language for reasoning on such structures would be the *indexed  $\lambda$ -calculus* together with *indexed reduction*, see [5].

## 9 Conclusions

**Contributions.** This paper addresses the problem of defining quantitative algebras, in the sense of Mardare *et al.*, capable of interpreting terms of higher-order calculi. Our contributions include both negative and positive results: on the one hand we identify the main mathematical obstacles to the construction of non-trivial quantitative higher-order algebras; on the other hand we introduce quantitative variants of the traditional notions of (weak)  $\lambda$ -algebras, together with a sound and complete syntax, and we show that, in spite of the limitations highlighted, intriguing notions of distance for the  $\lambda$ -calculus do indeed exist.

**Related Work.** Since [2], metric spaces have been exploited as an alternative, quantitative, framework to standard, domain-theoretic, denotational semantics [35, 4]. The possibility of giving a metric structure to *linear* or *affine* higher-order programs is known, since **Met** is an SMCC, even if not a CCC. In this sense it is worth recalling the work by de Amorim *et al.* [3], along with those of Reed and Pierce [33], as well as recent work by Dahlqvist and Neves [16]. Moreover, ultra-metrics have already been used to model PCF [21]. More recently, Pistone has given a precise account of cartesian closed structure in categories of generalized metric spaces [32]. In particular, it is known that if the quantale that captures distances can vary as the types vary, as for example in the so-called differential logical relations [17], categories of generalized metric spaces can become cartesian closed. The study of metric semantics for imperative and concurrent programming languages has a long tradition [19, 18]. However, this very sophisticated apparatus is not applicable to higher order programming languages.

Partial metrics have been well-studied since [9] and [26] (where they are called *M*-sets). [10] shows that these metrics are strongly related to Scott semantics. The setting of *quantaloid-enriched* categories [25, 34] provides an abstract unifying framework for the different metric structures discussed here. In this setting, [13, 14] provide a general characterization of exponentiable morphisms and objects in categories of (generalized) metric spaces.

Finally, a somehow related approach to quantitative reasoning is provided by the use of fuzzy logic to reason about degrees of similarity between programs, as spelled out in Zadeh's pioneering work [36, 37]. More recently, fuzzy algebraic theories in the style of Mardare *et al.* have been studied [12]. However, such theories seem to lack a compositionality condition comparable to the one expressed by axiom (Nexp), hence apparently diverging from the idea of interpreting programs as non-expansive functions.



**Perspectives.** Our focus here was on the simplest case, namely that of algebras *without* a barycentric structure, thus putting ourselves in a simpler setting than the one studied by Mardare *et al.* Indeed, a natural development of this work is to study quantitative algebras for  $\lambda$ -calculi enriched with operations having an intrinsically quantitative content, like e.g. probabilistic choice [15] or some form of differentiation [20, 11, 28]. Another direction to explore, already suggested by some of our models, is that of exploring quantitative algebras in categories of domains like, e.g. metric CPOs [3], or continuous Scott domains (especially in virtue of their close connection with partial metric spaces [10]).

Finally, the approaches of partial and approximate algebras open new lines of investigation: suitable approximate theories are called for, and moreover the distances on programs which arise are worth to be studied in depth.

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