Polynomial Termination Over ℕ Is Undecidable

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Abstract

In this paper we prove via a reduction from Hilbert's 10th problem that the problem whether the termination of a given rewrite system can be shown by a polynomial interpretation in the natural numbers is undecidable, even for rewrite systems that are incrementally polynomially terminating. We also prove that incremental polynomial termination is an undecidable property of terminating rewrite systems.

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1 Introduction

Proving termination of a rewrite system by using a polynomial interpretation over the natural numbers goes back to Lankford [10]. Two problems need to be addressed when using polynomial interpretations for proving termination, whether by hand or by a tool:

1. finding suitable polynomials for the function symbols,
2. showing that the induced order constraints on polynomials are valid.

Heuristics for the former problem are presented in [3, 18]. The latter problem amounts to ($\star$) proving $P(x_1, \dots, x_n) > 0$ for all natural numbers $x_1, \dots, x_n \in \mathbb{N}$, for polynomials $P \in \mathbb{Z}[x_1, \dots, x_n]$. This is known to be undecidable, as an easy consequence of Hilbert's 10th Problem, see e.g., Zantema [18, Proposition 6.2.11]. However, from the undecidability of problem 2 it does not immediately follow that (dis)proving polynomial termination is undecidable, since a decision procedure for problem 1 may exist which only produces decidable instances for problem 2.

In this paper we show that this is not the case, by proving the undecidability of the existence of a direct termination proof by a polynomial interpretation in ℕ by a reduction from ($\star$). This result is not surprising, but we are not aware of a proof of undecidability in the literature, and the construction is not entirely obvious. We strengthen the undecidability result to rewrite systems that can be shown terminating by an incremental polynomial interpretation in ℕ, where rules are not oriented all at once, but in stages (called *lexicographic combinations* in [18, Section 6.2.4]). We further show that the existence of an incremental polynomial termination proof is undecidable for terminating rewrite systems.

In the next section we recall the definitions of (incremental) polynomial termination over ℕ. In Section 3 we present the variations of Hilbert's 10th problem that we use to obtain our undecidability results. The latter are presented in detail in the subsequent three sections. The undecidability result in Section 4 was first announced at the *International Workshop on Termination* in 2021 [14]. We conclude in Section 7 with suggestions for future work.

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2 Preliminaries

We assume familiarity with term rewriting [2], but recall the definition of (incremental) polynomial termination over $\mathbb{N}$. Given a signature $\mathcal{F}$, a well-founded monotone $\mathcal{F}$-algebra $(\mathcal{A}, >)$ consists of a non-empty $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ and a well-founded order $>$ on the carrier A of $\mathcal{A}$ such that every algebra operation is strictly monotone in all its arguments, i.e., if $f \in \mathcal{F}$ has arity $n \geqslant 1$ then $f_{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) > f_{\mathcal{A}}(a_1, \ldots, b, \ldots, a_n)$ for all $a_1, \ldots, a_n, b \in A$ and $i \in \{1, \ldots, n\}$ with $a_i > b$. The induced order $>_{\mathcal{A}}$ on terms is a reduction order that ensures the termination of any compatible (i.e., $\ell >_{\mathcal{A}} r$ for all rewrite rules $\ell \to r$) term rewrite system (TRS for short) $\mathcal{R}$. We call $\mathcal{R}$ *polynomially terminating over* $\mathbb{N}$ if compatibility holds when the underlying algebra $\mathcal{A}$ is restricted to the set of natural numbers $\mathbb{N}$ with standard order $>_{\mathbb{N}}$ such that every n-ary function symbol f is interpreted as a monotone polynomial $f_{\mathbb{N}}$ in $\mathbb{Z}[x_1, \ldots, x_n]$ with $f_{\mathbb{N}}(0, \ldots, 0) \geqslant 0$. The latter condition is needed for $f_{\mathbb{N}}$ to be well-defined over $\mathbb{N}$. We use $\mathbb{N}_+$ to denote $\mathbb{N} \setminus \{0\}$.

Whereas well-founded monotone algebras are complete for termination, polynomial termination gives rise to a much more restricted class of TRSs. For instance, Hofbauer and Lautemann [8] proved that polynomially terminating TRSs induce a double-exponential upper bound on the derivational complexity. Polynomial interpretations can be used in an incremental fashion, extending their termination proving power. The idea is that in a first step a polynomial interpretation is used that orients all rewrite rules of a given TRS $\mathcal{R}$ weakly and at least one rule strictly. After removing the rules that are strictly oriented, the process is repeated. (This is of course not specific to polynomial interpretations and more generally known as proving termination via relative termination [6, Chapter 3.2].) When no rule remains, the incremental termination proof succeeds. In this case, $\mathcal{R}$ is called *incremental polynomially terminating over* $\mathbb{N}$. The following example is from [18, Example 6.2.21].

▶ **Example 1.** Consider the TRS $\mathcal{R}$ consisting of the rewrite rules

$$\mathsf{0} + y \to y \qquad \mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{0} \times y \to \mathsf{0} \qquad \mathsf{s}(x) \times y \to (x \times y) + y$$

The polynomial interpretation

$$\mathsf{0}_{\mathbb{N}} = 0 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 2 \qquad +_{\mathbb{N}}(x, y) = x + y + 2 \qquad \times_{\mathbb{N}}(x, y) = xy + 2x + 2y + 2$$

gives rise to the following order constraints on $\mathbb{N}$:

$$y + 2 > y \quad x + y + 4 = x + y + 4 \quad 2y + 2 > 0 \quad xy + 2x + 4y + 6 > xy + 2x + 3y + 4$$

So three of the four rules are oriented strictly. The exception is the rule $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$, which is turned into an equality. Changing the interpretation to

$$\mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad\qquad +_{\mathbb{N}}(x, y) = 2x + y$$

orients this rule strictly. Hence $\mathcal{R}$ is incremental polynomially terminating over $\mathbb{N}$. With some effort, it can be shown that $\mathcal{R}$ is not polynomially terminating over $\mathbb{N}$. So $\mathcal{R}$ resides in the middle ring in Figure 1.

3 Hilbert's 10th Problem

In 1901 David Hilbert published a list of 23 mathematical problems, all of which were unsolved at the time [7]. The tenth problem on the list asked for a procedure to solve Diophantine equations.

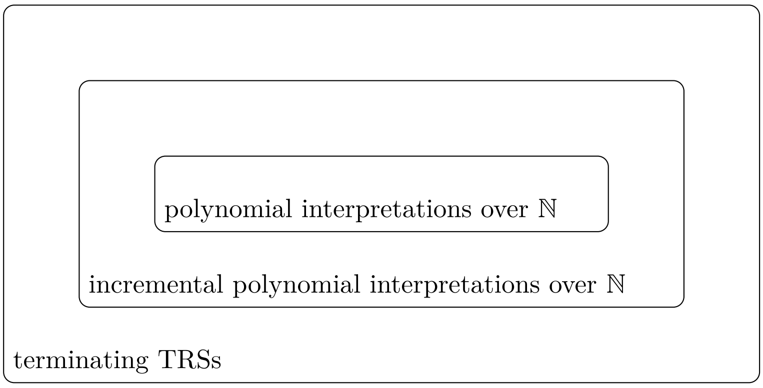

Figure 1 Polynomial termination hierarchy.

▶ **Problem 2** (Hilbert 10). *Given a polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$, determine if there exists $x_1, \ldots, x_n \in \mathbb{Z}$ such that $P(x_1, \ldots, x_n) = 0$.*

In 1970 Yuri Matiyasevich showed that recursively enumerable sets are diophantine [12]. From this it follows that Hilbert's 10th problem is undecidable [4].

▶ **Theorem 3.** *Hilbert's 10th problem is undecidable.*

In this paper we use the following three variations of Hilbert's 10th problem, all of which are undecidable. Like Hilbert's 10th problem, (2) and (3) are semi-decidable, in other words the "yes" instances can be answered in finite time. Due to the universal quantification this is not the case for (1), which is co-semi-decidable, meaning that the "no" instances can be answered in finite time.

▶ **Theorem 4.** *The following decision problems are undecidable.*

(1) *instance:* *a polynomial* $P \in \mathbb{Z}[x_1, \ldots, x_n]$
question: $P(x_1, \ldots, x_n) > 0$ *for all* $x_1, \ldots, x_n \in \mathbb{N}$*?*

(2) *instance:* *a polynomial* $P \in \mathbb{Z}[x_1, \ldots, x_n]$
question: $P(x_1, \ldots, x_n) = 0$ *for some* $x_1, \ldots, x_n \in \mathbb{N}_+$*?*

(3) *instance:* *a polynomial* $P \in \mathbb{Z}[x_1, \ldots, x_n]$
question: $P(x_1, \ldots, x_n) \geqslant 0$ *for some* $x_1, \ldots, x_n \in \mathbb{N}_+$*?*

Proof. This follows by a reduction from Problem 2. We show this for (3). The other statements can be shown in a similar way (cf. [18, Proposition 6.2.11]). Assume there exists a procedure to solve (3) and let $P \in \mathbb{Z}[x_1, \ldots, x_n]$ be some polynomial. We can modify the original question of Hilbert's 10th problem as follows:

$$\begin{aligned}
&\exists\, x_1, \ldots, x_n \in \mathbb{Z}\ P(x_1, \ldots, x_n) = 0 \\
&\iff \exists\, x_1, \ldots, x_n \in \mathbb{Z}\ \neg(P(x_1, \ldots, x_n)^2 > 0) \\
&\iff \exists\, x_1, \ldots, x_n \in \mathbb{Z}\ \neg(-P(x_1, \ldots, x_n)^2 < 0) \\
&\iff \exists\, x_1, \ldots, x_n \in \mathbb{Z}\ \ -P(x_1, \ldots, x_n)^2 \geqslant 0 \\
&\iff \exists\, a_1, \ldots, a_n \in \{-1, 0, 1\}\ \exists\, y_1, \ldots, y_n \in \mathbb{N}_+\ \ -P(a_1 y_1, \ldots, a_n y_n)^2 \geqslant 0
\end{aligned}$$

For each tuple $\vec{a} = (a_1, \ldots, a_n) \in \{-1, 0, 1\}^n$, we construct the polynomial $Q_{\vec{a}}(y_1, \ldots, y_n) = -P(a_1 y_1, \ldots, a_n y_n)^2$. From our assumption "$\exists\, y_1, \ldots, y_n \in \mathbb{N}_+\ Q_{\vec{a}}(y_1, \ldots, y_n) \geqslant 0$" is decidable for all $\vec{a}$. Since there only exist finitely many such tuples, this proves decidability of Problem 2. This obviously contradicts Theorem 3 and hence (3) is undecidable. ◀

Table 1 The TRS $\mathcal{R}$.

$\mathsf{f}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{f}(x)))$	(A)	$\mathsf{s}(\mathsf{s}(\mathsf{0})) \to \mathsf{q}(\mathsf{s}(\mathsf{0}))$	(G)
$\mathsf{q}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{f}(\mathsf{q}(x)))$	(B)	$\mathsf{s}(\mathsf{0}) \to \mathsf{q}(\mathsf{0})$	(H)
$\mathsf{f}(x) \to \mathsf{a}(x, x)$	(C)	$\mathsf{s}^5(\mathsf{0}) \to \mathsf{q}(\mathsf{s}(\mathsf{s}(\mathsf{0})))$	(I)
$\mathsf{s}(x) \to \mathsf{a}(\mathsf{0}, x)$	(D)	$\mathsf{q}(\mathsf{s}(\mathsf{s}(\mathsf{0}))) \to \mathsf{s}^3(\mathsf{0})$	(J)
$\mathsf{s}(x) \to \mathsf{a}(x, \mathsf{0})$	(E)	$\mathsf{s}(\mathsf{a}(x, x)) \to \mathsf{d}(x)$	(K)
$\mathsf{a}(\mathsf{q}(x), \mathsf{f}(x)) \to \mathsf{q}(\mathsf{s}(x))$	(F)	$\mathsf{s}(\mathsf{d}(x)) \to \mathsf{a}(x, x)$	(L)
		$\mathsf{s}(\mathsf{a}(\mathsf{q}(\mathsf{a}(x, y)), \mathsf{d}(\mathsf{a}(x, y)))) \to \mathsf{a}(\mathsf{a}(\mathsf{q}(x), \mathsf{q}(y)), \mathsf{d}(\mathsf{m}(x, y)))$	(M)
		$\mathsf{s}(\mathsf{a}(\mathsf{a}(\mathsf{q}(x), \mathsf{q}(y)), \mathsf{d}(\mathsf{m}(x, y)))) \to \mathsf{a}(\mathsf{q}(\mathsf{a}(x, y)), \mathsf{d}(\mathsf{a}(x, y)))$	(N)

4 Undecidability of Polynomial Termination

In this section we construct a family of TRSs $\mathcal{R}_P$ parameterized by polynomials $P \in \mathbb{Z}[x_1, \ldots, x_n]$ such that $\mathcal{R}_P$ is polynomially terminating over $\mathbb{N}$ if and only if $P(x_1, \ldots, x_n) > 0$ for all $x_1, \ldots, x_n \in \mathbb{N}$. The construction is based on techniques from [15], in which specific rewrite rules enforce the interpretations of certain function symbols. Our TRSs $\mathcal{R}_P$ consists of three parts: A fixed component $\mathcal{R}$, which is extended to $\mathcal{R}_k$ for some $k \in \mathbb{N}$ depending on the exponents in P, and a single rewrite rule that encodes the positiveness of P.

▶ **Definition 5.** *Given a polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$, the* TRS *$\mathcal{R}_P$ is defined over the signature consisting of a constant* $\mathsf{0}$*, unary function symbols* s, d, f, q, $\mathsf{p}_1, \ldots, \mathsf{p}_k$*, and binary function symbols* a *and* m*. Here k is the highest degree of an indeterminate in P.*

To encode the positiveness of P we need to constrain the possible interpretations of function symbols, in order to represent numbers, addition and multiplication. That is the purpose of the TRS $\mathcal{R}$, whose rules are presented in Table 1. It is a simplified and modified version of the TRS $\mathcal{R}_2$ in [15]. As will be shown later, this setup allows us to represent natural numbers as terms using the symbol $\mathsf{0}$ for the number 0 and s for the successor function. For the operations we have the binary symbols a for addition and m for multiplication. However, since multiplication is not strictly monotone on $\mathbb{N}$ we restrict the interpretation of m to $xy + x + y$, which suffices for the reduction. The remaining function symbols in $\mathcal{R}$ are not used to encode the positiveness of P, but are required for the construction to work. First we show that the mentioned interpretations prove termination of $\mathcal{R}$.

▶ **Lemma 6.** *The* TRS *$\mathcal{R}$ is polynomially terminating over $\mathbb{N}$.*

Proof. The well-founded algebra $(\mathbb{N}, >_{\mathbb{N}})$ with interpretations

$$\mathsf{0}_{\mathbb{N}} = 0 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{a}_{\mathbb{N}}(x, y) = x + y \qquad \mathsf{q}_{\mathbb{N}}(x) = x^2$$
$$\mathsf{d}_{\mathbb{N}}(x) = 2x \qquad \mathsf{f}_{\mathbb{N}}(x) = 4x + 6 \qquad \mathsf{m}_{\mathbb{N}}(x, y) = xy + x + y$$

is monotone and compatible with $\mathcal{R}$. Hence $\mathcal{R}$ is polynomially terminating. ◀

Note that this polynomial interpretation is found by the termination tool $\mathsf{T_TT_2}$ with the strategy `poly -direct -nl2 -ib 4 -ob 6`.

More importantly, to ensure termination in $(\mathbb{N}, >_{\mathbb{N}})$, the rewrite rules of $\mathcal{R}$ require that the interpretation of some of the function symbols is unique. The proof of the following lemma closely follows the reasoning in [15, Lemmata 4.4 and 5.2].

▶ **Lemma 7.** *Any monotone polynomial interpretation* $(\mathbb{N}, >_{\mathbb{N}})$ *compatible with* $\mathcal{R}$ *must interpret the function symbols* $\mathsf{0}$, s, d, a, m *and* q *as follows:*

$$\mathsf{0}_{\mathbb{N}} = 0 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{a}_{\mathbb{N}}(x, y) = x + y$$
$$\mathsf{d}_{\mathbb{N}}(x) = 2x \qquad \mathsf{m}_{\mathbb{N}}(x, y) = xy + x + y \qquad \mathsf{q}_{\mathbb{N}}(x) = x^2$$

Proof. Compatibility with (A) implies $\deg(\mathsf{f}_{\mathbb{N}}) \cdot \deg(\mathsf{s}_{\mathbb{N}}) \geqslant \deg(\mathsf{s}_{\mathbb{N}})^2 \cdot \deg(\mathsf{f}_{\mathbb{N}})$. This is only possible if $\deg(\mathsf{s}_{\mathbb{N}}) \leqslant 1$. Together with the strict monotonicity of $\mathsf{s}_{\mathbb{N}}$ we obtain $\deg(\mathsf{s}_{\mathbb{N}}) = 1$. Hence s must be interpreted by a linear polynomial: $\mathsf{s}_{\mathbb{N}}(x) = s_1 x + s_0$ with $s_1 \geqslant 1$ and $s_0 \geqslant 0$. The same reasoning applied to (B) yields $\mathsf{f}_{\mathbb{N}}(x) = f_1 x + f_0$ for some $f_1 \geqslant 1$ and $f_0 \geqslant 0$. The compatibility constraint imposed by rule (A) further gives rise to the inequality

$$f_1 s_1 x + f_1 s_0 + f_0 > f_1 s_1^2 x + f_0 s_1^2 + s_1 s_0 + s_0 \tag{1}$$

for all $x \in \mathbb{N}$. Since $s_1 \geqslant 1$ and $f_1 \geqslant 1$, this only holds if $s_1 = 1$. Simplifying (1) we obtain $f_1 s_0 > 2s_0$, which implies $s_0 > 0$ and $f_1 > 2$. If $\mathsf{q}_{\mathbb{N}}$ were linear, the same reasoning could be applied to (B) resulting in $f_1 = 1$, contradicting $f_1 > 2$. Hence $\mathsf{q}_{\mathbb{N}}$ is at least quadratic.

Next we turn our attention to the rewrite rules (C) – (F). Because $\mathsf{f}_{\mathbb{N}}$ is linear, compatibility with (C) and strict monotonicity of $\mathsf{a}_{\mathbb{N}}$ ensures $\deg(\mathsf{a}_{\mathbb{N}}) = 1$. Hence, $\mathsf{a}_{\mathbb{N}} = a_2 x + a_1 y + a_0$ with $a_2 \geqslant 1$, $a_1 \geqslant 1$ and $a_0 \geqslant 0$. From compatibility with rules (D) and (E) we obtain $a_1 = 1$ and $a_2 = 1$. Using the current shapes of $\mathsf{a}_{\mathbb{N}}$, $\mathsf{f}_{\mathbb{N}}$ and $\mathsf{s}_{\mathbb{N}}$, compatibility with rule (F) yields the inequality $\mathsf{f}_{\mathbb{N}}(x) + a_0 > \mathsf{q}_{\mathbb{N}}(x + s_0) - \mathsf{q}_{\mathbb{N}}(x)$ for all $x \in \mathbb{N}$. This can only be the case if $\deg(\mathsf{f}_{\mathbb{N}}(x) + a_0) \geqslant \deg(\mathsf{q}_{\mathbb{N}}(x + s_0) - \mathsf{q}_{\mathbb{N}}(x))$, which in turn simplifies to $1 \geqslant \deg(\mathsf{q}_{\mathbb{N}}(x)) - 1$. Hence $\mathsf{q}_{\mathbb{N}}(x) = q_2 x^2 + q_1 x + q_0$ with $q_2 \geqslant 1$. From monotonicity we also have $\mathsf{q}_{\mathbb{N}}(1) > \mathsf{q}_{\mathbb{N}}(0)$, which leads to $q_2 + q_1 \geqslant 1$.

To further constrain $\mathsf{s}_{\mathbb{N}}$ we consider the rewrite rule (G). The compatibility constraint gives rise to

$$\begin{aligned}
\mathsf{0}_{\mathbb{N}} + 2s_0 &> q_2(\mathsf{0}_{\mathbb{N}} + s_0)^2 + q_1(\mathsf{0}_{\mathbb{N}} + s_0) + q_0 \\
&= q_2 \mathsf{0}_{\mathbb{N}}^2 + q_2 s_0^2 + \mathsf{0}_{\mathbb{N}}(2q_2 s_0 + q_1) + q_1 s_0 + q_0 \\
&\geqslant q_2 s_0^2 + \mathsf{0}_{\mathbb{N}} + (1 - q_2)s_0 && (q_2 + q_1 \geqslant 1 \text{ and } q_0, q_2, s_0 \geqslant 1) \\
&= q_2 s_0(s_0 - 1) + \mathsf{0}_{\mathbb{N}} + s_0 \geqslant s_0^2 + \mathsf{0}_{\mathbb{N}} && (s_0 \geqslant 1)
\end{aligned}$$

Hence the inequality $2s_0 > s_0^2$ holds, which is only true if $s_0 = 1$. Therefore $\mathsf{s}_{\mathbb{N}}(x) = x + 1$. Compatibility with (D) now amounts to $x + 1 > \mathsf{0}_{\mathbb{N}} + x + a_0$, which implies $\mathsf{0}_{\mathbb{N}} = a_0 = 0$. At this point we have uniquely constrained $\mathsf{0}_{\mathbb{N}}$, $\mathsf{s}_{\mathbb{N}}$ and $\mathsf{a}_{\mathbb{N}}$. To fully constrain $\mathsf{q}_{\mathbb{N}}$ we turn to (H), which implies $q_0 = 0$, the rule (G), which together with monotonicity implies $2 > \mathsf{q}_{\mathbb{N}}(1) > 0$ and thus $\mathsf{q}_{\mathbb{N}}(1) = q_2 + q_1 = 1$, and the rules (I) and (J), which imply $5 > \mathsf{q}_{\mathbb{N}}(2) > 3$ and thus $\mathsf{q}_{\mathbb{N}}(2) = 4q_2 + 2q_1 = 4$. Consequently, $q_2 = 1$ and $q_1 = 0$. Hence $\mathsf{q}_{\mathbb{N}}(x) = x^2$. Compatibility with the rules (K) and (L) yields $x + x + 1 > \mathsf{d}_{\mathbb{N}}(x)$ and $\mathsf{d}_{\mathbb{N}}(x) + 1 > x + x$ which imply $\mathsf{d}_{\mathbb{N}}(x) = 2x$. Finally, compatibility with the rules (M) and (N) amounts to $(x + y)^2 + 2x + 2y + 1 > x^2 + y^2 + 2\mathsf{m}_{\mathbb{N}}(x, y) \geqslant (x + y)^2 + 2x + 2y$, which uniquely determines $\mathsf{m}_{\mathbb{N}}(x, y) = xy + x + y$. ◀

Using the previously fixed interpretations we can easily restrict the interpretations of any new symbols. By adding the two rules

$$\mathsf{s}(t) \to u \qquad\qquad \mathsf{s}(u) \to t$$

for some terms t and u, we enforce an equality constraint on the interpretations of t and u, assuming the system remains polynomially terminating.

To represent the exponents in the polynomial P we use the symbols p_i for $1 \leqslant i \leqslant k$, where k is the maximal exponent in P. To fix $\mathsf{p}_{i\mathbb{N}}(x) = x^i$, we add two rules per symbol, according to the following definition.

▶ **Definition 8.** *We define a family of* TRSs $(\mathcal{R}_k)_{k \geqslant 0}$ *as follows:*

$$\begin{aligned}
\mathcal{R}_0 &= \mathcal{R} \\
\mathcal{R}_1 &= \mathcal{R}_0 \cup \{\mathsf{s}(\mathsf{p}_1(x)) \to x,\ \mathsf{s}(x) \to \mathsf{p}_1(x)\} \\
\mathcal{R}_{k+1} &= \mathcal{R}_k \cup \left\{\begin{aligned} &\mathsf{s}(\mathsf{a}(\mathsf{p}_{k+1}(x), \mathsf{a}(x, \mathsf{p}_k(x)))) \to \mathsf{m}(x, \mathsf{p}_k(x)) \\ &\qquad\qquad \mathsf{s}(\mathsf{m}(x, \mathsf{p}_k(x))) \to \mathsf{a}(\mathsf{p}_{k+1}(x), \mathsf{a}(x, \mathsf{p}_k(x)))\end{aligned}\right\}
\end{aligned}$$

▶ **Lemma 9.** *For any $k \geqslant 0$, the* TRS $\mathcal{R}_k$ *is polynomially terminating over* $\mathbb{N}$ *if and only if* $\mathsf{p}_{i\mathbb{N}}(x) = x^i$ *for all* $1 \leqslant i \leqslant k$.

Proof. From Lemma 6 we know that $\mathcal{R}$ is polynomially terminating and the interpretations are unique due to Lemma 7. Hence the lemma holds for $\mathcal{R}_0$. For $k \geqslant 1$, the *if* direction holds, since the interpretations $\mathsf{p}_{i\mathbb{N}}$ are monotone and the polynomial interpretation is compatible with $\mathcal{R}_k$:

$$x + 1 > x \qquad\qquad x + 1 > x$$

for $\mathcal{R}_1 \setminus \mathcal{R}_0$ and

$$x^k + x + x^{k-1} + 1 > xx^{k-1} + x + x^{k-1} \qquad xx^{k-1} + x + x^{k-1} + 1 > xx^k + x + x^{k-1}$$

for $\mathcal{R}_k \setminus \mathcal{R}_{k-1}$. For the *only if* direction we show that compatibility with the additional rules implies $\mathsf{p}_{i\mathbb{N}}(x) = x^i$ for all $1 \leqslant i \leqslant k$. This is done by induction on k. For $k = 1$ the two rules in $\mathcal{R}_1 \setminus \mathcal{R}$ enforce $\mathsf{p}_{i\mathbb{N}}(x) + 1 > x$ and $x + 1 > \mathsf{p}_{i\mathbb{N}}(x)$. Hence $\mathsf{p}_{i\mathbb{N}}(x) = x$. For $k > 1$ the rules in $\mathcal{R}_k \setminus \mathcal{R}_{k-1}$ enforce $\mathsf{p}_{k\mathbb{N}}(x) = x \cdot \mathsf{p}_{k-1\mathbb{N}}(x)$ by the same reasoning. From the induction hypothesis we obtain $\mathsf{p}_{k-1\mathbb{N}}(x) = x^{k-1}$ and hence $\mathsf{p}_{k\mathbb{N}} = x^k$ as desired. ◀

The fixed interpretations can now be used to construct arbitrary polynomials. Since non-monotone operations, such as subtraction (negative coefficients) and multiplication, cannot serve as interpretations for function symbols, we model these using the difference of two terms. In the following we write $[t]_\mathbb{N}$ for the polynomial that is the interpretation of the term t, according to the interpretations stated in Lemmata 7 and 9.

▶ **Lemma 10.** *For any monomial $M = c\,x_1^{i_1} \cdots x_m^{i_m}$ with $i_1, \ldots, i_m > 0$ and $c \neq 0$ there exist terms ℓ_M and r_M over the signature of $\mathcal{R}_{\mathsf{max}(0,i_1,\ldots,i_m)}$, such that $M = [\ell_M]_\mathbb{N} - [r_M]_\mathbb{N}$ and $\mathcal{V}\mathsf{ar}(\ell_M) = \mathcal{V}\mathsf{ar}(r_M)$.*

Proof. First we assume the coefficient c is positive. We construct ℓ_M and r_M by induction on m. If $m = 0$ then $M = c$ and we take $\ell_M = \mathsf{s}^c(\mathsf{0})$ and $r_M = \mathsf{0}$. We trivially have $\mathcal{V}\mathsf{ar}(\ell_M) = \varnothing = \mathcal{V}\mathsf{ar}(r_M)$ and $[\ell_M]_\mathbb{N} - [r_M]_\mathbb{N} = c - 0 = M$. For $m > 0$ we have $M = M' x_m^{i_m}$ with $M' = c x_1^{i_1} \cdots x_{m-1}^{i_{m-1}}$. The induction hypothesis yields terms $\ell_{M'}$ and $r_{M'}$ with $M' = [\ell_{M'}]_\mathbb{N} - [r_{M'}]_\mathbb{N}$ and $\mathcal{V}\mathsf{ar}(\ell_{M'}) = \mathcal{V}\mathsf{ar}(r_{M'})$. Hence

$$\begin{aligned}
M = M' x_m^{i_m} &= [\ell_{M'}]_\mathbb{N} x_m^{i_m} - [r_{M'}]_\mathbb{N} x_m^{i_m} \\
&= (\mathsf{m}_\mathbb{N}([\ell_{M'}]_\mathbb{N}, x_m^{i_m}) - [\ell_{M'}]_\mathbb{N} - x_m^{i_m}) - (\mathsf{m}_\mathbb{N}([r_{M'}]_\mathbb{N}, x_m^{i_m}) - [r_{M'}]_\mathbb{N} - x_m^{i_m}) \\
&= (\mathsf{m}_\mathbb{N}([\ell_{M'}]_\mathbb{N}, \mathsf{p}_{i_m\mathbb{N}}(x_m)) + [r_{M'}]_\mathbb{N}) - (\mathsf{m}_\mathbb{N}([r_{M'}]_\mathbb{N}, \mathsf{p}_{i_m\mathbb{N}}(x_m)) + [\ell_{M'}]_\mathbb{N})
\end{aligned}$$

and thus we can take $\ell_M = \mathsf{a}(\mathsf{m}(\ell_{M'}, \mathsf{p}_{i_m}(x_m)), r_{M'})$ and $r_M = \mathsf{a}(\mathsf{m}(r_{M'}, \mathsf{p}_{i_m}(x_m)), \ell_{M'})$. Note that $\mathcal{V}\mathsf{ar}(\ell_M) = \mathcal{V}\mathsf{ar}(\ell_{M'}) \cup \{x_m\} \cup \mathcal{V}\mathsf{ar}(r_{M'}) = \mathcal{V}\mathsf{ar}(r_M)$.

If $c < 0$ then we take $\ell_M = r_{-M}$ and $r_M = \ell_{-M}$. We obviously have $\mathcal{V}\mathsf{ar}(\ell_M) = \mathcal{V}\mathsf{ar}(r_{-M}) = \mathcal{V}\mathsf{ar}(\ell_{-M}) = \mathcal{V}\mathsf{ar}(r_M)$. Moreover,

$$M = -(-M) = -([\ell_{-M}]_{\mathbb{N}} - [r_{-M}]_{\mathbb{N}}) = -([r_M]_{\mathbb{N}} - [\ell_M]_{\mathbb{N}}) = [\ell_M]_{\mathbb{N}} - [r_M]_{\mathbb{N}}$$ ◀

▶ **Definition 11.** *Let $P = M_1 + \cdots + M_{l-1} + M_l \in \mathbb{Z}[x_1, \ldots, x_n]$ be a sum of monomials. We define $\ell_P = \mathsf{a}(\ell_{M_1}, \cdots \mathsf{a}(\ell_{M_{l-1}}, \ell_{M_l}) \cdots)$ and $r_P = \mathsf{a}(r_{M_1}, \cdots \mathsf{a}(r_{M_{l-1}}, r_{M_l}) \cdots)$. Moreover, $\ell_0 = r_0 = \mathsf{0}$. We define the* TRS *$\mathcal{R}_P$ as the extension of $\mathcal{R}_k$ with the single rule $\ell_P \to r_P$. Here k is the maximal exponent occurring in P.*

Note that the rewrite rule $\ell_P \to r_P$ in $\mathcal{R}_P$ is well-defined; ℓ_P is not a variable and $\mathcal{V}\mathsf{ar}(\ell_P) = \mathcal{V}\mathsf{ar}(r_P)$ as a consequence of Lemma 10.

▶ **Example 12.** The polynomial $P = 2x^2y - xy + 3$ is first split into its monomials $M_1 = 2x^2y$, $M_2 = -xy$ and $M_3 = 3$. Hence we obtain the TRS $\mathcal{R}_{P_1} = \mathcal{R}_2 \cup \{\mathsf{a}(\ell_{M_1}, \mathsf{a}(\ell_{M_2}, \ell_{M_3})) \to \mathsf{a}(r_{M_1}, \mathsf{a}(r_{M_2}, r_{M_3}))\}$, where

$$\ell_{M_1} = \mathsf{a}(\mathsf{m}(\underbrace{\mathsf{a}(\mathsf{m}(\mathsf{s}^2(\mathsf{0}), \mathsf{p}_2(x)), \mathsf{0})}_{\ell_{2x^2}}, \mathsf{p}_1(y)), \underbrace{\mathsf{a}(\mathsf{m}(\mathsf{0}, \mathsf{p}_2(x)), \mathsf{s}^2(\mathsf{0}))}_{r_{2x^2}})$$

$$r_{M_1} = \mathsf{a}(\mathsf{m}(\underbrace{\mathsf{a}(\mathsf{m}(\mathsf{0}, \mathsf{p}_2(x)), \mathsf{s}^2(\mathsf{0}))}_{r_{2x^2}}, \mathsf{p}_1(y)), \underbrace{\mathsf{a}(\mathsf{m}(\mathsf{s}^2(\mathsf{0}), \mathsf{p}_2(x)), \mathsf{0})}_{\ell_{2x^2}})$$

$$\ell_{M_2} = \mathsf{a}(\mathsf{m}(\underbrace{\mathsf{a}(\mathsf{m}(\mathsf{0}, \mathsf{p}_1(x)), \mathsf{s}(\mathsf{0}))}_{r_x}, \mathsf{p}_1(y)), \underbrace{\mathsf{a}(\mathsf{m}(\mathsf{s}(\mathsf{0}), \mathsf{p}_1(x)), \mathsf{0})}_{\ell_x})$$

$$r_{M_2} = \mathsf{a}(\mathsf{m}(\underbrace{\mathsf{a}(\mathsf{m}(\mathsf{s}(\mathsf{0}), \mathsf{p}_1(x)), \mathsf{0})}_{\ell_x}, \mathsf{p}_1(y)), \underbrace{\mathsf{a}(\mathsf{m}(\mathsf{0}, \mathsf{p}_1(x)), \mathsf{s}(\mathsf{0}))}_{r_x})$$

$$\ell_{M_3} = \mathsf{s}^3(\mathsf{0}) \qquad r_{M_3} = \mathsf{0}$$

Note that in the terms ℓ_{M_2} and r_{M_2} the ℓ and r of the recursive call are switched since M_2 has a negative coefficient.

▶ **Theorem 13.** *For any polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$, the* TRS *$\mathcal{R}_P$ is polynomially terminating over $\mathbb{N}$ if and only if $P(x_1, \ldots, x_n) > 0$ for all $x_1, \ldots, x_n \in \mathbb{N}$.*

Proof. First suppose $\mathcal{R}_P$ is polynomially terminating over $\mathbb{N}$. So there exists a monotone polynomial interpretation in $(\mathbb{N}, >)$ that orients the rules of $\mathcal{R}_P$ from left to right. Let k be the maximum exponent in P. From Lemma 7 and Lemma 9 we infer that the interpretations of the function symbols $\mathsf{0}$, s, a, m, and p_i for $1 \leqslant i \leqslant k$ are fixed such that, according to Lemma 10, $P = [\ell_P]_{\mathbb{N}} - [r_P]_{\mathbb{N}}$. Since the rule $\ell_P \to r_P$ belongs to $\mathcal{R}_P$, $P(x_1, \ldots, x_n) > 0$ for all $x_1, \ldots, x_n \in \mathbb{N}$ by compatibility.

For the if direction, we assume that $P \in \mathbb{Z}[x_1, \ldots, x_n]$ satisfies $P(x_1, \ldots, x_n) > 0$ for all $x_1, \ldots, x_n \in \mathbb{N}$. By construction of $\ell_P \to r_P$ and Lemma 10, the interpretations in Lemma 7 and Lemma 9 orient the rule $\ell_P \to r_P$ from left to right. The same holds for rules $\mathcal{R}_n$. Hence $\mathcal{R}_P$ is polynomially terminating over $\mathbb{N}$. ◀

▶ **Corollary 14.** *It is undecidable whether a* TRS *is polynomially terminating over $\mathbb{N}$.*

Since the proof reduces polynomial termination to (1) from Theorem 4, which is not semi-decidable, the same holds for polynomial termination.

The construction of $\mathcal{R}_P$ may produce non-terminating systems. Take for example the polynomial $P_1 = -1$. The resulting TRS $\mathcal{R}_{P_1} = \mathcal{R} \cup \{\mathsf{0} \to \mathsf{s}(\mathsf{0})\}$ is obviously not terminating. Hence the question remains whether polynomial termination over $\mathbb{N}$ is undecidable for terminating TRSs. In the next section we show that this is indeed the case.

Another question is whether incremental polynomial termination over $\mathbb{N}$, where we take the lexicographic extension of the order induced by the polynomial interpretations, is decidable. The construction of $\mathcal{R}_P$ cannot be used to answer this question. Consider for instance the polynomial $P_2 = x$. We obtain $\ell_{P_2} = \mathsf{a}(\mathsf{m}(\mathsf{s}(\mathsf{0}), \mathsf{p}_1(x)), \mathsf{0})$ and $r_{P_2} = \mathsf{a}(\mathsf{m}(\mathsf{0}, \mathsf{p}_1(x)), \mathsf{s}(\mathsf{0}))$. As a result, the TRS $\mathcal{R}_{P_2} = \mathcal{R}_1 \cup \{\ell_{P_2} \to r_{P_2}\}$ is not polynomially terminating over $\mathbb{N}$ since $[\ell_{P_2}]_{\mathbb{N}} = 2x + 1 \not> x + 1 = [r_{P_2}]_{\mathbb{N}}$ for $x = 0$. However, if we take a second interpretation over $\mathbb{N}$ where the interpretation of m is changed to $\mathsf{m}_{\mathbb{N}}(x, y) = 2x + y$, then $[\ell_{P_2}]_{\mathbb{N}} = x + 2 > x + 1 = [r_{P_2}]_{\mathbb{N}}$ for all $x \in \mathbb{N}$. Hence $\mathcal{R}_{P_2}$ is incremental polynomially terminating over $\mathbb{N}$. In Section 6 we provide a different construction which permits to answer the question about incremental polynomial termination over $\mathbb{N}$

5 Polynomial Termination of Terminating Rewrite Systems

In the reduction in the previous section indeterminates in the input polynomial P are modeled as variables in the rewrite rule $\ell_P \to r_P$. In this and the next section we model indeterminates as unary function symbols. The following example illustrates how we intend to model the indeterminates as coefficients of the interpretation of the associated function symbol.

▶ **Example 15.** Suppose the interpretations of the function symbols $\mathsf{0}$, s, and a are already fixed to 0, the successor function, and addition. Moreover, let f be a unary function symbol whose interpretation is linear without any upper bound on the coefficients. The rewrite rules

$$\mathsf{s}(\mathsf{0}) \to \mathsf{X}(\mathsf{0}) \qquad \mathsf{s}(\mathsf{0}) \to \mathsf{Y}(\mathsf{0}) \qquad \mathsf{f}(x) \to \mathsf{X}(x) \qquad \mathsf{f}(x) \to \mathsf{Y}(x)$$

constrain the interpretations of the unary function symbols X and Y to homogeneous linear polynomials: $\mathsf{X}_{\mathbb{N}}(x) = cx$ and $\mathsf{Y}_{\mathbb{N}}(x) = dx$, where $c, d > 0$. The claim that the polynomial $P(x, y) = x^2 + xy - x - 3$ has a root in $\mathbb{N}_+$ is equivalent to the claim that the rules

$$\mathsf{s}(\mathsf{a}(\mathsf{X}(\mathsf{s}(\mathsf{0})), \mathsf{s}^3(\mathsf{0}))) \to \mathsf{a}(\mathsf{X}(\mathsf{X}(\mathsf{s}(\mathsf{0}))), \mathsf{X}(\mathsf{Y}(\mathsf{s}(\mathsf{0}))))$$
$$\mathsf{s}(\mathsf{a}(\mathsf{X}(\mathsf{X}(\mathsf{s}(\mathsf{0}))), \mathsf{X}(\mathsf{Y}(\mathsf{s}(\mathsf{0}))))) \to \mathsf{a}(\mathsf{X}(\mathsf{s}(\mathsf{0})), \mathsf{s}^3(\mathsf{0}))$$

can be oriented by a polynomial interpretation, assuming the interpretations are constrained as above. To see this we look at the induced compatibility constraint of the two rules:

$$c + 3 = c^2 + cd$$

After some rearranging $c, d \in \mathbb{N}_+$ take the place of x and y in the polynomial. Hence this equation has a solution if and only if the polynomial has a root in the positive natural numbers.

Note that natural numbers and addition are still modeled using the symbols $\mathsf{0}$, s and a, however multiplication of indeterminates (and a single coefficient) are now modeled using function composition. For example $2xy$ becomes $\mathsf{Y}(\mathsf{X}(\mathsf{s}(\mathsf{s}(\mathsf{0}))))$. To make this possible we constrain the possible interpretations of these symbols using the TRS $\mathcal{C}_P$.

Table 2 The TRS $\mathcal{C}$.

$\mathsf{f}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{f}(x)))$	(A)	$\mathsf{f}(x) \to \mathsf{a}(x,x)$	(D)	$\mathsf{s}(\mathsf{s}(\mathsf{0})) \to \mathsf{q}(\mathsf{s}(\mathsf{0}))$	(G)
$\mathsf{q}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{f}(\mathsf{q}(x)))$	(B)	$\mathsf{s}(x) \to \mathsf{a}(\mathsf{0},x)$	(E)	$\mathsf{s}(\mathsf{A}(x)) \to \mathsf{B}(x)$	(L)
$\mathsf{a}(\mathsf{q}(x),\mathsf{f}(x)) \to \mathsf{q}(\mathsf{s}(x))$	(C)	$\mathsf{s}(x) \to \mathsf{a}(x,\mathsf{0})$	(F)	$\mathsf{s}(\mathsf{B}(x)) \to \mathsf{A}(x)$	(M)

▶ **Definition 16.** *Given a polynomial $P \in \mathbb{Z}[x_1,\dots,x_n]$, the* TRS $\mathcal{C}_P$ *is defined over the signature consisting of a constant* $\mathsf{0}$*, unary function symbols* s, f, q, $\mathsf{X}_1,\dots,\mathsf{X}_n$, A, B, *and a binary function symbol* a*. It contains the rewrite rules presented in Table 2, which we denote by* $\mathcal{C}$*, as well as the* $2n$ *rules*

$$\mathsf{s}(\mathsf{0}) \to \mathsf{X}_i(\mathsf{0}) \quad (\mathrm{H}_i) \qquad\qquad \mathsf{f}(x) \to \mathsf{X}_i(x) \quad (\mathrm{I}_i)$$

for all $1 \leqslant i \leqslant n$*, which we denote by* $\mathcal{X}_n$*.*

The function symbols A and B in the rules (L) and (M), are not needed for modeling P or for constraining the interpretations of the other symbol, but will be used to prove incremental polynomial termination of the TRS later.

▶ **Lemma 17.** *The* TRS $\mathcal{C} \cup \mathcal{X}_n$ *is polynomially terminating over* $\mathbb{N}$*, for any* $n \geqslant 0$*.*

Proof. The interpretations

$$\begin{array}{llll} \mathsf{0}_{\mathbb{N}} = 0 & \mathsf{s}_{\mathbb{N}}(x) = x+1 & \mathsf{a}_{\mathbb{N}}(x,y) = x+y & \mathsf{q}_{\mathbb{N}}(x) = x^2 \\ \mathsf{A}_{\mathbb{N}}(x) = x & \mathsf{B}_{\mathbb{N}}(x) = x & \mathsf{f}_{\mathbb{N}}(x) = fx+f+2 & \mathsf{X}_{i\mathbb{N}}(x) = c_i x \quad \text{for } 1 \leqslant i \leqslant n \end{array}$$

for any $c_1,\dots,c_n > 0$ and with $f = \max(3,c_1,\dots,c_n)$ are compatible with the rules in $\mathcal{C} \cup \mathcal{X}_n$. For the rules in $\mathcal{C}$ this can be seen as follows:

$$[\mathsf{f}(\mathsf{s}(x))]_{\mathbb{N}} = fx+2f+2 > fx+f+4 = [\mathsf{s}(\mathsf{s}(\mathsf{f}(x)))]_{\mathbb{N}} \tag{A}$$
$$[\mathsf{q}(\mathsf{f}(x))]_{\mathbb{N}} = f^2x^2+2fx(f+2)+f^2+4f+4 > f^2x^2+f^2+3f+2 = [\mathsf{f}(\mathsf{f}(\mathsf{q}(x)))]_{\mathbb{N}} \tag{B}$$
$$[\mathsf{a}(\mathsf{q}(x),\mathsf{f}(x))]_{\mathbb{N}} = x^2+fx+f+2 > x^2+2x+1 = [\mathsf{q}(\mathsf{s}(x))]_{\mathbb{N}} \tag{C}$$
$$[\mathsf{f}(x)]_{\mathbb{N}} = fx+f+2 > 2x = [\mathsf{a}(x,x)]_{\mathbb{N}} \tag{D}$$
$$[\mathsf{s}(x)]_{\mathbb{N}} = x+1 > x = [\mathsf{a}(\mathsf{0},x)]_{\mathbb{N}} \tag{E}$$
$$[\mathsf{s}(x)]_{\mathbb{N}} = x+1 > x = [\mathsf{a}(x,\mathsf{0})]_{\mathbb{N}} \tag{F}$$
$$[\mathsf{s}(\mathsf{s}(\mathsf{0}))]_{\mathbb{N}} = 2 > 1 = [\mathsf{q}(\mathsf{s}(\mathsf{0}))]_{\mathbb{N}} \tag{G}$$
$$[\mathsf{s}(\mathsf{A}(x))]_{\mathbb{N}} = x+1 > x = [\mathsf{B}(x)]_{\mathbb{N}} \tag{L}$$
$$[\mathsf{s}(\mathsf{B}(x))]_{\mathbb{N}} = x+1 > x = [\mathsf{A}(x)]_{\mathbb{N}} \tag{M}$$

For the rules in $\mathcal{X}_n$ we have

$$[\mathsf{s}(\mathsf{0})]_{\mathbb{N}} = 1 > 0 = [\mathsf{X}_i(\mathsf{0})]_{\mathbb{N}} \tag{H_i}$$
$$[\mathsf{f}(x)]_{\mathbb{N}} = fx+f+2 > c_i x = [\mathsf{X}_i(x)]_{\mathbb{N}} \tag{I_i}$$

◀

Before we can formally define the two ground rules that model "$P(x_1,\dots,x_n) = 0$ for some $x_1,\dots,x_n \in \mathbb{N}_+$," we need a preliminary definition which associates terms with polynomials.

▶ **Definition 18.** *Given a polynomial*

$$P = \sum_{i=1}^{m} M_i + \sum_{j=1}^{k} N_j \in \mathbb{Z}[x_1, \dots, x_n]$$

such that the monomials $M_1, \dots, M_m$ have positive and the monomials $N_1, \dots, N_k$ have negative coefficients, we define $P_+ = M_1 + \cdots + M_m$ and $P_- = -(N_1 + \cdots + N_k)$. Given a monomial $M \in \mathbb{Z}[x_1, \dots, x_n]$ with positive coefficient, we define the term t_M inductively as follows:

$$t_M = \begin{cases} \mathsf{s}^c(\mathsf{0}) & \text{if } M = c \in \mathbb{N}_+ \\ \mathsf{X}^i(t_{M'}) & \text{if } M = M'x^i \end{cases}$$

Given a polynomial $P = M_1 + \cdots + M_l \in \mathbb{Z}[x_1, \dots, x_n]$ with positive coefficients, we define the term t_P inductively as follows:

$$t_P = \begin{cases} \mathsf{0} & \text{if } l = 0 \\ t_{M_1} & \text{if } l = 1 \\ \mathsf{a}(t_{M_1}, t_{P-M_1}) & \text{otherwise} \end{cases}$$

▶ **Example 19.** For the polynomial $P(x, y) = x^3 - 2xy^2 + 3y - 2$ we obtain

$$t_{P_+} = \mathsf{a}(\mathsf{X}(\mathsf{X}(\mathsf{X}(\mathsf{s}(\mathsf{0})))), \mathsf{Y}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{0}))))) \qquad t_{P_-} = \mathsf{a}(\mathsf{Y}(\mathsf{Y}(\mathsf{X}(\mathsf{s}(\mathsf{s}(\mathsf{0}))))), \mathsf{s}(\mathsf{s}(\mathsf{0})))$$

▶ **Definition 20.** *The* TRS $\mathcal{C}_P$ *is the extension of* $\mathcal{C} \cup \mathcal{X}_n$ *with the two ground rules*

$$\mathsf{A}(\mathsf{s}(t_{P_+})) \to \mathsf{B}(t_{P_-}) \quad \text{(J)} \qquad \mathsf{A}(\mathsf{s}(t_{P_-})) \to \mathsf{B}(t_{P_+}) \quad \text{(K)}$$

▶ **Theorem 21.** *For any polynomial $P \in \mathbb{Z}[x_1, \dots, x_n]$, the* TRS $\mathcal{C}_P$ *is polynomially terminating over $\mathbb{N}$ if and only if $P(x_1, \dots, x_n) = 0$ for some $x_1, \dots, x_n \in \mathbb{N}_+$. Moreover, $\mathcal{C}_P$ is incremental polynomially terminating over $\mathbb{N}$.*

Proof. First suppose that $\mathcal{C}_P$ is polynomially terminating over $\mathbb{N}$. By the same reasoning as in the proof of Lemma 7, compatibility with the rules (A) – (G) in Table 2 ensures

$$\mathsf{0}_{\mathbb{N}} = 0 \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad \mathsf{a}_{\mathbb{N}}(x, y) = x + y$$

Moreover, the interpretation of f must be linear, i.e., $\mathsf{f}_{\mathbb{N}}(x) = f_1 x + f_0$, and $f_0 > f_1 + 1$. The rules (L) and (M) mandate $\mathsf{A}_{\mathbb{N}}(x) = \mathsf{B}_{\mathbb{N}}(x)$ for all $x \in \mathbb{N}$. Importantly this also means $\mathsf{A}_{\mathbb{N}}(x) = \mathsf{B}_{\mathbb{N}}(y)$ implies $x = y$ for all $x, y \in \mathbb{N}$ due to monotonicity. The rules in $\mathcal{R}_n$ constrain the interpretations of the symbols $\mathsf{X}_1, \dots, \mathsf{X}_n$ to $\mathsf{X}_{i\mathbb{N}}(x) = c_i x$ with arbitrary values $c_1, \dots, c_n \in \mathbb{N}_+$. Hence $[t_M]_{\mathbb{N}} = c\, c_1^{i_1} \cdots c_n^{i_n}$ for a monomial $M = c\, x_1^{i_1} \cdots x_n^{i_n}$ with $c > 0$, and thus also $[t_Q]_{\mathbb{N}} = Q(c_1, \dots, c_n)$ for a polynomial with positive coefficients. Consequently, the two rules in Definition 20 induce the constraint

$$P_+(c_1, \dots, c_n) = P_-(c_1, \dots, c_n)$$

This constraint is satisfiable if and only if the polynomial $P_+(x_1, \dots, x_n) - P_-(x_1, \dots, x_n) = P(x_1, \dots, x_n)$ has a root in $\mathbb{N}_+$. Conversely, suppose $P(a_1, \dots, a_n) = 0$ for some $a_1, \dots, a_n \in \mathbb{N}_+$. From Lemma 17 we obtain that $\mathcal{C} \cup \mathcal{X}_n$ is polynomially terminating over $\mathbb{N}$. According to the proof of Lemma 17, we are free to choose the (positive) coefficients $c_1, \dots, c_n$ of $\mathsf{X}_{1\mathbb{N}}, \dots, \mathsf{X}_{n\mathbb{N}}$. By taking $c_i = a_i$ for $1 \leqslant i \leqslant n$, we ensure $[t_{P_+}]_\mathbb{N} = [t_{P_-}]_\mathbb{N}$ and hence $\mathcal{C}_P$ is polynomially terminating over $\mathbb{N}$.

For the second statement we start with the interpretations

$$\begin{array}{llll} \mathsf{0}_\mathbb{N} = 0 & \mathsf{s}_\mathbb{N}(x) = x & \mathsf{a}_\mathbb{N}(x,y) = x+y & \mathsf{q}_\mathbb{N}(x) = 2x^2 + x \\ \mathsf{f}_\mathbb{N}(x) = 2x+2 & \mathsf{A}_\mathbb{N}(x) = x & \mathsf{B}_\mathbb{N}(x) = x & \mathsf{X}_{i\mathbb{N}}(x) = x \quad \text{for } 1 \leqslant i \leqslant n \end{array}$$

that strictly orients (B), (C), (D) and (I$_i$):

$$[\mathsf{q}(\mathsf{f}(x))]_\mathbb{N} = 8x^2 + 18x + 10 > 8x^2 + 4x + 6 = [\mathsf{f}(\mathsf{f}(\mathsf{q}(x)))]_\mathbb{N} \tag{B}$$
$$[\mathsf{a}(\mathsf{q}(x), \mathsf{f}(x))]_\mathbb{N} = 2x^2 + 3x + 2 > 2x^2 + x = [\mathsf{q}(\mathsf{s}(x))]_\mathbb{N} \tag{C}$$
$$[\mathsf{f}(x)]_\mathbb{N} = 2x + 2 > 2x = [\mathsf{a}(x,x)]_\mathbb{N} \tag{D}$$
$$[\mathsf{f}(x)]_\mathbb{N} = 2x + 2 > x = [\mathsf{X}_i(x)]_\mathbb{N} \tag{I$_i$}$$

All other rules of $\mathcal{C}_P$ are weakly oriented. Note that $[t_M]_\mathbb{N} = 0$ for all monomials M with positive coefficient. Hence the rules (J) and (K) are turned into the identity constraint $0 = 0$. In the second step we use the interpretation

$$\begin{array}{llll} \mathsf{0}_\mathbb{N} = 0 & \mathsf{s}_\mathbb{N}(x) = 2x & \mathsf{a}_\mathbb{N}(x,y) = x+y & \mathsf{q}_\mathbb{N}(x) = x \\ \mathsf{A}_\mathbb{N}(x) = x+1 & \mathsf{B}_\mathbb{N}(x) = x+1 & \mathsf{f}_\mathbb{N}(x) = x^2 & \mathsf{X}_{i\mathbb{N}}(x) = x \quad \text{for } 1 \leqslant i \leqslant n \end{array}$$

which orients (L) and (M) strictly:

$$[\mathsf{s}(\mathsf{A}(x))]_\mathbb{N} = 2x + 2 > x + 1 = [\mathsf{B}(x)]_\mathbb{N} \tag{L}$$
$$[\mathsf{s}(\mathsf{B}(x))]_\mathbb{N} = 2x + 2 > x + 1 = [\mathsf{A}(x)]_\mathbb{N} \tag{M}$$

In the third step we change the interpretation of B:

$$\begin{array}{llll} \mathsf{0}_\mathbb{N} = 0 & \mathsf{s}_\mathbb{N}(x) = 2x & \mathsf{a}_\mathbb{N}(x,y) = x+y & \mathsf{q}_\mathbb{N}(x) = x \\ \mathsf{A}_\mathbb{N}(x) = x+1 & \mathsf{B}_\mathbb{N}(x) = x & \mathsf{f}_\mathbb{N}(x) = x^2 & \mathsf{X}_{i\mathbb{N}}(x) = x \quad \text{for } 1 \leqslant i \leqslant n \end{array}$$

This allows to orient (L) and (M) strictly:

$$[\mathsf{A}(\mathsf{s}(t_{P_+}))]_\mathbb{N} = 1 > 0 = [\mathsf{B}(t_{P_-})]_\mathbb{N} \tag{J}$$
$$[\mathsf{A}(\mathsf{s}(t_{P_-}))]_\mathbb{N} = 1 > 0 = [\mathsf{B}(t_{P_+})]_\mathbb{N} \tag{K}$$

The remaining rules (A), (E), (F), (G) and (H$_i$) are strictly oriented using the final interpretation:

$$\begin{array}{lll} \mathsf{0}_\mathbb{N} = 0 & \mathsf{a}_\mathbb{N}(x,y) = x+y & \mathsf{q}_\mathbb{N}(x) = x^2 \\ \mathsf{s}_\mathbb{N}(x) = x+1 & \mathsf{f}_\mathbb{N}(x) = 3x & \mathsf{X}_{i\mathbb{N}}(x) = x \quad \text{for } 1 \leqslant i \leqslant n \end{array}$$ ◀

▶ **Corollary 22.** *Polynomial termination over $\mathbb{N}$ is undecidable for incremental polynomially terminating* TRS*s.*

6 Incremental Polynomial Termination is Undecidable

The final result is the undecidability of incremental polynomial termination over $\mathbb{N}$. This time we model the indeterminates in the given polynomial as the degree of the interpretation of the associated function symbols. Due to monotonicity of the interpretations we cannot use polynomials of degree zero. We therefore limit the arguments of the polynomial P to $\mathbb{N}_+$ and use a reduction to (3) from Theorem 4. The idea behind modeling polynomials as degrees of interpretations is illustrated in the following example.

Table 3 The TRS $\mathcal{D}$.

$\mathsf{q}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{f}(\mathsf{q}(x)))$	(A)	$\mathsf{f}(x) \to \mathsf{a}(x,x)$	(C)	$\mathsf{f}(\mathsf{q}(x)) \to \mathsf{m}(x,x)$	(F)
$\mathsf{a}(\mathsf{q}(x), \mathsf{f}(\mathsf{f}(x))) \to \mathsf{q}(\mathsf{a}(x, \mathsf{f}(\mathsf{0})))$	(B)	$\mathsf{f}(x) \to \mathsf{m}(\mathsf{0}, x)$	(D)	$\mathsf{m}(x,x) \to \mathsf{q}(x)$	(G)
		$\mathsf{f}(x) \to \mathsf{m}(x, \mathsf{0})$	(E)		

▶ **Example 23.** Consider the symbols m and f where the interpretation of m is fixed as $\mathsf{m}_{\mathbb{N}}(x,y) = xy + x + y$ (like in Section 4) and $\mathsf{f}_{\mathbb{N}}(x) = dx + c$ is some linear polynomial with coefficients $d, c > 0$. To model that $P(x,y) = 2xy - x \geqslant 0$ for some $x, y \in \mathbb{N}_+$ we also add function symbols for each indeterminate. In this example X and Y. To model the (positive) coefficients of P we use the variable x for 1 together with the symbol m, which models addition on the level of the degrees of polynomials. Multiplication is modeled as function composition. The rule

$$\mathsf{f}(\mathsf{Y}(\mathsf{X}(\mathsf{m}(x,x)))) \to \mathsf{X}(x)$$

can be oriented only if

$$\deg([\mathsf{f}(\mathsf{Y}(\mathsf{X}(\mathsf{m}(x,x))))]_{\mathbb{N}}) = 2\deg(\mathsf{X}_{\mathbb{N}})\deg(\mathsf{Y}_{\mathbb{N}}) \geqslant \deg(\mathsf{X}_{\mathbb{N}}) = \deg([\mathsf{X}(x)]_{\mathbb{N}})$$

for some polynomials $\mathsf{X}_{\mathbb{N}}$ and $\mathsf{Y}_{\mathbb{N}}$ where $\deg(\mathsf{X}_{\mathbb{N}}), \deg(\mathsf{Y}_{\mathbb{N}}) > 0$. Since otherwise, there will always be some $x \in \mathbb{N}_+$ such that $[\mathsf{f}(\mathsf{Y}(\mathsf{X}(\mathsf{m}(x,x))))]_{\mathbb{N}} < [\mathsf{X}(x)]_{\mathbb{N}}$. Moreover the outermost f allows us to always chose a large enough d and c, such that the rule can be oriented if $2\deg(\mathsf{X}_{\mathbb{N}})\deg(\mathsf{Y}_{\mathbb{N}}) \geqslant \deg \mathsf{X}_{\mathbb{N}}$ independent of the exact shape of $\mathsf{X}_{\mathbb{N}}$ and $\mathsf{Y}_{\mathbb{N}}$. Orienting this rules is therefore possibly if and only if $P(x,y) \geqslant 0$ for some $x, y \in \mathbb{N}_+$.

To constrain the interpretations of the function symbols for this setup to work, we use the following TRS.

▶ **Definition 24.** *Given a polynomial $P \in \mathbb{Z}[x_1, \dots, x_n]$, the TRS $\mathcal{D}_P$ is defined over the signature consisting of a constant $\mathsf{0}$, unary function symbols f, q, $\mathsf{X}_1, \dots, \mathsf{X}_n$, and binary function symbols a and m. It contains the rewrite rules presented in Table 3, which we denote by $\mathcal{D}$, as well as the single ground rule*

$$\mathsf{f}(\mathsf{0}) \to \mathsf{m}(\mathsf{q}(\mathsf{0}), \mathsf{a}(\mathsf{0}, \mathsf{a}(\mathsf{X}_1(\mathsf{0}), \dots, \mathsf{a}(\mathsf{X}_{n-1}(\mathsf{0}), \mathsf{X}_n(\mathsf{0}))\dots))) \tag{X}$$

Note that all function symbols with the exception of f appear in the right-hand side of (X). The importance of this observation we will see later.

▶ **Definition 25.** *Given a monomial $M \in \mathbb{Z}[x_1, \dots, x_n]$ with positive coefficient, we define the term t_M inductively as follows:*

$$t_M = \begin{cases} x & \text{if } M = 1 \\ \mathsf{m}(x, t_{c-1}) & \text{if } M = c > 1 \\ \mathsf{X}^i(t_{M'}) & \text{if } M = M'x^i \end{cases}$$

Given a polynomial $P = M_1 + \cdots + M_l \in \mathbb{Z}[x_1, \dots, x_n]$ with $l \geqslant 1$ and positive coefficients, we define the term t_P inductively as follows:

$$t_P = \begin{cases} t_{M_1} & \text{if } l = 1 \\ \mathsf{m}(t_{M_1}, t_{P-M_1}) & \text{otherwise} \end{cases}$$

Note that $\mathcal{V}\mathsf{ar}(t_P) = \{x\}$ for any $P \in \mathbb{Z}[x_1, \dots, x_n]$. So $[t_P]_\mathbb{N}$ is a univariate polynomial.

▶ **Example 26.** For the monomial $M = 3xy^2$ and polynomial $P = 3x^2 + y + 2$ we have

$$t_M = \mathsf{Y}(\mathsf{Y}(\mathsf{X}(\mathsf{m}(x, \mathsf{m}(x, x)))))$$
$$t_P = \mathsf{m}(\mathsf{X}(\mathsf{X}(\mathsf{m}(x, \mathsf{m}(x, x)))), \mathsf{m}(\mathsf{Y}(x), \mathsf{m}(x, x)))$$

▶ **Lemma 27.** *If $\mathsf{m}_\mathbb{N}(x, y) = m_3xy + m_2x + m_1y + m_0$ with $m_3 > 0$ then*

$$\deg([t_M]_\mathbb{N}) = c \cdot \deg(\mathsf{X}_{1\mathbb{N}})^{i_1} \cdots \deg(\mathsf{X}_{n\mathbb{N}})^{i_n} = M(\deg(\mathsf{X}_{1\mathbb{N}}), \dots, \deg(\mathsf{X}_{n\mathbb{N}}))$$

for monomials $M = c\,x_1^{i_1} \cdots x_n^{i_n}$ with $c > 0$, and

$$\deg([t_P]_\mathbb{N}) = \deg([t_{M_1}]_\mathbb{N}) + \cdots + \deg([t_{M_k}]_\mathbb{N}) = P(\deg(\mathsf{X}_{1\mathbb{N}}), \dots, \deg(\mathsf{X}_{n\mathbb{N}}))$$

for polynomials $P = M_1 + \cdots + M_l$ with $l \geqslant 1$ and positive coefficients.

Proof. We prove the statement for monomials M. If $M = 1$ then $t_M = x$ and $[t_M]_\mathbb{N} = x$ and thus $\deg([t_M]_\mathbb{N}) = 1$. If $M = c > 1$ then $t_M = \mathsf{m}(x, t_{c-1})$ and, due to the assumption concerning the interpretation of m, $[t_M]_\mathbb{N} = m_3x[t_{c-1}]_\mathbb{N} + m_2x + m_1[t_{c-1}]_\mathbb{N} + m_0$ with $m_3 > 0$. We obtain $\deg([t_{c-1}]_\mathbb{N}) = c - 1$ from the induction hypothesis. Hence

$$\deg([t_M]_\mathbb{N}) = 1 + (c - 1) = c = M(\deg(\mathsf{X}_{1\mathbb{N}}), \dots, \deg(\mathsf{X}_{n\mathbb{N}}))$$

If $M = M'x^i$ then $t_M = \mathsf{X}^i(t_{M'})$. Without loss of generality we assume that $x = x_n$ and $M' \in \mathbb{Z}[x_1, \dots, x_{n-1}]$. We have $[t_M]_\mathbb{N} = (\mathsf{X}_{n\mathbb{N}})^i[t_{M'}]_\mathbb{N}$. The induction hypothesis yields

$$\deg([t_{M'}]_\mathbb{N}) = c \cdot \deg(\mathsf{X}_{1\mathbb{N}})^{i_1} \cdots \deg(\mathsf{X}_{n-1\mathbb{N}})^{i_{n-1}} = M'(\deg(\mathsf{X}_{1\mathbb{N}}), \dots, \deg(\mathsf{X}_{n-1\mathbb{N}}))$$

and thus $\deg([t_M]_\mathbb{N}) = c \cdot \deg(\mathsf{X}_{1\mathbb{N}})^{i_1} \cdots \deg(\mathsf{X}_{n\mathbb{N}})^{i_n} = M(\deg(\mathsf{X}_{1\mathbb{N}}), \dots, \deg(\mathsf{X}_{n\mathbb{N}}))$ by setting $i_n = i$. The statement for polynomials is an easy consequence of the one for monomials. ◀

▶ **Example 28.** Consider the polynomial $P = 2x^2 + y + 1$ and suppose $\mathsf{m}_\mathbb{N}(x, y) = xy + x + y$, $\mathsf{X}_\mathbb{N}(x) = x^2$ and $\mathsf{Y}_\mathbb{N}(x) = x^3$. We have

$$t_P = \mathsf{m}(\mathsf{X}(\mathsf{X}(\mathsf{m}(x, x))), \mathsf{m}(\mathsf{Y}(x), x))$$
$$[t_P]_\mathbb{N} = ((x^2 + 2x)^2)^2(x^3x + x^3 + x) + ((x^2 + 2x)^2)^2) + (x^3x + x^3 + x)$$

Note that $\deg([t_P]_\mathbb{N}) = 12 = P(2, 3)$. If we change the interpretations of X and Y to $\mathsf{X}_\mathbb{N}(x) = x^5$ and $\mathsf{Y}_\mathbb{N}(x) = x^4$ then

$$[t_P]_\mathbb{N} = ((x^2 + 2x)^5)^5(x^4x + x^4 + x) + ((x^2 + 2x)^5)^5) + (x^4x + x^4 + x)$$

and $\deg([t_P]_\mathbb{N}) = 55 = P(5, 4)$.

▶ **Definition 29.** *The* TRS $\mathcal{D}_P$ *is the extension of* $\mathcal{D} \cup \{(\mathrm{X})\}$ *with the single rule*

$$\mathsf{f}(t_{P_+}) \to t_{P_-} \tag{H}$$

Since t_0 is undefined in Definition 25, the TRS $\mathcal{D}_P$ is defined only when P contains both monomials with positive and with negative coefficients. Since Hilbert's 10th problem is trivially decidable for polynomials with only positive (negative) coefficients, this entails no loss of generality.

▶ **Example 30.** The TRS $\mathcal{D}_{x^2-2xy+3}$ consists of the rules in Table 3 extended with

$$\mathsf{f}(\mathsf{0}) \to \mathsf{m}(\mathsf{q}(\mathsf{0}), \mathsf{a}(\mathsf{0}, \mathsf{a}(\mathsf{X}(\mathsf{0}), \mathsf{Y}(\mathsf{0})))) \qquad \mathsf{f}(\mathsf{m}(\mathsf{X}(\mathsf{X}(x)), \mathsf{m}(x, \mathsf{m}(x, x)))) \to \mathsf{Y}(\mathsf{X}(\mathsf{m}(x, x)))$$

The following lemma is used in the proof of the main result of this section.

▶ **Lemma 31.** *If* $\deg(P) \geqslant \deg(Q)$ *for univariate polynomials* $P, Q \in \mathbb{Z}[x]$ *with positive coefficients then*

$$c \cdot P(x) + c > Q(x)$$

for some $c \in \mathbb{N}$ *and all* $x \in \mathbb{N}$.

Proof. Let k be the degree of Q. So $Q(x) = a_k x^k + \cdots + a_1 x^1 + a_0$ for some coefficients $a_0, \ldots, a_k \in \mathbb{N}$ with $a_k \neq 0$. Define $c = a_k + \cdots + a_1 + a_0 + 1$. We have

$$c \cdot P(x) + c \geqslant cx^k + c \geqslant Q(x) + 1 > Q(x)$$

for all $x \in \mathbb{N}$. ◀

▶ **Theorem 32.** *For any polynomial* $P \in \mathbb{Z}[x_1, \ldots, x_n]$ *with both positive and negative coefficients, the* TRS $\mathcal{D}_P$ *is incremental polynomially terminating over* $\mathbb{N}$ *if and only if* $P(a_1, \ldots, a_n) \geqslant 0$ *for some* $a_1, \ldots, a_n \in \mathbb{N}_+$.

Proof. For the only-if direction, suppose $\mathcal{D}_P$ is incremental polynomially terminating over $\mathbb{N}$. From (A) we infer $\deg(\mathsf{f}_\mathbb{N}) = 1$. So $\mathsf{f}_\mathbb{N}(x) = f_1 x + f_0$ with $f_1 > 0$. From rule (C) we obtain $\deg(\mathsf{a}_\mathbb{N}) = 1$ and thus $\mathsf{a}_\mathbb{N}(x, y) = a_2 x + a_1 y + a_0$ with $a_2, a_1 > 0$ and subsequently $f_1 > (a_2 + a_1) \geqslant 2$. Because $\mathcal{D}_P$ is incremental polynomially terminating, at least one of its rewrite rules is oriented strictly. This is possible only if the constant part of some interpretation function is positive. Now consider rule (X). Since it contains all function symbols, either $f_0 > 0$ or

$$[\mathsf{m}(\mathsf{q}(\mathsf{0}), \mathsf{a}(\mathsf{0}, \mathsf{a}(\mathsf{X}_1(\mathsf{0}), \ldots, \mathsf{a}(\mathsf{X}_{n-1}(\mathsf{0}), \mathsf{X}_n(\mathsf{0})) \ldots)))]_\mathbb{N} > 0$$

In both cases we obtain $[\mathsf{f}(\mathsf{0})]_\mathbb{N} > 0$. Consider rule (A) again. If $\mathsf{q}_\mathbb{N}(x)$ is linear then $f_1 = 1$, contradicting $f_1 \geqslant 2$. So $\deg(\mathsf{q}_\mathbb{N}) \geqslant 2$. From (B) we infer

$$a_1 \mathsf{f}_\mathbb{N}(\mathsf{f}_\mathbb{N}(x)) + a_0 \geqslant \mathsf{q}_\mathbb{N}(a_2 x + a_1 [\mathsf{f}(\mathsf{0})]_\mathbb{N} + a_0) - a_2 \mathsf{q}_\mathbb{N}(x)$$

Abbreviating $a_1 [\mathsf{f}(\mathsf{0})]_\mathbb{N} + a_0$ to d and letting $\mathsf{q}_\mathbb{N}(x) = q_k x^k + \cdots + q_1 x^1 + q_0$ with $k \geqslant 2$, the expression $\mathsf{q}_\mathbb{N}(a_2 x + d) - a_2 \mathsf{q}_\mathbb{N}(x)$ evaluates to

$$\begin{aligned}
\sum_{i=0}^{k} q_i (a_2 x + d)^i - a_2 \sum_{i=0}^{k} q_i x^i &= \sum_{i=0}^{k} q_i \sum_{j=0}^{i} \binom{i}{j} a_2^j x^j d^{i-j} - a_2 \sum_{i=0}^{k} q_i x^i \\
&= \sum_{i=0}^{k} q_i \left(a_2^i x^i + \sum_{j=0}^{i-1} \binom{i}{j} a_2^j x^j d^{i-j} \right) - a_2 \sum_{i=0}^{k} q_i x^i \\
&= \sum_{i=0}^{k} q_i \sum_{j=0}^{i-1} \binom{i}{j} a_2^j x^j d^{i-j} - \sum_{i=0}^{k} q_i (a_2^i - a_2) x^i
\end{aligned}$$

Note that $d > 0$ because $[\mathsf{f}(\mathsf{0})]_\mathbb{N} > 0$ and $a_2 > 0$. The degree of the left sum is $k - 1$ whereas the right sum has degree 0 if $a_2 = 1$ and k if $a_2 \neq 1$. Since the degree is bounded by $\deg(\mathsf{f}_\mathbb{N})^2 = 1$, we must have $a_2 = 1$. Hence

$$1 \geqslant \deg(\mathsf{q}_\mathbb{N}(a_2 x + a_1 [\mathsf{f}(\mathsf{0})]_\mathbb{N} + a_0) - a_2 \mathsf{q}_\mathbb{N}(x)) = \deg(\mathsf{q}_\mathbb{N}) - 1$$

and thus $\deg(\mathsf{q}_{\mathbb{N}}) = 2$. Rules (F) and (G) ensure $\deg(\mathsf{m}_{\mathbb{N}}(x,x)) = 2$ and thus we may write $\mathsf{m}_{\mathbb{N}}(x,y) = m_5x^2 + m_4y^2 + m_3xy + m_2x + m_1y + m_0$. Considering rule (D) yields

$$1 \geqslant \deg(\mathsf{m}_{\mathbb{N}}(\mathsf{0}_{\mathbb{N}}, x)) = \deg(m_5\mathsf{0}_{\mathbb{N}}^2 + m_4x^2 + (m_3\mathsf{0}_{\mathbb{N}} + m_1)x + m_2\mathsf{0}_{\mathbb{N}} + m_0)$$

and thus $m_4 = 0$. Similarly, rule (E) yields $m_5 = 0$. Hence $m_3 > 0$ for otherwise $\deg(\mathsf{m}_{\mathbb{N}}(x,x)) = 2$ does not hold. From rule (H) with $\deg(\mathsf{f}_{\mathbb{N}}) = 1$ we obtain $\deg([t_{P_+}]_{\mathbb{N}}) \geqslant \deg([t_{P_-}]_{\mathbb{N}})$. Subsequently applying Lemma 27 to the terms t_{P_+} and t_{P_-} results in

$$P_+(\deg(\mathsf{X}_{1\mathbb{N}}), \ldots, \deg(\mathsf{X}_{n\mathbb{N}})) \geqslant P_-(\deg(\mathsf{X}_{1\mathbb{N}}), \ldots, \deg(\mathsf{X}_{n\mathbb{N}}))$$

Hence $P(\deg(\mathsf{X}_{1\mathbb{N}}), \ldots, \deg(\mathsf{X}_{n\mathbb{N}})) \geqslant 0$ as desired.

For the if direction, suppose $P(a_1, \ldots, a_n) \geqslant 0$ for some $a_1, \ldots, a_n \in \mathbb{N}_+$. The interpretation

$$\begin{aligned}
\mathsf{0}_{\mathbb{N}} &= 0 & \mathsf{a}_{\mathbb{N}}(x,y) &= x + y & \mathsf{q}_{\mathbb{N}}(x) &= x^2 + x \\
\mathsf{f}_{\mathbb{N}}(x) &= fx + f & \mathsf{m}_{\mathbb{N}}(x,y) &= xy + x + y & \mathsf{X}_{i\mathbb{N}}(x) &= x^{a_i} \quad \text{for } 1 \leqslant i \leqslant n
\end{aligned}$$

with $f \geqslant 2$ orients the rules of $\mathcal{D} \cup \{\,(\mathrm{X})\,\}$ as follows:

$$\begin{aligned}
\mathsf{q}(\mathsf{f}(x)) &\to \mathsf{f}(\mathsf{f}(\mathsf{q}(x))) & (fx+f)^2 + fx + f &\geqslant f^2(x^2+x) + f^2 + f & \text{(A)} \\
\mathsf{a}(\mathsf{q}(x), \mathsf{f}(\mathsf{f}(x))) &\to \mathsf{q}(\mathsf{a}(x, \mathsf{f}(\mathsf{0}))) & x^2 + x + f^2x + f^2 + f &\geqslant (x+f)^2 + x + f & \text{(B)} \\
\mathsf{f}(x) &\to \mathsf{a}(x,x) & fx + f &> 2x & \text{(C)} \\
\mathsf{f}(x) &\to \mathsf{m}(\mathsf{0},x) & fx + f &> x & \text{(D)} \\
\mathsf{f}(x) &\to \mathsf{m}(x,\mathsf{0}) & fx + f &> x & \text{(E)} \\
\mathsf{f}(\mathsf{q}(x)) &\to \mathsf{m}(x,x) & f(x^2+x) + f &> x^2 + 2x & \text{(F)} \\
\mathsf{m}(x,x) &\to \mathsf{q}(x) & x^2 + 2x &\geqslant x^2 + x & \text{(G)} \\
\mathsf{f}(\mathsf{0}) &\to \mathsf{m}(\mathsf{q}(\mathsf{0}), \mathsf{a}(\mathsf{0}, \mathsf{a}(\mathsf{X}_1(\mathsf{0}), \ldots, \mathsf{a}(\mathsf{X}_{n-1}(\mathsf{0}), \mathsf{X}_n(\mathsf{0}))\ldots))) & f &> 0 & \text{(X)}
\end{aligned}$$

The assumption $P(a_1, \ldots, a_n) \geqslant 0$ in connection with Lemma 27 yields

$$0 \leqslant P_+(a_1, \ldots, a_n) - P_-(a_1, \ldots, a_n) = \deg([t_{P_+}]_{\mathbb{N}}) - \deg([t_{P_-}]_{\mathbb{N}})$$

Since $[t_{P_+}]_{\mathbb{N}}$ and $[t_{P_-}]_{\mathbb{N}}$ are univariate polynomials, we can apply Lemma 31. This yields a $c \in \mathbb{N}$ such that $c[t_{P_+}]_{\mathbb{N}} + c > [t_{P_-}]_{\mathbb{N}}$. Hence, by choosing $f = \max(c, 2)$, the rule (H) is strictly oriented. This concludes the first step in the incremental polynomial termination proof of $\mathcal{D}_P$. The interpretation

$$\begin{aligned}
\mathsf{0}_{\mathbb{N}} &= 0 & \mathsf{a}_{\mathbb{N}}(x,y) &= 2x + y & \mathsf{q}_{\mathbb{N}}(x) &= 3x + 2 \\
\mathsf{f}_{\mathbb{N}}(x) &= x + 1 & \mathsf{m}_{\mathbb{N}}(x,y) &= 2x + y + 3
\end{aligned}$$

orients the remaining rules (A), (B) and (G):

$$\begin{aligned}
\mathsf{q}(\mathsf{f}(x)) &\to \mathsf{f}(\mathsf{f}(\mathsf{q}(x))) & 3x + 5 &> 3x + 4 & \text{(A)} \\
\mathsf{a}(\mathsf{q}(x), \mathsf{f}(\mathsf{f}(x))) &\to \mathsf{q}(\mathsf{a}(x, \mathsf{f}(\mathsf{0}))) & 7x + 6 &> 6x + 5 & \text{(B)} \\
\mathsf{m}(x,x) &\to \mathsf{q}(x) & 3x + 3 &> 3x + 2 & \text{(G)}
\end{aligned}$$

Hence $\mathcal{D}_P$ is incremental polynomially terminating over $\mathbb{N}$. ◀

▶ **Corollary 33.** *Incremental polynomial termination over $\mathbb{N}$ is an undecidable property of finite* TRS*s.*

We do not know whether the TRS $\mathcal{D}_P$ is terminating (independent of P). Hence $\mathcal{D}_P$ cannot be used to strengthen the result of Corollary 33 to terminating TRSs. However, a small modification is sufficient to obtain this result.

▶ **Definition 34.** *The* TRS $\mathcal{D}'_P$ *is defined as* $\{\mathsf{D}^{\ell}_{r}(\ell) \to \mathsf{D}^{\ell}_{r}(r) \mid \ell \to r \in \mathcal{D}_P\}$.

So each rule $\ell \to r$ of $\mathcal{D}_P$ is placed under a designated unary function symbol D^{ℓ}_{r}. The indices ℓ and r ensure that different rules are rooted by different symbols. The proof of Theorem 32 is not affected by this modification, since monotonicity implies that $\mathsf{D}^{\ell}_{r}(x) > \mathsf{D}^{\ell}_{r}(y)$ if and only if $x > y$.

▶ **Lemma 35.** *The* TRS $\mathcal{D}'_P$ *is terminating.*

Proof. Let the signature of $\mathcal{D}'_P$ be denoted by $\mathcal{F}$. We prove termination using a well-founded monotone $\mathcal{F}$-algebra $\mathcal{A} = (\mathbb{N} \times \mathcal{F}, >_{\mathcal{A}})$, where $(n, f) >_{\mathcal{A}} (m, g)$ if $f = g$ and $n >_{\mathbb{N}} m$. The interpretation functions for the symbols in $\mathcal{D}_P$ are (with $1 \leqslant i \leqslant n$)

$$\mathsf{0}_{\mathcal{A}} = (1, 0) \qquad \mathsf{a}_{\mathcal{A}}((x, f), (y, g)) = (x + y + 1, \mathsf{a}) \qquad \mathsf{q}_{\mathcal{A}}((x, f)) = (x + 1, \mathsf{q})$$
$$\mathsf{f}_{\mathcal{A}}((x, f)) = (x + 1, \mathsf{f}) \quad \mathsf{m}_{\mathcal{A}}((x, f), (y, g)) = (x + y + 1, \mathsf{m}) \quad \mathsf{X}_{i\mathcal{A}}((x, f)) = (x + 1, \mathsf{X}_i)$$

Intuitively these interpretations keep track of the size and the root symbol of the term. The interpretations of the symbols D^{ℓ}_{r} ensure that they weigh more when appearing on the left and are defined as

$$\mathsf{D}^{\ell}_{r\mathcal{A}}((x, f)) = \begin{cases} (|r| \cdot x + 1, \mathsf{D}^{\ell}_{r}) & \text{if } f = \mathsf{root}(\ell) \\ (x + 1, \mathsf{D}^{\ell}_{r}) & \text{otherwise} \end{cases}$$

where $|r|$ denotes the size of the term r. One easily verifies that all rewrite rules in $\mathcal{D}'_P$ are oriented strictly with respect to $>_{\mathcal{A}}$; note that for right-hand sides $\mathsf{D}^{\ell}_{r}(r)$ the second case in the interpretation of D^{ℓ}_{r} applies, except when $r = t_1 = x$ in which case the rule is oriented based on the size of the terms alone. ◀

The second component in the interpretations in the above proof simulates root-labeling [16] and the lemma can also be shown using semantic root-labeling in connection with LPO [17, 13].

▶ **Corollary 36.** *Incremental polynomial termination over* ℕ *is an undecidable property of terminating* TRS*s.*

7 Conclusion

In this paper we proved the undecidability of polynomial termination over ℕ for TRSs that are incremental polynomially terminating over ℕ. We also proved that incremental polynomial termination over ℕ is an undecidable property of terminating TRSs. The proofs remain valid if we restrict to polynomial interpretations with natural numbers as coefficients. A simple tool that generates the TRSs $\mathcal{R}_P$, $\mathcal{C}_P$ and $\mathcal{D}_P$ given a polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$ is available[1] and useful for tool builders and competition organizers.

As possible future work regarding decidability we mention weakly monotone interpretations over ℕ as used in a dependency pairs setting [1]. Polynomial interpretations over ℚ and ℝ ([11, 15]) are also of interest. Moreover, matrix [5] and arctic [9] interpretations are under-explored as far as decidability issues are concerned.

[1] `https://github.com/fabeulous/pt-hilbert-encodings`

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