A Combinatorial Approach to Higher-Order Structure for Polynomial Functors

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Abstract

Polynomial functors are categorical structures used in a variety of applications across theoretical computer science; for instance, in database theory, denotational semantics, functional programming, and type theory. A well-known problem is that the bicategory of finitary polynomial functors between categories of indexed sets is not cartesian closed, despite its success and influence on denotational models and linear logic.

This paper introduces a formal bridge between the model of finitary polynomial functors and the combinatorial theory of generalised species of structures. Our approach consists in viewing finitary polynomial functors as free analytic functors, which correspond to free generalised species. In order to systematically consider finitary polynomial functors from this combinatorial perspective, we study a model of groupoids with additional logical structure; this is used to constrain the generalised species between them. The result is a new cartesian closed bicategory that embeds finitary polynomial functors.

2012 ACM Subject Classification Theory of computation → Categorical semantics; Theory of computation → Lambda calculus; Mathematics of computing → Combinatorics

Keywords and phrases Bicategorical models, denotational semantics, stable domain theory, linear logic, polynomial functors, species of structures, groupoids

Digital Object Identifier 10.4230/LIPIcs.FSCD.2022.31

Funding *Marcelo Fiore*: Research partially supported by EPSRC grant EP/V002309/1.
Zeinab Galal: Research partially supported by EPSRC grant EP/V002309/1 and by ANR grant ANR-20-CE48-0010.
Hugo Paquet: Research supported by a Royal Society University Research Fellowship.

1 Introduction

We introduce a formal bridge between two mathematical theories which have been influential in the context of programming language semantics:

1. The theory of *polynomial functors*, a popular categorification of the notion of polynomial function.
2. The theory of *generalised species* and analytic functors, due to Fiore, Gambino, Hyland, and Winskel, which provides higher-order notions of combinatorial structures.

The connection gives a new combinatorial perspective on polynomial functors. We exploit this in the paper to overcome the problem that polynomial functors do not form a cartesian closed model.

7th International Conference on Formal Structures for Computation and Deduction (FSCD 2022).
Editor: Amy P. Felty; Article No. 31; pp. 31:1–31:19
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Applications of polynomial functors in computer science are surprisingly varied, and include: models of dependent type theory [21, 48, 5], representation of data types [1, 4], implicit complexity theory [41], and dynamical systems [43]. As for semantics, polynomial functors support a well-known model of the lambda calculus, developed by Girard [26], who explicitly cites it as a catalyst for linear logic [25].

Generalised species [18] were put forward more recently as a general *bicategorical* framework for the study of substitution for combinatorial structures, generalising Joyal's prior work on species of structures [31, 32]. The bicategory of generalised species is cartesian closed, and thus provides a convenient basis for denotational semantics. It has become a prime example of a semantic model in which program symmetries are represented explicitly as 2-cells, and several lines of research are benefitting from this idea [47, 19, 40, 20, 42].

The connection between these two concepts is already understood in simple settings; we give a brief overview. On one side, we consider *finitary* polynomial functors $\mathbf{Set} \longrightarrow \mathbf{Set}$, which correspond to operations on sets of the form

$$X \longmapsto \sum_{n \in \mathbb{N}} A_n \times X^n \tag{1}$$

where the coefficients A_n are sets. On the other side, Joyal's species of structures are equivalent to *analytic functors* $\mathbf{Set} \longrightarrow \mathbf{Set}$, corresponding to operations of the form

$$X \longmapsto \sum_{n \in \mathbb{N}} F_n \underset{\mathfrak{S}_n}{\times} X^n \tag{2}$$

where the coefficients F_n are sets equipped with an action of the symmetric group $\mathfrak{S}_n$ on n elements, and the operator $\underset{\mathfrak{S}_n}{\times}$ performs a quotient of the product under this action. In special cases, when the actions on F_n are *free actions* (§5), the quotient is equivalently a set of the form $A_n \times X^n$, and so the analytic functor is also polynomial. Conversely, every finitary polynomial functor is analytic when its coefficients are regarded as freely generated actions.

Summary of contributions

We extend the correspondence between finitary polynomial functors and free analytic functors to a generalised setting: instead of functors between categories of indexed sets, we consider functors between full subcategories of presheaves over groupoids. Our first contribution is a logical device for constraining the actions on the coefficients of analytic functors: in particular one may require all actions to be free. We call this device a *kit* (§3).

We show that one can systematically consider analytic functors controlled by kits. This leads us to the construction of a 2-category whose morphisms we called *stable functors*. In the basic setting of endofunctors on sets, we recover the simple connection above: stable functors correspond to finitary polynomial functors.

We then push this further and consider higher-order structure in this bicategory. We introduce *stable species*, combinatorial structures constrained by kits, and show that these correspond to stable functors, just as generalised species correspond to analytic functors. This gives our second main contribution: we prove that stable species, and therefore the equivalent stable functors, form a cartesian closed bicategory.

This is significant because the bicategory of finitary polynomial functors between categories of indexed sets is not cartesian closed; for instance, Girard's lambda-calculus model cannot be extended directly to a typed lambda calculus. This situation has attracted a considerable amount of attention [45, 28, 12]; our approach has the advantage of making the combinatorics of the problem clear and explicit, via the kit on the function space in our bicategory.

Outline of the paper

We first (§2) give an introductory account of the connection between analytic and polynomial endofunctors on sets, including the representation of coefficients as species. Then, the formal development is organised as follows:

- We introduce *groupoids with kits*, demonstrating their purpose in controlling actions (§3) and identifying the important class of *Boolean kits*.
- We introduce a 2-category **Stable** whose objects are groupoids with Boolean kits and whose morphisms are called *stable functors* (§4.3). Stable functors are closely related to finitary polynomial functors, coinciding with them at discrete groupoids.
- We introduce a bicategory **SEsp** of *stable species of structures* (§6), a refinement of generalised species of structures, based on groupoids with Boolean kits. We establish that **SEsp** is cartesian closed (Theorem 22).
- We exhibit a biequivalence between **SEsp** and **Stable** (§7), deducing that **Stable** is a cartesian closed bicategory (Theorem 17).

Finally (§8) we mention related work in the area. We explain the influence of Taylor's *creeds* [45] on our work, and discuss connections with Berry's stable domain theory [9] and with Girard's linear logic [25].

2 Polynomial functors and analytic endofunctors on sets

At the simplest level, polynomial and analytic functors are defined on the category **Set** of sets and functions.

Polynomial functors

A function $p : E \to B$ between sets E and B determines an endofunctor on **Set** defined as

$$X \longmapsto \sum_{b \in B} X^{E_b}$$

where $E_b = p^{-1}\{b\}$ is the fibre of p over $b \in B$, and X^{E_b} is the set of functions $E_b \to X$. It is common (although not essential for this paper) to think of B as a set of operators, where the arity of an operator $b \in B$ is specified by (the cardinality of) E_b. In particular, one can restrict to *finitary* polynomial functors with finite arities. Every finitary polynomial functor is then naturally isomorphic to one of the form

$$X \longmapsto \sum_{n \in \mathbb{N}} F_n \times X^n \tag{3}$$

where each set F_n corresponds to the set of operators of arity n. The analogy with traditional polynomials is manifest in this representation. Finitary polynomial functors are determined by their action on finite input sets, in the same vein as continuous maps between domains in domain theory.

Analytic functors and Joyal species of structures

An endofunctor on **Set** is an *analytic functor* [32] if it is naturally isomorphic to one of the form

$$X \longmapsto \sum_{n \in \mathbb{N}} F(n) \underset{\mathfrak{S}_n}{\times} X^n, \tag{4}$$

where:

1. Each $F(n)$ is a set with a left action of the symmetric group $\mathfrak{S}_n$ on n elements. Concretely, this means that we have an assignment that to every permutation $\sigma \in \mathfrak{S}_n$ of the set $[n] = \{1, \ldots, n\}$ associates a permutation of the set $F(n)$ preserving identity and composition. We write $\sigma \cdot p$ for the action of $\sigma \in \mathfrak{S}_n$ on an element $p \in F(n)$.
2. The set $F(n) \underset{\mathfrak{S}_n}{\times} X^n$ is obtained by quotienting the product $F(n) \times X^n$ under the equivalence relation $\sim$ containing the pairs

$$(p, (x_{\sigma 1}, \ldots, x_{\sigma n})) \;\sim\; (\sigma \cdot p, (x_1, \ldots, x_n))$$

for all $\sigma \in \mathfrak{S}_n$, $p \in F(n)$ and $(x_1, \ldots, x_n) \in X^n$.

The notation $F(-)$ is justified, because the coefficients $F(n)$, *together with* the group actions, can be bundled into a functor $F : \mathbf{B} \longrightarrow \mathbf{Set}$ where $\mathbf{B}$ is the category whose objects are the natural numbers, and whose morphisms $m \to n$ are the bijections $[m] \to [n]$. The action of a permutation $\sigma \in \mathfrak{S}_n$ on the set $F(n)$ is then simply given by the functorial action $F(\sigma) : F(n) \to F(n)$.

The functor $F : \mathbf{B} \longrightarrow \mathbf{Set}$ is a *species of structures* (or just a *species*, with its elements referred to as *structures*) corresponding to the analytic functor (4). Every analytic functor has, up to isomorphism, a unique generating species, which may be recovered using so-called *weak generic elements* (§7). This combinatorial theory was developed by Joyal [31, 32], including the connection to polynomial functors as we explain next.

Polynomial functors are free analytic functors

Finitary polynomial functors correspond to free analytic functors and this gives an equivalence. The basic idea is as follows. Every set A generates a *free action* of a group G, given by the product set $A \times G$ with the action

$$\tau \cdot (a, \sigma) \overset{\text{def}}{=} (a, \tau\, \sigma) \tag{5}$$

for every $\tau \in G$ and $(a, \sigma) \in A \times G$. We extend this to polynomial functors. Consider the polynomial functor $X \longmapsto \sum_{n \in \mathbb{N}} A_n \times X^n$. Taking the free action generated by A_n of $\mathfrak{S}_n$, for every $n \in \mathbb{N}$, we obtain a species $\mathbf{B} \longrightarrow \mathbf{Set}$ given by

$$\begin{aligned} n &\longmapsto A_n \times \mathfrak{S}_n \\ (\tau : n \to n) &\longmapsto ((a, \sigma) \mapsto (a, \tau\, \sigma)) \end{aligned}$$

which, via the construction (4), generates an analytic functor. For this species, when taking the quotient in (4), we have

$$(A_n \times \mathfrak{S}_n) \underset{\mathfrak{S}_n}{\times} X^n \;\cong\; A_n \times X^n$$

and therefore recover the polynomial functor. Thus, every finitary polynomial functor is, in particular, analytic.

One can characterize the analytic functors that are polynomial in terms of the generating species. To do this, observe the following key property: for the free action defined in (5), every element has trivial stabilizer. Recall that the *stabilizer* of an element $p \in P$ with respect to the action of a group G is defined as $\mathrm{Stab}_G(p) \overset{\text{def}}{=} \{\sigma \in G \mid \sigma \cdot p = p\}$, a subgroup of G. Then, if $F : \mathbf{B} \longrightarrow \mathbf{Set}$ is a species such that for every $n \in \mathbb{N}$, the action of $\mathfrak{S}_n$ on $F(n)$ is free (in the sense that every structure in $F(n)$ has trivial stabilizer) then the associated analytic endofunctor on $\mathbf{Set}$ is polynomial.

Cartesian natural transformations between polynomial functors

We have described finitary polynomial functors as a subclass of analytic functors. Following Girard and others [26, 45], the natural transformations to be considered between them are as follows:

▶ **Definition 1.** *A **cartesian natural transformation** is a natural transformation for which every naturality square is a pullback.*

There are several justifications for this choice:

- With the interpretation of polynomial functors as arising from sets of operators with arities, a cartesian natural transformation corresponds to a mapping between operator sets that preserves arities.
- Cartesian natural transformations between polynomial functors are in bijection with (arbitrary) natural transformations between the associated free species.
- Cartesian natural transformations are a categorification of Berry's stable order in domain theory, a point of view that motivates our approach (see the discussion in §8).

The combinatorial and extensional views

So far, we have given definitions of polynomial and analytic functors in terms of coefficient species. This is the *combinatorial* (or *intensional*) view. A strength of the theory is that there is an alternative presentation: both kinds of functors may be characterised abstractly without reference to coefficients. This is the *extensional* view:

- Analytic endofunctors on sets are the finitary ones (i.e. filtered-colimit preserving) that preserve wide quasi-pullbacks [32].
- Polynomial endofunctors on sets are those that preserve wide pullbacks (equivalently, enjoy a local right adjoint property, see Definition 5) [44, 22].

We will generalise beyond endofunctors on sets and define our *stable functors* in the extensional style. But it is the combinatorial view, given by *stable species*, that explains the higher-order structure.

3 Groupoids with kits

We introduce kits, a structure for controlling group (and, more generally, groupoid) actions. As motivation for the general theory, the discussion in this section is only concerned with endofunctors on sets.

Species and stabilizers

Recall two key observations from the previous section:

- every species $\mathbf{B} \longrightarrow \mathbf{Set}$ induces an analytic endofunctor on sets; and
- this functor is polynomial if and only if the species is *free* (in the sense that all its structures have trivial stabilizer).

One key idea of this paper is to use subgroups to specify the extent to which a species may be free. Indeed, by specifying, for each $n \in \mathbf{B}$, a set $\mathcal{K}(n)$ of subgroups of $\mathbf{B}(n, n)$ that are to be regarded as *permitted stabilizers*, we may restrict to species with structures having only permitted stabilizers (Definition 4) and thereby identify a class of generalised polynomial functors. Appropriate such families $\mathcal{K} = \{\mathcal{K}(n)\}_{n \in \mathbf{B}}$ we will call kits (Definition 2).

As extreme special cases, one can take $\mathcal{K}(n)$ to contain all the subgroups of $\mathbf{B}(n,n)$ and recover the analytic functors; or, instead, take $\mathcal{K}(n)$ to consist only of the trivial subgroup, forcing the species to be free, and recover polynomial functors.

There is no need that the choice of permitted stabilizers be uniform across all $n \in \mathbf{B}$ as in the two examples above. But it is natural to require that permitted stabilizers are closed under conjugation, since for every structure $p \in F(n)$ and permutation $\sigma \in \mathbf{B}(n,n)$, the stabilizer subgroups of p and of $\sigma \cdot p$ are conjugate of each other: $\mathrm{Stab}(\sigma \cdot p) = \sigma \, \mathrm{Stab}(p) \, \sigma^{-1}$.

As a step towards the generalised model to come, we introduce kits on arbitrary groupoids. This brings us close to Taylor's *creeds* [45], see also §8.

▶ **Definition 2.** *A **kit** on a groupoid $\mathbb{A}$ is a family $\mathcal{A} = \{\, \mathcal{A}(a) \,\}_{a \in \mathbb{A}}$ where $\mathcal{A}(a)$ is a set of subgroups of $\mathbb{A}(a,a)$ closed under conjugation in the following sense:*

For all $a, a' \in \mathbb{A}$ and $\alpha \in \mathbb{A}(a,a')$, if $H \in \mathcal{A}(a)$ then $\alpha \, H \, \alpha^{-1} \in \mathcal{A}(a')$.

The proposition below provides important examples of kits.

▶ **Proposition 3.** *For a presheaf $F : \mathbb{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ on a groupoid $\mathbb{A}$, the family $\mathcal{S} = \{\, \mathcal{S}(a) \,\}_{a \in \mathbb{A}}$ with $\mathcal{S}(a) = \{\, \mathrm{Stab}_{\mathbb{A}(a,a)}(p) \mid p \in F(a) \,\}$ is a kit.*

At this stage, we are in a position to define a restricted notion of analytic endofunctor on **Set** parametrised by a kit on the groupoid $\mathbf{B}$. First we appropriately restrict species:

▶ **Definition 4.** *Let $\mathcal{K}$ be a kit on $\mathbf{B}$. A **$\mathcal{K}$-species** is a functor $F : \mathbf{B} \longrightarrow \mathbf{Set}$ such that every F-structure has stabilizer in $\mathcal{K}$.*

This way, every kit $\mathcal{K}$ gives rise to the subclass of analytic endofunctors on **Set** induced by $\mathcal{K}$-species. In this paper, we are targeting polynomial functors and, in accordance, our construction of stable species (Definition 19) induces the kit on $\mathbf{B}$ that consists of only the trivial subgroups.

4 Stable functors between categories of stable presheaves

A kit $\mathcal{A}$ on a groupoid $\mathbb{A}$ determines a full subcategory $\boldsymbol{\mathcal{S}}(\mathbb{A}, \mathcal{A})$ of the presheaf category $\boldsymbol{\mathcal{P}}(\mathbb{A}) = [\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$, whose objects we call *stable presheaves*. We will consider *stable functors* between categories of stable presheaves. These generalise finitary polynomial functors beyond endofunctors on **Set**. We begin with a brief overview of existing generalisations of polynomial functors and analytic functors which are relevant to the paper.

4.1 Generalising polynomial and analytic functors

Polynomial functors between categories of indexed sets

There is a rich theory of polynomial functors defined with respect to a *locally cartesian closed* category [22, 4]. When that category is **Set**, one obtains a notion of polynomial functor $\mathbf{Set}^I \longrightarrow \mathbf{Set}^J$, where I and J are sets. These can be given representations with coefficients in the style of (3). As we will later on study these coefficients using generalised species, for now we give a definition in the extensional style [22, §1.18]. This will form the basis for our notion of stable functor (Definition 10).

▶ **Definition 5.** *A functor* $F : \mathscr{C} \longrightarrow \mathscr{D}$ *is a* ***local right adjoint*** *if for every object* $C \in \mathscr{C}$*, the local functor*

$$F/C : \mathscr{C}/C \;\longrightarrow\; \mathscr{D}/F(C)$$

between slice categories, which transports an object $A \xrightarrow{a} C$ *to its image* $F(A) \xrightarrow{F(a)} F(C)$*, has a left adjoint.*

For sets I *and* J*, a functor* $\mathbf{Set}^I \longrightarrow \mathbf{Set}^J$ *is a* ***polynomial functor*** *if it is a local right adjoint.*

Polynomial functors between categories of indexed sets can be organised into a 2-category with indexing sets as objects, and cartesian natural transformations as 2-cells. This model has convenient structure: for instance, there is a canonical polynomial functor isomorphism $\mathbf{Set}^{I+J} \cong \mathbf{Set}^I \times \mathbf{Set}^J$ that exhibits $I + J$ as a cartesian product of I and J. Restricting to finitary polynomial functors, Girard (who called them *normal functors*) described a model of the lambda calculus [26]. His idea of representing programs as polynomials triggered a radical shift in perspective, leading to linear logic [25].

However, the bicategory of polynomial functors is not cartesian closed. One approach to circumvent this, pioneered by Lamarche [36], is to consider polynomial *functions* between domains rather than categories, where coefficients are elements of a suitable semiring. Here we propose a solution using generalised species of structures, which we present next.

Generalised species of structures and analytic functors

We consider analytic functors between presheaf categories over groupoids. These include categories of indexed sets $\mathbf{Set}^I$, viewing the set I as a discrete groupoid.

Recall the notion of a species $\mathbf{B} \longrightarrow \mathbf{Set}$ for an analytic functor on $\mathbf{Set}$ and consider the following two basic observations. First, the groupoid $\mathbf{B}$ is the symmetric strict monoidal completion $!\mathbf{1}$ of the terminal category $\mathbf{1}$ with one object and one morphism. Second, $\mathbf{Set}$ is isomorphic to the category of presheaves $\boldsymbol{\mathcal{P}}\mathbf{1}$ over $\mathbf{1}$. The approach of Fiore, Gambino, Hyland and Winskel [18] extends the basic notion of a species $\mathbf{B} \longrightarrow \mathbf{Set}$, corresponding to $!\mathbf{1} \longrightarrow \boldsymbol{\mathcal{P}}\mathbf{1}$, to a *generalised species*

$$!\mathbb{A} \longrightarrow \boldsymbol{\mathcal{P}}\mathbb{B} \tag{6}$$

for $\mathbb{A}$ and $\mathbb{B}$ groupoids (in fact, they can be arbitrary small categories but we do not use this generality here). Their main result is that these assemble into a bicategory of groupoids, generalised species, and natural transformations that is cartesian closed.

A generalised species $!\mathbb{A} \longrightarrow \boldsymbol{\mathcal{P}}\mathbb{B}$ induces an analytic functor $\boldsymbol{\mathcal{P}}\mathbb{A} \longrightarrow \boldsymbol{\mathcal{P}}\mathbb{B}$ between presheaf categories which we will recall in §7. This generalises Joyal's notion (4) to analytic functors between presheaf categories. These analytic functors have also been characterised extensionally [16].

4.2 Stable presheaves

We move towards the construction of a 2-category (Proposition 16) whose objects are groupoids with appropriate kits and whose morphisms are functors between full subcategories of presheaves with permitted stabilizers in kits (or *quantitative domains* [45]).

▶ **Definition 6.** *Let* $\mathbb{A}$ *be a groupoid equipped with a kit* $\mathcal{A}$*. A presheaf* $F : \mathbb{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ *is an* ***$\mathcal{A}$-stable presheaf*** *if every element of* F *has stabilizer in* $\mathcal{A}$*. The* ***category of $\mathcal{A}$-stable presheaves on*** $\mathbb{A}$*, a full subcategory of* $\boldsymbol{\mathcal{P}}(\mathbb{A})$*, is denoted* $\boldsymbol{\mathcal{S}}(\mathbb{A}, \mathcal{A})$*.*

Observe, in particular, that one may recover the whole presheaf category by permitting all subgroups: $\boldsymbol{\mathcal{S}}(\mathbb{A}, \mathcal{A}) = \boldsymbol{\mathcal{P}}(\mathbb{A})$ for the maximal kit $\mathcal{A}$ (that is, the one consisting of all subgroups of endomorphisms).

The category $\boldsymbol{\mathcal{S}}(\mathbb{A}, \mathcal{A})$ has a convenient and intuitive characterisation in terms of sums and quotients, which fits well with our goal of controlling quotients in polynomials. In the discussion below, we will use the following basic elements from the theory of presheaves:

- The *representable* presheaves on $\mathbb{A}$ are those which are naturally isomorphic to ones of the form $\mathbb{A}(-, a) : \mathbb{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ for some object $a \in \mathbb{A}$.
- There is a functor $\mathrm{y} : \mathbb{A} \longrightarrow \boldsymbol{\mathcal{P}}(\mathbb{A}) : a \mapsto \mathbb{A}(-, a)$, the *Yoneda embedding*, that is full and faithful.
- Presheaf categories have all small limits and colimits, which are calculated pointwise in terms of those in **Set**.
- Every presheaf is a canonical colimit of representable presheaves. In fact, the Yoneda embedding exhibits $\boldsymbol{\mathcal{P}}\mathbb{A}$ as the small colimit completion of $\mathbb{A}$.
- For presheaves on groupoids this last point can be strengthened considerably: every presheaf on a groupoid is a sum of quotients of representable presheaves by subgroups. In fact, for a groupoid $\mathbb{A}$, $\boldsymbol{\mathcal{P}}\mathbb{A}$ is the coproduct completion of the quotients of representable presheaves by subgroups (Theorem 9).

The above quotients are special colimits as follows: for an object a of a groupoid $\mathbb{A}$, the quotient of $\mathrm{y}(a)$ by a subgroup H of $\mathbb{A}(a, a)$ is the colimit $q : \mathrm{y}(a) \to \mathrm{y}(a)_{/H}$ of the diagram $H \hookrightarrow \mathbb{A} \overset{\mathrm{y}}{\hookrightarrow} \boldsymbol{\mathcal{P}}\mathbb{A}$ as depicted below:

$$\overset{\underset{(h \in H)}{\mathrm{y}(h)}}{\circlearrowleft}\ \mathrm{y}(a) \overset{q}{\dashrightarrow} \mathrm{y}(a)_{/H} \tag{7}$$

Concretely, the presheaf $\mathrm{y}(a)_{/H}$ maps an object x to the quotient of $\mathrm{y}(a)(x) = \mathbb{A}(x, a)$ under the equivalence relation $\sim_H$ given by $\alpha' \sim_H \alpha$ if and only if $\alpha' \alpha^{-1} \in H$. Quotienting under the trivial subgroup has no effect: $\mathrm{y}(a)_{/\{\mathrm{id}_a\}} \cong \mathrm{y}(a)$; while quotienting under the full group of endomorphisms gives the presheaf $\mathrm{y}(a)_{/\mathbb{A}(a,a)}$ that maps x to a singleton when $x \cong a$ and to the empty set otherwise.

Quotients and stabilizers are closely related.

▶ **Proposition 7.** *Let $\mathbb{A}$ be a groupoid and let H be a subgroup of $\mathbb{A}(a, a)$ for $a \in \mathbb{A}$. For all $x \in \mathbb{A}$ and $\alpha \in \mathrm{y}(a)_{/H}(x)$, $\mathrm{Stab}(\alpha) = H$.*

In particular, quotients of representable presheaves by subgroups in a kit are stable presheaves:

▶ **Corollary 8.** *Let $\mathcal{A}$ be a kit on a groupoid $\mathbb{A}$. For $a \in \mathbb{A}$ and $H \in \mathcal{A}(a)$, $\mathrm{y}(a)_{/H} \in \boldsymbol{\mathcal{S}}(\mathbb{A}, \mathcal{A})$.*

More generally, we have a representation theorem for stable presheaves as follows:

▶ **Theorem 9.** *Let $\mathbb{A}$ be a groupoid equipped with a kit $\mathcal{A}$. Then, assuming the axiom of choice, every presheaf X in $\boldsymbol{\mathcal{S}}(\mathbb{A}, \mathcal{A})$ is isomorphic to a sum of quotients of representable presheaves by subgroups in $\mathcal{A}$; that is,*

$$X \cong \sum_{i \in I} \mathrm{y}(a_i)_{/H_i}$$

for some I-indexed family $\{ (a_i, H_i) \}_{i \in I}$ of objects $a_i \in \mathbb{A}$ and groups $H_i \in \mathcal{A}(a_i)$. Conversely, every presheaf of this form is in $\boldsymbol{\mathcal{S}}(\mathbb{A}, \mathcal{A})$.

Therefore, kits provide a concrete way of restricting the coproduct completion of quotients of representable presheaves by subgroups, yielding the stable presheaves.

4.3 Stable functors

We will consider *stable functors* (Definition 10) of type $\boldsymbol{\mathcal{S}}(\mathbb{A},\mathcal{A}) \longrightarrow \boldsymbol{\mathcal{S}}(\mathbb{B},\mathcal{B})$, our notion of finitary polynomial functor generalised to categories of stable presheaves. Our definition is extensional and abstract enough to be stated generally:

▶ **Definition 10.** *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. We call a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$* ***stable*** *if it satisfies the following conditions:*

- *F is a local right adjoint.*
- *F is finitary (that is, preserves filtered colimits).*
- *F preserves regular epimorphisms.*

We comment on the definition. The local right adjoint condition is an extensional presentation of polynomial functors (recall Definition 5). The restriction to finitary functors is natural in the context of semantic models of higher-order computation; it is also essential because our proof of cartesian closure (Theorems 22 and 32) of the bicategory **Stable** (introduced in Proposition 16 below) is based on a representation by means of finitary coefficients in the style of (4) and (6). The preservation of regular epimorphisms crucially ensures the preservation of quotient maps (7). This condition is not necessary for finitary polynomial functors of type $\mathbf{Set}^I \longrightarrow \mathbf{Set}^J$, which always preserve regular epimorphisms (as do arbitrary such polynomial functors assuming the axiom of choice).

▶ **Example 11.** In a cartesian closed and extensive category, such as a topos, the finite product and finite coproduct functors are stable. In connection to this, we discuss the prototypical non-stable, and hence non-sequential, function from Berry's stable domain theory [9]. To this end, let S be the Sierpinski space $(0 \subset 1)$ and, for $\mathrm{Bool} = \{\mathbf{f},\mathbf{t}\}$, consider the *parallel-or* function $\mathit{por} : S^{\mathrm{Bool}} \times S^{\mathrm{Bool}} \to S^{\mathrm{Bool}}$ defined as the least monotone function such that: $\mathit{por}\big(\{\mathbf{f}\},\{\mathbf{f}\}\big) = \{\mathbf{f}\}$ and $\mathit{por}\big(\{\mathbf{t}\},\{\,\}\big) = \mathit{por}\big(\{\,\},\{\mathbf{t}\}\big) = \{\mathbf{t}\}$.

Parallel-or is not realisable as a stable functor at the categorical level in the strong sense that there is no stable functor $F : \mathbf{Set}^{\mathrm{Bool}} \times \mathbf{Set}^{\mathrm{Bool}} \longrightarrow \mathbf{Set}^{\mathrm{Bool}}$ such that $F\big(0,0\big) = 0$ and $F\big(\mathrm{y}(\mathbf{t}),0\big) = F(0,\mathrm{y}(\mathbf{t})) = F(\mathrm{y}(\mathbf{t}),\mathrm{y}(\mathbf{t})) = \mathrm{y}(\mathbf{t})$. Indeed, such functors do not preserve the pullback

$$\begin{array}{ccccc} & & \big(\mathrm{y}(\mathbf{t}),\mathrm{y}(\mathbf{t})\big) & & \\ & \nearrow & & \nwarrow & \\ \big(\mathrm{y}(\mathbf{t}),0\big) & & & & \big(0,\mathrm{y}(\mathbf{t})\big) \\ & \nwarrow & & \nearrow & \\ & & \big(0,0\big) & & \end{array}$$

and thus induce local functors $F/(\mathrm{y}(\mathbf{t}),\mathrm{y}(\mathbf{t})) : \mathbf{Set}^{\mathrm{Bool}} \times \mathbf{Set}^{\mathrm{Bool}}/(\mathrm{y}(\mathbf{t}),\mathrm{y}(\mathbf{t})) \longrightarrow \mathbf{Set}^{\mathrm{Bool}}/\mathrm{y}(\mathbf{t})$ that fail to be right adjoints.

However, the generalisation from domains to categories allows for an intensional quantitative interpretation of parallel or. Indeed, for $K : \mathbf{Set} \longrightarrow S$ the *collapse* functor mapping a set to 0 if it is empty and to 1 otherwise, we have a stable functor

$$P(X,Y) = (X_{\mathbf{f}} \times Y_{\mathbf{f}}, X_{\mathbf{t}} + Y_{\mathbf{t}})$$

lifting *por* as follows:

$$\begin{array}{ccc} \mathbf{Set}^{\mathrm{Bool}} \times \mathbf{Set}^{\mathrm{Bool}} & \xrightarrow{\;P\;} & \mathbf{Set}^{\mathrm{Bool}} \\ {\scriptstyle K^{\mathrm{Bool}} \times K^{\mathrm{Bool}}}\big\downarrow & & \big\downarrow{\scriptstyle K^{\mathrm{Bool}}} \\ S^{\mathrm{Bool}} \times S^{\mathrm{Bool}} & \xrightarrow[\mathit{por}]{} & S^{\mathrm{Bool}} \end{array}$$

5 Boolean kits and the 2-category of stable functors

Kits and negation

We have deliberately introduced a general notion of kit to emphasise its fundamental role in controlling stabilizers and quotients. In practice, however, one need impose conditions on kits to ensure desirable structure in the induced categories of stable presheaves (see Proposition 15). This may be done in several ways and we concentrate here on a principled approach based on a notion of *logical negation* described next.

▶ **Proposition 12.** *Let $\mathcal{A}$ be a kit on a groupoid $\mathbb{A}$. The following definition, for $a \in \mathbb{A}$,*

$$\mathcal{A}^{\perp}(a) = \left\{ H \;\mid\; H \text{ is a subgroup of } !\mathbb{A}(u,u) \text{ satisfying: for all } G \in \mathcal{A}(a),\ G \cap H = \{\mathrm{id}_a\} \right\}$$

yields a kit $\mathcal{A}^{\perp}$ called the ***negation*** *(or* ***dual****) of $\mathcal{A}$.*

▶ **Definition 13.** *A* ***Boolean kit*** *is a kit $\mathcal{A}$ such that*

$$\mathcal{A}^{\perp\perp} = \mathcal{A}. \tag{8}$$

Boolean kits have useful closure properties:

▶ **Lemma 14.** *Let $\mathcal{A}$ be a Boolean kit on a groupoid $\mathbb{A}$. For every $a \in \mathbb{A}$, the set $\mathcal{A}(a)$ is closed under subgroups and under directed unions.*

From here, one can show that the category of stable presheaves induced by a Boolean kit has a rich structure inherited from the presheaf category.

▶ **Proposition 15.** *Let $\mathcal{A}$ be a kit on a groupoid $\mathbb{A}$. The category $\boldsymbol{\mathcal{S}}(\mathbb{A},\mathcal{A})$ is closed under isomorphisms and coproducts taken in $\boldsymbol{\mathcal{P}}(\mathbb{A})$. If the kit $\mathcal{A}$ is Boolean, then $\boldsymbol{\mathcal{S}}(\mathbb{A},\mathcal{A})$ additionally inherits filtered colimits and all nonempty limits from $\boldsymbol{\mathcal{P}}(\mathbb{A})$. Furthermore, the terminal presheaf is in $\boldsymbol{\mathcal{S}}(\mathbb{A},\mathcal{A})$ if and only if $\boldsymbol{\mathcal{S}}(\mathbb{A},\mathcal{A}) = \boldsymbol{\mathcal{P}}(\mathbb{A})$.*

The 2-category of Boolean kits and stable functors

The conditions defining stable functors (Definition 10) are preserved under composition, and identity functors are stable. Just like for polynomial endofunctors on sets, we will consider *cartesian* natural transformations (Definition 1) between stable functors.

▶ **Proposition 16.** *The following data forms a 2-category, called* **Stable***:*

- objects $(\mathbb{A},\mathcal{A})$*: groupoids with Boolean kits.*
- 1-cells $(\mathbb{A},\mathcal{A}) \to (\mathbb{B},\mathcal{B})$*: stable functors $\boldsymbol{\mathcal{S}}(\mathbb{A},\mathcal{A}) \longrightarrow \boldsymbol{\mathcal{S}}(\mathbb{B},\mathcal{B})$.*
- 2-cells*: cartesian natural transformations.*

Compositions and identities in **Stable** *are defined as for functors and natural transformations.*

The rest of the paper is devoted to the development and study of a bicategory of *stable species* of structures (Proposition 21) that provides an equivalent combinatorial view of **Stable**. The ultimate objective is our main theorem:

▶ **Theorem 17.** **Stable** *is a cartesian closed bicategory.*

Note the distinction between 2-categories, in which morphisms compose strictly, and bicategories, in which there are structural 2-cells in place of associativity and identity laws. This terminology extends to cartesian closed structure: **Stable** is cartesian as a 2-category but cartesian closed as a bicategory, since currying and uncurrying are up to isomorphism.

6 The cartesian closed bicategory of stable species

As a path to Theorem 17, this section introduces *stable species*: a combinatorial presentation of stable functors. Our methodology relies on generalised species [16], which we will enrich with kits.

The bicategory of generalised species

A *generalised species* [18] from $\mathbb{A}$ to $\mathbb{B}$ is a functor $\mathbb{B}^{\mathrm{op}} \times !\mathbb{A} \longrightarrow \mathbf{Set}$ (or, equivalently, a functor $!\mathbb{A} \longrightarrow \mathcal{P}\mathbb{B}$) where $!\mathbb{A}$ is the *symmetric strict monoidal completion* of $\mathbb{A}$. The construction of $!\mathbb{A}$ generalises the groupoid $\mathbf{B}$, which arises as $!\mathbf{1}$, and is motivated by the desire of passing from linear to cartesian higher-order structure.

The objects of $!\mathbb{A}$ are finite sequences $\langle a_1, \dots, a_n \rangle$ ($n \in \mathbb{N}$) of objects of $\mathbb{A}$ and a morphism

$$\alpha : \langle a_1, \dots, a_m \rangle \longrightarrow \langle b_1, \dots b_n \rangle$$

consists of a pair $\big(\underline{\alpha}, (\alpha_i)_{i \in [n]}\big)$ where $\underline{\alpha} \in \mathbf{B}(m, n)$ and $\alpha_i : a_i \to b_{\underline{\alpha}(i)}$ is a morphism in $\mathbb{A}$ for every $i \in [m]$. The concatenation of sequences gives a monoidal tensor $(u, v) \longmapsto u \otimes v$ for $!\mathbb{A}$ having the empty list as monoidal unit.

It is helpful to understand the bicategory of generalised species in terms of *profunctors* (alternatively, *bimodules* or *distributors*) [50, 6, 7, 37]. A profunctor from $\mathbb{A}$ to $\mathbb{B}$, denoted $\mathbb{A} \nrightarrow \mathbb{B}$, is a functor $\mathbb{B}^{\mathrm{op}} \times \mathbb{A} \longrightarrow \mathbf{Set}$. Small categories, profunctors, and natural transformations form a symmetric monoidal (compact) closed bicategory $\mathbf{Prof}$ with tensor product $\times$ and internal hom $\mathbb{A} \multimap \mathbb{B} := \mathbb{A}^{\mathrm{op}} \times \mathbb{B}$.

A generalised species from $\mathbb{A}$ to $\mathbb{B}$ is then simply a profunctor $!\mathbb{A} \nrightarrow \mathbb{B}$ and, in fact, the bicategory of species is a coKleisli bicategory for $!$ as a pseudo-comonad on $\mathbf{Prof}$ [18, 13]. The identity species $\mathrm{id}_{\mathbb{A}} : !\mathbb{A} \nrightarrow \mathbb{A}$ is the functor mapping a pair $(a, u) \in \mathbb{A}^{\mathrm{op}} \times !\mathbb{A}$ to the set

$$!\mathbb{A}(\langle a \rangle, u) \;\cong\; \begin{cases} \mathbb{A}(a, a') & \text{, if } u = \langle a' \rangle \\ \varnothing & \text{, otherwise} \end{cases}$$

The composition of species $F : !\mathbb{A} \nrightarrow \mathbb{B}$ and $G : !\mathbb{B} \nrightarrow \mathbb{C}$ is the species $G \circ F : !\mathbb{A} \nrightarrow \mathbb{C}$ that maps a pair $(c, u) \in \mathbb{C}^{\mathrm{op}} \times !\mathbb{A}$ to the set

$$\int^{v = \langle b_1, \dots, b_n \rangle \in !\mathbb{B}} G(c, v) \times \int^{u_1, \dots, u_n \in !\mathbb{A}} \prod_{i=1}^{n} F(b_i, u_i) \times !\mathbb{A}(u_1 \otimes \cdots \otimes u_n, u)$$

We call $\mathbf{Esp}$ the bicategory of groupoids, generalised species, and natural transformations. This is the restriction to groupoids of the bicategory of generalised species of structures defined in [18], whose objects can be arbitrary small categories.

Stable species of structures

We now introduce a new bicategory $\mathbf{SEsp}$ (Proposition 21) whose objects are groupoids with kits and whose morphisms are generalised species with action restricted by the kits; we call these *stable species*.

We start by extending the $!$ construction to kits, so that for every groupoid with kit $(\mathbb{A}, \mathcal{A})$ we can set $!(\mathbb{A}, \mathcal{A}) \stackrel{\text{def}}{=} (!\mathbb{A}, !\mathcal{A})$. To define $!\mathcal{A}(u)$ for an object $u = \langle a_1, \dots, a_n \rangle \in !\mathbb{A}$, we need a preliminary definition. Recall that an endomorphism α on u in $!\mathbb{A}$ is a pair consisting of a permutation $\underline{\alpha} \in \mathfrak{S}_n$ and a sequence $(\alpha_i : a_i \to a_{\underline{\alpha}(i)})_{i \in [n]}$ of morphisms in $\mathbb{A}$. For every $i \in [n]$, define the endomorphism loop_i^{α} on a_i in $\mathbb{A}$ as the composite

$$a_i \xrightarrow{\alpha_i} a_{\underline{\alpha}(i)} \xrightarrow{\alpha_{\underline{\alpha}(i)}} a_{\underline{\alpha}^2(i)} \longrightarrow \cdots \longrightarrow a_{\underline{\alpha}^{o(i)-1}(i)} \xrightarrow{\alpha_{\underline{\alpha}^{o(i)-1}(i)}} a_{\underline{\alpha}^{o(i)}(i)} = a_i$$

where $o(i)$ is the smallest positive integer such that $\underline{\alpha}^{o(i)}(i) = i$. Equivalently, $o(i)$ is the length of the cycle containing i in the disjoint cycle decomposition of the permutation $\underline{\alpha}$.

▶ **Definition 18.** *Let $(\mathbb{A}, \mathcal{A})$ be a groupoid with a kit. For $u = \langle a_1, \ldots, a_n \rangle \in !\mathbb{A}$, we define*

$$!\mathcal{A}(u) \stackrel{def}{=} \{\, H \mid H \text{ is a subgroup of } !\mathbb{A}(u,u) \text{ satisfying } \forall\, \alpha \in H.\, \forall\, i \in [n].\, \mathrm{loop}_i^\alpha \in \bigcup \mathcal{A}(a_i) \,\}^{\perp\perp}.$$

The closure under double dual directly ensures that we obtain a Boolean kit $!\mathcal{A} = \{\, !\mathcal{A}(u) \,\}_{u \in !\mathbb{A}}$ on $!\mathbb{A}$.

We now define stable species between groupoids with kits. As for the $\mathcal{K}$-species of Definition 4, our definition involves stabilizers controlled by kits. Note that, for a generalised species $F : !\mathbb{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathbb{B}$, the stabilizer $\mathrm{Stab}_F(p)$ of an element $p \in F(b,u)$ is a subgroup of $\mathbb{B}(b,b) \times !\mathbb{A}(u,u)$.

▶ **Definition 19.** *Let $(\mathbb{A}, \mathcal{A})$ and $(\mathbb{B}, \mathcal{B})$ be groupoids with kits. A* ***stable species*** *from $(\mathbb{A}, \mathcal{A})$ to $(\mathbb{B}, \mathcal{B})$ is a generalised species $F : !\mathbb{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathbb{B}$ such that, for every $b \in \mathbb{B}$, $u \in !\mathbb{A}$ and $p \in F(b,u)$, if $(\beta, \alpha) \in \mathrm{Stab}_F(p)$ then:*

$$\alpha \in \bigcup !\mathcal{A}(u) \Rightarrow \beta \in \bigcup \mathcal{B}(b) \quad \textit{and} \quad \beta \in \bigcup \mathcal{B}^\perp(b) \Rightarrow \alpha \in \bigcup (!\mathcal{A})^\perp(u). \tag{9}$$

In special cases we recover previous concepts:

Stable presheaves. The initial groupoid $\mathbf{0}$ has a unique, empty, Boolean kit. Moreover, $!\mathbf{0}$ is the terminal groupoid, which also admits a unique Boolean kit. It follows that stable species from $(\mathbf{0}, \varnothing)$ to any $(\mathbb{A}, \mathcal{A})$ correspond to presheaves in $\boldsymbol{\mathcal{S}}(\mathbb{A}, \mathcal{A})$.

Free Joyal species. Let $\mathcal{K}_1$ be the unique Boolean kit on the terminal groupoid $\mathbf{1}$. Unfolding Definition 18, we get that $!\mathcal{K}_1$ is the maximal kit (consisting of all subgroups) on $!\mathbf{1} = \mathbf{B}$ with dual kit $(!\mathcal{K}_1)^\perp$ the minimal Boolean kit (consisting of trivial subgroups). Then, the stable species from $(\mathbf{1}, \mathcal{K}_1)$ to $(\mathbf{1}, \mathcal{K}_1)$ coincide with the free species $\mathbf{B} \longrightarrow \mathbf{Set}$.

More generally, by considering pairs of endomorphisms of the form (β, id_u) and (id_b, α) in the two implications of (9) above, and observing that Boolean kits always contain identities, we obtain the following two properties:

▶ **Lemma 20.** *Let F be a stable species from $(\mathbb{A}, \mathcal{A})$ to $(\mathbb{B}, \mathcal{B})$ with $\mathcal{A}$ and $\mathcal{B}$ Boolean.*

($\mathcal{B}$-stability) *The corresponding functor $F : !\mathbb{A} \to \boldsymbol{\mathcal{P}}(\mathbb{B})$ factors through the inclusion $\boldsymbol{\mathcal{S}}(\mathbb{B}, \mathcal{B}) \hookrightarrow \boldsymbol{\mathcal{P}}(\mathbb{B})$.*

($!\mathcal{A}$-freeness) *For $\alpha \in !\mathbb{A}(u,u)$, if $F\alpha \in \boldsymbol{\mathcal{P}}\mathbb{B}(Fu, Fu)$ fixes an element of the presheaf Fu, then $\alpha \in (!\mathcal{A})^\perp(u)$.*

One verifies the following directly:

▶ **Proposition 21.** *The following data forms a bicategory, called* **SEsp***:*

- objects $(\mathbb{A}, \mathcal{A})$*: groupoids with Boolean kits;*
- 1-cells $(\mathbb{A}, \mathcal{A}) \to (\mathbb{B}, \mathcal{B})$*: stable species* $!(\mathbb{A}, \mathcal{A}) \longrightarrow\!\!\!\!\!\!\mapsto (\mathbb{B}, \mathcal{B})$*;*
- 2-cells*: natural transformations.*

Compositions and identities in **SEsp** *are defined as for generalised species.*

Cartesian closed structure in the bicategory of stable species

The cartesian closed categorical structure of **SEsp** extends that in **Esp** without difficulty:

Cartesian products. The bicategory **Esp** has finite products given by the finite sum of groupoids. For a finite family of groupoids with kits $\{ (\mathbb{A}_i, \mathcal{A}_i) \}_{i \in [n]}$, the sum groupoid $\mathbb{A}_1 + \cdots + \mathbb{A}_n$ has boolean kit $\mathcal{A}_1 + \cdots + \mathcal{A}_n$ defined as $(\mathcal{A}_1 + \cdots + \mathcal{A}_n)(\imath_i(a)) = \mathcal{A}_i(a)$ where $\imath_i$ is the coproduct inclusion. This contruction endows **SEsp** with a finite product structure.

Higher-order structure. We recall the situation in **Esp** [18]. For groupoids $\mathbb{A}$ and $\mathbb{B}$, the function space $\mathbb{A} \Rightarrow \mathbb{B}$ is defined as $!\mathbb{A} \multimap \mathbb{B}$. The proof of cartesian closure relies on a fundamental canonical equivalence

$$!\mathbb{A} \times !\mathbb{B} \simeq !(\mathbb{A} + \mathbb{B})$$

given by the composite

$$!\mathbb{A} \times !\mathbb{B} \xrightarrow{!(\imath_1) \times !(\imath_2)} !(\mathbb{A} + \mathbb{B}) \times !(\mathbb{A} + \mathbb{B}) \xrightarrow{\otimes} !(\mathbb{A} + \mathbb{B})$$

and the symmetric monoidal extension of the functor

$$\mathbb{A} + \mathbb{B} \to !\mathbb{A} \times !\mathbb{B} : \begin{cases} \imath_1(a) \longmapsto (\langle a \rangle, \varnothing) & \text{, for } a \in \mathbb{A} \\ \imath_2(b) \longmapsto (\varnothing, \langle b \rangle) & \text{, for } b \in \mathbb{B} \end{cases}$$

With this equivalence, we have a chain of equivalences of hom-categories as follows:

$$\mathbf{Esp}(\mathbb{C}, \mathbb{A} \Rightarrow \mathbb{B}) = \mathbf{Prof}(!\mathbb{C}, !\mathbb{A} \multimap \mathbb{B}) \cong \mathbf{Prof}(!\mathbb{C} \times !\mathbb{A}, \mathbb{B}) \simeq \mathbf{Prof}(!(\mathbb{C} + \mathbb{A}), \mathbb{B}) = \mathbf{Esp}(\mathbb{C} + \mathbb{A}, \mathbb{B})$$

This provides the required *currying/uncurrying* pseudo-natural equivalence, which we must extend to **SEsp**. This we achieve by setting $(\mathbb{A}, \mathcal{A}) \Rightarrow (\mathbb{B}, \mathcal{B}) = (\mathbb{A} \Rightarrow \mathbb{B}, \mathcal{A} \Rightarrow \mathcal{B})$ where $(\mathcal{A} \Rightarrow \mathcal{B})(u, b)$ consists of all the subgroups H of $!\mathbb{A}(u, u) \times \mathbb{B}(b, b)$ such that (9) is satisfied for every $(\alpha, \beta) \in H$.

▶ **Theorem 22.** *The bicategory* **SEsp** *is cartesian closed.*

▶ Remark. To establish the theorem in full rigour we first develop a theory of profunctors underlying stable species. This relies on a notion of *stabilised profunctor* between groupoids with Boolean kits and on the fact that the ! construction is a linear exponential pseudo-comonad over an associated $\star$-autonomous bicategory **SProf**. This is a fairly technical development but along expected lines, given the linear exponential pseudo-comonad structure of ! on **Prof** [18, 13] and our construction of Boolean kits based on negation. This will be developed fully in a companion paper, exploring in depth the connections with linear logic and linear negation.

7 The biequivalence between stable species and stable functors

We show that stable species and stable functors are alternative presentations of the same model. Formally, we establish a bicategorical equivalence **SEsp** $\simeq$ **Stable** (Theorem 32). From stable species to stable functors, we study the class of analytic functors induced by stable species and show that they are stable. In the opposite direction, we show how to recover the coefficients of a stable species from a stable functor in terms of generic factorisations.

7.1 From stable species to stable functors

The *analytic functor* induced by a generalised species $P : !\mathbb{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathbb{B}$ is the functor $\boldsymbol{\mathcal{P}}(\mathbb{A}) \longrightarrow \boldsymbol{\mathcal{P}}(\mathbb{B})$ that maps a presheaf $X \in \boldsymbol{\mathcal{P}}(\mathbb{A})$ to the presheaf on $\mathbb{B}$ defined, for $b \in \mathbb{B}$, as

$$b \longmapsto \int^{u=\langle a_1,\ldots,a_n\rangle \in !\mathbb{A}} P(b,u) \times \prod_{i=1}^{n} X(a_i) \tag{10}$$

This formula, and indeed the earlier one for Joyal species (4), are obtained through the universal construction of *left Kan extension*:

▶ **Definition 23.** *For groupoids $\mathbb{A}, \mathbb{B}$, and a generalised species $P : !\mathbb{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathbb{B}$, the analytic functor described pointwise above is a left Kan extension of P along the functor $s_{\mathbb{A}} : !\mathbb{A} \longrightarrow \boldsymbol{\mathcal{P}}(\mathbb{A}) : \langle a_1, \ldots, a_n\rangle \longmapsto \sum_{i=1}^{n} \mathrm{y}(a_i)$, as on the left below:*

For $\mathbb{A} = \mathbb{B} = \mathbf{1}$ and P viewed as a species $\mathbf{B} \longrightarrow \mathbf{Set}$, this corresponds to the diagram on the right above, where j is the inclusion functor, and induces the formula (4)*.*

Given a stable species from $(\mathbb{A}, \mathcal{A})$ to $(\mathbb{B}, \mathcal{B})$, one can apply the formula (10) to obtain a stable functor $\boldsymbol{\mathcal{S}}(\mathbb{A}, \mathcal{A}) \longrightarrow \boldsymbol{\mathcal{S}}(\mathbb{B}, \mathcal{B})$.

▶ **Proposition 24.** *Let $P : !(\mathbb{A}, \mathcal{A}) \longrightarrow\!\!\!\!\!\!\mapsto (\mathbb{B}, \mathcal{B})$ in* **SEsp***. The restricted left Kan extension $\boldsymbol{\mathcal{S}}(\mathbb{A}, \mathcal{A}) \hookrightarrow \boldsymbol{\mathcal{P}}(\mathbb{A}) \xrightarrow{\mathrm{Lan}_{s_{\mathbb{A}}} P} \boldsymbol{\mathcal{P}}(\mathbb{B})$ factors through the inclusion $\boldsymbol{\mathcal{S}}(\mathbb{B}, \mathcal{B}) \hookrightarrow \boldsymbol{\mathcal{P}}(\mathbb{B})$. Furthermore, the resulting functor $\widetilde{P} : \boldsymbol{\mathcal{S}}(\mathbb{A}, \mathcal{A}) \longrightarrow \boldsymbol{\mathcal{S}}(\mathbb{B}, \mathcal{B})$ is stable.*

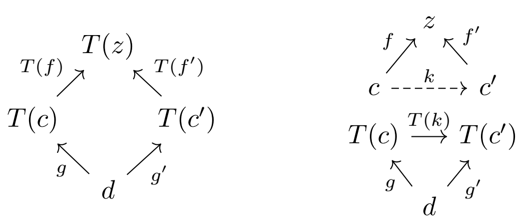
To extend the mapping $\widetilde{(-)}$ to a functor $\mathbf{SEsp}\big((\mathbb{A}, \mathcal{A}), (\mathbb{B}, \mathcal{B})\big) \longrightarrow \mathbf{Stable}\big((\mathbb{A}, \mathcal{A}), (\mathbb{B}, \mathcal{B})\big)$, we verify that natural transformations between stable species are mapped to cartesian natural transformations between stable functors:

▶ **Proposition 25.** *Let $(\mathbb{A}, \mathcal{A}), (\mathbb{B}, \mathcal{B})$ be groupoids with Boolean kits and let $f : P \to Q$ be a natural transformation between stable species $P, Q : !(\mathbb{A}, \mathcal{A}) \longrightarrow\!\!\!\!\!\!\mapsto (\mathbb{B}, \mathcal{B})$. The natural transformation $\widetilde{f} : \widetilde{P} \to \widetilde{Q} : \boldsymbol{\mathcal{S}}(\mathbb{A}, \mathcal{A}) \longrightarrow \boldsymbol{\mathcal{S}}(\mathbb{B}, \mathcal{B})$, canonically induced by left Kan extension, is cartesian.*

7.2 From stable functors to stable species

We have given the components of a pseudo-functor $\widetilde{(-)} : \mathbf{SEsp} \longrightarrow \mathbf{Stable}$. We proceed to show that it is part of a biequivalence, and construct a pseudo-inverse $\mathbf{Stable} \longrightarrow \mathbf{SEsp}$. We rely on a characterisation of stable functors in terms of *generic morphisms*:

▶ **Definition 26.** *Let $T : \mathscr{C} \to \mathscr{D}$ be a functor between categories $\mathscr{C}$ and $\mathscr{D}$. A morphism $g : d \to T(c)$ in $\mathscr{D}$ is called* **generic** *if, for every commuting square as on the left below*



there exists a unique morphism $k : c \to c'$ in $\mathscr{C}$ making the two triangles on the right above commute. The functor T is said to ***admit generic factorisations*** *if every morphism $h : d \to T(z)$ has a factorisation $h = T(f) \circ g$ for some $f : c \to z$ and $g : d \to T(c)$ with g generic.*

Generic morphisms (elsewhere known as *candidates* [45] or *strict generic* [49], and a strict version of the *generic elements* considered in [32, 16]) provide the elements in the construction of a stable species from a stable functor. They also correspond to the *normal forms* studied in [26, 28]. The presence of generic factorisations is closely related to the existence of local left adjoints, and we have the following:

▶ **Proposition 27.** *Let $(\mathbb{A}, \mathcal{A})$ and $(\mathbb{B}, \mathcal{B})$ be groupoids with Boolean kits. A functor from $\boldsymbol{\mathcal{S}}(\mathbb{A}, \mathcal{A})$ to $\boldsymbol{\mathcal{S}}(\mathbb{B}, \mathcal{B})$ is stable if and only if admits generic factorisations, is finitary, and preserves regular epimorphisms.*

We use this to show that the following operation provides a pseudo-inverse to $P \longmapsto \widetilde{P}$:

▶ **Definition 28.** *For $(\mathbb{A}, \mathcal{A})$ and $(\mathbb{B}, \mathcal{B})$ groupoids with Boolean kits, the* ***trace*** *of a functor $T : \boldsymbol{\mathcal{S}}(\mathbb{A}, \mathcal{A}) \longrightarrow \boldsymbol{\mathcal{S}}(\mathbb{B}, \mathcal{B})$ is the generalised species $\mathrm{Tr}(T) : !\mathbb{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{B}$ with object mapping*

$$(b, u) \longmapsto \left\{\, g : \mathrm{y}_{\mathbb{B}}(b) \to T(s_{\mathbb{A}} u) \text{ in } \boldsymbol{\mathcal{P}}(\mathbb{B}) \;\middle|\; g \text{ is generic} \,\right\}$$

and functorial action given by composition (which is well-defined because genericity is invariant under isomorphism).

▶ **Proposition 29.** *For a functor T in* $\mathbf{Stable}\big((\mathbb{A}, \mathcal{A}), (\mathbb{B}, \mathcal{B})\big)$*, the species* $\mathrm{Tr}(T)$ *is in* $\mathbf{SEsp}\big((\mathbb{A}, \mathcal{A}), (\mathbb{B}, \mathcal{B})\big)$*.*

Cartesian natural transformations preserve and reflect generic morphisms [49]. Thus, one can immediately extend the trace operation to 2-cells:

▶ **Proposition 30.** *For a cartesian natural transformation $f : S \to T$ in* $\mathbf{Stable}\big((\mathbb{A}, \mathcal{A}), (\mathbb{B}, \mathcal{B})\big)$*, the mapping*

$$\mathrm{Tr}(f)_{(b,u)} : \big(\, g : \mathrm{y}_{\mathbb{B}}(b) \to S(s_{\mathbb{A}} u) \,\big) \;\longmapsto\; \big(\, f_{s_{\mathbb{A}}(u)} \circ g : \mathrm{y}_{\mathbb{B}}(b) \to T(s_{\mathbb{A}} u) \,\big) \qquad (b \in \mathbb{B}, u \in !\mathbb{A})$$

provides the components of a natural transformation $\mathrm{Tr}(f) : \mathrm{Tr}(S) \to \mathrm{Tr}(T)$*.*

It remains to exhibit local equivalences between the corresponding hom-categories:

▶ **Lemma 31.**
1. *For a stable species $F \in$* $\mathbf{SEsp}\big((\mathbb{A}, \mathcal{A}), (\mathbb{B}, \mathcal{B})\big)$*,* $\mathrm{Tr}(\widetilde{F}) \cong F$*.*
2. *For a stable functor $S \in$* $\mathbf{Stable}\big((\mathbb{A}, \mathcal{A}), (\mathbb{B}, \mathcal{B})\big)$*,* $\widetilde{\mathrm{Tr}(S)} \cong S$*.*

▶ **Theorem 32.** *There is a biequivalence of bicategories* $\mathbf{SEsp} \simeq \mathbf{Stable}$*.*

As biequivalences preserve cartesian closed structure, our main Theorem 17 is a corollary of Theorem 22.

8 Conclusion

We have defined a new cartesian closed bicategorical model that we have presented independently in combinatorial and extensional forms, respectively as:

- stable species of structures between groupoids with Boolean kits, and
- stable functors between categories of stable presheaves over groupoids with Boolean kits.

Restricting our extensional model to discrete groupoids, one precisely obtains finitary polynomial functors $\mathbf{Set}^I \to \mathbf{Set}^J$ between categories of indexed sets, corresponding to Girard's *normal functors* [26]. Polynomial functors of this type (finitary or not) are typically represented in combinatorial form as *polynomials* in **Set**; namely, diagrams $(I \leftarrow E \rightarrow B \rightarrow J)$ of sets and functions representing operators with sorted (or coloured) arities (§2) [22, 4]. In the finitary case it is not hard to translate between these and our stable species representation.

However, extending the above to incorporate higher-order structure seems to require moving on to the general context of groupoids: even when I and J are discrete groupoids, the function space $I \Rightarrow J = !I^{\mathrm{op}} \times J$ is *not* discrete, and to remain within polynomials one must make explicit and control the groupoid action.

In recent work, Finster, Lucas, Mimram and Seiller [12] present another groupoid model, which they describe in the language of homotopy type theory. The relationship with our model should be considered, also in connection to the work of Kock et al. [33, 23].

Connections with stable domain theory

Our model provides a form of generalised domain theory in which continuous functions between domains are generalised to finitary functors between domain-like categories (see e.g. [3]); this fits in the general research programme outlined in [29]. Specifically, we have a generalised form of *stable* domain theory in the sense of Berry [9] and Girard [24]; stable functions are finitary local right adjoints between stable domains and Berry's stable order amounts to unique degenerate cartesian natural transformations.

In our endeavour, we follow Lamarche [35] and Taylor [45, 46], who in the 1980s pioneered the categorification of stable domain theory. Taylor's work is especially relevant: he introduced *creeds*, a combinatorial structure on groupoids used to control actions at higher order, and even raised the thought of a connection with Joyal's ideas [45, page 172]. Whilst our Boolean kits are rather different from creeds, the present work recasts these ideas in a modern structural combinatorial setting and bicategorical language, suggesting new avenues for research.

Further work on bicategorical models of linear logic and differentiation

The bicategory **SEsp** is obtained from a bicategorical model of classical linear logic **SProf**, whose theory we will present in a future paper. A question we are investigating is whether this model can be obtained through a bicategorical glueing construction by means of an orthogonality technique [30].

Another promising direction is the study of formal differentiation. In this respect, there are likely connections with several lines of work, including: differentiation for polynomial functors in type theory [39, 2, 27, 15]; differentiation for analytic functors in combinatorics [8, 34]; and differential linear logic [11, 14, 17, 10] of which **SProf** is a bicategorical model.

For combinatorial species, formal differentiation gives rise to formal integration and to the study of differential equations from that perspective [38]. Free species, which can always be integrated [34], are most useful in that context; our bicategory **SEsp** is a promising setting for extending these notions to a higher-order logical setting.

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