A Mathematical Comparison Between Response-Time Analysis and Real-Time Calculus for Fixed-Priority Preemptive Scheduling

Victor Pollex
INCHRON AG, Erlangen, Germany

Frank Slomka
Institute of Embedded Systems/Real-Time Systems, Faculty of Engineering and Computer Science, Universität Ulm, Germany

Abstract

Fixed-priority preemptive scheduling is a popular scheduling scheme for real-time systems. This is accompanied by a vast amount of research on how to analyse and check whether these systems satisfy their real-time requirements. Two methods that emerged from this research are the response-time analysis and the real-time calculus. These two methods have been compared empirically on the basis of several abstract systems showing that for some systems one method gives better results than the other and for other systems both methods appear to give the same results. However, empirical analyses inherently contain uncertainty. To get a definitive answer we compare both methods mathematically and we show that both methods give the same results for systems that use fixed-priority preemptive scheduling and independent tasks.

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1 Introduction

For real-time systems it is necessary to verify that they meet their real-time requirements. Two of the methods that have emerged to verify these systems are the response-time analysis (RTA) and the real-time calculus (RTC). The response-time analysis originates from a proof [12, Theorem 5] that shows when a real-time system that periodically runs a set of independent tasks will always produce results on time. Whereas the origin of the real-time calculus is a mathematical framework [6, 7] to find a bound for the delay that a data stream is subjected to when flowing through a packet switched network.

Over time large amounts of work was produced regarding the response-time analysis and the real-time calculus that covers, among other things, different scheduling algorithms, different patterns on how tasks recur, and dependencies between tasks as well as empirical comparisons.

In distributed real-time systems where the activation of tasks can follow a complex pattern, the real-time calculus, due to its more expressive model, appears to produce the same or better results than the response-time analysis [14, Benchmark 1]. When the distributed real-time system is a feedback loop, then both methods appear to produce the same results [14, Benchmark 2]. However, when the distributed real-time system has a cyclic dependency or there are data dependencies, then the response-time analysis appears to produce the same or better results than the real-time calculus [14, Benchmark 3 and 4].

All of these comparisons are empirical. Therefore, we only have an indication that one method might always produce the same or better results than the other. However, we do not know for certain. So, to get a definitive answer we compare these methods formally. By means of a mathematical proof we show that both methods produce the same results when the real-time system uses fixed-priority preemptive scheduling and their tasks are independent.

Structure of the Paper

The remainder of this paper has the following structure: First we describe the related work in Section 2. Then we introduce the models and the analyses of the response-time analyses and the real-time calculus in Section 3. Subsequently, in Section 4 we describe the assumptions that we use and formally compare the response-time analysis with the real-time calculus by means of a mathematical proof. We conclude the paper with a summary in Section 5.

2 Related Work

In their seminal paper [12] Liu and Layland introduce a sufficient test for real-time systems that run a set of independent tasks which recur periodically, have an implicit deadline, and are subject to the fixed-priority preemptive scheduling algorithm. For the same type of real-time systems, Joseph and Pandya improve the analysis in [9] by supplying an exact test. Moreover, their test is not only suitable for tasks with implicit deadlines, but also for tasks with restricted deadlines. Lehoczky presents in [11] a further improvement to the previous test by extending the exact test to also include tasks with arbitrary deadlines, which then Tindell et al. further extend in [25] to allow tasks to have a release jitter. Similarly, Audsley et al. improve in [1] the exact test where they assume that the set of tasks have implicit deadlines, but they allow the tasks to block internally and have a release jitter. In [24] Tindell and Clark provides a test that combines all of these improvements. Audsley et al. present in [2] an historic perspective on fixed-priority preemptive scheduling. In [16] Richter presents an abstract representation of the bounds on how often the tasks recur. Based on which, Schliecker et al. introduce in [18] the multiple event busy time as a generalization of the concept of busy period which many response-time analyses use.

The work of Cruz [6, 7] is considered seminal for network calculus which is a mathematical framework to find bounds for the latency that network components cause on bit streams [10]. Fidler presents in [8] a comprehensive survey of the models that the network calculus uses Based on the network calculus, Thiele et al. introduce in [23] the models for the real-time calculus, how to get these models from a recurring real-time task, and they describe a schedulability test with these models for systems that use a fixed-priority preemptive scheduling algorithm. Chakraborty et al. refine in [5] the real-time calculus to calculate tighter bounds and apply it to scheduling networks. Wandeler improves the real-time calculus further in [26].

In [14] Perathoner et al. use several small abstract systems to empirically benchmark various formal performance analyses with these systems. Among the analyses are the response-time analysis and the real-time calculus. They show that neither of these two analyses always outperforms the other. But rather it depends on the characteristics of the system under analysis whether one outperforms the other. However, this is an empirical comparison. We compare them mathematically instead.

Naedele et al. present in [13] a schedulability test with the real-time calculus for a system that uses a fixed-priority preemptive scheduling algorithm. They indicate that it is possible to derive the test in [25] from their schedulability test. Similarly, Pollex et al. show in [15] a generalization of the response-time analysis with the help of the real-time calculus. They exemplarily derive the analysis in [25] from the real-time calculus. However, we use the more general schedulability test from [17] for our comparison.

In [17] Schliecker presents the multiple event busy time, how to derive it for fixed-priority preemptive scheduling, and an accompanying analysis as an extension of the work in [25]. They also show how to derive a multiple event busy time from the service curves of the real-time calculus. However, there is no discussion how the multiple event busy times, the one extended from [25] and the other derived from the service curve of the real-time calculus, relate to each other. We show that they are in fact identical.

Boyer and Roux propose in [3, 4] a model which can embed the models that the network calculus and the response-time analysis use, therefore making it possible to analyse a system that uses both models. However, they only look into how to interface between the different models and not how the individual analyses compare. Furthermore, much of the mathematical background that they use assumes real-valued functions that have the extended non-negative real numbers as domain and co-domain. We generalise some of them, where we assume mappings that use partially ordered sets or lattices as domain and co-domain.

Depending whether the real-time systems use fixed or dynamic priority scheduling, the existing analyses differ because of the different mathematical requirements on the models. In [19, 20] Slomka and Sadeghi introduce a new mathematical framework based on mathematical tools from electric engineering to analyse real-time systems. This new mathematical framework makes it possible to describe an unified analysis for real-time systems that use fixed and/or dynamic priority scheduling. They also describe existing analyses like [25] and analyses for real-time systems with dynamic priority scheduling with the new mathematical framework. Based on this mathematical framework, Slomka and Sadeghi show in [21] preliminary work for investigating the relationship between the response-time analysis and the real-time calculus. They sketch possible similarities, however they do not express the real-time calculus with their new mathematical framework, let alone compare them.

3 Models and Analyses

To analyse a real-time system we need to appropriately model it. Since we compare the response-time analyses with the real-time calculus, we first describe the assumptions and notations that both analyses have in common followed by a running example that we use to illustrate the concepts of both analyses. Second, we present the common mathematical concepts that both analyses use. Third, we restate the notation and the analyses themselves as presented in [17] and [26] for the response-time analysis and the real-time calculus, respectively.

3.1 Common Assumptions and Notation

We assume that the real-time system has a single core processor that uses a fixed-priority scheduler which allows a higher priority task to preempt a lower priority task at any time. The set of tasks Γ that is assigned to the processor has n tasks τ_i, where $i \in \{1, \ldots, n\}$. We

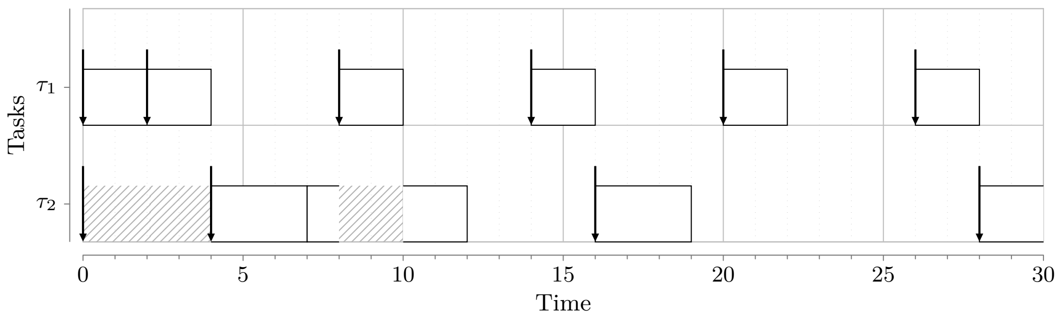


Figure 1 Worst-case schedule of the system described in Example 1.

only consider events that cause the system to release a job of a task which the system then puts into the ready queue of the scheduler. The scheduler is work-conserving, i.e. whenever a job of a task is in the ready queue, the scheduler assigns a job to the processor to execute it. Each task τ_i has a unique priority which defines a strict order on the set of tasks Γ. We use the index of τ_i to also represent the priority of the task. A lower numerical value of the index means that the task has a higher priority, i.e. task τ_3 has a higher priority than task τ_8. The tasks are independent from each other. There are no data dependencies, temporal dependencies, or any other dependencies between them. Also, the jobs of the tasks do not use any shared resources other than the processor.

To illustrate the concepts of both analyses we use the following running example of a real-time system.

▶ **Example 1.** Let the system consist of two tasks $\Gamma := \{\tau_1, \tau_2\}$, where τ_1 has a higher priority than τ_2. Task τ_1 releases a job every $p_1 := 6$ time units, has a release jitter of $j_1 := 4$ time units, and the processor needs $c_1^+ := 2$ time units to process each of its jobs. Similarly, task τ_2 releases a job every $p_2 := 12$ time units, has a release jitter of $j_2 := 8$ time units, and the processor needs $c_2^+ := 3$ time units to process each of its jobs.

Figure 1 shows the worst-case schedule for this system. Task τ_1 releases a job at time points 0, 2, and from then on every p_1 time units. Similarly, task τ_2 releases a job at time points 0, 4, and from then on every p_2 time units. The processor completes the first job of task τ_2 at 7 time units and the second job at 12 time units. So, the length of the time interval for the first two jobs from their release to when the processor completes them is $7 - 0 = 7$ and $12 - 4 = 8$ time units, respectively. Because at 12 time units an interval starts where no jobs are pending and therefore the processor is idle, we can conclude that a job of task τ_2 will never need more than 8 time units from the time it was released until the processor completes it.

3.2 Common Mathematical Notation and Definitions

First we introduce common mathematical notation and definitions. Then we present three mathematical definitions which are fundamental to many lemmas on which the theorem of our main contribution bases.

The set of positive integers and non-negative integers is $\mathbb{N}$ and $\mathbb{N}_0$, respectively. Furthermore, the set of real numbers, the extended real numbers (includes $-\infty$ and ∞), and the non-negative real numbers is $\mathbb{R}$, $\overline{\mathbb{R}}$, and $\mathbb{R}_0^+$, respectively.

▶ **Definition 2** (Monotonicity). *Let $f: X \to Y$ be a mapping from a partially ordered set X to a partially ordered set Y, then f is* isotone *or* antitone *if*

$$\forall x_1, x_2 \in X : x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2) \text{ or} \tag{1a}$$

$$\forall x_1, x_2 \in X : x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2), \text{ respectively.} \tag{1b}$$

Isotone mappings are also called *order-preserving* or in case of functions *increasing* or *non-decreasing*. Similarly, antitone mappings are also called *order-reversing*, *decreasing*, or *non-increasing*.

▶ **Definition 3** (Directional Continuity). *Let $f: X \to \mathbb{R}$ be a function from a subset X of the real numbers, then f is* continuous on the left *or* right *at $x \in X$ if*

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall \xi \in X \cap (x - \delta, x) : |f(\xi) - f(x)| < \epsilon \text{ or} \tag{2a}$$

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall \xi \in X \cap (x, x + \delta) : |f(\xi) - f(x)| < \epsilon, \text{ respectively.} \tag{2b}$$

If f is continuous on the left or right at every element of X, then f is called continuous on the left or right, respectively. Alternatively, f can be called left-continuous or right-continuous.

The models use several functions like the event load function (Definition 8) or the arrival curves (Definition 11) which in general are increasing, but not strictly increasing. Therefore, their inverse functions do not necessarily exist, but their closely related pseudo-inverse do. We define the pseudo-inverse of a function with the help of its contour set.

▶ **Definition 4** (Contour Set). *Let $f: X \to Y$ be a mapping from a set X to a partially ordered set Y, then the lower contour set $X_{f \leq y}$ and the upper contour set $X_{y \leq f}$ of f at $y \in Y$ are*

$$X_{f \leq y} := \{x \in X : f(x) \leq y\} \text{ and} \tag{3a}$$

$$X_{y \leq f} := \{x \in X : y \leq f(x)\}\,. \tag{3b}$$

▶ **Definition 5** (Pseudo-Inverse). *Let $f: X \to Y$ be a mapping from a subset X of a complete lattice L to a partially ordered set Y, then the pseudo-inverse $f^{\underline{-1}}: Y \to L$ and $f^{\overline{-1}}: Y \to L$ at $y \in Y$ are*

$$f^{\underline{-1}}(y) := \inf X_{y \leq f} \text{ and} \tag{4a}$$

$$f^{\overline{-1}}(y) := \sup X_{f \leq y} \tag{4b}$$

with the convention that $\inf \emptyset = \sup X$ *and* $\sup \emptyset = \inf X$.

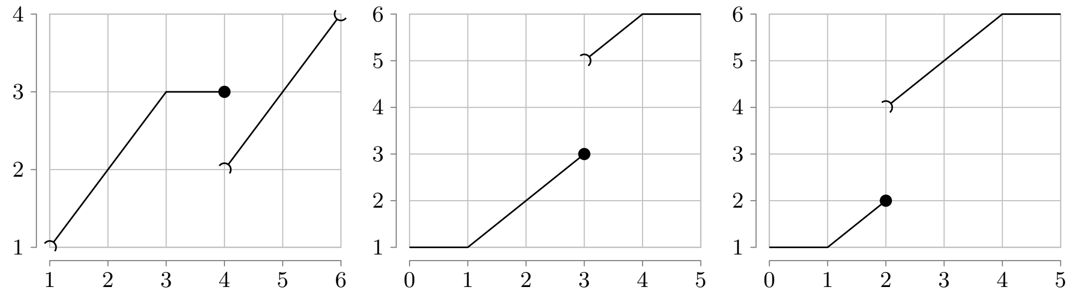
Note that we do not require that f is increasing as in [4, Definition 5]. This makes the new result in Lemma 50 possible. Also, note that the image of the pseudo-inverses $f^{\underline{-1}}$ and $f^{\overline{-1}}$ is a subset of the complete lattice L and not a subset of X. The reason is that the pseudo-inverses are the infimum and supremum of subsets of X. These do not necessarily have to be in X, but they are in L. To illustrate this, we use the following example.

▶ **Example 6.** Let $f: I \to \mathbb{R}$ be a function from the open interval $I := (1, 6)$ to the real numbers where $f(x) = x$ when $x \in (1, 3]$, $f(x) = 3$ when $x \in (3, 4]$, and $f(x) = x - 2$ when $x \in (4, 6)$. See Figure 2a for a plot of f. Table 1 shows the contour sets and their respective pseudo-inverses for any $y \in \mathbb{R}$. Lastly, Figures 2b and 2c show the plots for the pseudo-inverses $f^{\underline{-1}}$ and $f^{\overline{-1}}$. As Table 1 shows, the pseudo-inverses attain the values 1 and 6, which are not in I.

Table 1 Resulting contour sets, $I_{y \le f}$ and $I_{f \le y}$, and their respective pseudo-inverses, $f^{\underline{-1}}(y)$ and $f^{\overline{-1}}(y)$, for any $y \in \mathbb{R}$ for function f defined in Example 6.

$y \in$	$I_{y \le f}$	$f^{\underline{-1}}(y)$
$(-\infty, 1]$	I	1
$(1, 2]$	$[y, 6)$	y
$(2, 3]$	$[y, 4] \cup [y+2, 6)$	y
$(3, 4)$	$[y+2, 6)$	$y+2$
$[4, \infty)$	$\emptyset$	6

$y \in$	$I_{f \le y}$	$f^{\overline{-1}}(y)$
$(-\infty, 1]$	$\emptyset$	1
$(1, 2]$	$(1, y]$	y
$(2, 3)$	$(1, y] \cup (4, y+2]$	$y+2$
$[3, 4)$	$(1, y+2]$	$y+2$
$[4, \infty)$	I	6



(a) Plot of f. **(b)** Plot of $f^{\underline{-1}}$. **(c)** Plot of $f^{\overline{-1}}$.

Figure 2 Plot of the functions f defined in Example 6 and its pseudo-inverses $f^{\underline{-1}}$ and $f^{\overline{-1}}$.

▶ **Definition 7** (Deconvolution). *Let $f \colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $g \colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be increasing functions. The deconvolution in inf-plus $\oslash$ and in sup-plus $\overline{\oslash}$ are defined as follows [10, Definition 3.1.13 and 3.2.2]:*

$$(f \oslash g)(x) := \sup_{0 \le \xi} \{ f(x+\xi) - g(\xi) \} \tag{5a}$$

$$(f \,\overline{\oslash}\, g)(x) := \inf_{0 \le \xi} \{ f(x+\xi) - g(\xi) \} \tag{5b}$$

3.3 Response-Time Analysis

Over time, events occur that release jobs of tasks. To capture the density of these events for a task τ_i, the response-time analysis uses two functions, the event load function η_i^+ and the event distance function δ_i^-.

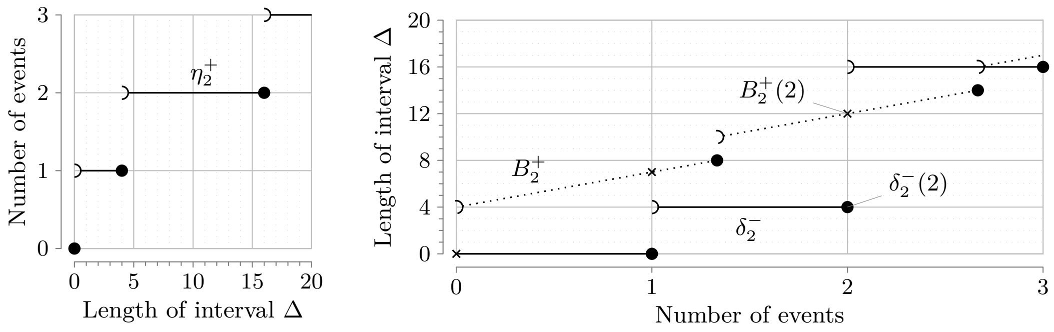
▶ **Definition 8** (Event Load Function). *Confer [17, p. 53, (3.3)]. The upper event load function for task τ_i maps a length of a time interval to an upper bound of the number of events that can occur in any time interval of that length and is denoted by*

$$\eta_i^+ : \mathbb{R}_0^+ \to \mathbb{N}_0. \tag{6}$$

▶ **Definition 9** (Event Distance Function). *Confer [17, p. 53, (3.1)]. The minimum event distance function for task τ_i maps a number of events to a lower bound of the length of a time interval in which at least that amount of events occur and is denoted by*

$$\delta_i^- : \mathbb{R}_0^+ \to \mathbb{R}_0^+. \tag{7}$$

Given q events, every interval in which at least q events occur has a length of at least $\delta_i^-(q)$. Or, in any interval with a length smaller than $\delta_i^-(q)$ less than q events occur.



(a) The event load function η_2^+ of task τ_2.

(b) The event distance function δ_2^- and the multiple event busy time function B_2^+ of task τ_2.

Figure 3 The concepts of the response-time analysis applied on the system described in Example 1.

Both the event load function and the event distance function are closely related, such that we can derive one function from the other. The relationship between these functions is that one function is essentially the pseudo-inverse of the other. Commonly, we have the upper event load function and from that we derive the minimum event distance function with (cf. [17, p. 54, (3.7)])

$$\delta_i^- := \underline{\eta_i^{+-1}}. \tag{8}$$

Additionally, a task has a worst-case execution-time c_i^+ which describes the maximum amount of processor time without any interference of higher priority task that a job needs for the processor to execute it.

The worst-case response time r_i^+ of task τ_i is bounded by (cf. [17, p. 64, (3.22)])

$$r_i^+ \leq \max_{q \in \mathbb{N}_0} \left\{ B_i^+(q) - \delta_i^-(q) \right\}. \tag{9}$$

Equation (9) uses the multiple event busy time function $B_i^+ : \mathbb{N}_0 \to \mathbb{R}_0^+$, cf. [17, p. 63, Definition 3.6]. For fixed priority preemptive scheduling the multiple event busy time function is (cf. [17, p. 64, (3.23)])

$$B_i^+(q) = \min_{\Delta \in \mathbb{R}_0^+} \left\{ \Delta : \Delta = q \cdot c_i^+ + \sum_{j=1}^{i-1} \left(\eta_j^+(\Delta) \cdot c_j^+ \right) \right\}. \tag{10}$$

In (10) we explicitly specify the smallest fix-point to resolve any possible mathematical ambiguity, because that is how [17, p. 64, (3.23)] is intended.

▶ **Example 10.** Given the system in Example 1 we exemplarily derive the various functions of the response-time analysis for it.

For task τ_1 the worst-case execution time is the same as given in the example system, i.e. $c_1^+ = 2$. Because task τ_1 has the highest priority, the multiple event busy time function is $B_1^+(q) = q \cdot c_1^+$ according to Equation (10). With the release of jobs every $p_1 = 6$ time units and a release jitter of $j_1 = 4$ time units, the event load function is $\eta_1^+(\Delta) = \left\lceil \frac{\Delta + j_1}{p_1} \right\rceil$ for $\Delta > 0$ and $\eta_1^+(\Delta) := 0$ for $\Delta = 0$. Now that we have the event load function we derive the event distance function according to Equation (8), i.e. $\delta_1^-(q) = \max\{0, \lceil q - 1 \rceil \cdot p_1 - j_1\}$.

Similarly, for task τ_2 the worst-case execution time is the same as in the example system, i.e. $c_2^+ = 3$. According to Equation (8) the multiple event busy time function B_2^+ has to consider the interference from the higher priority task τ_1 which results in $B_2^+(q) = q \cdot c_2^+ + \left\lceil \frac{q \cdot c_2^+ + 8}{4} - 1 \right\rceil \cdot c_1^+$ for $q > 0$ and $B_2^+(q) = 0$ for $q = 0$. Task τ_2 releases jobs every $p_2 = 12$ time units and has a release jitter of $j_2 = 8$ time units, therefore its event load functions is $\eta_2^+(\Delta) = \left\lceil \frac{\Delta + j_2}{p_2} \right\rceil$ for $\Delta > 0$ and $\eta_2^+(\Delta) = 0$ for $\Delta = 0$. Deriving the event distance function from the event load function according to Equation (8) results in $\delta_2^-(q) = \max\{0, \lceil q-1 \rceil \cdot p_2 - j_2\}$.

Now that we have both the multiple event busy time function B_2^+ and the event distance function η_2^+ for task τ_2 we can calculate the upper bound for the worst-case response-time, see Figure 3b for a plot with both of these functions. The worst-case response-time for task τ_2 is not greater than $r_2^+ \leq \max_{q \in \mathbb{N}_0} \left\{ B_i^+(q) - \delta_i^-(q) \right\} = \max\{0-0, 7-0, 12-4, \ldots\} = 8$ time units. Note that we do not necessarily need to compute all the values of the multiple event busy time function B_2^+. In a schedulable system there will be a point when there are no pending jobs of task τ_2 that could defer the execution of any of its following jobs, cf. [17, p. 72, Theorem 3.9].

3.4 Real-Time Calculus

The real-time calculus models a task with the Greedy Processing Component which has event-based arrival curves $\overline{\alpha}_i$ and resource-based service curves β_i as input.

▶ **Definition 11** (Event-Based Arrival Curves). *Confer [26, p. 16, Def. 1] and [26, p. 73, Def. 3]. Let $R[s,t)$ denote the number of events that occur in the interval $[s,t)$, where $s \in \mathbb{R}_0^+$ is a point in time before $t \in \mathbb{R}_0^+$, i.e. $s \leq t$. Then, the event-based lower arrival curve $\overline{\alpha}^- : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and the event-based upper arrival curve $\overline{\alpha}^+ : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy for every point in time $t \in \mathbb{R}$ and every length of interval $\Delta \in \mathbb{R}_0^+$ the property:*

$$\overline{\alpha}^-(\Delta) \leq R[t, t+\Delta) \leq \overline{\alpha}^+(\Delta) \tag{11}$$

▶ **Definition 12** (Resource-Based Service Curves). *Confer [26, p. 19, Def. 2] and [26, p. 73, Def. 6]. Let $C[s,t)$ denote the amount of resources that are available in the interval $[s,t)$, where $s \in \mathbb{R}_0^+$ is a point in time before $t \in \mathbb{R}_0^+$, i.e. $s \leq t$. Then, the resource-based lower service curve $\beta^- : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and the resource-based upper service curve $\beta^+ : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy for every point in time $t \in \mathbb{R}$ and every length of interval $\Delta \in \mathbb{R}_0^+$ the property:*

$$\beta^-(\Delta) \leq C[t, t+\Delta) \leq \beta^+(\Delta) \tag{12}$$

The arrival curves $\overline{\alpha}^-$ and $\overline{\alpha}^+$ map a length of a time interval to a lower, respectively upper, bound of the amount of events that can occur in any interval of that length which causes the system to release jobs. Similarly, the service curves β^- and β^+ map a length of a time interval to a lower, respectively upper, bound of the amount of resources that are available in any interval of that length to execute any pending jobs. However, the arrival curves in Definition 11 are event-based, which map from a length of an interval to a number of events. But, the service curves in Definition 12 are resource-based, which map from a length of an interval to an amount of resources. Number of events and amount of resources are not comparable. So, we need to transform at least one of them into the other. Which we do with workload curves.

▶ **Definition 13** (Workload Curves). *Confer [26, p. 74, Def. 7]. Let $W(u)$ denote the amount of resources that are necessary to process u consecutive events, then the lower workload curve $\gamma^- : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and the upper workload curve $\gamma^+ : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy for $u \in \mathbb{R}_0^+$ and $v \in \mathbb{R}_0^+$ consecutive events, where $u \leq v$, the property:*

$$\gamma^-(v-u) \leq W(v) - W(u) \leq \gamma^+(v-u) \tag{13}$$

To transform the resource-based lower service curve into its event-based form, we compose ($\circ$) it with the pseudo-inverse of the upper workload curve. Confer [26, p. 74, (4.6)] and [26, p. 75, (4.11)].

$$\overline{\beta}^- = \overline{\gamma^{+^{-1}}} \circ \beta^- \tag{14}$$

Similarly, we transform the event-based upper arrival curve into its resource-based form by composing it with the upper workload curve. Confer [26, p. 75, (4.8)].

$$\alpha^+ = \gamma^+ \circ \overline{\alpha}^+ \tag{15}$$

With the arrival curve and the service curve in either the resource-based form or the event-based form, we can describe an upper bound on the delay of task τ_i. This is the longest time that the processor needs to execute a job of task τ_i. From the time the event occurred which released the job until the job was completely executed. We describe this upper bound with the notion of the largest horizontal distance between functions.

▶ **Definition 14** (Largest horizontal distance between functions). *Confer [10, p. 154, (3.21)]. Let $f, g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be increasing functions, then the largest horizontal distance $\leftrightarrow$ between f and g is*

$$f \leftrightarrow g := \sup_{\lambda \in \mathbb{R}_0^+} \left\{ \inf_{\mu \in \mathbb{R}_0^+} \{\mu : f(\lambda) \leq g(\lambda + \mu)\} \right\} \tag{16}$$

With the event-based upper arrival curve $\overline{\alpha}_i^+$ and lower service curve $\overline{\beta}_i^-$ of task τ_i, we can express the upper bound of the delay of task τ_i by means of the largest horizontal distance between functions and is (cf. [26, p. 26, (2.11)])

$$\overline{\alpha}_i^+ \leftrightarrow \overline{\beta}_i^- . \tag{17}$$

However, we need the event-based lower service curve $\overline{\beta}_i^-$ for task τ_i. Given a task τ_i with its corresponding arrival and service curves, then the remaining lower service curve is (cf. [26, p. 23, (2.10) or p. 201, (A.15)])

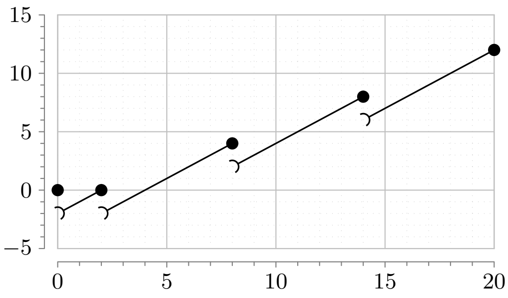
$$\beta_j^-(\Delta) = \sup_{0 \leq \lambda \leq \Delta} \left\{ \beta_i^-(\lambda) - \alpha_i^+(\lambda) \right\} \text{ or } \beta_j^- = (\beta_i^- - \alpha_i^+)^{\nearrow}. \tag{18}$$

In the latter part of Equation (18) we apply an increasing operator, which is defined as

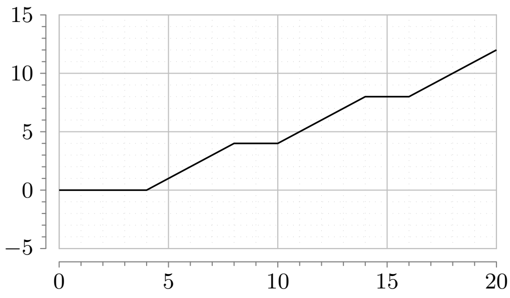
▶ **Definition 15** (Increasing operator). *Let $\mathbb{F}$ be the set of functions $f : \mathbb{R}_0^+ \to \mathbb{R}$ and I_x the interval $[0, x]$ for a non-negative real number x, then the increasing operator $^{\nearrow} : \mathbb{F} \to \mathbb{F}$ is*

$$f^{\nearrow}(x) := \sup_{\xi \in I_x} \{f(\xi)\} \tag{19}$$

The increasing operator transforms a function f into an increasing function and it is a closure operator.



(a) Plot of $\beta_1^- - \alpha_1^+$.



(b) Plot of $\beta_2^- = (\beta_1^- - \alpha_1^+)^{\nearrow}$.

Figure 4 The resource-based lower service curve β_2^- of task τ_2 for the system described in Examples 1 and 19 according to Equation (18) by applying the increasing operator.

▶ Remark 16. Let $\mathbb{F}$ be the set of functions $f\colon \mathbb{R}_0^+ \to \mathbb{R}$, then

$$f^{\nearrow} \text{ is increasing} \tag{20}$$

Proof. Let I_x the interval $[0, x]$ for a non-negative real number x, $x_1, x_2 \in \mathbb{R}_0^+$ with $x_1 \le x_2$, then $I_{x_1} \subseteq I_{x_2}$, therefore $f^{\nearrow}(x_1) = \sup_{\xi \in I_{x_1}} \{f(\xi)\} \le \sup_{\xi \in I_{x_2}} \{f(\xi)\} = f^{\nearrow}(x_2)$. ◀

▶ Remark 17 (Increasing closure). Let $\mathbb{F}$ be the set of functions $f\colon \mathbb{R}_0^+ \to \mathbb{R}$, then the increasing operator $^{\nearrow}\colon \mathbb{F} \to \mathbb{F}$ is a closure operator on the partially ordered set $(\mathbb{F}, \le)$, where $\le$ is the pointwise order on functions.

Proof. Let $f, g \in \mathbb{F}$ be functions and I_x the interval $[0, x]$ for a non-negative real number x, then

(a) $f \le f^{\nearrow}$: Because x is an element in I_x it follows that $f(x) \le \sup_{\xi \in I_x} \{f(\xi)\} = f^{\nearrow}(x)$.

(b) $f \le g \Rightarrow f^{\nearrow} \le g^{\nearrow}$: Follows directly from Equation (19).

(c) $f^{\nearrow\nearrow} = f^{\nearrow}$: Follows from Equations (19) and (20), i.e. $f^{\nearrow\nearrow}(x) = \sup_{\xi \in I_x} \{f^{\nearrow}(\xi)\} = f^{\nearrow}(x)$.

Because the increasing operator $^{\nearrow}$ satisfies (a)–(c), it is a closure operator. ◀

For the case of fixed-priority preemptive scheduling the lower available service curve β_i^- is the lower remaining service curve of the next higher priority task τ_{i-1}. See Figure 4 that shows the available service curve β_2^- of task τ_2 or the lower remaining service curve of task τ_1 for the system described in Examples 1 and 19. When $i = 1$ then the lower available service curve of the highest priority task τ_1 is equal to the lower available service curve to the entire scheduler itself, i.e. $\beta_1^- = \beta^-$.

Similarly to the horizontal distance between functions (Definition 14) we define the vertical distance between functions.

▶ **Definition 18** (Largest vertical distance of functions). *Confer [10, p. 154, (3.20)]. Let $f, g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be increasing functions, then the largest vertical distance $\updownarrow$ between f and g is*

$$f \updownarrow g := \sup_{\xi \in \mathbb{R}_0^+} \{f(\xi) - g(\xi)\} = \sup(f - g) \tag{21}$$

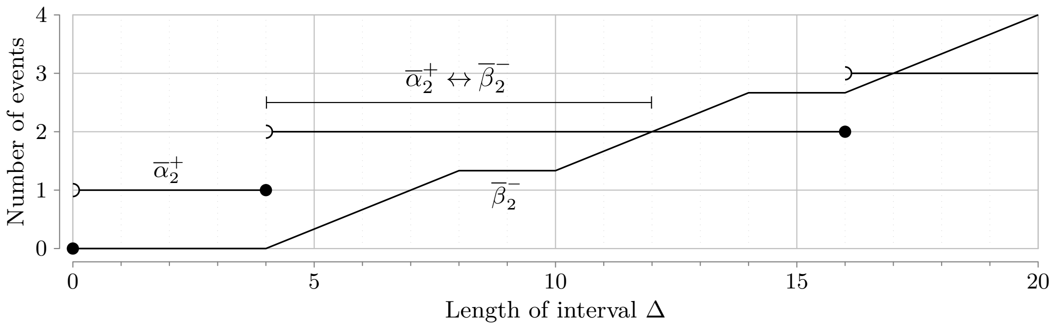


Figure 5 The event-based upper arrival curve $\overline{\alpha}_2^+$, the event-based lower service curve $\overline{\beta}_2^-$, and the upper bound of the delay $\overline{\alpha}_2^+ \leftrightarrow \overline{\beta}_2^-$ for τ_2 of the system described in Example 1.

▶ **Example 19.** Given the system in Example 1 we exemplarily derive the various functions of the real-time calculus for it.

For task τ_1 be event-based upper arrival curve is $\overline{\alpha}_1^+(\Delta) = \left\lceil \frac{\Delta+j_1}{p_1} \right\rceil$ for $\Delta > 0$ and $\overline{\alpha}_1^+(\Delta) = 0$ for $\Delta = 0$. Because task τ_1 has the highest priority, the resource-based lower service curve is $\beta_1^-(\Delta) = \Delta$. Lastly, the upper workload curve is $\gamma_1^+(q) = q \cdot c_1^+$. Similarly, for task τ_2 the event-based upper arrival curve is $\overline{\alpha}_2^+(\Delta) = \left\lceil \frac{\Delta+j_2}{p_2} \right\rceil$ for $\Delta > 0$ and $\overline{\alpha}_2^+(\Delta) = 0$ for $\Delta = 0$. According to Equation (18) the resource-based lower service curve is $\beta_2^-(\Delta) = \sup_{0 \le \lambda \le \Delta} \left\{ \beta_1^-(\lambda) - \alpha_1^+(\lambda) \right\}$, see Figure 4b for a plot of β_2^-, and the upper workload curve is $\gamma_2^+(q) = q \cdot c_2^+$.

Figure 5 shows the event-based upper arrival curve $\overline{\alpha}_2^+$ and the event-based lower service curve $\overline{\beta}_2^-$ of task τ_2. From those we derive the upper bound of the delay $\overline{\alpha}_2^+ \leftrightarrow \overline{\beta}_2^-$ which is $12 - 4 = 8$ time units.

4 Formal Comparison of the RTA with the RTC

In this section we formally compare the upper bound for the worst-case response-time, Equation (9), that the response-time analysis uses with the upper bound for the delay, Equation (17), that the real-time calculus uses. For a fair comparison we must ensure identical initial conditions, therefore we make the following assumptions:

▶ Assumption 20. For mathematical reasons, every curve ($\overline{\alpha}_i^+$, β^-, and γ^+) is increasing and not bounded above, lower curves (β^-) are right-continuous, and upper curves ($\overline{\alpha}_i^+$ and γ^+) are left-continuous.

For the event-based upper arrival curve $\overline{\alpha}_i^+$ this means that an interval of greater length exists where at least as many events occur than in any interval of the same or smaller length and if the system keeps running for all eternity, an infinite amount of events will occur.

▶ Assumption 21. Jobs of tasks do not starve, every job finishes after a finite amount of time. For a set of n tasks, we express this by assuming that the available service for the lowest priority task τ_n is not bounded above, i.e. for all $r \in \mathbb{R}_0^+$ a $\Delta \in \mathbb{R}_0^+$ exists such that $r < \beta_n^-(\Delta)$.

This implies that the resource-based lower service curve β_i^- for every task $i \in \{1, \ldots, n\}$ is not bounded above.

▶ Assumption 22. The event load function and the event-based upper arrival curve for a task τ_i are equal, i.e. $\eta_i^+ = \overline{\alpha}_i^+$, because both model exactly the same.

Examples for how to define the event load function for common event models are in [16, p. 50] or in [26, p. 16, Ex. 1]. All those definitions satisfy Assumption 20.

▶ Assumption 23. An implicit assumption of the response-time analysis is that one unit of processor time is available per time unit. Therefore, the lower bound of the available resources that a fixed-priority preemptive scheduler has is $\beta^-(\Delta) = \Delta$, cf. [26, p. 20, Ex. 2].

The resource-based lower service curve $\beta^-(\Delta) = \Delta$ is not bounded above and is continuous. Therefore, it is also right-continuous and thus satisfies Assumption 20.

▶ Assumption 24. Furthermore, the response-time analysis assumes that every job of a task τ_i needs at most c_i^+ processor time to execute. So, we have for the upper workload curve $\gamma_i^+(q) = q \cdot c_i^+$.

This also satisfies Assumption 20.

Under Assumptions 20–24 we provide our main contribution, Theorem 31, a proof that the upper bound for the worst-case response-time of the response-time analysis, Equation (9), and the upper bound for the delay of the real-time calculus, Equation (17), are equal for every task in a set of independent tasks as described in Section 3.1.

We divide the proof of Theorem 31 into several steps and begin by revisiting Figures 3b and 5. These show the functions that the response-time analysis and the real-time calculus use to model the system in Example 1. Both analyses calculate the same upper bound for the worst-case response-time or delay. It appears that the functions in Figure 3b are the pseudo-inverse of the functions in Figure 5. So, there seem to exist a relation between the horizontal distance between functions, Equation (16), and the vertical distance between functions, Equation (21). This turns out to be the case, Lemma 25. Next, we need to compare the pseudo-inverse of the event-based upper arrival curve $\overline{\alpha}_i^{+\underline{-1}}$ and the pseudo-inverse of the event-based lower service curve $\overline{\beta}_i^{-\underline{-1}}$ with the event distance function δ_i^- and the multiple event busy time function B_i^+. The former is straight forward, Remark 26, however the latter is more challenging. For that we need to determine what the pseudo-inverse of the resource-based lower service curve $\beta_i^{-\underline{-1}}$ is, Lemma 27. From it, we then have to derive the pseudo-inverse of the event-based lower service curve $\overline{\beta}_i^{-\underline{-1}}$, Lemma 28. With that we can compare the pseudo-inverse of the event-based lower service curve $\overline{\beta}_i^{-\underline{-1}}$ with the multiple event busy time function B_i^+, Lemma 29. Lastly, we need to verify that the different domains of the pseudo-inverse of the event-based lower service curve $\overline{\beta}_i^{-\underline{-1}}$ and the multiple event busy time function B_i^+ do not affect the comparison, Lemma 30. After all these steps we can finally prove the equality of the upper bound of the worst-case response-time, Equation (9), and the upper bound for the delay, Equation (17), in Theorem 31.

▶ **Lemma 25.** *Let $f, g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be increasing functions that are not bounded above, then the largest horizontal distance between f and g is equal to the largest vertical distance between $g^{\underline{-1}}$ and $f^{\underline{-1}}$.*

$$f \leftrightarrow g = g^{\underline{-1}} \updownarrow f^{\underline{-1}} \tag{22}$$

Proof. f and g are both increasing, so we use the equality of Equation (52). Both functions f and g are also not bounded above, therefore we use Equation (49), Definition 7, and Definition 18.

$$f \leftrightarrow g \overset{(52)}{=} (g \,\overline{\oslash}\, f)^{\underline{-1}}(0) \overset{(49)}{=} (g^{\underline{-1}} \oslash f^{\underline{-1}})(0) \overset{(5a)}{=} \sup_{\xi \in \mathbb{R}_0^+} \left\{ g^{\underline{-1}}(\xi) - f^{\underline{-1}}(\xi) \right\} \overset{(21)}{=} g^{\underline{-1}} \updownarrow f^{\underline{-1}} \qquad ◀$$

▶ Remark 26. Let Γ be a set of n independent tasks as described in Section 3.1 and let Assumption 22 hold, then for any task τ_i of Γ the event distance function δ_i^- is equal to the pseudo-inverse of the event-based upper arrival curve $\overline{\alpha}_i^{+\underline{-1}}$.

$$\delta_i^- = \overline{\alpha}_i^{+\underline{-1}} \tag{23}$$

Proof. The minimum event distance function is equal to the pseudo-inverse of the upper event load function Equation (8). And the upper event load function is equal to the event-based upper service curve, Assumption 22.

$$\delta_i^- \overset{(8)}{=} \eta_i^{+\underline{-1}} = \overline{\alpha}_i^{+\underline{-1}}$$ ◀

▶ **Lemma 27.** *Let Γ be a set of n independent tasks as described in Section 3.1 and let Assumptions 20 and 21 hold, then the pseudo-inverse of the resource-based lower service curve $\beta_i^{-\underline{-1}}$ of task τ_i is*

$$\beta_i^{-\underline{-1}}(r) = \min_{\Delta\in\mathbb{R}_0^+}\left\{\Delta : \Delta = \beta^{-\underline{-1}}\left(r + \sum_{j=1}^{i-1}\alpha_j^+(\Delta)\right)\right\} \tag{24}$$

Proof. Let $f_i(\lambda) := \beta^-(\lambda) - \sum_{j=1}^{i-1}\alpha_j^+(\lambda)$ and $g_{i,r}(\Delta) := \beta^{-\underline{-1}}\left(r + \sum_{j=1}^{i-1}\alpha_j^+(\Delta)\right)$.

(a) β^- is upper semi-continuous: Follows from Assumption 20 and Lemma 36.

(b) α_i^+ is lower semi-continuous: Follows from Assumption 20 and Lemma 36.

(c) $-\alpha_i^+$ is upper semi-continuous: Follows from (b) and Lemma 34.

(d) f_i is upper semi-continuous: Follows from (a) and (c) and Lemma 35.

(e) $\beta_i^-(\Delta) = \sup_{0\le\lambda\le\Delta}\{f_i(\lambda)\}$, the resource-based lower service curve of task τ_i is the increasing closure of f_i: Follows from Equation (50).

(f) f_i is not bounded above: Let x be a non-negative real number and I_x be the interval $[0, x]$. Because f_i is upper semi-continuous, (d), and I_Δ is a compact set, then f_i achieves its maximum in I_Δ. Therefore, for every $\Delta \in \mathbb{R}_0^+$ a $\lambda \in I_\Delta$ exists such that $\beta_i^-(\Delta) = f_i(\lambda)$. From Assumption 21 we have that β_i^- is not bounded above, so for every $r \in \mathbb{R}_0^+$ there exist a $\Delta \in \mathbb{R}_0^+$ and subsequently a $\lambda \in \mathbb{R}_0^+$ such that $r < \beta_i^-(\Delta) = f_i(\lambda)$. Therefore, f_i is not bounded above.

(g) $g_{i,r}$ is increasing: Follows from Assumption 20 and Equation (42a) and that increasing functions are closed under addition and composition.

(h) $r \le f_i(\Delta) \Leftrightarrow \Delta \ge g_{i,r}(\Delta)$: Follows from Assumption 20 and Equation (41)

$$r \le f_i(\Delta) \Leftrightarrow r \le \beta^-(\Delta) - \sum_{j=1}^{i-1}\alpha_j^+(\Delta) \Leftrightarrow \beta^-(\Delta) \ge r + \sum_{j=1}^{i-1}\alpha_j^+(\Delta)$$
$$\overset{(41)}{\Leftrightarrow} \Delta \ge \beta^{-\underline{-1}}\left(r + \sum_{j=1}^{i-1}\alpha^+(\Delta)\right) \Leftrightarrow \Delta \ge g_{i,r}(\Delta)$$

(i) $g_{i,r}$ has a smallest fix-point: f_i is not bounded above, (f), so we have that for any $r \in \mathbb{R}_0^+$ a $\Delta \in \mathbb{R}_0^+$ exists such that

$$f_i(\Delta) > r \Leftrightarrow \beta^-(\Delta) - \sum_{j=1}^{i-1}\alpha_j^+(\Delta) > r \Leftrightarrow \beta^-(\Delta) > r + \sum_{j=1}^{i-1}\alpha_j^+(\Delta)$$

holds. This implies that $\beta^{-}(\Delta) \geq r + \sum_{j=1}^{i-1} \alpha_j^{+}(\Delta)$. Because of Assumption 20, we apply Equation (41), and we get $\Delta \geq \beta^{-\underline{-1}}\left(r + \sum_{j=1}^{i-1} \alpha_j^{+}(\Delta)\right) = g_{i,r}(\Delta)$. Let I be the closed interval $[0, \Delta]$ of real numbers, then I is a complete lattice. Because of (g) and $g_{i,r}(\Delta) \leq \Delta$, the restriction of $g_{i,r}$ to I maps to itself, i.e. $g_r(I) \subseteq I$. Therefore, according to Lemma 56, g_r has a smallest fix-point in I.

Equation (24) follows from Equation (47), because of (d) and (e), (h), and (i).

$$\begin{aligned}\beta_i^{-\underline{-1}}(r) &\overset{(47)}{=} f_i^{\underline{-1}}(r) \overset{(4a)}{\underset{(3b)}{=}} \inf_{\Delta \in \mathbb{R}_0^+} \{\Delta : r \leq f_i(\Delta)\} \overset{(h)}{=} \inf_{\Delta \in \mathbb{R}_0^+} \{\Delta : \Delta \geq g_{i,r}(\Delta)\} \\ &\overset{(i)}{=} \min_{\Delta \in \mathbb{R}_0^+} \{\Delta : \Delta = g_{i,r}(\Delta)\} = \min_{\Delta \in \mathbb{R}_0^+} \left\{\Delta : \Delta = \beta^{-\underline{-1}}\left(r + \sum_{j=1}^{i-1} \alpha^{+}(\Delta)\right)\right\}\end{aligned}$$ ◀

▶ **Lemma 28.** *Let Γ be a set of n independent tasks as described in Section 3.1 and let Assumptions 20 and 21 hold, then the pseudo-inverse of the event-based lower service curve $\overline{\beta}_i^{-\underline{-1}}$ of task τ_i is*

$$\overline{\beta}_i^{-\underline{-1}}(q) = \min_{\Delta \in \mathbb{R}_0^+} \left\{\Delta : \Delta = \beta^{-\underline{-1}}\left(\gamma_i^{+}(q) + \sum_{j=1}^{i-1} \alpha_j^{+}(\Delta)\right)\right\} \tag{25}$$

Proof. We expand the event-based lower service curve into the composition of the pseudo-inverse of the upper workload curve and the resource-based lower service curve. The pseudo-inverse of the upper workload curve is increasing, Equation (42b), and right-continuous, Equation (43b). So, we expand the pseudo-inverse of the composition according to Equation (45b). The upper workload curve is increasing and left-continuous according to Assumption 20, therefore the two pseudo-inverse operations cancel each other out, Equation (46a).

$$\overline{\beta}_i^{-\underline{-1}} \overset{(14)}{=} \left(\gamma_i^{+\overline{-1}} \circ \beta_i^{-}\right)^{\underline{-1}} \overset{(45b)}{=} \beta_i^{-\underline{-1}} \circ \gamma_i^{+\overline{-1}\underline{-1}} \overset{(46a)}{=} \beta_i^{-\underline{-1}} \circ \gamma_i^{+}$$

We satisfy the antecedents of Lemma 27, and so we get

$$(\beta_i^{-\underline{-1}} \circ \gamma_i^{+})(q) \overset{(24)}{=} \min_{\Delta \in \mathbb{R}_0^+} \left\{\Delta : \Delta = \beta^{-\underline{-1}}\left(\gamma_i^{+}(q) + \sum_{j=1}^{i-1} \alpha_j^{+}(\Delta)\right)\right\}$$ ◀

▶ **Lemma 29.** *Let Γ be a set of n independent tasks as described in Section 3.1 and let Assumptions 20–24 hold, then for any task τ_i of Γ the multiple event busy time function B_i^{+} is equal of the pseudo-inverse of the event-based lower service curve $\overline{\beta}_i^{-\underline{-1}}$*

$$B_i^{+} = \overline{\beta}_i^{-\underline{-1}} \tag{26}$$

Proof. First we abbreviate some expressions. Let $f_i(q, \Delta) := \beta^{-\underline{-1}}\left(\gamma_i^{+}(q) + \sum_{j=1}^{i-1} \alpha_j^{+}(\Delta)\right)$ and $g_i(q, \Delta) := q \cdot c_i^{+} + \sum_{j=1}^{i-1} \left(\eta_j^{+}(\Delta) \cdot c_j^{+}\right)$.

(a) $\overline{\beta}_i^{-\underline{-1}}(q) = \min_{\Delta \in \mathbb{R}_0^+} \{\Delta : \Delta = f_i(q, \Delta)\}$: Follows from Lemma 28.

(b) $\beta^{-\underline{-1}}(r) = r$: Follows from Assumption 23 and Definition 5.

(c) $f_i(q,\Delta) = g_i(q,\Delta)$: Follows from (b), Equation (15), Assumption 24, and Assumption 22

$$
\begin{aligned}
f_i(q,\Delta) = \beta^{-\underline{-1}}\left(\gamma_i^+(q) + \sum_{j=1}^{i-1}\alpha_j^+(\Delta)\right) &\overset{(b)}{=} \gamma_i^+(q) + \sum_{j=1}^{i-1}\alpha_j^+(\Delta) \\
&\overset{(15)}{=} \gamma_i^+(q) + \sum_{j=1}^{i-1}\gamma_j^+\left(\overline{\alpha}_j^+(\Delta)\right) = q\cdot c_i^+ + \sum_{j=1}^{i-1}\left(\overline{\alpha}_j^+(\Delta)\cdot c_j^+\right) \\
&= q\cdot c_i^+ + \sum_{j=1}^{i-1}\left(\eta_j^+(\Delta)\cdot c_j^+\right) = g_i(q,\Delta)
\end{aligned}
$$

(d) $B_i^+(q) = \min_{\Delta\in\mathbb{R}_0^+}\{\Delta : \Delta = g_i(q,\Delta)\}$: Follows from Equation (10).

Equation (26) follows directly from (a), (c), and (d)

$$
\overline{\beta}_i^{-\underline{-1}}(q) \overset{(a)}{=} \min_{\Delta\in\mathbb{R}_0^+}\{\Delta : \Delta = f_i(q,\Delta)\} \overset{(c)}{=} \min_{\Delta\in\mathbb{R}_0^+}\{\Delta : \Delta = g_i(q,\Delta)\} \overset{(d)}{=} B_i^+(q)
$$ ◀

▶ **Lemma 30.** *Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $g\colon \mathbb{R}_0^+ \to \mathbb{N}_0$, then*

$$
\max_{n\in\mathbb{N}_0}\left\{f^{\underline{-1}}(n) - g^{\underline{-1}}(n)\right\} = \sup_{r\in\mathbb{R}_0^+}\left\{f^{\underline{-1}}(r) - g^{\underline{-1}}(r)\right\} \tag{27}
$$

Proof. Let $h := f^{\underline{-1}} - g^{\underline{-1}}$,$b_X := \sup_{x\in X}\{h(x)\}$, r a non-negative real number and $n := \lceil r\rceil$ a non-negative integer.

(a) $b_{\mathbb{N}_0} \le b_{\mathbb{R}_0^+}$: Follows from $\mathbb{N}_0$ being a subset of $\mathbb{R}_0^+$.

(b) $f^{\underline{-1}}(r) \le f^{\underline{-1}}(n)$: Follows from $r \le n$ and Equation (42a).

(c) $g^{\underline{-1}}(r) = g^{\underline{-1}}(n)$: Follows from Definitions 4 and 5 and because the co-domain of g are the non-negative integers

$$
g^{\underline{-1}}(r) = \inf_{x\in\mathbb{R}_0^+}\{x : r \le g(x)\} = \inf_{x\in\mathbb{R}_0^+}\{x : \lceil r\rceil \le g(x)\} = g^{\underline{-1}}(\lceil r\rceil) = g^{\underline{-1}}(n)
$$

(d) $h(r) \le h(n)$: Follows from (b) and (c)

(e) $b_{\mathbb{R}_0^+} \le b_{\mathbb{N}_0}$: Follows because for every $r \in \mathbb{R}_0^+$ an $n \in \mathbb{N}_0$ exists such that $h(r) \le h(n)$.

This Lemma follows from (a) and (e). ◀

▶ **Theorem 31.** *Let Γ be a set of n independent tasks as described in Section 3.1 and let Assumptions 20–24 hold, then for any task τ_i of Γ the upper bound for the worst-case response time from the response-time analysis is equal to the upper bound for the delay from the real-time calculus.*

$$
\max_{q\in\mathbb{N}_0}\left\{B_i^+(q) - \delta_i^-(q)\right\} = \overline{\alpha}_i^+ \leftrightarrow \overline{\beta}_i^- \tag{28}
$$

Proof. First we substitute the multiple event busy time function B_i^+ with the pseudo-inverse of the event-based lower service curve $\overline{\beta}_i^{-\underline{-1}}$, Lemma 29 and the event distance function δ_i^- with the pseudo-inverse of the event-based upper the arrival curve $\overline{\alpha}_i^{+\underline{-1}}$, Remark 26.

$$
\max_{q\in\mathbb{N}_0}\left\{B_i^+(q) - \delta_i^-(q)\right\} \overset{(26)}{=} \max_{q\in\mathbb{N}_0}\left\{\overline{\beta}_i^{-\underline{-1}}(q) - \delta_i^-(q)\right\} \overset{(23)}{=} \max_{q\in\mathbb{N}_0}\left\{\overline{\beta}_i^{-\underline{-1}}(q) - \overline{\alpha}_i^{+\underline{-1}}(q)\right\}
$$

Next we interchange the maximum max with the supremum sup and change the set from the non-negative integers $\mathbb{N}_0$ to the non-negative real numbers $\mathbb{R}_0^+$ (Lemma 30). This results in the vertical distance between the pseudo-inverse of the event-based lower service curve $\overline{\beta}_i^{-\overline{-1}}$ and the pseudo-inverse of the event-based upper arrival curve $\overline{\alpha}_i^{+\overline{-1}}$ by Definition 18:

$$\max_{q\in\mathbb{N}_0}\left\{\overline{\beta}_i^{-\overline{-1}}(q) - \overline{\alpha}_i^{+\overline{-1}}(q)\right\} \overset{(27)}{=} \sup_{\lambda\in\mathbb{R}_0^+}\left\{\overline{\beta}_i^{-\overline{-1}}(\lambda) - \overline{\alpha}_i^{+\overline{-1}}(\lambda)\right\} \overset{(21)}{=} \overline{\beta}_i^{-\overline{-1}} \updownarrow \overline{\alpha}_i^{+\overline{-1}}$$

Finally, we apply the equality between the vertical distance of the pseudo-inverse functions and the horizontal distance between the functions, Lemma 25, so that we ultimately get Equation (17), the upper bound for the delay.

$$\overline{\beta}_i^{-\overline{-1}} \updownarrow \overline{\alpha}_i^{+\overline{-1}} \overset{(22)}{=} \overline{\alpha}_i^{+} \leftrightarrow \overline{\beta}_i^{-}$$ ◀

5 Summary

We looked into the existing analyses for real-time systems with a single processor that uses the fixed-priority preemptive scheduling algorithm to process a set of independent tasks that do not share any resources other than the processor. One is the response-time analysis that Schliecker presents in [17] and the other is the real-time calculus that Wandeler describes in [26]. Both use abstract event models and produce upper bounds on the amount of time that the processor needs to complete the tasks. The existing empirical comparisons could only give us indications as how these two analyses compare.

However, we can now give a definite answer. We gave a mathematical proof that both analyses produce for the investigated type of systems identical upper bounds. So, from a mathematical point of view both analyses are equivalent and regarding the results it does not matter which analysis is used. However, a different criteria, like run-time complexity, can favour one over the other.

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A Properties of Semi-Continuous Functions

▶ **Definition 32** (Lower and Upper Limit). *Let $f\colon X \to \mathbb{R}$ be a function from a subset X of the real numbers, then the lower respectively upper limit of f at an accumulation point x for X is*

$$\liminf_{\xi\to x} f(\xi) := \sup_{r>0} \inf_{\xi\in X} \{f(\xi) : 0 < |\xi - x| < r\} \text{ respectively} \tag{29a}$$

$$\limsup_{\xi\to x} f(\xi) := \inf_{r>0} \sup_{\xi\in X} \{f(\xi) : 0 < |\xi - x| < r\}\,. \tag{29b}$$

Note that in Definition 32 the domain X of f has no further restrictions. In particular X does not have to be dense.

▶ **Definition 33** (Semi-Continuity). *Let $f\colon X \to \overline{\mathbb{R}}$ be a function from a subset X of the real numbers to the extended real numbers, then f is lower respectively upper semi-continuous at $x \in X$ if*

$$f(x) \le \liminf_{\xi\to x} f(\xi) \text{ respectively} \tag{30a}$$

$$f(x) \ge \limsup_{\xi\to x} f(\xi). \tag{30b}$$

If f is lower respectively upper semi-continuous at every element $x \in X$, then we call f a lower respectively upper semi-continuous function.

▶ **Lemma 34** (Duality of Semi-Continuity). *Let $f\colon X \to \overline{\mathbb{R}}$ be a function from a subset X of the real numbers to the extended real numbers, then*

$$f \text{ is lower semi-continuous} \Leftrightarrow -f \text{ is upper semi-continuous} \tag{31}$$

Proof. Let x be an element of X, then it follows from Definition 33, i.e.

$$f(x) \le \liminf_{\xi\to x} f(\xi) \Leftrightarrow -f(x) \ge -\liminf_{\xi\to x} f(\xi) \Leftrightarrow -f(x) \ge \limsup_{\xi\to x} -f(\xi)$$ ◀

▶ **Lemma 35** (Semi-Continuous Functions are Closed under Addition). *Let $f, g\colon X \to \overline{\mathbb{R}}$ be lower respectively upper semi-continuous functions from a subset X of the real numbers to the extended real numbers, then $f + g$ is lower respectively upper semi-continuous, if no $x \in X$ exists such that $f(x) + g(x)$ is of the type $-\infty + \infty$.*

Proof. Let x be an element of X, then if follows from Definition 33, i.e.

$$f(x) + g(x) \leq \liminf_{\xi \to x} f(x) + \liminf_{\xi \to x} g(x) \leq \liminf_{\xi \to x} f(x) + g(x) \text{ respectively}$$

$$f(x) + g(x) \geq \limsup_{\xi \to x} f(x) + \limsup_{\xi \to x} g(x) \geq \limsup_{\xi \to x} f(x) + g(x).$$ ◀

▶ **Lemma 36.** *Let $f \colon I \to \mathbb{R}$ be an increasing function from an interval I of the real numbers, then*

$$f \text{ is lower semi-continuous} \qquad \text{if } f \text{ is left-continuous} \tag{32a}$$

$$f \text{ is upper semi-continuous} \qquad \text{if } f \text{ is right-continuous} \tag{32b}$$

Proof. Let $x \in I$ and f be left-continuous respectively right-continuous, then

$$\liminf_{\xi \to x} f(\xi) = \sup_{r>0} \inf_{\xi \in I} \{f(\xi) : 0 < |\xi - x| < r\} \geq \sup_{r>0} f(x - r) = f(x)$$

$$\limsup_{\xi \to x} f(\xi) = \inf_{r>0} \sup_{\xi \in I} \{f(\xi) : 0 < |\xi - x| < r\} \leq \inf_{r>0} f(x + r) = f(x)$$ ◀

B Properties of Pseudo-Inverses

▶ **Lemma 37.** *Confer [3, Proposition 6, (6) and (7)]. Let $f \colon X \to Y$ be a mapping from a subset X of a complete lattice L to a partially ordered set Y, then for an element $x \in X$ and an element $y \in Y$*

$$y \leq f(x) \Rightarrow f^{\underline{-1}}(y) \leq x \text{ and} \tag{33a}$$

$$f(x) \leq y \Rightarrow x \leq f^{\overline{-1}}(y). \tag{33b}$$

Proof of (33a). Let $y \leq f(x)$, then $x \in X_{y \leq f}$, therefore $f^{\underline{-1}}(y) = \inf X_{y \leq f} \leq x$. ◀

Proof of (33b). Let $f(x) \leq y$, then $x \in X_{f \leq y}$, therefore $x \leq \sup X_{f \leq y} = f^{\overline{-1}}(y)$. ◀

▶ **Lemma 38.** *Confer [3, Proposition 6, (14) and (12)]. Let $f \colon X \to Y$ be a mapping from a subset X of a complete lattice L to a totally ordered set Y, then for an element $x \in X$ and an element $y \in Y$*

$$x < f^{\underline{-1}}(y) \Rightarrow f(x) < y \text{ and} \tag{34a}$$

$$f^{\overline{-1}}(y) < x \Rightarrow y < f(x). \tag{34b}$$

Proof of (34a). Let $x < f^{\underline{-1}}(y)$, then the partial order of X implies that $\neg(f^{\underline{-1}}(y) \leq x)$. We can then apply Equation (33a) whose contraposition implies $\neg(y \leq f(x))$. The total order of Y then implies $f(x) < y$. ◀

Proof of (34b). Let $f^{\overline{-1}}(y) < x$, then the partial order of X implies that $\neg(x \leq f^{\overline{-1}}(y))$. We can then apply Equation (33b) whose contraposition implies $\neg(f(x) \leq y)$. The total order of Y then implies $y < f(x)$. ◀

▶ **Lemma 39.** *Confer [3, Proposition 6, (8), (9), and (10)]. Let $f \colon X \to Y$ be an isotone mapping from a totally ordered subset X of a complete lattice L to a partially ordered set Y, then for an element $x \in X$ and an element $y \in Y$*

$$f(x) < y \Rightarrow x \leq f^{\underline{-1}}(y) \text{ and} \tag{35a}$$

$$y < f(x) \Rightarrow f^{\overline{-1}}(y) \leq x. \tag{35b}$$

$$f^{\underline{-1}}(y) < x \Rightarrow y \leq f(x) \tag{36}$$

Proof of (35a). Let $f(x) < y$ and let ξ be an element of $X_{y \leq f}$, then we have $f(x) < y \leq f(\xi)$. This implies $x < \xi$, because f is isotone and X is totally ordered. Therefore, x is a lower bound of $X_{y \leq f}$, so $x \leq \inf X_{y \leq f} = f^{\underline{-1}}(y)$. ◀

Proof of (35b). Let $y < f(x)$ and let ξ be an element of $X_{f \leq y}$, then we have $f(\xi) \leq y < f(x)$. This implies $\xi < x$, because f is isotone and X is totally ordered. Therefore, x is an upper bound of $X_{f \leq y}$, so $f^{\overline{-1}}(y) = \sup X_{f \leq y} \leq x$. ◀

Proof of (36). Let $f^{\underline{-1}}(y) < x$, the infimum of $X_{y \leq f}$ is less than x. Then a $\xi \in X_{y \leq f}$ must exist such that $f^{\underline{-1}}(y) \leq \xi < x$, due to how the infimum is defined and because X is a totally ordered set. On the one hand from $\xi \in X_{y \leq f}$ follows that $y \leq f(\xi)$, on the other hand from $\xi < x$ follows $f(\xi) \leq f(x)$, because f is isotone, subsequently $y \leq f(x)$ holds. ◀

▶ **Lemma 40.** *Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a function that is not bounded above, then the image of $f^{\underline{-1}}$ is a subset of the non-negative real numbers.*

$$f^{\underline{-1}}(\mathbb{R}_0^+) \subseteq \mathbb{R}_0^+, \tag{37}$$

Proof. Let $\mathbb{R}_0^+ \cup \{\infty\}$ be the complete lattice of which $\mathbb{R}_0^+$ is a subset of and let y be in $\mathbb{R}_0^+$. Since f is not bounded above, a non-negative real number x exists ($x \in \mathbb{R}_0^+$) that satisfies $y < f(x)$.

On the one hand $y < f(x)$ implies $f^{\underline{-1}}(y) \leq x$ according to (33a). On the other hand the relation $\mathbb{R}_0^+ \supseteq X_{y \leq f}$ implies $0 = \inf \mathbb{R}_0^+ \leq \inf X_{y \leq f} = f^{\underline{-1}}(y)$. Therefore $f^{\underline{-1}}(y)$ is a non-negative real number. ◀

▶ **Lemma 41.** *Confer [3, Proposition 6, (17), and (18)]. Let $f\colon X \to \mathbb{R}$ be an increasing and left-continuous function from a subset X of the real numbers $\mathbb{R}$, then for an element $x \in X$ and an element $y \in \mathbb{R}$*

$$y < f(x) \Rightarrow f^{\overline{-1}}(y) < x \tag{38a}$$

$$x \leq f^{\overline{-1}}(y) \Rightarrow f(x) \leq y \tag{38b}$$

Proof. Let $y < f(x)$. Because f is left-continuous a $\delta > 0$ exists for $\varepsilon = f(x) - y > 0$ such that $|f(x) - f(\xi_1)| < \varepsilon$ holds for any $\xi_1 \in X$ where $x - \delta < \xi_1 < x$. From $|f(x) - f(\xi_1)| < f(x) - y$ it follows that $y < f(\xi_1)$. Let ξ_2 be an element of $X_{f \leq y}$, hence $f(\xi_2) \leq y < f(\xi_1)$ holds. Since f is increasing and X is totally ordered it follows that $\xi_2 < \xi_1$, therefore ξ_1 is an upper bound of $X_{f \leq y}$. So $\sup X_{f \leq y} = f^{\overline{-1}}(y) \leq \xi_1 < x$. The contraposition $x \leq f^{\overline{-1}}(y) \Rightarrow f(x) \leq y$ follows directly. ◀

▶ **Lemma 42.** *Confer [3, Proposition 6, (15), and (16)]. Let $f\colon X \to \mathbb{R}$ be an increasing and right-continuous function from a subset X of the real numbers $\mathbb{R}$, then for an element $x \in X$ and an element $y \in \mathbb{R}$*

$$f(x) < y \Rightarrow x < f^{\underline{-1}}(y) \tag{39a}$$

$$f^{\underline{-1}}(y) \leq x \Rightarrow y \leq f(x) \tag{39b}$$

Proof. Let $f(x) < y$. Because f is right-continuous a $\delta > 0$ exists for $\varepsilon = y - f(x) > 0$ such that $|f(\xi_1) - f(x)| < \varepsilon$ holds for any $\xi_1 \in X$ where $x < \xi_1 < x + \delta$. From $|f(\xi_1) - f(x)| < y - f(x)$ it follows that $f(\xi_1) < y$. Let ξ_2 be an element of $X_{y \leq f}$, hence $f(\xi_1) < y \leq f(\xi_2)$ holds. Since f is increasing and X is totally ordered it follows that $\xi_1 < \xi_2$, therefore ξ_1 is a lower bound of $X_{y \leq f}$. So $x < \xi_1 \leq f^{\underline{-1}}(y) = \inf X_{y \leq f}$. The contraposition $f^{\underline{-1}}(y) \leq x \Rightarrow y \leq f(x)$ follows directly. ◀

▶ **Lemma 43.** *Let $f\colon X \to \mathbb{R}$ be an increasing and left-continuous functions from a subset X of the real numbers $\mathbb{R}$, then for an element $x \in X$ and an element $y \in \mathbb{R}$*

$$f(x) \leq y \Leftrightarrow x \leq f^{\overline{-1}}(y) \tag{40}$$

Proof. Follows directly from Equations (33b) and (38b).

$$f(x) \leq y \overset{(33b)}{\Rightarrow} x \leq f^{\overline{-1}}(y) \overset{(38b)}{\Rightarrow} f(x) \leq y$$ ◀

▶ **Lemma 44.** *Let $f\colon X \to \mathbb{R}$ be an increasing and right-continuous function from a subset X of the real numbers $\mathbb{R}$, then for an element $x \in X$ and an element $y \in \mathbb{R}$*

$$y \leq f(x) \Leftrightarrow f^{\underline{-1}}(y) \leq x \tag{41}$$

Proof. Follows directly from Equations (33a) and (39b).

$$y \leq f(x) \overset{(33a)}{\Rightarrow} f^{\underline{-1}}(y) \leq x \overset{(39b)}{\Rightarrow} y \leq f(x)$$ ◀

▶ **Lemma 45** (Isotone pseudo-inverse)**.** *Confer [3, Proposition 3]. Let $f\colon X \to Y$ be a mapping from a subset X of a complete lattice L to a partially ordered set Y, then*

$$f^{\underline{-1}} \textit{ is isotone} \tag{42a}$$

$$f^{\overline{-1}} \textit{ is isotone} \tag{42b}$$

Proof of (42a). Confer [10, p. 131, Theorem 3.1.2]. Let y_1 and y_2 be elements of Y with $y_1 \leq y_2$ and let x be an element of $X_{y_2 \leq f}$. Then, we have $f(x) \geq y_2 \geq y_1$, so x is also an element of $X_{y_1 \leq f}$. Therefore $X_{y_1 \leq f}$ is a superset of $X_{y_2 \leq f}$ and subsequently $f^{\underline{-1}}(y_1) = \inf X_{y_1 \leq f} \leq \inf X_{y_2 \leq f} = f^{\underline{-1}}(y_2)$. ◀

Proof of (42b). Let y_1 and y_2 be elements of Y with $y_1 \leq y_2$ and let x be an element of $X_{f \leq y_1}$. Then, we have $f(x) \leq y_1 \leq y_2$, so x is also an element of $X_{f \leq y_2}$. Therefore $X_{f \leq y_1}$ is a subset of $X_{f \leq y_2}$ and subsequently $f^{\overline{-1}}(y_1) = \sup X_{f \leq y_1} \leq \sup X_{f \leq y_2} = f^{\overline{-1}}(y_2)$. ◀

▶ **Lemma 46** (Directional continuity of pseudo-inverse)**.** *Confer [3, Proposition 5]. Let $f\colon I \to \mathbb{R}$ be an increasing function from an interval I of the real numbers $\mathbb{R}$, then*

$$f^{\underline{-1}} \textit{ is left-continuous} \tag{43a}$$

$$f^{\overline{-1}} \textit{ is right-continuous} \tag{43b}$$

Proof of (43a). Let y be a real number and $\varepsilon > 0$.

Case $f^{\underline{-1}}(y) - \varepsilon < \inf I$. For any $\delta > 0$ and for any $\upsilon \in (y - \delta, y)$ we have $f^{\underline{-1}}(y) - \varepsilon < \inf I \leq f^{\underline{-1}}(\upsilon)$. Therefore, we get $f^{\underline{-1}}(y) - f^{\underline{-1}}(\upsilon) < \varepsilon$.

Case $\inf I \leq f^{\underline{-1}}(y) - \varepsilon$. Note that $f^{\underline{-1}}(y) - \varepsilon < f^{\underline{-1}}(y) \leq \sup I$. Because I is an interval of the real numbers, a real number $\xi \in I$ exists such that $f^{\underline{-1}}(y) - \varepsilon < \xi < f^{\underline{-1}}(y)$. By applying Equation (34a) $\xi < f^{\underline{-1}}(y)$ implies $f(\xi) < y$. Let $\delta := y - f(\xi)$, then for all $\upsilon \in \mathbb{R} \cap (y - \delta, y)$ we have $y - \delta = f(\xi) < \upsilon$. Through Equation (35a) this implies $\xi \leq f^{\underline{-1}}(\upsilon)$. So, together with $f^{\underline{-1}}(y) - \varepsilon < \xi$ we get $f^{\underline{-1}}(y) - f^{\underline{-1}}(\upsilon) < \varepsilon$. ◀

Proof of (43b). Let y be a real number and $\varepsilon > 0$.

Case $\sup I < f^{\overline{-1}}(y) + \varepsilon$. For any $\delta > 0$ and for any $\upsilon \in (y, y+\delta)$ we have $f^{\overline{-1}}(\upsilon) \leq \sup I < f^{\overline{-1}}(y) + \varepsilon$. Therefore, we get $f^{\overline{-1}}(\upsilon) - f^{\underline{-1}}(y) < \varepsilon$.

Case $f^{\overline{-1}}(y) + \varepsilon \leq \sup I$. Note that $\inf I \leq f^{\overline{-1}}(y) < f^{\overline{-1}}(y) + \varepsilon$. Because I is an interval of the real numbers, a real number $\xi \in I$ exists such that $f^{\overline{-1}}(y) < \xi < f^{\overline{-1}}(y) + \varepsilon$. By applying Equation (34b) $f^{\overline{-1}}(y) < \xi$ implies $y < f(\xi)$. Let $\delta := f(\xi) - y$, then for all $\upsilon \in \mathbb{R} \cap (y, y+\delta)$ we have $\upsilon < f(\xi) = y + \delta$. Through Equation (35b) this implies $f^{\underline{-1}}(\upsilon) \leq \xi$. So, together with $\xi < f^{\overline{-1}}(y) + \varepsilon$ we get $f^{\overline{-1}}(\upsilon) - f^{\overline{-1}}(y) < \varepsilon$. ◀

▶ **Lemma 47** (Pseudo-inverse operator is antitone)**.** *Let $f\colon X \to Y$ and $g\colon X \to Y$ be mappings from a subset X of a complete lattice L to a partially order set Y, then*

$$f \leq g \Rightarrow f^{\underline{-1}} \geq g^{\underline{-1}} \tag{44a}$$

$$f \leq g \Rightarrow f^{\overline{-1}} \geq g^{\overline{-1}} \tag{44b}$$

Proof of (44a). Let $f \leq g$, y an element of Y, and x an element of $X_{y \leq f}$, then $y \leq f(x) \leq g(x)$, so $X_{y \leq f} \subseteq X_{y \leq g}$, and subsequently $f^{\underline{-1}}(y) = \inf X_{y \leq f} \geq \inf X_{y \leq g} = g^{\underline{-1}}(y)$. ◀

Proof of (44b). Let $f \leq g$, y and element of Y, and x an element of $X_{g \leq y}$, then $y \geq g(x) \geq f(x)$, so $X_{g \leq y} \subseteq X_{f \leq y}$, and subsequently $f^{\overline{-1}}(y) = \sup X_{f \leq y} \geq \sup X_{g \leq y} = g^{\overline{-1}}(y)$. ◀

▶ **Lemma 48** (Pseudo-inverse of a composition)**.** *Confer [3, Proposition 6, (25) and (24)]. Let $f\colon X \to Y$ be a function from a subset X of a complete lattice L to a subset Y of the real numbers and let $g\colon Y \to \mathbb{R}$ be an increasing function, then*

$$(g \circ f)^{\overline{-1}} = (f^{\overline{-1}} \circ g^{\overline{-1}}) \qquad \textit{if } g \textit{ is left-continuous} \tag{45a}$$

$$(g \circ f)^{\underline{-1}} = (f^{\underline{-1}} \circ g^{\underline{-1}}) \qquad \textit{if } g \textit{ is right-continuous} \tag{45b}$$

Proof of (45a). Let z be a real number and let g be left-continuous, then we can apply Equation (40) and we get

$$(g \circ f)^{\overline{-1}}(z) \overset{(4b)}{=} \sup_{x \in X} \{x : g(f(x)) \leq z\} \overset{(40)}{=} \sup_{x \in X} \left\{x : f(x) \leq g^{\overline{-1}}(z)\right\} \overset{(4b)}{=} (f^{\overline{-1}} \circ g^{\overline{-1}})(z) \quad ◀$$

Proof of (45b). Let z be a real number and let g be right-continuous, then we can apply Equation (41) and we get

$$(g \circ f)^{\underline{-1}}(z) \overset{(4a)}{=} \inf_{x \in X} \{x : z \leq g(f(x))\} \overset{(41)}{=} \inf_{x \in X} \left\{x : g^{\underline{-1}}(z) \leq f(x)\right\} \overset{(4a)}{=} (f^{\underline{-1}} \circ g^{\underline{-1}})(z) \quad ◀$$

▶ **Lemma 49** (The Pseudo-Inverse Operators are inverse to each other)**.** *Let $f\colon X \to \mathbb{R}$ be an increasing function from a subset X of $\overline{\mathbb{R}}$, then*

$$f^{\overline{-1}\underline{-1}} = f \qquad \textit{if } f \textit{ is left-continuous} \tag{46a}$$

$$f^{\underline{-1}\overline{-1}} = f \qquad \textit{if } f \textit{ is right-continuous} \tag{46b}$$

Proof of (46a). Let x be an element of X and let f be left-continuous, then we can apply Equation (40) and we get

$$f^{\overline{-1}\underline{-1}}(x) \overset{(4a)}{=} \inf X_{x \leq f^{\overline{-1}}} \overset{(3b)}{=} \inf_{y \in \mathbb{R}} \left\{y : x \leq f^{\overline{-1}}(y)\right\} \overset{(40)}{=} \inf_{y \in \mathbb{R}} \{y : f(x) \leq y\} = f(x) \quad ◀$$

Proof of (46b). Let x be an element of X and let f be right-continuous, then we can apply Equation (41) and we get

$$f^{\underline{-1}\,\overline{-1}}(x) \overset{(4b)}{=} \sup X_{f^{\underline{-1}} \leq x} \overset{(3a)}{=} \sup_{y\in\mathbb{R}} \{y : f^{\underline{-1}}(y) \leq x\} \overset{(41)}{=} \sup_{y\in\mathbb{R}} \{y : y \leq f(x)\} = f(x) \qquad ◀$$

▶ **Lemma 50.** *Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an upper semi-continuous function and $g := f^{\nearrow}$ be the increasing closure of f, then*

$$f^{\underline{-1}} = g^{\underline{-1}} \tag{47}$$

Proof. Let x be a non-negative real number and let I_x be the interval $[0, x]$, then

(a) $f \leq g$: Follows directly from g being the increasing closure of f, Remark 17.

(b) $f^{\underline{-1}} \leq g^{\underline{-1}}$: f is upper semi-continuous, I_x is a compact set, therefore f achieves its maximum in I_x and an element $x_0 \in I_x$ with $x_0 \leq x$ exists where $f(x_0) = g(x)$.
Let y be a non-negative real number. Then, for every element $x \in X_{y\leq g}$ there exists a $x_0 \leq x$ where $y \leq g(x) = f(x_0)$. Therefore, x_0 is an element of $X_{y\leq f}$ and subsequently $f^{\underline{-1}}(y) = \inf X_{y\leq f} \leq \inf X_{y\leq g} = g^{\underline{-1}}(y)$.

(c) $f^{\underline{-1}} \geq g^{\underline{-1}}$: Follows from (a) and (44a).

Equation (47) follows from (b) and (c). ◀

C Properties of Deconvolution

▶ **Lemma 51** (Monotonicity of Deconvolution)**.** *Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an increasing function and $g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ a function, then*

$$(f \,\overline{\oslash}\, g) \text{ is increasing} \tag{48}$$

Proof. Let x_1 and x_2 be non-negative real numbers ($x_1, x_2 \in \mathbb{R}_0^+$) such that $x_1 \leq x_2$.

From $x_1 \leq x_2$ follows that $x_1 + \xi \leq x_2 + \xi$ holds for every $\xi \in \mathbb{R}_0^+$ and subsequently $f(x_1 + \xi) \leq f(x_2 + \xi)$, since f is increasing. Furthermore $f(x_1 + \xi) - g(\xi) \leq f(x_2 + \xi) - g(\xi)$ holds for every $\xi \in \mathbb{R}_0^+$, therefore

$$(f \,\overline{\oslash}\, g)(x_1) = \inf_{\xi\in\mathbb{R}_0^+} \{f(x_1 + \xi) - g(\xi)\} \leq \inf_{\xi\in\mathbb{R}_0^+} \{f(x_2 + \xi) - g(\xi)\} = (f \,\overline{\oslash}\, g)(x_2) \qquad ◀$$

▶ **Theorem 52.** *Let $f, g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be increasing functions that are not bounded above, then*

$$(f \,\overline{\oslash}\, g)^{\underline{-1}} = (f^{\underline{-1}} \oslash g^{\underline{-1}}) \tag{49}$$

Proof. Let $y \in \mathbb{R}_0^+$ be a non-negative real number.

(a) $f^{\underline{-1}}(\mathbb{R}_0^+) \subseteq \mathbb{R}_0^+$: Follows from f being not bounded above and Equation (37).

(b) $g^{\underline{-1}}(\mathbb{R}_0^+) \subseteq \mathbb{R}_0^+$: Follows from g being not bounded above and Equation (37).

i.e. the images of $f^{\underline{-1}}$ and $g^{\underline{-1}}$ are subsets of the non-negative real numbers, therefore $(f^{\underline{-1}} \oslash g^{\underline{-1}})$ is well-defined.

Part 1 shows that $(f \,\overline{\oslash}\, g)^{\underline{-1}} \leq (f^{\underline{-1}} \oslash g^{\underline{-1}})$:
Let x be a non-negative real number such that $x < (f \,\overline{\oslash}\, g)^{\underline{-1}}(y)$. According to (34a) this implies $(f \,\overline{\oslash}\, g)(x) < y$ and due to (5b) a non-negative real number ξ exists that satisfies $f(x + \xi) - g(\xi) < y$. Furthermore a non-negative real number v exists such that $f(x + \xi) - y < v \leq g(\xi)$, because $g(\xi)$ is a non-negative real number.

On the one hand f is increasing, therefore $f(x+\xi) < y+\upsilon$ implies $x+\xi \leq f^{\underline{-1}}(y+\upsilon)$ according to (35a), so $x \leq f^{\underline{-1}}(y+\upsilon)-\xi$. On the other hand $\upsilon \leq g(\xi)$ implies $g^{\underline{-1}}(\upsilon) \leq \xi$ according to (33a). This results altogether in $x+g^{\underline{-1}}(\upsilon) \leq f^{\underline{-1}}(y+\upsilon)-\xi+g^{\underline{-1}}(\upsilon) \leq f^{\underline{-1}}(y+\upsilon)$ and due to (a) and (b) ultimately in $x \leq f^{\underline{-1}}(y+\upsilon) - g^{\underline{-1}}(\upsilon) \leq \sup_{\lambda\in\mathbb{R}_0^+}\{f^{\underline{-1}}(y+\lambda) - g^{\underline{-1}}(\lambda)\} = (f^{\underline{-1}} \oslash g^{\underline{-1}})(y)$.

In conclusion, the set $X_{l,y} := \left\{x \in \mathbb{R}_0^+ : x < (f \mathbin{\overline{\oslash}} g)^{\underline{-1}}(y)\right\}$ is a subset of the set $X_{r,y} := \left\{x \in \mathbb{R}_0^+ : x \leq (f^{\underline{-1}} \oslash g^{\underline{-1}})(y)\right\}$. Since $(f \mathbin{\overline{\oslash}} g)^{\underline{-1}}(y) = \sup X_{l,y}$ and $\sup X_{r,y} = (f^{\underline{-1}} \oslash g^{\underline{-1}})(y)$ this implies $(f \mathbin{\overline{\oslash}} g)^{\underline{-1}}(y) \leq (f^{\underline{-1}} \oslash g^{\underline{-1}})(y)$.Furthermore y is chosen arbitrarily, hence $(f \mathbin{\overline{\oslash}} g)^{\underline{-1}} \leq (f^{\underline{-1}} \oslash g^{\underline{-1}})$.

Part 2 shows that $(f \mathbin{\overline{\oslash}} g)^{\underline{-1}} \geq (f^{\underline{-1}} \oslash g^{\underline{-1}})$:
Let x be a non-negative real number that satisfies $x < (f^{\underline{-1}} \oslash g^{\underline{-1}})(y)$. Due to (5a) a non-negative real number υ exists such that $x < f^{\underline{-1}}(y+\upsilon) - g^{\underline{-1}}(\upsilon)$. Furthermore a non-negative real number ξ exists that satisfies $g^{\underline{-1}}(\upsilon) < \xi < f^{\underline{-1}}(y+\upsilon) - x$ due to (a) and (b).

On the one hand $x+\xi < f^{\underline{-1}}(y+\upsilon)$ implies $f(x+\xi) < y+\upsilon$ according to (34a). On the other hand g is increasing, therefore $g^{\underline{-1}}(\upsilon) < \xi$ implies $\upsilon \leq g(\xi)$ according to (36). This results altogether in $f(x+\xi) < y+g(\xi)$ and thus $(f \mathbin{\overline{\oslash}} g)(x) \leq f(x+\xi) - g(\xi) < y$. Because f is increasing, so is $(f \mathbin{\overline{\oslash}} g)$ according to (48), therefore $(f \mathbin{\overline{\oslash}} g)(x) < y$ implies $x \leq (f \mathbin{\overline{\oslash}} g)^{\underline{-1}}(y)$ according to (35a).

In conclusion the set $X_{r,y} := \left\{x \in \mathbb{R}_0^+ : x < (f^{\underline{-1}} \oslash g^{\underline{-1}})(y)\right\}$ is a subset of the set $X_{l,y} := \left\{x \in \mathbb{R}_0^+ : x \leq (f \mathbin{\overline{\oslash}} g)^{\underline{-1}}(y)\right\}$. Since $(f \mathbin{\overline{\oslash}} g)^{\underline{-1}}(y) = \sup X_{l,y}$ and $\sup X_{r,y} = (f^{\underline{-1}} \oslash g^{\underline{-1}})(y)$ this implies $(f \mathbin{\overline{\oslash}} g)^{\underline{-1}}(y) \geq (f^{\underline{-1}} \oslash g^{\underline{-1}})(y)$.Furthermore y is chosen arbitrarily, hence $(f \mathbin{\overline{\oslash}} g)^{\underline{-1}} \geq (f^{\underline{-1}} \oslash g^{\underline{-1}})$.

Combining both parts yields the desired equality $(f \mathbin{\overline{\oslash}} g)^{\underline{-1}} = (f^{\underline{-1}} \oslash g^{\underline{-1}})$. ◀

D Other Properties

▶ **Lemma 53.** *Confer [15, Lemma 1]. Let Γ be a set of n independent tasks as described in Section 3.1 and let Assumption 20 hold, then the resource-based lower service curve β_i^- of task τ_i is*

$$\beta_i^-(\Delta) = \sup_{0\leq\lambda\leq\Delta}\left\{\beta^-(\lambda) - \sum_{j=1}^{i-1}\alpha_j^+(\lambda)\right\} \tag{50}$$

Proof. See [15, Lemma 1]. ◀

▶ **Lemma 54.** *Confer [10, p. 154][1]. Let $f, g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be increasing functions, then*

$$f \leftrightarrow g = \inf_{\mu\in\mathbb{R}_0^+}\left\{\mu : \forall\lambda \in \mathbb{R}_0^+, f(\lambda) \leq g(\lambda+\mu)\right\} \tag{51}$$

Proof. Let $A_\lambda := \left\{\mu \in \mathbb{R}_0^+ : f(\lambda) \leq g(\lambda+\mu)\right\}$, $d(\lambda) := \inf A_\lambda$, and $B := \bigcap_{\lambda\in\mathbb{R}_0^+} A_\lambda$, then $f \leftrightarrow g = \sup_{\lambda\in\mathbb{R}_0^+}\{d(\lambda)\}$ and $\inf_{\mu\in\mathbb{R}_0^+}\left\{\mu : \forall\lambda \in \mathbb{R}_0^+, f(\lambda) \leq g(\lambda+\mu)\right\} = \inf B$.

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[1] Le Boudec and Thiran textually state Equation (51) in [10, p. 154] without proof and mistakenly refer to the vertical deviation [10, p. 154, (3.20)] and not the horizontal deviation [10, p. 154, (3.21)].

Case 1. The set A_λ is empty for some $\lambda \in \mathbb{R}_0^+$.

(a) $f \leftrightarrow g = \infty$: For that $\lambda \in \mathbb{R}_0^+$ we have $d(\lambda) = \inf A_\lambda = \inf \emptyset = \infty$, therefore $f \leftrightarrow g = \sup_{\lambda \in \mathbb{R}_0^+} \{d(\lambda)\} = \infty$.

(b) $\inf_{\mu \in \mathbb{R}_0^+} \{\mu : \forall \lambda \in \mathbb{R}_0^+, f(\lambda) \leq g(\lambda + \mu)\} = \infty$: Because there is a $\lambda \in \mathbb{R}_0^+$ where A_λ is the empty set, the set $B = \bigcap_{\lambda \in \mathbb{R}_0^+} A_\lambda = \emptyset$ is also empty. Therefore,

$$\inf_{\mu \in \mathbb{R}_0^+} \{\mu : \forall \lambda \in \mathbb{R}_0^+, f(\lambda) \leq g(\lambda + \mu)\} = \inf B = \inf \emptyset = \infty.$$

Case 2. The set A_λ is not empty for every $\lambda \in \mathbb{R}_0^+$.

(c) A_λ is an interval with $\sup A_\lambda = \infty$: Follows directly from g being increasing, i.e. let $\lambda \in \mathbb{R}_0^+$ and $\mu \in A_\lambda$, then for any $\xi \in \mathbb{R}_0^+$ such that $\xi \geq \mu$ we have $g(\lambda + \xi) \geq g(\lambda + \mu)$, therefore $\xi \in A_\lambda$ and $\sup A_\lambda = \infty$.

(d) B is an interval: Follows directly from (c), an intersection of intervals is an interval.

(e) $\mu \in B \Rightarrow f \leftrightarrow g \leq \mu$: $\mu \in B \Rightarrow \forall \lambda \in \mathbb{R}_0^+ : \mu \in A_\lambda \Rightarrow \forall \lambda \in \mathbb{R}_0^+ : d(\lambda) \leq \mu \Rightarrow \sup_{\lambda \in \mathbb{R}_0^+} \{d(\lambda)\} \leq \mu \Rightarrow f \leftrightarrow g \leq \mu$

(f) $\mu \notin B \Rightarrow \mu \leq f \leftrightarrow g$: $\mu \notin B \Rightarrow \exists \lambda \in \mathbb{R}_0^+ : \mu \notin A_\lambda \Rightarrow \exists \lambda \in \mathbb{R}_0^+ : \mu \leq d(\lambda) \leq \sup_{\lambda \in \mathbb{R}_0^+} \{d(\lambda)\} = f \leftrightarrow g$

Equation (51) follows from (d)–(f), i.e.

$$f \leftrightarrow g = \inf B = \inf_{\mu \in \mathbb{R}_0^+} \{\mu : \forall \lambda \in \mathbb{R}_0^+, f(\lambda) \leq g(\lambda + \mu)\}$$ ◀

▶ **Lemma 55.** *Let $f, g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be increasing functions, then the horizontal distance between them is*

$$f \leftrightarrow g = (g \mathbin{\overline{\oslash}} f)^{\underline{-1}}(0) \tag{52}$$

Proof. f and g are increasing, so we can use the equality of Equation (51). After some rearranging we apply the definition for the deconvolution in inf-plus, Equation (5b), and for the pseudo-inverse, Equation (4a).

$$\begin{aligned}
f \leftrightarrow g &\overset{(51)}{=} \inf_{\mu \in \mathbb{R}_0^+} \{\mu : \forall \lambda \in \mathbb{R}_0^+, f(\lambda) \leq g(\lambda + \mu)\} \\
&= \inf_{\mu \in \mathbb{R}_0^+} \{\mu : \forall \lambda \in \mathbb{R}_0^+, 0 \leq g(\lambda + \mu) - f(\lambda)\} \\
&= \inf_{\mu \in \mathbb{R}_0^+} \left\{\mu : 0 \leq \inf_{\lambda \in \mathbb{R}_0^+} \{g(\lambda + \mu) - f(\lambda)\}\right\} \\
&\overset{(5b)}{=} \inf_{\mu \in \mathbb{R}_0^+} \{\mu : 0 \leq (g \mathbin{\overline{\oslash}} f)(\mu)\} \overset{(4a)}{=} (g \mathbin{\overline{\oslash}} f)^{\underline{-1}}(0)
\end{aligned}$$ ◀

▶ **Lemma 56.** *Confer [22, p. 286, Lemma 1]. Let $f : A \to A$ be an increasing function from and into a complete lattice A and let P be the set of fix-points of f, then P is not empty, P is a complete lattice and*

$$\sup P = \sup_{x \in A} \{x : f(x) \geq x\} \in P \tag{53a}$$

$$\inf P = \inf_{x \in A} \{x : f(x) \leq x\} \in P \tag{53b}$$

Proof. See [22, p. 286, Lemma 1] ◀