Set Membership with Two Classical and Quantum Bit Probes

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Abstract

We study the classical and quantum bit-probe versions of the static set membership problem : Given a subset, S ($|S| \leq n$) of a universe, $\mathcal{U}$ ($|\mathcal{U}| = m \gg n$), represent it as a binary string in memory so that the query "Is x in S?" ($x \in \mathcal{U}$) can be answered by making at most t probes into the string. Let $s_A(m, n, t)$ denote the minimum length of the bit string in any scheme that solves this static set membership problem. We show that for $n \geq 4$

$$s_A(m, n, t = 2) = \begin{cases} \mathcal{O}(m^{1-1/(n-1)}) & \text{if } n = 0 \pmod 3; \\ \mathcal{O}(m^{1-1/n}) & \text{if } n = 1, 2 \pmod 3; \\ \mathcal{O}(m^{6/7}) & \text{if } n = 8, 9. \end{cases}$$

These bounds are shown using a common scheme that is based on a graph-theoretic observation on orienting the edges of a graph of high girth. For all $n \geq 4$, these bounds substantially improve on the previous best bounds known for this problem, some of which required elaborate constructions [4]. Our schemes are explicit. A lower bound of the form $s_A(m, n, 2) = \Omega(m^{1-\frac{1}{\lfloor n/4 \rfloor}})$ was known for this problem. We show an improved lower bound of $s_A(m, n, 2) = \Omega(m^{1-\frac{2}{n+3}})$; this bound was previously known only for $n = 3, 5$ [5, 6, 2, 7, 4].

We consider the quantum version of the problem, where access to the bit-string $b \in \{0, 1\}^s$ is provided in the form of a quantum oracle that performs the transformation $\mathcal{O}_b : |i\rangle \mapsto (-1)^{b_i} |i\rangle$. Let $s_Q(m, n, 2)$ denote the minimum length of the bit string that solves the above set membership problem in the quantum model (with adaptive queries but no error). We show that for all $n \leq m^{1/8}$, we have $s_{QA}(m, n, 2) = \mathcal{O}(m^{7/8})$. This upper bound makes crucial use of Nash-William's theorem [10] for decomposing a graph into forests. This result is significant because, prior to this work, it was not known if quantum schemes yield any advantage over classical schemes. We also consider schemes that make a small number of quantum *non-adaptive* probes. In particular, we show that the space required in this case, $s_{QN}(m, n = 2, t = 2) = O(\sqrt{m})$ and $s_{QN}(m, n = 2, t = 3) = O(m^{1/3})$; in contrast, it is known that two non-adaptive classical probes yield no savings. Our quantum schemes are simple and use only the fact that the XOR of two bits of memory can be computed using just one quantum query to the oracle.

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1 Introduction

We consider the problem of representing small subsets S of a universe $[m] = \{1, 2, \ldots, m\}$ in memory as a bit string so that membership queries of the form "Is $x \in S$?" can be answered with a small number of bit probes to the memory [9]. This is a fundamental question that asks how much a very sparse string can be compressed if we want to extract the bits of the original string efficiently from its compressed version. One natural way of representing sets in memory is the characteristic vector, which uses m bits of memory and answers membership queries using a single bit probe. Since there are only $O(\binom{m}{n})$ sets of size at most n (assume $n \ll m$) one might hope to represent them using $O(\log_2 \binom{m}{n}) \approx O(n \log m)$ bits of memory. However, compression to near the information-theoretic limits comes with a cost: membership can no longer be determined by reading just a small number of bits. To describe the trade-offs between efficiency of compression and the effort for extraction (measured as the number of bit probes), we will use the following notation [11]. Let $s_N(m, n, t)$ denote the minimum number of bits in a scheme that can represent sets of size at most n and answer membership queries by probing at most t bits of the memory non-adaptively (that is, the probes are made in parallel). We write $s_A(m, n, t)$ if the scheme is adaptive; we use the subscript Q if the scheme makes quantum queries (zero-error), which can be adaptive or non-adaptive [18], and write $s_{QA}(m, n, t)$ and $s_{QN}(m, n, t)$. Clearly,

$$s_N(m, n, t) \geq \left\{ \begin{array}{c} s_A(m, n, t) \\ s_{QN}(m, n, t) \end{array} \right\} \geq s_{QA}(m, n, t)$$

Radhakrishnan, Sen and Venkatesh [18] obtained lower bounds, which for the range of parameters of interest to us ($n \leq \sqrt{m}$, t constant) implies the following.

$$s_{QA}(m, n, t) = \Omega(m^{1/t} n^{1-1/t}).$$

(A similar lower bound in the classical setting was shown by Buhrman *et al.* [9].) Note that this bound shows that if the space is compressed to $O(n \log m)$, then $t = \Omega(\log m)$. Furthermore, if $t = 1$ no compression is possible even for $n = 1$; it also shows that $s_Q(m, 1, 2) = \Omega(\sqrt{m})$, for which there is a matching upper bound $s_N(m, n = 1, 2) = O(\sqrt{m})$. The first, non-trivial case is $n = 2$ and $t = 2$, where the above bound implies that $s_{QA}(m, 2, 2) = \Omega(\sqrt{m})$. It is known that this bound is not tight for classical schemes; better lower bound are known: $s_N(m, 2, 2) = m$ and $s_A(m, 2, 2) = \Omega(m^{4/7})$ [9, 19]. Remarkably, it is known that $s_A(m, 3, 2) = \Theta(m^{2/3})$ [13, 5]. Two-probe classical schemes have been constructed for representing small sets in several works starting with Alon and Feige [1] (see, e.g., [4, 2, 7, 12, 13, 15, 17], where sets of specific sizes are considered); the following upper and lower bounds was obtained by Garg and Radhakrishnan [12].

$$\Omega(m^{1-\frac{1}{\lfloor n/4 \rfloor}}) \leq s_A(m, n, t = 2) \leq \mathcal{O}(m^{1-\frac{1}{4n+1}}). \tag{1}$$

which roughly characterizes the space requirement for the problem, and, in particular, establishes that no savings over the standard characteristic vector representation can be expected if $n \geq \log m$. We show the following.

▶ **Theorem 1** (Result 1). *For $n \geq 4$,*

$$s_A(m, n, t = 2) = \begin{cases} \mathcal{O}(m^{1-1/(n-1)}) & \text{if } n = 0 \pmod 3; \\ \mathcal{O}(m^{1-1/n}) & \text{if } n = 1, 2 \pmod 3; \\ \mathcal{O}(m^{6/7}) & \text{if } n = 8, 9. \end{cases}$$

The above bounds improve the bounds for all other values of n (see Figure 1) for a comparison. Unlike the previous works, where different constructions (some of which quite intricate) were invented for different set sizes, our result is obtained using a unified approach based on graphs of high girth. (For $n = 2, 3$, the construction matches the currently best bounds.) To obtain a scheme for a set of a given size, one just plugs in the best available result for high-girth graph and obtains the bound claimed above. More importantly for us, the method used here inspires a quantum scheme that yields Result 3 below. We also obtain improved lower bounds.

▶ **Theorem 2** (Result 2). *For all odd n ($3 \leq n \leq \log m$), $s_A(m, n, t = 2) = \Omega(m^{1-2/(n+3)})$.*

This bound matches the best bound known earlier for $n = 3$ (see [13]) and $n = 5$ (see [3]), and improves the current best lower bound (see Equation (1) above) for all larger values of n. We follow the approach of [12], who consider the graph underlying a two-probe scheme, and show that if it is dense then it must contain a forbidden configuration. We make better use of the structure of the underlying graph to force the existence of forbidden configurations.

We now describe our results in the quantum model. As stated above, our classical approach helps develop a new quantum schemes.

▶ **Theorem 3** (Result 3). $s_{QA}(m, n = m^{1/8}, t = 2) = \mathcal{O}(m^{7/8})$.

This result is especially significant because it shows that, unlike in the classical case, two probes give substantial savings over the characteristic vector representation for sets substantially larger than $\log m$ (see the remark above following Equation (1)). Before this work, quantum schemes were not known to provide significant savings over classical schemes. Our quantum scheme is also based on dense graphs that are locally sparse, this time we do not make use of high girth. Instead, we invoke a result of Nash-Williams [10] on covering the edges of a graph with two forests. After this, our construction uses only the following basic fact from quantum computation (Deutsch's algorithm [16]): the parity of two bits of memory can be computed using just one quantum probe. In fact, only the second probe in our scheme is truly quantum. This result opens the possibility that the lower bounds of $\sqrt{mn}$ cited above is perhaps achievable for quantum schemes. We show in fact that for $n = 2$, the lower bound can be matched using *non-adaptive* constructions.

▶ **Theorem 4** (Result4).

$$s_{QN}(m, n = 2, t = 2) = \mathcal{O}(\sqrt{m});$$
$$s_{QN}(m, n = 2, t = 3) = \mathcal{O}(m^{1/3}).$$

The query scheme is simple. The query scheme for $(n = 2, t = 2)$ on input x computes for locations $\ell_1(x), \ell_2(x), \ell_3(x), \ell_4(x)$, and returns "Yes" iff the bits at the first two locations are different and the bits at the last two locations are different, that is, we use an AND of two inequality computations, each of which requires just one quantum probe. A similar query scheme that uses an AND of three inequality computations gives the three-probe non-adaptive quantum scheme. We also obtain non-adaptive two-probe schemes with sublinear space for storing sets with more elements (see Appendix D). These bounds are interesting because no non-adaptive two-probe classical scheme exists with sublinear space [9].

2 Classical two-probe adaptive schemes

In this section we establish Theorem 1. Our two-probe adaptive schemes are based on dense graphs of high girth. We first specify the storage scheme and the query schemes based on an underlying graph. Then, to complete the proof, we will show the following: (i) if the underlying graph has high girth, then there is an assignment of values to the memory such that all queries are answered correctly; (ii) the available explicit constructions of dense graphs of high girth in the literature yield the claimed bounds.

▶ **Definition 5** (Classical (G, K)-scheme). *Let G be a directed graph with N vertices and M edges; let K be a positive integer. We refer to the following as a (G, K)-scheme. The storage consists of two bit arrays, A and B. To answer a membership query the decision tree will make the first probe to array A and the second probe to array B.*

Edge array: *An array $A : E(G) \to \{0, 1\}$, indexed by edges of G.*

Vertex array: *A two dimensional array $B : V \times [K] \to \{0, 1\}$.*

Elements: *We identify our universe of elements $[m]$ with a subset of $E(G) \times [K]$ (so we must ensure that the graph has at least m/K edges); thus, each element $x \in [m]$ will be referred to as (e, i).*

Query: *We represent an edge of G as an ordered pair of the form $e = (v_0, v_1)$ with the convention that $v_0 < v_1$. To process the query for the element $x = (e, i)$, we read $A[e]$ (first probe); then we return the value $B[v_{A[e]}, i]$ (by making the second probe into B). In other words, we may think of $A[e]$ as a bit that orients the edge e towards either its smaller vertex or the larger vertex; depending on this bit, the second probe is made into the array B corresponding to the vertex towards which the edge points. Note that this scheme is adaptive: the second probe depends on the first.*

Space: *We will ensure that $MK \geq m$. The space used by this scheme is then $NK + M$ bits (NK for the N vertex array and K for the edge array). By choosing the graph G and the parameter K appropriately we will show that our schemes use small space.*

The following lemma provides the connection between dense graphs of high girth and efficient two-probe adaptive schemes.

▶ **Lemma 6.** *Let G be a graph with N vertices M edges and girth g such that $n \leq \lfloor \frac{3g}{4} \rfloor$ and $M \leq m$. Then, $s_A(m, n, 2) \leq M + N\lceil m/M \rceil$.*

Before we present the proof of this lemma formally, let us derive from it the bounds claimed in Theorem 1. Since every graph has a bipartite subgraph that includes at least half the edges, it is sufficient to restrict attention to bipartite graphs, and hence to graphs whose girth is even. The smallest even number g such that $n \leq \lfloor \frac{3g}{4} \rfloor$ is given by

$$g(n) = \begin{cases} 4\lceil n/3 \rceil & n = -1, 0 \pmod 3; \\ 4\lceil n/3 \rceil - 2 & n = 1 \pmod 3. \end{cases} \tag{2}$$

Now, suppose that for a girth g, there are constant $c(g)$ $d(g)$ and $\tau(g)$, such that for all large L, there is a graph with at most $c(g)L$ vertices, girth g and $d(g)L^{1+\tau(g)}$ edges. Then, taking a graph with $N = \Theta(m^{1/(1+2\tau(g(n)))})$ vertices and $M = \Theta(m^{(1+\tau)/(1+2\tau)})$ edges in Lemma 6, we obtain the following.

▶ **Corollary 7.** $s_A(m, n, 2) = \mathcal{O}(m^{(1+\tau(g(n)))/(1+2\tau(g(n)))})$.

In particular, by using the current best constructions of dense graphs with large girth we obtain the following bounds for $s_A(m, n, 2)$. For example, we may take $\tau(6) = 1/2$, $\tau(8) = 1/3$ and $\tau(12) = 1/5$ based on graphs described Wenger [20]. (In Appendix A, we explain in what sense these constructions, and hence the resulting schemes, are explicit.)

Proof of Lemma 6. Fix a graph G with N vertices and M edges as in the statement of the theorem. Consider the (G, K)-scheme with $K = \lceil m/M \rceil$. Clearly the space used by the scheme is $N + KM = N + M\lceil m/M \rceil$. It remains to show that there is an assignment to the edge and vertex arrays of this scheme so that every query is answered correctly. Fix a set S

n	girth $g(n)$	$\tau(g(n))$	Our bound ($m^{(1+\tau)/(1+2\tau)}$)	Previous best bound
2,3	4	1	$\mathcal{O}(m^{2/3})$	$\mathcal{O}(m^{2/3})$ [13]
4	6	1/2	$\mathcal{O}(m^{3/4})$ (using [20])	$\mathcal{O}(m^{5/6})$ [6, 4]
5,6	8	1/3	$\mathcal{O}(m^{4/5})$ (using [20])	$\mathcal{O}(m^{5/6})$ (for $n = 5$ [4])
7	10	1/5	$\mathcal{O}(m^{6/7})$ (using [8])	$\downarrow$
8,9	12	1/5	$\mathcal{O}(m^{6/7})$ (using [8])	$\downarrow$
$n = 3r - 2$	$4r - 2$	$1/(3r - 4)$	$\mathcal{O}(m^{1-\frac{1}{n}})$ (using [14])	$\mathcal{O}(m^{1-1/(4n+1)})$ [12]
$n = 3r - 1, 3r$	$4r$	$1/(3r - 3)$	$\mathcal{O}(m^{1-\frac{1}{n}})$, $\mathcal{O}(m^{1-\frac{1}{n-1}})$ (using [14])	$\mathcal{O}(m^{1-1/(4n+1)})$ [12]

Figure 1 Our upper bounds use explicit constructions of graphs of large girth available in the literature (see Appendix A).

of at most n elements; recall that the elements of the universe have the form (e, i), where e is an edge of the graph and $i \in [K]$. We will assign values to the two arrays in two steps. First the edge array A will be assigned values. Recall that this assignment corresponds to assigning directions to the edges. We will show below how this is to be done. Assuming this we show how the array B is initialized. To start with, we initialize array B with zeros. Now for each element $(e, i) \in S$ (say $e = (v_0, v_1)$ where $v_0 < v_1$), if $A[e] = 0$ we set $B[v_0, i] = 1$, otherwise we set $B[v_1, i] = 1$. This assignment ensures that the query scheme described above will answer correctly for each element in S, so there are no *false negatives*, no matter what initial orientation of the edges is chosen. The key idea is to choose an orientation that avoids *false positives*; we must ensure that the value in the array A are set in such a way that an element not in S and an element in S do not make second probes to the same location in array B. Definition 8 below formally describes such a *safe orientation*. Here edges e such that $(e, i) \in S$ for some i are colored GREEN and the other edges are colored RED. Thus, there are at most n GREEN edges. Our choice of colors RED and GREEN are based on the following consideration. Some edges support elements in the set, some others do not support any such element. We chose to regard edges with elements in the set as GREEN, because the eventual answer to the query in that case is 'Yes'. In our definition of safe orientation, RED edges and GREEN edges are not allowed to point to a common vertex. Two GREEN edges are not allowed to point to the same vertex either but two RED edges are allowed to. We warn the reader that our choice of colors might be confusing, because GREEN edges are more restrictive/dangerous than RED edges! Then, Lemma 9 below shows that the graph G above has a safe orientation. It follows that our query scheme answers all the queries correctly. ◀

▶ **Definition 8** (Safe orientation). *Suppose H is a graph whose edges are colored RED and GREEN. We say that an orientation of edges is* safe *if every vertex with an incoming GREEN edge has only one edge coming into it.*

▶ **Lemma 9.** *Suppose H is a graph with even girth $g \geq 4$ and $n \leq \lfloor 3g/4 \rfloor$ GREEN edges. Then, G has a safe orientation. (This claim should have a simple proof, but we have not been able to find one that covers all cases succinctly.)*

Preliminaries: In the following, suppose H is a graph with even girth g and $n \leq \lfloor 3g/4 \rfloor$ GREEN edges. To find the necessary orientation, we will proceed by induction on the size of H (its total number of edges plus vertices). For the base case, note that a graph with no edges clearly has a safe orientation. For the induction step, we will identify an initial set of vertices V' such that all edges that have at least one end point in V' can be safely oriented towards a vertex in V'. We then delete V' and the edges incident on it, and use induction to extend this orientation to the rest of H. To identify the set V', we will use a *breadth first* *search* procedure formally described below. This procedure produces a *breadth first search tree* or a *breadth first search forest* as usual, but we need to impose the following condition on it.

> If a vertex w in the tree is connected to its parent by a RED edge, then all of w's children are connected to w using GREEN edges; thus, in any root to leaf path in the tree, RED edges do not appear consecutively.

To enforce this, when a RED edge is added to the tree, we will mark the vertex it leads to RED; then when we visit this vertex, we only explore vertices connected to it by GREEN edges. If a GREEN edge is added to the tree, we mark the new vertex GREEN; then when we visit this vertex, we explore all its edges, whether GREEN or RED. The formal code is presented in Algorithm 1. (This is reminiscent of the breadth first search procedure employed by certain matching algorithms to discover augmenting paths; there one alternately explores either only the matched edge or all edges). As a first attempt we might want to orient the

Algorithm 1 Breadth-First Search (BFS).

```
Input  : A non-empty subset Z ⊆ V(H)
Output : A BFS forest rooted at the vertices in Z
Q = empty queue ;
push all elements of Z into Q and mark them GREEN;
while Q is non-empty do
    v = pop(Q);
    if v is marked GREEN then
        push all unmarked neighbors w of v into Q;
        // now assign them colors as follows
        if {v, w} is GREEN then
            add {v, w} to the forest, and mark w GREEN
        end
        else
            mark w RED
        end
    end
    if v is marked RED then
        push all unmarked neighbors w of v with {v, w} GREEN into Q;
        add {v, w} to the forest and mark w GREEN;
    end
end
```

edges of the forest away from the roots and hope to extend this orientation to the other edges that have at least one end point in the forest. If this gives a valid orientation for these edges, we let V' be the vertex set of the forest, and proceed as above. Unfortunately, this straightforward method may run into problems; this motivates the following definition.

▶ **Definition 10** (Blocking edge, see Figure 2). *In the forest constructed by BFS, we say that a non-tree edge is a blocking edge if (i) it is a non-tree* GREEN *edge both of whose end points are visited during BFS, or (ii) it is a non-tree* RED *edge with both end points marked* GREEN*.*

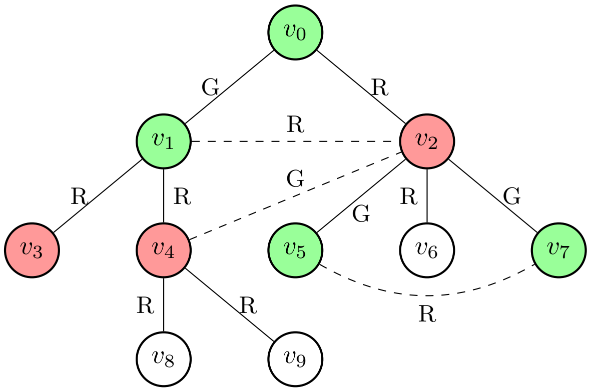


Figure 2 The BFS tree: (v_1, v_2) is not a blocking edge, (v_2, v_4) and (v_5, v_7) are blocking edges.

We will see that if there are no blocking edges, then the above strategy will work; otherwise, H has a cycle with many GREEN edges, and we will be able to exploit that.

▶ **Definition 11** (GREEN-dominated cycle, see Figure 3). *We say that a cycle in H is* GREEN-dominated *if all but (perhaps) one of its RED edges are followed by a* GREEN edge.

We will establish the following two lemmas below.

▶ **Lemma 12.** *Suppose H has no* GREEN*-dominated cycle. Suppose V' is the set of vertices of H visited by BFS starting at a vertex v_0. Then there is a safe orientation of edges of H incident on V'.*

▶ **Lemma 13.** *Suppose H has a* GREEN*-dominated cycle C. Let H' be the graph obtained by deleting from H all edges of C. Let V' be the vertices visited by BFS in H' starting with the vetex set $V(C)$ of the cycle C. Then, there is a safe orientation of edges of H incident on V'.*

Let us use these lemmas to complete the proof of Lemma 9.

Proof of the Lemma 9. If H has no GREEN-dominated cycle, then by Lemma 12, we obtain an initial set of vertices V' and an orientation of all edges incident on it. If H has a GREEN-dominated cycle, then by Lemma 13 we obtain an initial set of vertices V' and an orientation of all edges incident on it. We extend this orientation to the remaining edges of the graph by deleting V' and all edges incident on it, and applying induction to the resulting subgraph induced by the vertex set $V \setminus V'$. ◀

We now return to the unproved lemmas.

Proof of Lemma 12. Let v_0 be an arbitrary vertex. Consider the tree produced by BFS starting with $Z = \{v_0\}$. We claim that there is no blocking edge for the resulting tree. For suppose $e = \{a, b\}$ is a blocking edge. Let v be the least common ancestor of a and b, and recall that in the paths that connect v to a and v to b, no RED edge is followed by a RED edge. Let C be the cycle formed by taking the path from v to a followed by e and then the path from b back to v. If e is GREEN, then this cycle is GREEN-dominated, contradicting our assumption. If e is RED, then by the definition of blocking edge, both a and b are marked GREEN, that is, the tree edges connecting them to their parents are GREEN (note that e is not a back edge because both its vertices are GREEN). Again the

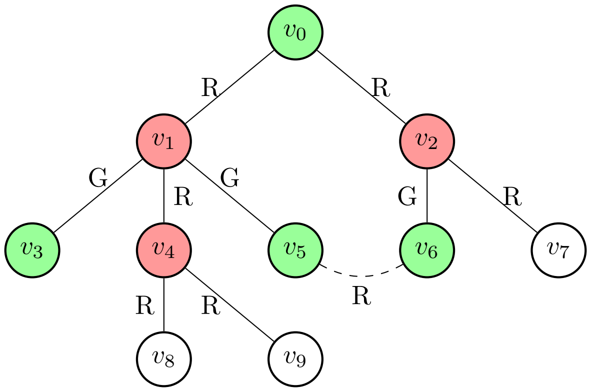


Figure 3 The edge is (v_5, v_6) is a blocking edge and $(v_0, v_1, v_5, v_6, v_2, v_0)$ is the resulting GREEN-dominated cycle, even though it has more RED edges than GREEN edges.

cycle C is GREEN-dominated, contradicting our assumption. Thus, the tree has no blocking edges. Let V' be the vertices visited by BFS. Orient all tree edges away from the root v_0. The remaining edges incident on V' (which cannot be GREEN) have at least one vertex marked RED, because they are not blocking. Orient them towards that RED vertex. It can be verified that the GREEN edges that received an orientation are all tree edges, and are oriented towards distinct GREEN vertices. The RED edges are oriented towards RED vertices. So all edges incident on V' can be oriented safely. ◀

Proof of Lemma 13. Fix a GREEN-dominated cycle C in H. Suppose it has ℓ_1 edges (for some $\ell_1 \geq g$) of which say n_1 are GREEN. Then,

$$n_1 \geq \lceil (\ell_1 - 1)/2 \rceil \geq g/2, \tag{3}$$

because g is even. First, suppose the resulting BFS forest has no blocking edges, then let V' be the vertices of this BFS forest. We orient the edges in C so that it becomes a directed cycle (we may choose either of the two ways to do this). Then, we orient all tree edges away from the roots in the BFS forest. Note that all other edges incident on V' must necessarily be RED; each such edge has at least one RED vertex in V'. We orient each such edge towards a RED vertex, and obtain the desired safe orientation.

Next, suppose there is a blocking edge $e = \{a, b\}$. If a and b belong to the same tree of the forest, then e and the paths from a and b to their least common ancestor form a GREEN-dominated cycle, consisting of say $\ell_2 \geq g$ edges of which n_2 are GREEN. Then,

$$n_2 \geq \lceil (\ell_2 - 1)/2 \rceil \geq g/2. \tag{4}$$

From Equation (3) and Equation (4), we obtain, $n \geq n_1 + n_2 \geq g$, contradicting our assumption that $n \leq 3g/4$. So, we may assume that a and b belong to different trees of the forest. Then, travelling from the root r_1 of a's tree to a, crossing over along e to b, and then travelling to the root r_2 of the tree of b, we obtain a path P^*, where no RED edge is followed by a RED edge; in particular, every RED edge except perhaps the last, is followed by a GREEN edge. Suppose this path has ℓ_3 edges of which n_3 are GREEN. We have the following.

$$\ell_3 \geq g - \lfloor \ell_1/2 \rfloor; \qquad (G \text{ has girth } g) \tag{5}$$

$$2n_3 \geq \ell_3 - 1. \tag{6}$$

From Equation (3), Equation (5) and Equation (6), we obtain

$$2(n_1 + n_3) \geq g + 2\lceil(\ell_1 - 1)/2\rceil - \lfloor \ell_1/2 \rfloor - 1 \tag{7}$$
$$= g + \lceil(\ell_1 - 1)/2\rceil - 1 \tag{8}$$
$$\geq 3g/2 - 1. \tag{9}$$

If $n > n_1 + n_3$ (that is, there is some GREEN edge outside $C \cup P^*$), then we obtain $n \geq n_1 + n_3 + 1 \geq 3g/4 + 1/2$, contradicting our assumption $n \leq 3g/4$. So, we may assume that all GREEN edges in the graph are contained in $C \cup P^*$. We think of $C \cup P^*$ as a set of three edge disjoint paths, P_1, P_2 and P_3, connecting r_1 to r_2, where $P_1 \cup P_2 = C$ and $P_3 = P^*$. Let V^* be the set of vertices in $P_1 \cup P_2 \cup P^*$. We will show that the graph induced by V^* can be safely oriented. Then, we will orient all other (necessarily RED) edges of G towards a vertex not in V^* to obtain a safe orientation of the *entire* graph H, and conclude that the lemma holds with $V' = V(H)$.

First, we show that we may assume that each of the three cycles $C = P_1 \cup P_2$, $P_1 \cup P_3$ and $P_2 \cup P_3$ is chordless. Since C is GREEN-dominated, it has at most $2n + 1$ edges. If it had a chord, we would get a cycle with at most $n \leq 3g/4 < g$ edges, a contradiction. Thus, the cycle $C = P_1 \cup P_2$ has no chord. Next, using a similar argument we show that the other two cycles are chordless. We first observe that both P_1 and P_2 have GREEN edges. If P_1 has no GREEN edges, then it can have at most two edges, and the at least $g/2$ GREEN edges of C all lie in P_2. Also, P_3 has at least $g - 2$ edges. Then, the number of GREEN edges in P_2 is at least $\max\{|P_2|, g\}/2$ (because C is GREEN-dominated), and similarly the number of GREEN edges in P_3 is at least $\lceil(|P_3| - 1)/2\rceil \geq g/2 - 1$. Thus,

$$3/4g \geq n \geq \max\{|P_2|, g\}/2 + \lceil(|P_3| - 1)/2\rceil \geq g - 1; \tag{10}$$

that is, $g = 4$, $|P_2| \leq 4$ and $|P_3| \leq 3$. Thus, $P_2 \cup P_3$ is a cycle with at most 7 vertices and it cannot have a chord because $g = 4$. Thus, we may assume that P_1 has at least one GREEN edge, that is, $P_2 \cup P_3$ has at most $n - 1$ GREEN edges. Let k_2 be the number of GREEN edges in P_2 and k_3 the number of GREEN edges in P_3. Since every RED edge in P_3, except perhaps one is followed by a GREEN edges, the number of edges in P_3 is at most $2k_2 + 2$. Then, the total number of edges in $P_2 \cup P_3 \leq (2k_2 + 2) + (2k_3 + 1)$ (the second term comes from Equation (6)), that is, at most $2n + 1$ edges in all. If $P_2 \cup P_3$ has a chord, then we have a cycle of length at most n, which, as we saw earlier, is not possible. Similarly, $P_2 \cup P_3$ has not chord.

So the graph induced by V^* consists of three disjoint paths, with no chords across them. If any path has two consecutive RED edges, then we may orient them towards each other and be left with a graph consisting of a cycle with two dangling paths, which can be oriented safely. Similarly, if some two paths start with RED edges or end with RED edges, then these edges can be oriented towards each other, and the remaining edges (which form a tree) can be oriented safely. We are left with the case where on all paths a RED edge is followed by a GREEN edge, and at both ends (r_1 and r_2), two of the paths start with GREEN edges. We will show that this is impossible. For otherwise, there is path (say, P_3), which has GREEN edges at both ends, so $|P_3|$ has at least $(|P_3| + 1)/2$ GREEN edges. For the remaining paths, either some path has both ends GREEN, or both paths have one end GREEN. In either case, they together have at least $(|P_1| + |P_2|)/2$ GREEN edges. Note that $2(|P_1| + |P_2| + |P_3|) \geq 3g$, because H has girth at least g. Thus, the total number of GREEN edges is $n \geq (|P_1| + |P_2| + |P_3| + 1)/2 \geq (3g/2 + 1)/2 > 3g/4$, contradicting our assumption that $n \leq 3g/4$. ◀

3 Quantum adaptive schemes

In this section, we establish Theorem 3. Our quantum scheme is based on a graph and is similar in some respects to the classical scheme described above. The main difference is in the second probe, which now computes the XOR of two bits of memory.

▶ **Definition 14** (Quantum (G, K)-scheme). *Let G be a directed graph with N vertices and M edges; let K be a positive integer. We refer to the following as a quantum (G, K)-scheme. The storage consists of three bit arrays, A, B_0 and B_1. To answer a membership query, the quantum decision tree first probes array A (this probe is classical) and then computes the XOR of two bits in either B_0 or B_1, using just one quantum probe.*

Edge array: *An array $A : E(G) \to \{0, 1\}$, indexed by edges of G.*

Vertex arrays: *Two two-dimensional arrays $B_0, B_1 : V \times [K] \to \{0, 1\}$, indexed by elements of the form (v, i).*

Elements: *As before, we identify our universe of elements $[m]$ with a subset of $E(G) \times [K]$; thus, each element $x \in [m]$ will be referred to as (e, i).*

Query: *Let the query be "Is x in S?", where $x = (e, i)$; suppose $e = \{x_0, x_1\}$. To process this query for we read $A[e]$ (first probe); then, based on the value of $A[e]$, we return either $B_0[(v_0, i)] + B_0[(v_1, i)] \pmod 2$ or $B_1[(v_0, i)] + B_1[(v_1, i)] \pmod 2$. In other words, the first probe directs us to either array B_0 or B_1; we then return the XOR of the bits in the i-th location in the rows corresponding to the two vertices of e.*

Space: *We will ensure that $MK \geq m$, to accommodate all elements of the universe. The space used by this scheme is then $2NK + M$ bits. By choosing the graph G and the parameter K appropriately we will show that our schemes uses small space.*

The main idea is the following. To store the set S in the data structure, we partition the edges of G using the 0-1 assignment to the array A. Let G_0 be the graph induced by the edges that are assigned 0 in A, and let G_1 be the graph induced by the edges assigned 1. Now, the bits of the arrays B_0 and B_1 must be assigned in such a way that certain XORs of two bits evaluate to 1 and others evaluate to 0. This leads to a system of linear equations in the bits of the arrays B_0 and B_1. To ensure that this system has a solution, we ensure that if $A[e] = 0$, then e is not in any cycle in G_0, and similarly, if $A[e] = 1$, then e is not in any cycle in G_1. It is then easy to see that the required assignment to the arrays B_0 and B_1 exists. To ensure that the edges of G can be partitioned in G_0 and G_1 to satisfy the requirements imposed by the set S, we will start with the graph G that is dense but locally sparse in the following sense, and use a theorem of Nash-Williams.

▶ **Definition 15** (Locally sparse graph). *A graph G is (k, α)-locally sparse if for every subsets $V' \subseteq V$ with $4 \leq |V'| \leq k$ vertices, the induced subgraph on V' has at most $\alpha|V'|$ edges.*

▶ **Lemma 16.** *If G has N vertices, M edges and is $(4n, 5/4)$-locally sparse, then*

$$s_{QA}(m, n, t = 2) \leq M + 2N\lceil \frac{m}{M} \rceil.$$

Before we present the proof of this lemma, let us see how this leads to Theorem 3. We will need a family of dense locally sparse graphs, whose existence we establish in Appendix C using a routine probabilistic argument.

▶ **Lemma 17.** *For all large N there is a $(4N^{1/6}, 5/4)$-locally sparse graph with N vertices and $\Omega(N^{7/6})$ edges.*

We set $N = m^{3/4}$, and plugging in the graph promised by Lemma 17 in Lemma 16, obtain $s_{QA}(m, m^{1/8}, 2) = O(m^{7/8})$, as claimed in Theorem 3. It remains to establish Lemma 16.

Proof of Lemma 16. Fix G with the given parameters. We now describe how the three arrays in our quantum scheme are assigned values. Recall that we view elements of $[m]$ as pairs (e, i). Edges of G for which there is an element of the form $(e, i) \in S$ will be called GREEN; the other edges of G will be called RED. Say, there are $\ell \leq n$ GREEN edges. We will construct a sets of vertices $D_0, D_1, D_2, \ldots$ by adding one vertex at a time. Let $D_0 \subseteq V(G)$ be the union of the GREEN edges; thus D_0 has at most 2ℓ vertices. To obtain D_{i+1} from D_i, add to D_i a new vertex that has at least two edges leading into D_i; if no such vertex exists, stop. We claim that this process stops before $2n$ vertices are added, for otherwise, the graph induced by D_{2n}, a set of size at most $2n + 2\ell \leq 4n$ vertices, has at least $4n + \ell$ edges. Since G is $(4n, 5/4)$-locally sparse, we have $(5/4)(2n + 2\ell) > 4n + \ell$, implying $\ell > n$, a contradiction. Let D be the set of vertices when the above process stops.

▷ Claim 18. The subgraph induced by D can be split into two disjoint forests.

We will justify this claim below (using Nash-Williams theorem). Let us assume it and complete the proof. Let the two forests guaranteed by this claim be F_1 and F_2. We set $A[e] = 0$ for all edges $e \in F_1$ and all edges that connect D to $V \setminus D$. Let G_0 be the subgraph of G with vertex set $V(G)$ that consist of edges e such that $A[e] = 0$. Let G_1 be the subgraph with vertex set $V(G)$ and all edges not included in G_0. Note that the connected components of G_1 are either in the forest F_2 or consist of RED edges with both end points in $V \setminus D$. Now, we are ready to describe the assignment to arrays B_0 and B_1. As stated above the constraints imposed by the GREEN and RED edges give a system of equational constraints; since G_0 has no cycle, it is easy to see that these constraints can all be satisfied by assigning B_0 appropriately. In G_1 again, the edges corresponding to F_2 do not induce a cycle, so the constraints imposed by them can be satisfied by assigning appropriate values to the rows of B_2 corresponding to vertices in D. The remaining edges share no vertex with the edges of F_2, and consist only of RED edges. So we assign zeroes to all rows of B_2 corresponding to vertices in $V(G) \setminus D$.

It remains to verify Claim 18. Since $|D| \leq 4n$, every subset D' of D (with $|D'| \geq 4$) induces at most $(5/4)|D'|$ edges; since $|D| \geq 4$, we have $(5/4)|D'| \leq 2(|D'| - 1)$. Note that the number of edges in any graph with at most $1 \leq \ell \leq 4$ vertices is at most $2\ell - 2$. So we may invoke Theorem 19 below and justify the claim. ◀

▶ **Theorem 19** (Nash-Williams (see Theorem 3.5.4 in [10]).). *Let $H = (V, E)$ be an undirected graph such that for each non-empty subset $X \subseteq V$, the number of edges with both end points in X is at most $2(|X| - 1)$. Then E can be partitioned as $E = E_1 \cup E_2$ such that (V, E_1) and (V, E_2) are both forests.*

4 Lower bounds for classical schemes

In this section, we present our justification for Theorem 2.

Canonical query schemes. Consider an (m, n, s, t)-scheme. Let us use M to denote the array of s-bits into which probes are made. With each element $x \in [m]$ of the universe such a scheme associates three addresses $(a(x), b(x), c(x)) \in [s]^3$, where the first probe is made to location $a(x)$; if the bit there is 0, then the second probe is made to $b(x)$, otherwise the probe is made to $c(x)$. We will assume that that the query scheme has the following *canonical form.* On query "Is x in S?", the answer is determined as follows: if $M[a(x)] = 1$, then return $M[b(x)]$, else return $M[(c(x)]$, where 0 is treated as false/no and 1 as true/yes. We refer to such schemes as *canonical schemes.* It is easy to see that by at most doubling the memory a scheme can be made canonical.

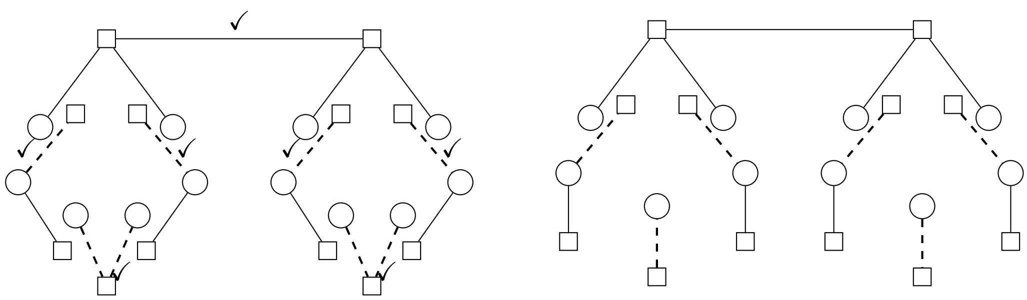
▶ **Proposition 20.** *If there is an $(m, n, s, t = 2)$-scheme, then there is an $(m, n, 2s, t = 2)$-canonical scheme.*

From now on we will assume that the scheme is canonical.

▶ **Definition 21** (The bipartite graph associated with a scheme)**.** *With the scheme σ, we associate a directed bipartite graph G_σ with vertex sets B and C, whose elements we refer to using $[s]$. For each $x \in [m]$ we add an edge $e(x) = (b(x), c(x))$ in G_σ with label $(x, a(x))$. The value $a(x)$ will be called the color of the edge; so $e(x)$ and $e(y)$ have the same color if and only if the first probes for the two queries "Is x in S?" and "Is y in S?" are made to the same location, namely $a(x) = a(y)$. We will use $\vec{e}$ to refer the oriented version of the edges of G_σ. We say that two distinct oriented edges, $\vec{e}_1$ and $\vec{e}_2$ are parallel if (i) $\vec{e}_1$ and $\vec{e}_2$ have the same color, and (ii) they are both oriented in the same direction (both from B to C or C to B).*

To store a set S, one must find an assignment to the locations where the first probes are made. This amounts to choosing an orientation for each color, and orienting the edges either from B to C or from C to B; the values in the array is then essentially forced because the protocol is assumed to be canonical. For this assignment to be valid, in the resulting directed graph, we must have the property that if $x \in S$ and $y \notin S$, then $\mathsf{head}(\vec{e}(x)) \neq \mathsf{head}(\vec{e}(y))$. We refer to such an orientation as a *safe* orientation for S.

We obtain our lower bound by establishing that if the data structure uses very small space, then there is a set S of size at most n edges, whose edges cannot be oriented safely.



(a) A forbidden configuration.

(b) The problem case.

Figure 4 $n = 7$.

Forbidden configuration. Fix n of the form $2\ell - 1$. Please refer to Figure 4a. Most edges in the figure come in pairs of solid and dotted edges, which are placed parallel to each other. For each edge, one vertex is a circle and the other is a square, to indicate that one of them comes from B and the other from C (we do not specify which is which). The elements labelling the edges are all distinct; however, we allow the edges to have the same color even if they are not explicitly depicted as being parallel. We say that such graph F is forbidden configuration of order n, if there is a subset $S \subseteq [m]$ ($|S| \leq n$) of elements appearing in the labels on the edges of F such that F is not safe for S. For example, Figure 4a is a forbidden configuration of order $n = 7$, where the set S is indicated by ✓. Our lower bound result Theorem 2 follows immediately from the following lemma.

▶ **Lemma 22.** *Fix an odd n ($3 \leq n \leq \log m$) and an $(m, n, s, t = 2)$-scheme σ. If G_σ does not contain any forbidden configuration of order n, then $s \geq cm^{1-2/(n+3)}$, for a constant $c > 0$ independent of n.*

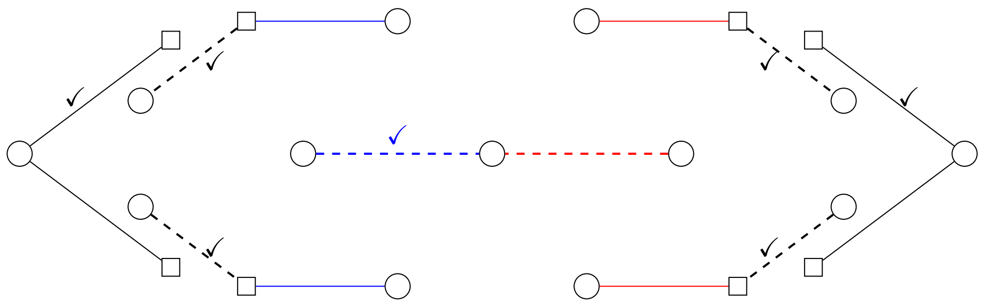


Figure 5 The second forbidden configuration ($n = 7$).

Proof Sketch. Consider $(m, n, s, t = 2)$-scheme σ. In G_σ we have m edges. Then, (on average) a vertex has degree about m/s and each colour has m/s edges. We start from an edge $e = (v, w)$, and from each of v and w, we build a tree as follows. Let us start with $v_0 = v$. We have m/s choices for an edge; for each choice of edge e of the form (v_0, v_1), we jump to an edge of the same colour (parallel to the first one), that takes us from a vertex v_1' to a vertex v_2. We continue this process, alternately expanding to a neighbor or jumping to a parallel edge for k steps in all. If we set k such that $(m/s)^k > s$, we obtain a "cycle". (Note that this is not a cycle in the usual graph-theoretic sense, because we jump from an edge to a parallel edges in alternate steps.) We delete these edges from the graph, and repeat this for the other vertex w of the starting edge e. Let us illustrate this for $n = 7$. Suppose $s \ll m^{1-2/(n+3)} = m^{4/5}$, that is, $(m/s) \gg m^{1/5}$. In particular, with $k = 4$, we have $(m/s)^k \gg m^{4/5} \gg s$, and at some point a vertex must repeat (we have only $2s$ vertices). A situation in such a case, with two cycles hanging off an edge corresponds to Figure 4a. We allow the two cycles to share vertices, but the edges involved must be distinct. Now, to see that this configuration has no safe orientation, first choose either orientation for the top edge e, say towards left. Then, the directions of all edges are forced in that cycle and we soon find edges corresponding to an element in the set and another corresponding to an element not in the set that point to the same vertex.

Unfortunately, there are other cases to consider, besides the ideal case of two cycles attached to an edge as in Figure 4a: (i) the cycles may not form right at the top, instead we might have to allow an initial path leading to the cycle; (ii) the cycle may not end with two tree edges pointing into it. Instead, it might be formed when two paths of a tree jump on to the same parallel edge. The first case presents no real difficulties; in fact, in this case the resulting configuration is not safe for even smaller sets. The second case presents genuine difficulties. For example, we might encounter a situation depicted in Figure 4b, where the three edges at the bottom are parallel. Note that all tree edges in this case can be forced away from their roots to obtain a safe orientation. The idea now is the following. If we encounter such a cycle, we put it aside and mark the middle vertex at the bottom as its terminal vertex. We have removed only a minuscule number of edges from the graph, so we can continue the exploration for a forbidden structure in the remaining graph. If we ever find a configuration corresponding to Figure 4a, we are done. Otherwise, we accumulate many edge disjoint bad cycles. Soon enough (by the time $s + 1$ bad cycles are encountered), two of them must have the same terminal vertex. We put these bad cycles together (as illustrated in Figure 5) and again obtain a forbidden configuration. The discussion above uses $k = 4$ and $n = 7$ for illustration, but the same argument applies for other k, and, in general, yields a configuration without any safe orientation for a set of size $n = 2k - 1$, whenever, $s \ll m^{1-1/(k+1)} = m^{1-2/(n+3)}$. The detailed argument will appear in the full version of the paper. ◀

5 Quantum non-adaptive schemes for $n = 2$ and $t = 2, 3$

In this section, we show that the lower bound in 1 is tight for two cases: $s_{QN}(m, n = 2, t = 2) = O(\sqrt{m})$ and $s_{QN}(m, n = 2, t = 3) = O(m^{1/3})$; the schemes we give are non-adaptive and only use the fact that the XOR of two bits can be computed using one quantum query. The proofs are algebraic.

5.1 Case $t = 2$

We identify $[m]$ with $A \times B$, where each of the sets has about $\sqrt{m}$ elements; A and B are disjoint. We will have two array indexed by A (we call them X_1 and X_2) and two arrays indexed by B (we call them Y_1 and Y_2).

Query: On receiving the element $x = (x_1, x_2) \in A \times B$, the algorithm returns

$$(X_1[x_1] + Y_1[x_2])(X_2[x_1] + Y_2[x_2]) \pmod 2,$$

which is a polynomial of degree two. Note that both $X_1[a] + Y_1[b]$ and $X_2[a] + Y_2[b]$ can be computed in parallel with one quantum query each. Thus, the scheme requires only two non-adaptive queries.

Storage: Given a pair of elements $\{\alpha_1, \alpha_2\} \subseteq [m]$, we need to show how values will be assigned to the four arrays: X_1, X_2, Y_1, Y_2. It will be easier to describe and analyse our storage algebraically. We view X_1, X_2 as functions from A to $\{0, 1\}$ and Y_1, Y_2 as functions from B to $\{0, 1\}$. For $a \in A$, let $\delta_a : A \to \{0, 1\}$ be defined by $\delta_a(z) = 1$ iff $z = a$; similarly for $b \in B$, let $\delta_b : B \to \{0, 1\}$ be defined by $\delta_b(z) = 1$ iff $z = b$. We have three cases based on the number of components $\ell \in \{0, 1, 2\}$ where α_1 and α_2 agree.

$\ell = 2$: We have only one element (a, b). We set $X_1 \equiv \delta_a$, $Y_1 = 0$, $X_2 \equiv 0$ and $Y_2 \equiv \delta_b$. The query polynomial reduces to the monomial $\delta_a(x_1)\delta_b(x_2)$, which is what we want.

$\ell = 1$: Say the set is $\{(a, b), (a', b)\}$. We set $X_1 \equiv \delta_a + \delta_{a'}$, $Y_1 \equiv 0$, $X_2 \equiv 0$ and $Y_2 \equiv \delta_b$. The query polynomial reduces to $(\delta_a(x_1) + \delta_{a'}(x_2))\delta_b(z_2) = \delta_a(z_1)\delta_b(z_2) + \delta_{a'}(z_1)\delta_b(z_2)$, which is what we want.

$\ell = 0$: Say the set is $\{(a, b), (a', b')\}$. We set $X_1 \equiv \delta_a$, $Y_1 \equiv \delta_{b'}$, $X_2 \equiv \delta_{a'}$ and $Y_2 \equiv \delta_b$. The query polynomial evaluates $(\delta_a(z_1)+\delta_{b'}(z_2))(\delta_{a'}(z_1)+\delta_b(z_2)) = \delta_a(z_1)\delta_b(z_2)+\delta_{a'}(z_1)\delta_{b'}(z_2) \pmod 2$, which is what we want.

5.2 Case $t = 3$

Let us identify $[m]$ with $A \times B \times C$, where each of the sets has roughly $m^{1/3}$ elements; we will assume that A, B and C are disjoint. We have six arrays, two indexed by A (we call them X_1 and X_2), two indexed by B (we call them Y_1 and Y_3) and two indexed by C (we call them Z_2 and Z_3); the subscripts indicate which query probes the corresponding array.

Query: On receiving the element $e = (x, y, z)$, the algorithm returns

$$(X_1[x] + Y_1[y])(X_2[x] + Z_2[z])(Y_3[y] + Z_3[z]) \pmod 2,$$

which is a polynomial of degree 3.

Storage: Given a pair of elements $\{\alpha_1, \alpha_2\}$, we need to show how values will be assigned to the six arrays. Let $\alpha = (a, b, c)$ and $\beta = (a', b', c')$. We define functions of the form $\delta_a : A \to \{0, 1\}$, $\delta_b : B \to \{0, 1\}$ and $\delta_c : C \to \{0, 1\}$ as before. Also 0 and 1 when denoting functions correspond to the all 0's and the all 1's functions. We have four cases, depending on the number of places $\ell \in \{0, 1, 2, 3\}$ where α_1 and α_2 agree.

$\ell = 3$: The set has only one element (a, b, c), say. The arrays are as follows. $X_1 \equiv \delta_a$; $Z_2 \equiv \delta_c$; $Y_3 \equiv \delta_b$, the other three arrays are 0. So, the query function becomes $(\delta_a(x) + 0)(\delta_c(z) + 0)(\delta_b(y) + 0) = \delta_a(x)\delta_b(y)\delta_c(z)$, which yields 1 iff $(x, y, z) = (a, b, c)$.

$\ell = 2$: Say $\alpha_1 = (a, b, c)$ and $\alpha_2 = (a, b, c')$. We set $X_1 \equiv \delta_a$, $Y_1 \equiv 0$, $X_2 \equiv 0$, $Z_2 \equiv \delta_c + \delta_{c'}$, $Y_3 \equiv \delta_b$ and $Z_3 \equiv 0$. Then, the query function becomes $(\delta_a(x)+0)(\delta_c(z)+\delta_{c'}(z))(\delta_b(y)+0)$, which reduces to $\delta_a(x)\delta_b(y)\delta_c(z) + \delta_a(x)\delta_b(y)\delta_{c'}(z)$, that is, the function that evaluates to 1 precisely when the input is (a, b, c) or (a, b, c'). The other cases are symmetric.

$\ell = 1$: Say $\alpha_1 = (a, b, c)$ and $\alpha_2 = (a, b', c')$. We set $X_1 \equiv 1 + \delta_a$, $Y_1 \equiv \delta_b + \delta_{b'}$, $X_2 \equiv 1 + \delta_a$, $Z_2 = \delta_c + \delta_{c'}$, $Y_3 = \delta_b$ $Z_3 = \delta_{c'}$. Our query polynomial then evaluates to

$$(1 + \delta_a + \delta_b + \delta_{b'})(1 + \delta_a + \delta_c + \delta_{c'})(\delta_b + \delta_{c'}), \tag{11}$$

where, to simplify notation, we just write δ_a instead of $\delta_a(x)$, etc. Applying the rule $gh = (g + h + 1)h$ twice, we obtain $(\delta_c + \delta_{b'})(1 + \delta_a + \delta_c + \delta_{c'})(\delta_b + \delta_{c'})$. Then, combining the first and last factors, we obtain, $(\delta_b\delta_c + \delta_{b'}\delta_{c'})(1 + \delta_a + \delta_c + \delta_{c'})$. Expanding this mod 2, we obtain $(\delta_b\delta_c + \delta_{b'}\delta_{c'})\delta_a$, which yields 1 iff $x \in \{(a, b, c), (a, b', c')\}$, as required.

$\ell = 0$ (the two elements differ on all coordinates): We set $X_1 \equiv \delta_a$, $X_2 \equiv \delta_{a'}$, $Y_1 \equiv \delta_{b'}$, $Y_2 \equiv \delta_b$, $Z_1 \equiv \delta_{c'}$, $Z_2 \equiv \delta_c$. The query expression evaluates to

$$(\delta_a + \delta_{b'})(\delta_{a'} + \delta_c)(\delta_b + \delta_{c'}).$$

Focus on the middle factor. If we pick $\delta_{a'}$ from that factor, then we are forced to pick $\delta_{b'}$ from the previous, which forces us to pick $\delta_{c'}$ from the last (to avoid getting 0); if we pick δ_c from the middle factor, then we are forced to pick δ_b from the last and then δ_a from the first. All other terms are 0. The final expression with two terms is $\delta_a\delta_b\delta_c + \delta_{a'}\delta_{b'}\delta_{c'}$, as required.

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A Explicit construction of graphs

We say that a graph on L vertices is explicitly, if the adjacency matrix of L can be constructed in polynomial time in L.

- We will use the construction due to Wenger [20] to exhibit explicit graphs with girths 8. Wenger constructs a graph $H_k(p)$ with $2p^k$ vertices and $2p^{k+1}$ edges, and shows that if p is prime, then $H_3(p)$ has girth at least 8. In the bipartite graph $H_k(p)$ the vertices are represented as k-tuples of numbers $\{0, \ldots, p\}$ and two vertices are connected based on a simple arithmetic check involving addition and multiplication modulo p. In our application, given a number L, we set p to be the smallest prime that is at least $L^{1/(k+1)}$. Then, $H_k(p)$ has at most $2^{k+2}L$ vertices and at least $2L^{1+1/k}$ edges. Thus, we obtain graphs with the following parameters: $g = 8$, $c(g = 8) = 2^5$, $\tau(g = 8) = 1/3$. Our application for $g = 8$ (see Figure 1) uses these parameters.
- For girth 12, we use a construction due to Benson [8]. Theorem 2 of the paper presents an algebraic construction where the graph is obtained by considering point-line incidences for points and lines of a quardic surface in the projective 6-space $P(6, q)$. The degree of each vertex of this graph is $q + 1$. On page 1093, the number of vertices in this graph is computed to be $(q + 1)(1 + q^2 + q^4)$. So, to get the graph suitable for our applications, we take q to be the smallest prime such that $(q + 1)(1 + q^2 + q^4) \geq L$ and use this construction. Then, it is easy to see that the number of vertices in this graph is $\mathcal{O}(L)$ and the number of edges is at least $L^{1+1/5}$.
- Lazebnik, Ustimenko and Woldar [14] exhibit dense graphs for various values of girth. Their Corollary 3.3 shows graphs with girth at least $2s + 2$, with $v \leq 2q^{(3s-3)/2}$ vertices if s is odd and at most $2q^{(3s-2)/2}$ vertices if s is even. The graph has $\frac{1}{2}vq$ edges. To construct the graphs for our application, fix L and let q be the smallest prime larger than $L^{2/(3s-3)}$ or $L^{2/(3s-2)}$ (depending on whether s is odd or even) and consider the graph obtained from the above construction. If the graph has fewer than L vertices, then we put together disjoint copies of it, to obtain one with number of vertices between L and $2L$. It can be verified that this graph has $O(L)$ vertices and $\Omega(L^{1+2/(3s-3)})$ or $\Omega(L^{1+2/(3s-2)})$ edges (depending on whether s is odd or even). In our application (see Figure 1), we use graphs with girth $4r$ and $4r - 2$. Setting $2s + 2 = 4r$, i.e., $s = 2r - 1$ (an odd number), we obtain a graph with $\Omega(L^{1+1/(3r-3)})$ edges; similarly setting $2s + 2 = 4r - 2$, i.e., $s = 2r - 2$ (an even number), we obtain a graph with $\Omega(L^{1+1/(3r-4)})$ edges.

B Examples that show Lemma 9 is tight

The bound shown Lemma 9 is tight in the following sense: for each positive even integer g, there exists a bipartite graph with girth g and $\lfloor 4g/3 \rfloor + 1$ GREEN edges that cannot be safely oriented. For example, the graph consisting of three edge-disjoint s-t paths, each of length $2k$, has girth $g = 4k$; but we can designate a set of $n = 3k + 1$ edges GREEN for which the graph has no safe orientation. For this graph $n = 3k + 1$ and $\lfloor 3g/4 \rfloor = 3k$. A similar example, with three edge-disjoint paths of length $2k + 1$, shows that the above lemma is tight for $g = 4k + 2$. Figure 6 shows these examples for $k = 2$.

C Proof of Lemma 17

Consider the random graph on N vertices where each edge is picked independently with probability $p = (1/50)N^{-5/6}$. The probability that G is not $(4N^{1/6}, 5/4)$-locally sparse is at most (we use the union bound over the choice of subsets of size $\ell \leq 4N^{1/6}$, and for each set over all choice of 1.24ℓ edges for simplicity we ignore floors and ceilings):

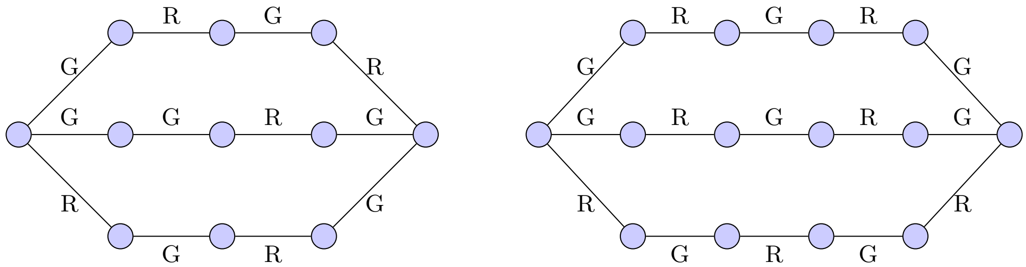

Figure 6 Examples of graphs with girth $g = 10$ and $g = 8$ that cannot be safely oriented.

$$\begin{aligned}
\sum_{\ell=5}^{4N^{1/6}} \binom{N}{\ell}\binom{\ell^2}{5/4\ell} p^{(5/4)\ell} &\leq \sum_{\ell=5}^{4N^{1/6}} \left(\frac{eN}{\ell}\right)^{\ell} \left(\frac{e\ell^2}{5/4\ell}\right)^{(5/4)\ell} p^{(5/4)\ell} \\
&\leq \sum_{\ell=5}^{4N^{1/6}} 8^{\ell} \left(N\ell^{1/4}\right)^{\ell} p^{(5/4)\ell} \\
&\leq \sum_{\ell=5}^{4N^{1/6}} \left(\frac{1}{7}\right)^{\ell} \left(N\ell^{1/4}\right)^{\ell} N^{-(5/6)(5/4)\ell}.
\end{aligned}$$

By considering terms corresponding to (say) $\ell < N^{1/12}$ and $\ell \geq N^{1/12}$ separately, we see that the last sum is $o(1)$. Thus, with high probability there is no set of size up to $4N^{1/6}$ that violates the local sparsity condition. Furthermore, with high probability the number of edges in the graph is at least $pN^2/2 = \Omega(N^{7/6})$. Thus, there exists an $(4N^{1/6}, 5/4)$-sparse graph with $\Omega(N^{7/6})$ edges.

D Non-adaptive quantum bounds

We give an upper bound on $s_Q(m, n = 2k+1, t = 2)$ for the non-adaptive classical scheme, where $k \in \mathbb{N}$. The arrangement of the element and bits is similar to the classical adaptive scheme we described in Section 2. The first probe is on the corresponding edge array and the second is an equality probe on the rows corresponding to the vertices of the edge. We AND the two probes to answer membership queries. We obtain

$$s_Q(m, n = 2k+1, t = 2) = \begin{cases} \mathcal{O}(v^{1+\frac{4}{3n-9}}) & \text{if } 4 | (n+1); \\ \mathcal{O}(v^{1+\frac{4}{3n-7}}) & \text{if } 4 \nmid (n+1). \end{cases} \tag{12}$$

▶ **Definition 23** (Non-adaptive Quantum (G, K)-scheme). *Let G be an un-directed graph with N vertices and M edges; let K be a positive integer. We refer to the following as a (non adaptive) quantum (G, K)-scheme. The storage consists of two bit arrays, A and B. To answer a membership query the decision tree will make the first probe to array A and the second probe to array B.*

Edge array: *An array $A : E(G) \to \{0, 1\}$, indexed by edges of G.*

Vertex array: *A two dimensional array $B : V \times [K] \to \{0, 1\}$.*

Elements: *We identify our universe of elements $[m]$ with a subset of $E(G) \times [K]$ (so we must ensure that the graph has at least m/K edges); thus, each element $x \in [m]$ will be referred to as (e, i).*

Query: *We represent an edge of* G *as an ordered pair of the form* $e = (v_0, v_1)$. *To process the query for the element* $x = (e, i)$, *we return the value* $A[e] \cdot (B[v_0, i] \oplus B[v_1, i])$.

Space: *We will ensure that* $MK \geq m$. *The space used by this scheme is then* $NK + M$ *bits (*NK *for the* N *vertex array and* K *for the edge array). By choosing the graph* G *and the parameter* K *appropriately we will show that our schemes use small space.*

As in the classical adaptive setting, an edges e is coloured GREEN if $(e, i) \in S$ for some i. Values can be assigned consistently to the arrays if there is no cycle in the graph consisting entirely of GREEN edges. This idea is formalized in the lemma below.

▶ **Lemma 24.** *Let* G *be a graph with* N *vertices* M *edges and girth* g *such that* $n < g$. *Then,* $s_A(m, n, 2) \leq M + N\lceil m/M \rceil$.