Testability and Local Certification of Monotone Properties in Minor-Closed Classes

Louis Esperet ✉ 🏠 📵
Univ. Grenoble Alpes, CNRS, Laboratoire G-SCOP, Grenoble, France

Sergey Norin ✉ 🏠 📵
Department of Mathematics and Statistics, McGill University, Montreal, Canada

─── **Abstract** ───

The main problem in the area of graph property testing is to understand which graph properties are *testable*, which means that with constantly many queries to any input graph G, a tester can decide with good probability whether G satisfies the property, or is far from satisfying the property. Testable properties are well understood in the dense model and in the bounded degree model, but little is known in sparse graph classes when graphs are allowed to have unbounded degree. This is the setting of the *sparse model*.

We prove that for any proper minor-closed class $\mathcal{G}$, any monotone property (i.e., any property that is closed under taking subgraphs) is testable for graphs from $\mathcal{G}$ in the sparse model. This extends a result of Czumaj and Sohler (FOCS'19), who proved it for monotone properties with finitely many forbidden subgraphs. Our result implies for instance that for any integers k and t, k-colorability of K_t-minor free graphs is testable in the sparse model.

Elek recently proved that monotone properties of bounded degree graphs from minor-closed classes that are closed under disjoint union can be verified by an approximate proof labeling scheme in constant time. We show again that the assumption of bounded degree can be omitted in his result.

1 Introduction

1.1 Property testing

We say that a graph G is ε*-far from some property* $\mathcal{P}$ if one needs to modify at least $\varepsilon|E(G)|$ of its adjacencies (replacing edges by non-edges and vice-versa) in order to obtain a graph satisfying $\mathcal{P}$. A property is *testable* if for any graph G, a tester can decide with good probability whether G satisfies $\mathcal{P}$ or is ε-far from $\mathcal{P}$, by only making a constant number of

queries to a given representation of G (i.e., the number of queries depends only on ε and $\mathcal{P}$, but is independent of the input graph G). The tester has *one-sided error* if it always gives the correct answer when $G \in \mathcal{P}$, and *two-sided error* otherwise.

In the *dense graph model* [28], there is a good understanding of which properties are testable with two-sided error [3] and one-sided error [5, 4]. In *the bounded degree model* [29], a sequence of papers [8, 15, 32] culminated in a proof that every property is testable with two-sided error within any *hyperfinite* graph family (this includes for instance any proper minor-closed class) [37]. The bounded degree assumption is crucial for obtaining this result and it has since then been an important open problem to obtain testability results in the weaker *sparse model*, which does not assume that the maximum degree is bounded [14, 16]. In this more general model there are two types of queries: given a vertex v, we can query the degree $d(v)$ of v in G; we can also query the i-th neighbor of v, for $1 \le i \le d(v)$ (all these queries are assumed to take constant time). In this model, much less is known: it was proved that bipartiteness is testable within any minor-closed class in [14], while already in the bounded degree model many simple properties are not testable in general graph classes [29], so the restriction to a sparse structured class such as a proper minor-closed class is very natural in this context. The interested reader is referred to the book of Goldreich [27] for more results and references on property testing, and especially Chapter 10 in the book, which focuses on the general graph model.

Instead of working in the sparse model as defined above, it will be enough to restrict ourselves to a single type of query: given a vertex v, we query a random neighbor of v, uniformly among the neighbors of v. Following [16], we say we make queries to the *random neighbor oracle*. Note that this type of queries can clearly be implemented is the sparse model, so this is a restriction of the model (see [16] for a comparison between these two models, and a third one were we are allowed to query a constant number of *distinct* random neighbors of a given vertex). The following was recently proved by Czumaj and Sohler [16].

▶ **Theorem 1** ([16])**.** *For every proper minor-closed class $\mathcal{G}$, and any finite family $\mathcal{H}$, the property of being $\mathcal{H}$-free for graphs from $\mathcal{G}$ is testable with one-sided error in the sparse model, where only queries to the random neighbor oracle are allowed.*

Here we say that a graph is *H-free* if it does not contain H as a subgraph, and $\mathcal{H}$-*free* if it is H-free for every $H \in \mathcal{H}$. Our main result is an extension of Theorem 1 to any *monotone* property, that is any property closed under taking subgraphs.

▶ **Theorem 2.** *For every proper minor-closed class $\mathcal{G}$, and any monotone property $\mathcal{P}$, the property of satisfying $\mathcal{P}$ for graphs from $\mathcal{G}$ is testable with one-sided error in the sparse model, where only queries to the random neighbor oracle are allowed.*

Note that for any monotone property $\mathcal{P}$ there is a (possibly infinite) family of graphs $\mathcal{H}$ such that $\mathcal{P}$ is precisely the property of being $\mathcal{H}$-free. This family $\mathcal{H}$ can be simply defined as the class of all the graphs that do not satisfy $\mathcal{P}$, or as the class of all the graphs that do not satisfy $\mathcal{P}$ and are minimal with this property (with respect to the subgraph relation). It follows that Theorem 2 is the natural generalization of Theorem 1, where we remove the assumption that $\mathcal{H}$ is finite. This can be seen as an analogue of the situation in the dense graph model: it was first proved that the property of being H-free (or $\mathcal{H}$-free for finite $\mathcal{H}$) was testable in this model [2], and then only much later was this extended to all monotone classes by Alon and Shapira [5]. Note that many natural monotone properties, such as being planar or k-colorable for some $k \ge 2$, do not have a finite set of minimal forbidden subgraphs. So there is a fundamental gap between being H-free and being $\mathcal{H}$-free for infinite $\mathcal{H}$.

1.2 Local certification

We now describe our second main result, which is obtained by extending the methods used in the proof of Theorem 2. We start by introducing the setting of this result: The problem of *local certification*.

In this part, all graphs are assumed to be connected. The vertices of any n-vertex graph G are assumed to be assigned distinct (but otherwise arbitrary) identifiers $(\mathrm{id}(v))_{v \in V(G)}$ from $\{1, \ldots, \mathrm{poly}(n)\}$. In the remainder of this section, all graphs are implicitly labelled by these distinct identifiers (for instance, whenever we talk about a subgraph H of a graph G, we implicitly refer to the corresponding labelled subgraph of G). We follow the terminology introduced by Göös and Suomela [30].

Proofs

A *proof* for a graph G is a function $P : V(G) \to \{0, 1\}^*$ (G is considered as a labelled graph, so the proof P is allowed to depend on the identifiers of the vertices of G). The binary words $P(v)$ are called *certificates*. The *size* of P is the maximum size of a certificate $P(v)$, for $v \in V(G)$.

Local verifiers

A *verifier* $\mathcal{A}$ is a function that takes a graph G, a proof P for G, and a vertex $v \in V(G)$ in input, and outputs an element of $\{0, 1\}$. We say that v *accepts* the instance if $\mathcal{A}(G, P, v) = 1$ and that v *rejects* the instance if $\mathcal{A}(G, P, v) = 0$.

Consider an integer $r \geq 0$, a graph G, a proof P for G, and a vertex $v \in V(G)$. Let $B_r(v)$ denote the set of vertices at distance at most r from v in G. We denote by $G[v, r]$ the subgraph of G induced by $B_r(v)$, and similarly we denote by $P[v, r]$ the restriction of P to $B_r(v)$.

A verifier $\mathcal{A}$ is *local* if there is a constant $r \geq 0$, such that for any $v \in G$, $\mathcal{A}(G, P, v) = \mathcal{A}(G[v, r], P[v, r], v)$. In other words, the output of v only depends on the ball of radius r centered in v, for any vertex v of G. The constant r is called the *local horizon* of the verifier.

Proof labelling schemes

For an integer $r \geq 0$, an *r-round proof labelling scheme* for a graph class $\mathcal{G}$ is a prover-verifier pair $(\mathcal{P}, \mathcal{A})$, with the following properties.

r-round: $\mathcal{A}$ is a local verifier with local horizon at most r.

Completeness: If $G \in \mathcal{G}$, then $P = \mathcal{P}(G)$ is a proof for G such that for any vertex $v \in V(G)$, $\mathcal{A}(G, P, v) = 1$.

Soundness: If $G \notin \mathcal{G}$, then for every proof P' for G, there exists a vertex $v \in V(G)$ such that $\mathcal{A}(G, P', v) = 0$.

In other words, upon looking at its ball of radius r (labelled by the identifiers and certificates), the local verifier of each vertex of a graph $G \in \mathcal{G}$ accepts the instance, while if $G \notin \mathcal{G}$, for every possible choice of certificates, the local verifier of at least one vertex rejects the instance.

The *complexity* of the labelling scheme is the maximum size of a proof $P = \mathcal{P}(G)$ for an n-vertex graph $G \in \mathcal{F}$. If we say that the complexity is $O(f(n))$, for some function f, the $O(\cdot)$ notation refers to $n \to \infty$. See [23, 30] for more details on proof labelling schemes and local certification in general.

It was proved in [25] that planar graphs have a 1-round proof labelling scheme of complexity $O(\log n)$, and that this complexity is optimal. The authors of [25] asked whether this can be extended to any proper minor-closed class. This was indeed extended in [24] to graphs embeddable in a fixed surface (see also [22] for a short proof), to graphs avoiding some small minors in [10], and more generally to any minor-closed class of bounded tree-width in [26] (in the last result, the complexity is $O(\log^2 n)$ instead of $O(\log n)$ in the other results mentioned here).

For $\varepsilon > 0$, define an *r-round ε-approximate proof labelling scheme* for some class $\mathcal{G}$ exactly as in the definition of *r*-round proof labelling scheme above, except that in the soundness part, the condition "If $G \notin \mathcal{G}$" is replaced by "If G is ε-far from $\mathcal{G}$" [13]. A graph class $\mathcal{G}$ is *summable* if for any $G_1, G_2 \in \mathcal{G}$, the disjoint union of G_1 and G_2 is also in $\mathcal{G}$. Elek recently proved the following result [21].

▶ **Theorem 3** ([21]). *For any $\varepsilon > 0$ and integer $D \geq 0$, and any monotone summable property $\mathcal{P}$ of a proper minor-closed class $\mathcal{G}$, there are constants $r \geq 0$ and $K \geq 0$ such that the class of graphs from $\mathcal{P}$ with maximum degree at most D has an r-round ε-approximate proof labelling scheme of complexity at most K.*

A natural problem is whether the bounded degree assumption in Elek's result can be omitted (Elek's proof crucially relies on this assumption). We prove that the bounded degree assumption can indeed be omitted.

▶ **Theorem 4.** *For any $\varepsilon > 0$ and any monotone summable property $\mathcal{P}$ of a proper minor-closed class $\mathcal{G}$, there are constants $r \geq 0$ and $K \geq 0$ such that $\mathcal{P}$ has an r-round ε-approximate proof labelling scheme of complexity at most K.*

We indeed prove a far-reaching generalization of this result (whose statement was suggested by Elek to the authors), concerning graph classes with bounded asymptotic dimension.

Asymptotic dimension

Given a graph G and an integer $r \geq 1$, we denote by G^r the graph obtained from G by adding edges between any pair of vertices at distance at most r in G. The *weak diameter* of a set S of vertices of G is the maximum distance in G between two vertices of S.

For an integer $d \geq 0$, a class of graphs $\mathcal{G}$ has *asymptotic dimension* at most d if there is a function $D : \mathbb{N} \to \mathbb{N}$ such that for any integer $r \geq 1$, any graph $G \in \mathcal{G}$ has a $(d+1)$-coloring of its vertex set such that any monochromatic[1] component of G^r has weak diameter at most $D(r)$ in G.

This notion was introduced by Gromov [31] in the more general context of metric spaces. In the specific case of graphs, it was proved that classes of bounded tree-width have asymptotic dimension at most 1, and proper minor-closed classes have asymptotic dimension at most 2 [9]. It was also proved that *d*-dimensional grids and families of graphs defined by the intersection of certain objects (such as unit balls) in $\mathbb{R}^d$ have asymptotic dimension d [9]. On the other hand, it is known that any class of bounded degree expanders has infinite asymptotic dimension (see [33]).

We will prove the following generalization of Theorem 4.

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[1] A *monochromatic component* in a colored graph G is a connected component of a subgraph of G induced by one of the color classes.

▶ **Theorem 5.** *For any $\varepsilon > 0$ and any monotone summable property $\mathcal{P}$ of a class $\mathcal{G}$ of bounded asymptotic dimension, there are constants $r \geq 0$ and $K \geq 0$ such that $\mathcal{P}$ has an r-round ε-approximate proof labelling scheme of complexity at most K.*

Note that a monotone property $\mathcal{P}$ is summable if and only if all minimal forbidden subgraphs for $\mathcal{P}$ are connected. This includes for instance minor-closed classes whose minimal forbidden minors are connected, such as planar graphs, K_t-minor free graphs for any $t \geq 2$, graphs of bounded tree-width, graphs of bounded tree-depth, and graphs of bounded Colin de Verdière parameter.

Natural examples of non summable properties include toroidal graphs (or more generally graphs embeddable on any fixed surface other than the sphere). For monotone properties that are not necessarily summable, we prove the following.

▶ **Theorem 6.** *For any $\varepsilon > 0$ and any monotone property $\mathcal{P}$ of a proper minor-closed class $\mathcal{G}$, $\mathcal{P}$ has a 1-round ε-approximate proof labelling scheme of complexity $O(\log n)$.*

While the complexity of the scheme guaranteed by Theorem 6 is not constant as in Elek's result [21] and Theorem 4, we do not require any bounded degree assumption (as in Theorem 4), and a local horizon of 1 is sufficient. More importantly, the fact that $\mathcal{P}$ is not necessarily summable requires a completely different set of techniques, much closer from the tools used to prove Theorem 2. Theorem 6 can be thought of as an approximate answer to the question of [25] on the local certification of minor-closed classes.

Organization of the paper

We start with some preliminary results in Section 2. In Section 3, we prove the main technical contribution of this paper, a result showing that if a graph from some minor-closed class is far from a monotone property $\mathcal{P}$, then it contains linearly many edge-disjoint subgraphs of bounded size that are not in $\mathcal{P}$. In Section 4 we deduce Theorem 2 from this result and Theorem 1. Theorems 4 and 6 are proved in Section 5. We conclude in Section 6 with some remarks.

2   Preliminaries

Minor-closed classes

We denote the number of vertices of a graph G by $\mathsf{v}(G)$, and its number of edges by $\mathsf{e}(G)$. A class of graphs $\mathcal{G}$ is *minor-closed* if any minor of a graph from $\mathcal{G}$ is also in $\mathcal{G}$. A class is *proper* if it does not contain all graphs. The following was proved by Mader [35].

▶ **Theorem 7** ([35]). *For any proper minor-closed class $\mathcal{G}$, there is a constant C such that for any graph $G \in \mathcal{G}$, $\mathsf{e}(G) \leq C\,\mathsf{v}(G)$.*

Tree-depth

Given a rooted tree T, the *closure* of T is the graph obtained from T by adding edges between each vertex and its ancestors in the tree. The *height* of a rooted tree is the maximum number of vertices on a root-to-leaf path in the tree. The *tree-depth* of a connected graph G is the maximum height of a rooted tree T such that G is a subgraph of the closure of T, and the tree-depth of a graph G, denoted by $\mathrm{td}(G)$, is the maximum tree-depth of its connected components (equivalently, it is equal to the maximum height of a rooted *forest* F such that G is a subgraph of the closure of F).

The following was implicitly proved by Dvořák and Sereni [20] (in the proof below the actual definition of tree-width is not needed, so we omit it).

▶ **Theorem 8** ([20]). *For every proper minor-closed class $\mathcal{G}$ and every $\delta > 0$ there exists $d = d_8(\mathcal{G}, \delta) \in \mathbb{N}$ and $s = s_8(\mathcal{G}, \delta) \in \mathbb{N}$ satisfying the following. For every $G \in \mathcal{G}$ there exist $X_1, X_2, \ldots, X_s \subseteq V(G)$ such that*

-   *for any $1 \leq i \leq s$, $\mathrm{td}(G[X_i]) \leq d$, and*
-   *every $v \in V(G)$ belongs to at least $(1 - \delta)s$ of the sets X_i.*

Proof. Let $t = \lceil \frac{2}{\delta} \rceil$. It was proved in [17] that there is a constant $k = k(t, \mathcal{G})$ such that any graph $G \in \mathcal{G}$ has a partition of its vertex set into t classes $Y_1, \ldots, Y_t$, such that the union of any $t - 1$ classes Y_i induces a graph of tree-width at most k. In particular, if we define $Z_i := V(G) \setminus Y_i$ for any $1 \leq i \leq t$, then each graph $G[Z_i]$ has tree-width at most k and each vertex v lies in $t - 1 = (1 - \frac{1}{t})t$ sets Z_i. Dvořák and Sereni [20, Theorem 31] proved[2] that for every integer k and real $\delta > 0$, there are integers $r = r(k, \delta)$ and $d = d(k, \delta)$ such that for any graph H of tree-width at most k, H has a cover of its vertex set by r sets $X_1, \ldots, X_r$, such that each $H[X_i]$ has tree-depth at most d and each vertex lies in at least $(1 - \frac{\delta}{2})r$ sets X_i. Applying this result to $H = G[Z_i]$ for any $1 \leq i \leq t$, we obtain rt sets $X'_1, \ldots, X'_{rt}$ of vertices of G, such that the subgraph $G[X'_i]$ induced by each of them has tree-depth at most d and each vertex of G lies in at least $(1 - \frac{\delta}{2})r \cdot (1 - \frac{1}{t})t \geq (1 - \delta)rt$ sets X'_i. Thus d and $s = rt$ satisfy the conditions of the theorem. ◀

We deduce the following useful result.

▶ **Corollary 9.** *For every proper minor-closed class $\mathcal{G}$ and every $\varepsilon > 0$ there exists $d = d_9(\mathcal{G}, \varepsilon) \in \mathbb{N}$ satisfying the following. For every $G \in \mathcal{G}$ there exist $F \subseteq E(G)$ such that $|F| \leq \varepsilon \, \mathsf{e}(G)$ and $\mathrm{td}(G \setminus F) \leq d$.*

Proof. Let $\delta = \frac{\varepsilon}{2}$. We show that $d = d_8(\mathcal{G}, \delta)$ satisfies the corollary. Indeed, for $G \in \mathcal{G}$ let $X_1, X_2, \ldots, X_s \subseteq V(G)$ be as in Theorem 8. Let $F_i = E(G) \setminus E(G[X_i])$ for $i \in [s]$, then $\mathrm{td}(G \setminus F_i) \leq d$. Moreover, every edge belongs to at most $2\delta s$ sets F_i, so

$$\frac{1}{s} \sum_{i=1}^{s} |F_i| \leq \frac{1}{s} \cdot 2\delta s \cdot \mathsf{e}(G) = \varepsilon \, \mathsf{e}(G).$$

Thus, by averaging, $|F_i| \leq \varepsilon \, \mathsf{e}(G)$ for some i, and $F = F_i$ satisfies the corollary. ◀

Note that the conclusion of Corollary 9 can be shown to hold in greater generality than in the context of minor-closed classes. For instance, any class in which all graphs can be made of bounded *tree-width* by removing an arbitrarily small fraction of edges also have this property (see [20]). This includes all graphs of bounded *layered tree-width* (see [18, 39]). Typical non minor-closed examples of such classes are families of graphs that can be embedded on a fixed surface, with a bounded number of crossings per edge [19]. However, since the proof of Theorem 1 itself strongly relies on edge-contractions (and thus on the graph class $\mathcal{G}$ being minor-closed), Theorem 2 does not seem to be easily extendable beyond minor-closed classes.

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[2] The property that s is bounded independently of G does not appear explicitly in the statement of their theorem, but readily follows from their proof. This will only be needed in Section 5.

3   Bounded size obstructions

3.1   General properties

A graph property $\mathcal{P}$ is a graph class that is closed under isomorphism. It will be convenient to write that $G \in \mathcal{P}$ instead of "G satisfies $\mathcal{P}$" in the remainder of the paper. A graph H is *minimally not in* $\mathcal{P}$ if $H \notin \mathcal{P}$ and any proper subgraph of H is in $\mathcal{P}$.

We will use the following result of Nešetřil and Ossona de Mendez (Lemma 6.13 in [36]) [3].

▶ **Lemma 10** ([36]). *For every integer $d \geq 1$ and every property $\mathcal{P}$, there exists $N = N_{10}(d, \mathcal{P})$ such that if H is minimally not in $\mathcal{P}$ and $\mathrm{td}(H) \leq d$ then $\mathsf{v}(H) \leq N$.*

3.2   Colorability

The conclusion of Lemma 10 is quite strong but it does not give explicit bounds on $N_{10}(d, \mathcal{P})$. For completeness, we give such an explicit bound when $\mathcal{P}$ is the property of being k-colorable. The specific question of whether 3-colorability of planar graphs was testable in the sparse model was raised by Christian Sohler at the Workshop on Local Algorithms (WOLA) in 2021. A positive answer to this question directly follows from Theorem 2, but the lemma below allows us to give an explicit bound on the query complexity of testing k-colorability in minor-closed classes (see Section 6).

Given a graph H and two vertex subsets $A, B \subseteq V(H)$, we say that (A, B) is a *proper separation* of H if $A \cup B = V(H)$, $A \setminus B$ and $B \setminus A$ are both non-empty, and there are no edges between $A \setminus B$ and $B \setminus A$ in H. We say that a graph H is *split* if there exists a proper separation (A, B) of H and an isomorphism $\phi : A \to B$ between $H[A]$ and $H[B]$ such that $\phi(v) = v$ for every $v \in A \cap B$ (see Figure 1 for an example). Equivalently, a split graph can be obtained by taking two copies of some smaller graph and, for a proper subset of vertices, identifying the two copies of the vertex subset with each other.

A connected graph H is *unsplit* if it is not split. Note that minimally non-k-colorable graphs are unsplit.
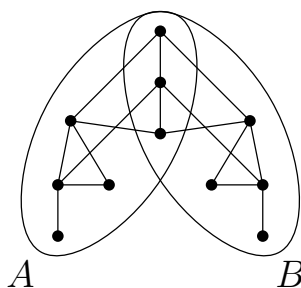
Figure 1 A split graph H and the corresponding proper separation (A, B) of H. The isomorphism $\phi : A \to B$ is the reflection symmetry with respect to the vertical axis.

The *tower function* is defined as $\mathsf{twr}(0) = 0$ and $\mathsf{twr}(i + 1) = 2^{\mathsf{twr}(i)}$ for any integer $i \geq 0$.

▶ **Lemma 11.** *For every integer $d \geq 1$, there exists $N = N_{11}(d) = \mathsf{twr}(O(d))$ such that if H is an unsplit graph with $\mathrm{td}(H) \leq d$ then $\mathsf{v}(H) \leq N$.*

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[3] The version we use here only needs Q to be a singleton in the statement of Lemma 6.13 in [36].

Proof. Define $k_d := 1$ and for any $0 \le i \le d - 1$, let $k_i := (2^{\binom{k_{i+1}}{2} + d \cdot k_{i+1}}) k_{i+1} + 1$.

Choose a rooted tree T of height at most $d + 1$ rooted at some vertex r, such that H is a subgraph of the closure of T. For each $v \in V(H)$, let T_v be the subtree of T rooted at v (consisting of v and all its descendants), and let H_v be the subgraph of H induced by $V(T_v)$. Define the *level* $\ell(v) := \mathrm{dist}_T(r, v) \in [0, d]$.

We prove by induction on $d - i$ that $\mathsf{v}(H_v) \le k_i$ for every v with $\ell(v) = i$. The base case $i = d$ trivially holds, as $k_d = 1$.

For the induction step, let $u_1, \dots, u_m$ be all the children of a vertex v with $\ell(v) = i$, and let A_v be the set consisting of v and its ancestors. Then $|A_v| = i + 1$. For each $j \in [m]$, let $H_j^+ = H[A_v \cup V(H_{u_j})]$. Note that there are at most $2^{\binom{k_{i+1}}{2} + d \cdot k_{i+1}}$ distinct (labelled) graphs G on at most $|A_v| + k_{i+1} \le d + k_{i+1}$ vertices such that the subgraph of G induced by the first $|A_v|$ vertices is isomorphic to $H[A_v]$. As $\mathsf{v}(H_{u_j}) \le k_{i+1}$, if $m > 2^{\binom{k_{i+1}}{2} + d \cdot k_{i+1}}$ there exist $j \ne j'$ and an isomorphism $\phi : V(H_j^+) \to V(H_{j'}^+)$ such that $\phi(w) = w$ for every $w \in A_v$. Such an isomorphism would imply that H is split, and so $m \le 2^{\binom{k_{i+1}}{2} + d \cdot k_{i+1}}$. It follows that $\mathsf{v}(H_v) \le m k_{i+1} + 1 \le (2^{\binom{k_{i+1}}{2} + d \cdot k_{i+1}}) k_{i+1} + 1 = k_i$, as desired.

By taking $N := k_0$ we obtain that $\mathsf{v}(H) \le N$. It can be checked from the definition of $(k_i)_{0 \le i \le d}$ that $N = k_0$ is at most a tower function of $O(d)$. ◄

3.3 A linear Erdős-Posá property

We use Corollary 9 and Lemma 10 to deduce the following result, which is the main technical contribution of this paper.

▶ **Theorem 12.** *For every proper minor-closed class $\mathcal{G}$, every $\varepsilon > 0$, and every property $\mathcal{P}$, there exists $\delta > 0$ and an integer N such that for every $G \in \mathcal{G}$ either*

- *there exists $F \subseteq E(G)$ with $|F| \le \varepsilon \, \mathsf{e}(G)$ such that $G \setminus F$ is in $\mathcal{P}$, or*
- *there exist edge-disjoint subgraphs $G_1, \dots, G_m$ of G that are not in $\mathcal{P}$, such that $m \ge \delta \, \mathsf{e}(G)$ and for every $1 \le i \le m$, $\mathsf{v}(G_i) \le N$.*

Proof. By Theorem 7, there exists C such that $\mathsf{e}(G) \le C \mathsf{v}(G)$ for every $G \in \mathcal{G}$. Let $d := d_9(\mathcal{G}, \varepsilon/2)$ and let $N := N_{10}(d, \mathcal{P})$. We show that $\delta := \frac{\varepsilon}{2NC}$ satisfies the theorem.

Let $G_1, \dots, G_m$ be a maximal collection of edge-disjoint subgraphs of G that are not in $\mathcal{P}$, and such that $\mathsf{v}(G_i) \le N$. If $m \ge \delta \mathsf{e}(G)$ the theorem holds, so we assume that $m < \delta \mathsf{e}(G)$.

Let $F' = \bigcup_{i=1}^m E(G_i)$. Then

$$|F'| \le C \sum_{i=1}^m \mathsf{v}(G_i) \le CmN < cN\delta \mathsf{e}(G) \le \frac{\varepsilon}{2} \mathsf{e}(G).$$

Let $G' = G \setminus F'$. By the choice of d, it follows from Corollary 9 that there exists $F'' \subseteq E(G')$ such that $|F''| \le \frac{\varepsilon}{2} \mathsf{e}(G)$ and $\mathrm{td}(G' \setminus F'') \le d$.

Let $G'' = G' \setminus F''$. Suppose first that G'' is not in $\mathcal{P}$, and let H be a minimal subgraph of G'' that is not in $\mathcal{P}$. As H is minimally not in $\mathcal{P}$, it follows from Lemma 10 that $\mathsf{v}(H) \le N$. Thus adding H to the collection $G_1, \dots, G_m$ contradicts its maximality.

It follows that G'' is in $\mathcal{P}$, but $G'' = G \setminus F$, where $F = F' \cup F''$ and $|F| \le \varepsilon \, \mathsf{e}(G)$, and so the theorem holds. ◄

4 Property testing in the sparse model

The model

As alluded to in the introduction, we work in the sparse model, only using queries to the random neighbor oracle. That is, given an input graph G, the tester only does a constant number of queries to the input, all of the following type: given a vertex v, return a random neighbor of v (uniformly at random among all the neighbors of v in G). The vertex v itself can be taken to be a random vertex of G, but does not need to. The computation of a random vertex of G and a random neighbor of a given vertex of G are assumed to take constant time in this model.

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let $\mathcal{G}$ be a proper-minor class and $\mathcal{P}$ be a monotone property. Let ε be given. Let $\delta > 0$ and N be obtained by applying Theorem 12 to $\mathcal{G}$, $\mathcal{P}$ and ε, and let $\mathcal{H}$ be the (finite) set of all graphs of at most N vertices that are not in $\mathcal{P}$. We now run the tester of Theorem 1 for testing whether a graph $G \in \mathcal{G}$ is $\mathcal{H}$-free or δ-far from being $\mathcal{H}$-free.

Assume first that $G \in \mathcal{P}$. If G contains a graph $H \in \mathcal{H}$ as a subgraph, then since $\mathcal{P}$ is monotone, we have $H \in \mathcal{P}$, which is a contradiction. Hence, G is $\mathcal{H}$-free, and it follows that the one-sided tester of Theorem 1 accepts G with probability 1. Assume now that G is ε-far from $\mathcal{P}$. By Theorem 12, there exist at least $\delta\mathsf{e}(G)$ edge-disjoint subgraphs of G that are all in $\mathcal{H}$, and thus one needs to remove at least one edge in each of these $\delta\mathsf{e}(G)$ edge-disjoint subgraphs to obtain an $\mathcal{H}$-free graph. As $\mathcal{P}$ is monotone, G is δ-far from being $\mathcal{H}$-free, and it follows that the tester of Theorem 1 rejects G with probability at least $\frac{2}{3}$, as desired. This concludes the proof of Theorem 2.   ◀

5 Local Certification

We recall that in this part, all graphs are assumed to be connected.

5.1 Summable properties

Before we prove Theorem 5, we will need the following consequence of a result of Brodskiy, Dydak, Levin and Mitra [12] (obtained by taking $r = 2$ in Theorem 2.4 in their paper). This can be seen as an analogue of Theorem 8 where tree-depth is replaced by the weaker notion of weak diameter, while proper minor-closed classes are replaced by the more general classes of bounded asymptotic dimension.

▶ **Theorem 13** ([12]). *Let $\mathcal{G}$ be a class of graphs of bounded asymptotic dimension and let $\delta > 0$ be a real number. Then there exist two constants $D = D_{13}(\mathcal{G}, \delta) \in \mathbb{N}$ and $s = s_{13}(\mathcal{G}, \delta) \in \mathbb{N}$ satisfying the following. For every $G \in \mathcal{G}$ there exist $X_1, X_2, \ldots, X_s \subseteq V(G)$ such that*

- *for any $1 \leq i \leq s$, each connected component of $G[X_i]$ has weak diameter at most D in G, and*
- *every $v \in V(G)$ belongs to at least $(1 - \delta)s$ of the sets X_i.*

It can be noted that if a subset S of vertices of a graph G is such that $G[S]$ has bounded tree-depth, then $G[S]$ has bounded diameter [36], and thus S has bounded weak diameter in G. It follows that in the special case of proper minor-closed classes, Theorem 8 implies Theorem 13 in a strong form.

Proof of Theorem 5. Fix any real number $\varepsilon > 0$ and an integer $d \geq 0$. Let $\mathcal{G}$ be a class of bounded asymptotic dimension, and let $\mathcal{P}$ be a monotone summable property of $\mathcal{G}$. Let $\delta = \frac{\varepsilon}{2}$. By Theorem 13, there exist two constants $D = D_{13}(\mathcal{G}, \delta) \in \mathbb{N}$ and $s = s_{13}(\mathcal{G}, \delta) \in \mathbb{N}$ satisfying the following. For every $G \in \mathcal{G}$ there exist $X_1, X_2, \ldots, X_s \subseteq V(G)$ such that

- for any $1 \leq i \leq s$, each component of $G[X_i]$ has weak diameter at most D in G, and
- every $v \in V(G)$ belongs to at least $(1 - \delta)s$ of the sets X_i.

For any $v \in V(G)$, we define the proof $P(v)$ as (a binary representation of) the set of indices $I(v) \subseteq \{1, 2, \ldots, s\}$ such that $v \in X_i$. This proof has constant size (depending only of $\mathcal{P}$ and ε).

For every vertex v, the local verifier $\mathcal{A}(G, P, v)$ first checks that $I(v)$ contains at least $(1 - \delta)s$ integers from $\{1, 2, \ldots, s\}$. If this is not the case, then v rejects the instance. In the remainder, we call a *monochromatic component of color i* a maximal connected subset of vertices v of G such that $i \in I(v)$. We omit the color if it is irrelevant in the discussion. Note that each vertex v belongs to $|I(v)|$ monochromatic components. For each vertex v, $\mathcal{A}(G, P, v)$ checks that the subgraph of G induced by the vertices $u \in B_r(v)$ is in $\mathcal{P}$, for $r = 2D + 1$, and that all monochromatic components of G containing v have weak diameter at most D (this can be clearly done as v has access to the subgraph of G induced by its ball of radius $r = 2D + 1$). If this is the case, then v accepts the instance, and otherwise v rejects the instance.

It follows from the definition of our scheme and the monotonicity of $\mathcal{P}$ that for any $G \in \mathcal{P}$, the local verifier $\mathcal{A}(G, P, v)$ of each vertex v of G accepts the instance. Consider now a graph G and a proof P' such that for each vertex v, $\mathcal{A}(G, P', v) = 1$. The proof P' assigns a subset $I(v)$ of indices of $\{1, \ldots, s\}$ to each vertex v of G, such that $|I(v)| \geq (1 - \delta)s$. For each $1 \leq i \leq s$, let X_i be the subset of vertices v of G such that $i \in I(v)$. Let C be a connected component of some $G[X_i]$ (that is, C is a monochromatic component of color i), for some $1 \leq i \leq s$. Since all vertices of C accept the instance, C has weak diameter at most D in G. It follows that C is contained in some ball of radius D in G, and thus (since $\mathcal{P}$ is monotone, and each ball of radius $r = 2D + 1 \geq D$ induces a graph of $\mathcal{P}$), C lies in $\mathcal{P}$. As $\mathcal{P}$ is summable, $G[X_i]$ also lies in $\mathcal{P}$.

It follows from the proof of Corollary 9, that if we set $F_i = E(G) \setminus E(G[X_i])$ for any $1 \leq i \leq s$, then the property that every vertex $v \in V(G)$ belongs to at least $(1 - \delta)s$ of sets X_i implies that there is an index $1 \leq i \leq s$ such that $|F_i| \leq \varepsilon\,\mathsf{e}(G)$. By the paragraph above $G \setminus F_i$ satisfies $\mathcal{P}$, and thus G is ε-close from $\mathcal{P}$ (we say that a graph is *ε-close* from $\mathcal{P}$ if it is not ε-far from $\mathcal{P}$). In the contrapositive, we have proved that if G is ε-far from $\mathcal{P}$, then there is at least one vertex v such that $\mathcal{A}(G, P', v) = 0$, as desired. ◀

Using the fact that proper minor-closed classes have asymptotic dimension at most 2 [9], we immediately obtain Theorem 4 as a corollary. Note that for the same purpose we could also use an earlier (and simpler) result of Ostrovskii and Rosenthal [38], who proved that for every integer t, the class of K_t-minor free graphs has asymptotic dimension at most 4^t. We could also use Theorem 8 without any reference to asymptotic dimension, as Theorem 8 implies Theorem 13 for proper minor-closed classes (see the discussion before the proof of Theorem 5).

5.2   Non necessarily summable properties

We now consider proof labelling schemes of complexity $O(\log n)$, rather than $O(1)$. To prove Theorem 6, we will need the following recent result of Bousquet, Feuilloley and Pierron [11].

▶ **Theorem 14** ([11]). *For every integer $d \geq 1$ and every first-order sentence φ, the class of graphs of tree-depth at most d satisfying φ has a 1-round proof labelling scheme of complexity $O(\log n)$.*

Although we will not need it, it is worth noting that the complexity in their result is of order $O(d \log n + f(d, \varphi))$. Observe also that checking whether a graph is $\mathcal{H}$-free for some fixed finite family $\mathcal{H}$ can be expressed by a first-order formula. In particular, it follows directly from Lemma 10 that checking whether $\mathrm{td}(G) \leq d$ can be expressed by a first-order formula (and thus certified with local horizon 1 with labels of size $O(\log n)$ per vertex).

The final ingredient that we will need is the ability to certify a rooted spanning tree, together with the children/parent relationship in this tree, with certificates of $O(\log n)$ bits per vertex (see [1, 6, 34] for the origins of this classical scheme). The prover gives the identifier $\mathrm{id}(r)$ of the root r of T to each vertex v of G, as well as $d_T(v, r)$, its distance to r in T, and each vertex v distinct from the root is also given the identifier of its parent $p(v)$ in T. The local verifier at v starts by checking that v agrees with all its neighbors in G on the identity of the root r of T. If so, if $v \neq r$, v checks that $d_T(v, r) = d_T(p(v), r) + 1$. It can be checked that all vertices accept the instance if and only T is a rooted spanning tree of G. Moreover, once the rooted spanning tree T has been certified, each vertex of G knows its parent and children (if any) in T.

We are now ready to prove Theorem 6.

Proof of Theorem 6. The beginning of the proof proceeds exactly as in the proof of Theorem 4. Fix any real number $\varepsilon > 0$. Let $\mathcal{G}$ be a proper minor-closed class, and let $\mathcal{P}$ be a monotone (not necessarily summable) property of $\mathcal{G}$. Let $\delta = \frac{\varepsilon}{2}$. By Theorem 8, there exist $d = d_8(\mathcal{G}, \delta) \in \mathbb{N}$ and $s = s_8(\mathcal{G}, \delta) \in \mathbb{N}$ satisfying the following. For every $G \in \mathcal{G}$ there exist $X_1, X_2, \ldots, X_s \subseteq V(G)$ such that

- for any $1 \leq i \leq s$, $\mathrm{td}(G[X_i]) \leq d$, and
- every $v \in V(G)$ belongs to at least $(1 - \delta)s$ of the sets X_i.

By Lemma 10, there exists a constant $N = N_{10}(d, \mathcal{P})$ such that if H is minimally not in $\mathcal{P}$ and $\mathrm{td}(H) \leq d$ then $\mathsf{v}(H) \leq N$. Let $\mathcal{H}$ be the (finite) set of all graphs of at most N vertices that are not in $\mathcal{P}$.

For any $v \in V(G)$, the proof $P(v)$ contains (a binary representation of) the set of indices $I(v) \subseteq \{1, \ldots, s\}$ such that $v \in X_i$. This part of the proof has constant size (depending only of $\mathcal{P}$ and ε). As in the proof of Theorem 4, the local verifier at each vertex v checks that $|I(v)| \geq (1 - \delta)s$, and rejects the instance if this does not hold.

For each $1 \leq i \leq s$, we do the following. In each connected component C of $G[X_i]$, we consider a rooted spanning tree T_C of C, with root r_C, and certify it using certificates of $O(\log n)$ bits per vertex. It follows from Theorem 14 that any first-order property of $G[C]$ can be certified with certificates of size $O(\log n)$ bits per vertex (as all the components C are vertex-disjoint, combining all these certificates and schemes still results in a scheme with labels of $O(\log n)$ bits per vertex). In particular we can certify that $\mathrm{td}(G[C]) \leq d$ (this is a first-order property). Let $\mathcal{H}'$ be the class of all (non-empty) graphs obtained from a graph $H \in \mathcal{H}$ by deleting an arbitrary subset of connected components of H (note that if all the graphs of $\mathcal{H}$ are connected, $\mathcal{H} = \mathcal{H}'$). Observe that all the graphs of $\mathcal{H}'$ have size at most N (which is a constant independent of the size of G). Then, for any $H' \in \mathcal{H}'$, we certify that $G[C]$ is H'-free or contains a copy of H' using Theorem 14, and store this information at the vertex r_C in a constant-size binary array $b(r_C)$, whose entries are indexed by all the graphs of $\mathcal{H}'$ (where the entry of $b(r_C)$ corresponding to some $H' \in \mathcal{H}'$ is equal to 1 if and only if C contains a copy of H' as a subgraph).

It remains to aggregate this information along some rooted spanning tree T of G (which can itself be certified with certificates of $O(\log n)$ bits per vertex). We do this as follows, for every $1 \le i \le s$. For a vertex v of the rooted tree T, the subtree of T rooted in v is denoted by T_v. For each vertex v of G, let $\mathcal{C}_v$ be the set of components C of $G[X_i]$ such that r_C lies in T_v. Then the proof $P(v)$ contains a binary array $c(v)$, whose entries are indexed by the graphs H' of $\mathcal{H}'$. The array $c(v)$ is defined as follows: for any $H' \in \mathcal{H}'$, the entry of $c(v)$ corresponding to H' is equal to 1 if and only if H' is a disjoint union of (non necessarily connected) graphs $H'_1, H'_2, \ldots, H'_k \in \mathcal{H}'$ such that each H'_i appears in a different component of $\mathcal{C}_v$. The consistency of the binary arrays $c(v)$ is verified locally as follows. For each vertex v of G, the local verifier at v considers the binary arrays $c(u)$, for all children u of v (and the binary array $b(v)$, if v is equal to some root r_C). For any $H' \in \mathcal{H}'$, the local verifier at v checks whether H' can be written as a disjoint union of graphs $H'_1, H'_2, \ldots, H'_k \in \mathcal{H}'$ such that each H'_i appears in a different array among the children of v (plus in $b(v)$, if v is a root of some component C). The local verifier at v then checks whether this is consistent with the entry corresponding to H' in $c(v)$. Clearly, all the vertices accept if and only if the information is consistent along the spanning tree, and it follows that the local verifier at the root r can check for each $H \in \mathcal{H} \subseteq \mathcal{H}'$, whether the entry of $c(r)$ corresponding to H is equal to 0 or 1. It follows that the local verifier at r can check whether $G[X_i]$ is $\mathcal{H}$-free (and accept the instance if and only if this is the case).

It follows from the definition of our scheme that for any $G \in \mathcal{P}$, the local verifier of each vertex of G accepts the instance.

Consider now some graph G together with some proof P' such that the local verifier $\mathcal{A}(G, P', v)$ at each vertex v of G accepts the instance. For any $1 \le i \le s$, let X_i be the set of vertices v such that $i \in I(v)$ (where $I(v)$ is given by the proof $P'(v)$), and let $F_i = E(G) \setminus E(G[X_i])$. As in the proof of Theorem 4, the property that every vertex $v \in V(G)$ belongs to at least $(1 - \delta)s$ of sets X_i implies that there is an index $1 \le i \le s$ such that $|F_i| \le \varepsilon \mathsf{e}(G)$.

By the properties of the local certificates, each component of $G \setminus F_i = G[X_i]$ has tree-depth at most d, and thus $G \setminus F_i$ has tree-depth at most d. Moreover, our local certificates imply that $G \setminus F_i$ is $\mathcal{H}$-free. Since $\mathcal{P}$ is monotone, $G \setminus F_i$ is in $\mathcal{P}$. It follows that G is ε-close from $\mathcal{P}$. Taking the contrapositive, this shows that if a graph is ε-far from $\mathcal{P}$, then at least one local verifier will reject the instance. This concludes the proof of Theorem 6.   ◄

6   Conclusion

In this paper we proved that for any proper minor-closed class $\mathcal{G}$, using constantly many queries to the random neighbor oracle, a tester can decide with good probability whether an input graph $G \in \mathcal{G}$ satisfies some fixed monotone property $\mathcal{P}$, or is ε-far from $\mathcal{P}$. Given the level of generality of the result it is to be expected that no explicit bounds on the query complexity are given. However, we can give explicit estimates on the query complexity for specific properties. For instance, it follows from the bounds of [20, Corollary 35], combined with Lemma 11 and our proof of Theorem 2, that 3-colorability can be tested in planar graphs with $\mathsf{twr}(\mathrm{poly}(1/\varepsilon))$ queries to the random neighbor oracle. This can be extended to testing k-colorability in K_t-minor free graphs, for any k and t, at the expense of a significant increase in the height of the tower function, by combining the results of [20] with the main result of [17] (the bounds there are not explicit as a function of t, but can be made explicit using results from the Graph Minor series). This is to be compared with the main result of [14], that 2-colorability can be tested with $2^{2^{\mathrm{poly}(1/\varepsilon)}}$ queries in planar graphs. It is a

natural problem to understand whether these properties can be tested with $\mathrm{poly}(1/\varepsilon)$ queries to the random neighbor oracle, and more generally to develop techniques for proving finer lower bounds on the query complexity of monotone properties in this model (see [7] for recent results in this direction in the bounded degree model).

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