A Study of Weisfeiler–Leman Colorings on Planar Graphs

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Abstract

The Weisfeiler–Leman (WL) algorithm is a combinatorial procedure that computes colorings on graphs, which can often be used to detect their (non-)isomorphism. Particularly the 1- and 2-dimensional versions 1-WL and 2-WL have received much attentionattention, due to their numerous links to other areas of computer science.

Knowing the expressive power of a certain dimension of the algorithm usually amounts to understanding the computed colorings. An increase in the dimension leads to finer computed colorings and, thus, more graphs can be distinguished. For example, on the class of planar graphs, 3-WL solves the isomorphism problem. However, the expressive power of 2-WL on the class is poorly understood (and, in particular, it may even well be that it decides isomorphism).

In this paper, we investigate the colorings computed by 2-WL on planar graphs. Towards this end, we analyze the graphs induced by edge color classes in the graph. Based on the obtained classification, we show that for every 3-connected planar graph, it holds that: a) after coloring all pairs with their 2-WL color, the graph has fixing number 1 with respect to 1-WL, or b) there is a 2-WL-definable matching that can be used to transform the graph into a smaller one, or c) 2-WL detects a connected subgraph that is essentially the graph of a Platonic or Archimedean solid, a prism, a cycle, or a bipartite graph $K_{2,\ell}$. In particular, the graphs from case (a) are identified by 2-WL.

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1 Introduction

The Weisfeiler–Leman (WL) algorithm [41] is a combinatorial procedure that given a graph G, computes a coloring on G which respects (and sometimes also detects) the symmetries in the graph. Its most prominent application is in theoretical [6, 8, 31] and practical approaches [2, 10, 30, 35, 36] to the graph isomorphism problem. The original algorithm by Weisfeiler and Leman is the 2-dimensional version and it colors pairs of vertices. Its generalization yields for every natural number k the k-dimensional WL algorithm k-WL, which iteratively refines a coloring of vertex k-tuples by aggregating local structural information encoded in the colors. Its final output is a coloring that is *stable* with respect to the criterion for partitioning the color classes, and graphs with different final colorings are never isomorphic.

Over the decades, fascination for the algorithm has persisted. This is to a large extent due to the discovery of numerous connections to other areas in computer science that are still being explored. For example, the algorithm has close links to linear and semidefinite programming [4, 5, 25], homomorphism counting [11, 12], and machine learning [1, 21, 37, 39, 43]. Its expressive power can be characterized via winning strategies for the players in a particular type of Ehrenfeucht-Fraïssé game [8, 27]. Moreover, it is known that two graphs receive different final colorings with respect to k-WL if and only if the graphs can be distinguished via a formula in the counting-logic fragment C^{k+1} [8, 29].

In this work, we focus on the original version 2-WL, as introduced by Weisfeiler and Leman [41]. Besides the connections outlined for k-WL above, 2-WL has a precise correspondence to coherent configurations (see, e.g., [9]). Despite the simple and very natural concept behind the algorithm, its behavior is not well-understood and there is an extensive line of study to capture its expressive power. For example, one branch of research aims at understanding which graph properties can be detected by 2-WL. In this direction, Fürer [16] as well as Arvind et al. and Fuhlbrück et al. [3, 15] obtained insights concerning the ability of 2-WL to detect and count small subgraphs. Furthermore, the algorithm is able to detect 2-separators in graphs and implicitly computes the decomposition of a graph into its 3-connected components [32].

A related line of research analyzes which graphs are *identified* by 2-WL, i.e., on which graphs 2-WL serves as a complete isomorphism test. Positive examples include interval graphs [13] and distance-hereditary graphs [17] as well as almost all regular graphs [7]. In the light of the upper bound of 3 on the dimension of the algorithm needed to identify all planar graphs [34], there is hope that the class of planar graphs can eventually be added to the list. Towards a complete characterization of the expressive power of 2-WL, Fuhlbrück, Köbler, and Verbitsky [14] developed an algorithmic characterization of the graphs of color class size at most 4 that are identified by 2-WL.

Our Contribution. In this work, we investigate 2-WL on planar graphs. We are interested in analyzing the stable output coloring computed by 2-WL and deducing symmetries and other properties of the input graph from properties of the coloring.

As a starting point, we precisely characterize the planar graphs in which all edges receive the same color with respect to 2-WL. Since the coloring that 2-WL computes is preserved by automorphisms, edge-transitive planar graphs clearly fall into this category. As our first main result, we prove the converse of this statement: every planar graph in which all edges receive the same color with respect to 2-WL is edge-transitive. To show the implication, we reprove the classification of edge-transitive planar graphs (see, e.g., [26]) building solely on the 2-WL coloring.

Using the classification, we continue to analyze the WL coloring on general planar graphs. Since, by [32], the algorithm 2-WL implicitly computes the graph decomposition into 3-connected components, understanding 2-WL on planar graphs essentially amounts to a study of 3-connected planar graphs. Here, we can exploit a theorem due to Whitney [42], which says that all embeddings of a 3-connected planar graph are combinatorially equivalent.

Our focus lies on the following three tasks: (i) classify the subgraphs induced by edges of the same 2-WL color that can occur, (ii) analyze how these subgraphs interleave, and (iii) establish connections to properties of the entire graph G.

Let G be a 3-connected planar graph and let $C_E(G)$ denote the set of 2-WL colors that correspond to edges of G. For every $c \in C_E(G)$, denote by $G[c]$ the subgraph induced by all edges of 2-WL color c. To describe our results, it turns out to be useful to partition edge colors into three types depending on the number of faces per connected component of $G[c]$. We say that c has *Type I* if every connected component of $G[c]$ has one face, *Type II* if every connected component of $G[c]$ has two faces, and *Type III* if every connected component of $G[c]$ has at least three faces. (By the properties of 2-WL, these types indeed cover all cases that can occur.)

First, we analyze the graphs induced by edge colors c of Type III. It is not hard to see that every edge in such a graph $G[c]$ receives the same 2-WL color (when applying 2-WL to $G[c]$), and thus, by our classification, $G[c]$ is edge-transitive. However, it turns out that much stronger statements are possible, since, in the end, many edge-transitive planar graphs cannot appear as a graph $G[c]$. For example, we show that $G[c]$ is always connected. As our central result for colors of Type III, we obtain a precise classification of the possible graphs $G[c]$. An interesting consequence of this classification is that the automorphism group $\mathrm{Aut}(G)$ of G is always isomorphic to a subgroup of $\mathrm{Aut}(G[c])$. More precisely, we show that fixing the images of all vertices of $G[c]$ uniquely determines the image of every vertex of G under any automorphism of G. Hence, by only looking at the subgraph induced by a single edge color of Type III, we obtain strong insights about the symmetries of the entire graph.

On the other side of the spectrum, we prove that if all edge colors are of Type I, then G has *fixing number* at most 1, where the fixing number is the minimum number of vertices that need to be fixed pointwise so that the identity mapping is the only automorphism of G. It is known that 3-connected planar graphs have fixing number at most 3, and there is a complete characterization of those graphs of fixing number exactly 3 [34]. In our analysis of 3-connected planar graphs G in which all edge colors are of Type I, we only use 1-WL to prove that G has no non-trivial automorphisms after fixing a certain single vertex. This implies that 2-WL identifies all such graphs.

If neither of the above cases applies, then there is an edge color of Type II. Let us first remark that the graphs of many Archimedean solids fall into this category (while the edge colors in the graph of all Platonic solids are of Type III). In such a situation, the graph of the Archimedean solid is defined by edge colors c, d where one of the two colors has Type II. With this in mind, towards solving task (ii), we analyze how edge colors of Type II interleave with other edge colors. More precisely, similarly as for Type III, we aim at identifying a connected subgraph defined by two colors $c, d \in C_E(G)$, where c has Type II, that corresponds to one of the Archimedean solids, or stems from a small number of infinite graph families. We remark that, similar to the case of edge colors of Type III, if we have such a subgraph $G[c, d]$, then $\mathrm{Aut}(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(G[c, d])$.

We show that either this goal can be achieved, or G has fixing number 1 or there is a *WL-definable matching*. Such a matching is given by an edge color c such that $G[c]$ is a matching graph (i.e., every vertex has degree 1) and the endpoints of every edge receive different colors. Such matchings also play a crucial role in the analysis of 2-WL on graphs of color class size 4 [14], and contracting all matching edges preserves many crucial properties related to WL such as the stable coloring, identifiability by WL, as well as the automorphism group of G. As a result, finding a WL-definable matching is beneficial since we can proceed to a smaller graph without affecting the problem at hand.

Towards the WL Dimension of Planar Graphs. The WL dimension of a graph class $\mathcal{C}$ is the minimal k such that k-WL identifies every graph from $\mathcal{C}$, i.e., k-WL serves as a complete isomorphism test for the class $\mathcal{C}$. Many classes of graphs are known to have a finite WL dimension, for example, interval graphs [13], graphs of bounded rank-width [24] as well as graphs of bounded genus [22] and, more generally, all graph classes that exclude a fixed graph as a minor [19, 20].

For planar graphs, the quest for bounds on their WL dimension was initiated by Immerman already over three decades ago [28]. In a first step, Grohe [18] proved that the dimension is finite. Analyzing Grohe's proof in detail, Redies [38] showed an upper bound of 14 on the WL dimension of planar graphs. This was further improved in [34], where it is shown

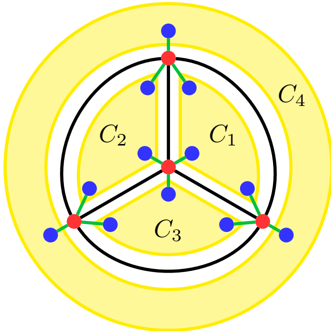


Figure 1 The visualization shows a 3-connected planar graph G and an edge color $c \in C_E(G)$ (shown in black) such that $G[c]$ is isomorphic to K_4. Also, C_1, C_2, C_3, C_4 are the vertex sets of the connected components of $G - V(G[c])$. Using regularity constraints specific to the case $G[c] \cong K_4$, one can show that the graphs induced by the C_i are all indistinguishable by 2-WL. More strongly, it actually holds for all $i, j \in [4]$, $v \in N(C_i)$ and $w \in N(C_j)$ that $\{\{\chi(v, v') \mid v' \in N(v) \cap C_i\}\} = \{\{\chi G(w, w') \mid w' \in N(w) \cap C_j\}\}$, where χ denotes the 2-WL coloring. (In the picture, this multiset contains one green edge, i.e., every $v \in V(G[c])$ has exactly one neighbor via a green edge in each adjacent C_i.)

that already 3-WL identifies all planar graphs, thus narrowing down the WL dimension of planar graphs to 2 or 3. Moreover, it was recently shown that a constant dimension of the WL algorithm suffices to identify all planar graphs in a logarithmic number of refinement rounds [23], extending previous results for 3-connected planar graphs [40]. Still, the task to determine the precise WL-dimension of the class of planar graphs remains open. A central motivation for our work is to find out whether 2-WL identifies every planar graph.

Our results suggest an inductive approach to this question. Indeed, building on the fact that 2-WL is able to detect the decomposition into 3-connected components [32], we can restrict our attention to 3-connected graphs. Given a 3-connected planar graph G, by combining the results described above, we always obtain that G has one of the following:

(A) fixing number 1 under 1-WL, i.e., individualizing a single vertex and performing 1-WL (after coloring all pairs with their 2-WL color) results in a discrete coloring,

(B) a WL-definable matching, or

(C) a connected subgraph induced by at most two edge colors that corresponds to a Platonic or Archimedean solid or stems from a small number of infinite graph families.

In Case A, the graph G is identified by 2-WL. In Case B, we can follow the strategy outlined in [14] and move to a smaller graph by contracting the definable matching. Therefore, determining the WL dimension of the class of planar graphs boils down to defeating Case C. In this case, we obtain a connected subgraph H that is defined by at most two edge colors c and d and which we can classify precisely. Let $C_1, \ldots, C_s$ denote the vertex sets of the connected components of $G - V(H)$, the graph G with the vertices in H removed (see also Figure 1). Also, let G' be a second graph that cannot be distinguished from G by 2-WL. Let H' denote the subgraph of G' induced by c and d and let $C'_1, \ldots, C'_s$ denote the vertex sets of the connected components of $G' - V(H')$. Presupposing by induction that the statement holds for smaller graphs, we may assume that 2-WL identifies the subgraphs induced by $C_1, \ldots, C_s$. This implies that $G[C_i]$ is isomorphic to $G'[C'_i]$ for all $i \in [s]$ (possibly after reordering the sets $C'_1, \ldots, C'_s$). It is not hard to see that 2-WL identifies H and, thus, H is isomorphic to H'. Now, ideally, we want to glue all these partial isomorphisms together to obtain a global isomorphism from G to G'. Towards this end, it is our intuition that the options for the interplay between H and the sets C_i are extremely limited due to G being planar and H being defined by few edge colors, which enforces strong regularity conditions on the interaction between H and the surrounding graph. A formalization of such a strategy for all subcases that can appear in Case C should yield that 2-WL identifies every planar graph.

2 Preliminaries

Graphs. An (undirected) *graph* is a pair $G = (V(G), E(G))$ of a finite *vertex set* $V(G)$ and an *edge set* $E(G) \subseteq \{\{u, v\} \mid u \neq v \in V(G)\}$. Unless stated explicitly otherwise, graphs are undirected. For a directed graph G', we write $\mathsf{undir}(G')$ to denote its undirected version. For $v, w \in V(G)$, we also write vw as a shorthand for $\{v, w\}$. The *neighborhood* of v in G is denoted by $N_G(v)$ and the *degree* of v in G is $\deg_G(v) := |N_G(v)|$. If the graph G is clear from the context, we usually omit the index and simply write $N(v)$ and $\deg(v)$. For $W \subseteq V(G)$, we also define $N(W) := \left(\bigcup_{v\in W} N(v)\right) \setminus W$. We denote by $G[W]$ the *induced subgraph* of G on the vertex set W, and define $G - W := G[V(G) \setminus W]$. A set $S \subseteq V(G)$ is a *separator* of G if $G - S$ has more connected components than G. A k*-separator* of G is a separator of G of size k. The graph G is k*-connected* if it is connected and has no separator of size at most $k - 1$.

In our definitions of vertex sets of graphs, we use the notation $\uplus$ to denote a formal disjoint union. More precisely, for sets V and W, the set $V \uplus W$ contains $|V| + |W|$ vertices, one distinct copy of each vertex in V and one distinct copy of each vertex in W. (For ease of notation, we refer to the vertices by their original names in V and W instead of renaming them first.)

A *vertex-colored graph* is a tuple (G, λ) where G is a graph and $\lambda\colon V(G) \to C$ is a *vertex coloring*, a mapping from $V(G)$ into some set C of colors. We define the set of *arcs* of a graph G as $A(G) := \{(v, v) \mid v \in V(G)\} \cup \{(v, w) \mid \{v, w\} \in E(G)\}$. Observe that for each $vw \in E(G)$, there are the two arcs (v, w), (w, v). An *arc-colored graph* is a tuple (G, λ), where G is a graph and $\lambda\colon A(G) \to C$ is a mapping from $A(G)$ into some set C of colors. Similarly, a *pair-colored graph* is a tuple (G, λ), where G is a graph and $\lambda\colon (V(G))^2 \to C$ is a mapping into some set of colors C.

Typically, the set C is chosen to be an initial segment $[n]$ of the natural numbers. We say a coloring λ is *discrete* if it is injective, i.e., all color classes have size 1. Finally, for a coloring λ and distinct vertices $v_1, \dots, v_\ell$, we denote by $(G, \lambda, v_1, \dots, v_\ell)$ the colored graph where each v_i for $i \in [\ell]$ is individualized. To be more precise, if λ is a vertex coloring, then $(G, \lambda, v_1, \dots, v_\ell) := (G, \widetilde{\lambda})$ where $\widetilde{\lambda}(v_i) = (1, i)$ for all $i \in [\ell]$, and $\widetilde{\lambda}(v) = (0, \lambda(v))$ for all $v \in V(G) \setminus \{v_1, \dots, v_\ell\}$. The definitions for arc and pair colorings are analogous. We generally assume that all graphs are arc-colored even if not explicitly stated. Every (uncolored) graph can be interpreted as an arc-colored graph by assigning to every diagonal arc (v, v) the color 1 and assigning to every non-diagonal arc the color 2.

A graph is called *planar* if it can be embedded into the plane $\mathbb{R}^2$. A *plane graph* is a graph embedded into the plane. As the following statement shows, all plane realizations of a planar graph have the same number of faces, i.e., regions bounded by edges.

▶ **Theorem 1** (Euler's formula). *Let G be a connected plane graph with n vertices, m edges, and f faces. Then $n - m + f = 2$.*

We will also fall back on the following famous theorem due to Whitney.

▶ **Theorem 2** (Whitney's theorem [42]). *Up to homeomorphism, a 3-connected planar graph has a unique embedding into the plane.*

The theorem allows us to speak about faces of 3-connected planar graphs as abstract objects, since it implies that in a 3-connected planar graphs, the set of faces does not depend on a specific embedding and thus, the faces can be viewed as combinatorial objects associated with G and are uniquely defined by their sets of vertices $V(F)$ and the edges $E(F)$ bounding F. We will therefore not draw a clear distinction between this combinatorial view and the topological view of F as a region and just use whichever is most suitable for our purpose.

The Weisfeiler–Leman Algorithm. Let $\chi_1, \chi_2 \colon (V(G))^k \to C$ be colorings of the k-tuples of vertices of a graph G. We say χ_1 *refines* χ_2, denoted $\chi_1 \preceq \chi_2$, if $\chi_1(\bar{v}) = \chi_1(\bar{w})$ implies $\chi_2(\bar{v}) = \chi_2(\bar{w})$ for all $\bar{v}, \bar{w} \in (V(G))^k$. The colorings χ_1 and χ_2 are *equivalent*, denoted $\chi_1 \equiv \chi_2$, if $\chi_1 \preceq \chi_2$ and $\chi_2 \preceq \chi_1$.

Given a graph G, the algorithm 1-WL iteratively computes an isomorphism-invariant coloring of the vertices of G. In this work, we actually require an extension of 1-WL, which also takes arc colors into account. For an arc-colored graph (G, λ), we define the initial coloring computed by the algorithm via $\chi^1_0[G](v) := \lambda(v, v)$ for all $v \in V(G)$. This coloring is refined via $\chi^1_{i+1}[G](v) := (\chi^1_i[G](v), \mathcal{M}_i(v))$, where $\mathcal{M}_i(v)$ is a multiset defined as

$$\mathcal{M}_i(v) := \left\{\!\!\left\{ \left(\chi^1_i[G](w), \lambda(v, w), \lambda(w, v)\right) \;\middle|\; w \in N_G(v) \right\}\!\!\right\}.$$

By definition, $\chi^1_{i+1}[G] \preceq \chi^1_i[G]$ holds for all $i \geq 0$. Hence, there is a minimal value i_∞ such that $\chi^1_{i_\infty}[G] \equiv \chi^1_{i_\infty+1}[G]$. We call $\chi^1_{i_\infty}[G]$ the *stable* coloring of G and denote it by $\chi^1_{\mathsf{WL}}[G]$. The algorithm 1-WL takes an arc-colored graph (G, λ) as input and returns $\chi^1_{\mathsf{WL}}[G]$.

We can also apply 1-WL to a pair-colored graph (G, λ). This can be done by defining $\widetilde{\lambda}(v_1, v_2) := (1, \lambda(v_1, v_2))$ for all $v_1, v_2 \in V(G)$ with $v_1v_2 \in E(G)$, and $\widetilde{\lambda}(v_1, v_2) := (0, \lambda(v_1, v_2))$ for all $v_1, v_2 \in V(G)$ with $v_1v_2 \notin E(G)$. Then we define $\chi^1_{\mathsf{WL}}[G, \lambda] := \chi^1_{\mathsf{WL}}[H, \widetilde{\lambda}]$ where H is a complete graph on vertex set $V(G)$.

Next, we describe the *k-dimensional Weisfeiler–Leman algorithm* (k-WL) for $k \geq 2$. For an input graph G, let $\chi^k_0[G] \colon (V(G))^k \to C$ be the coloring where each tuple is colored with the isomorphism type of its underlying ordered subgraph. $v_i = v_j \Leftrightarrow v'_i = v'_j$ and $v_iv_j \in E(G) \Leftrightarrow v'_iv'_j \in E(G)$. If the graph comes equipped with a coloring, the initial coloring $\chi^k_0[G]$ also takes the input coloring into account. More formally, for an arc coloring λ, for $\chi^k_0[G](v_1, \ldots, v_k) = \chi^k_0[G](v'_1, \ldots, v'_k)$ to hold, we have the additional conditions $\lambda(v_i, v_j) = \lambda(v'_i, v'_j)$ for all $i, j \in [k]$ with $(v_i, v_j) \in A(G)$. For a pair coloring λ, we have the additional conditions $\lambda(v_i, v_j) = \lambda(v'_i, v'_j)$ for all $i, j \in [k]$.

We then recursively define the coloring $\chi^k_i[G]$ obtained after i rounds of the algorithm. For $\bar{v} = (v_1, \ldots, v_k) \in (V(G))^k$, let $\chi^k_{i+1}[G](\bar{v}) := \left(\chi^k_i[G](\bar{v}), \mathcal{M}_i(\bar{v})\right)$, where

$$\mathcal{M}_i(\bar{v}) := \left\{\!\!\left\{ \left(\chi^k_i[G](\bar{v}[w/1]), \ldots, \chi^k_i[G](\bar{v}[w/k])\right) \;\middle|\; w \in V(G) \right\}\!\!\right\}$$

and $\bar{v}[w/i] := (v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_k)$ is the tuple obtained from substituting the i-th entry of $\bar{v}$ with w. Again, there is a minimal i_∞ such that $\chi^k_{i_\infty}[G] \equiv \chi^k_{i_\infty+1}[G]$, and we set $\chi^k_{\mathsf{WL}}[G] := \chi^k_{i_\infty}[G]$.

The algorithm k-WL takes a (pair- or arc-)colored graph G as input and returns $\chi^k_{\mathsf{WL}}[G]$. Given graphs G and H, the algorithm *distinguishes* G and H if $\{\!\{\chi^k_{\mathsf{WL}}[G](\bar{v}) \mid \bar{v} \in (V(G))^k\}\!\} \neq \{\!\{\chi^k_{\mathsf{WL}}[H](\bar{w}) \mid \bar{w} \in (V(H))^k\}\!\}$. Also, k-WL *identifies* G if it distinguishes G from every other non-isomorphic graph.

▶ **Definition 3.** *Let G be a graph and let $k \geq 2$. Then* k-WL determines arc orbits on G *if for every $(v_1, v_2) \in A(G)$, every graph H, and every $(w_1, w_2) \in A(H)$ such that $\chi^k_{\mathsf{WL}}[G](v_1, v_2, \ldots, v_2) = \chi^k_{\mathsf{WL}}[H](w_1, w_2, \ldots, w_2)$, there is an isomorphism $\varphi\colon G \cong H$ such that $\varphi(v_i) = w_i$ holds for both $i \in \{1, 2\}$.*

Moreover, k-WL determines pair orbits of G *if for all $v_1, v_2 \in V(G)$, every graph H, and all $w_1, w_2 \in V(H)$ such that $\chi^k_{\mathsf{WL}}[G](v_1, v_2, \ldots, v_2) = \chi^k_{\mathsf{WL}}[H](w_1, w_2, \ldots, w_2)$, there is an isomorphism $\varphi\colon G \cong H$ such that $\varphi(v_i) = w_i$ holds for both $i \in \{1, 2\}$.*

Observe that if k-WL determines arc or pair orbits of G, then it identifies G. Indeed, if for a second graph H, there is no isomorphism from G to H, the multisets of χ^k_{WL}-colors in the two graphs must be disjoint by Definition 3.

3 Edge-Transitive Planar Graphs

In this section, we classify planar graphs where all edges receive the same color with respect to 2-WL. We call an undirected graph G *edge-transitive* if for all $uv, u'v' \in E(G)$, there is an automorphism $\varphi\colon V(G) \to V(G)$ with $\varphi(u) = u'$ and $\varphi(v) = v'$. It is well-known that there are only nine edge-transitive connected planar graphs of minimum degree 3 [26]. Based on this result, one can easily classify all edge-transitive planar graphs. Clearly, all of these graphs have the property that all edges receive the same color with respect to 2-WL. In this section, we show the converse of this statement, i.e., every planar graph in which all edges receive the same color with respect to 2-WL is edge-transitive. Towards this goal, we reprove the classification from [26] relying only on 2-WL colors. More precisely, the main result in this section is the following theorem (see also Figure 2). Since 2-WL colors directed pairs and it may happen that a pair (u, v) receives a different color than (v, u), it is more convenient to consider directed graphs and demand that all directed edges receive the same color (rather than saying the pair of colors for both orientations is the same for all undirected edges).

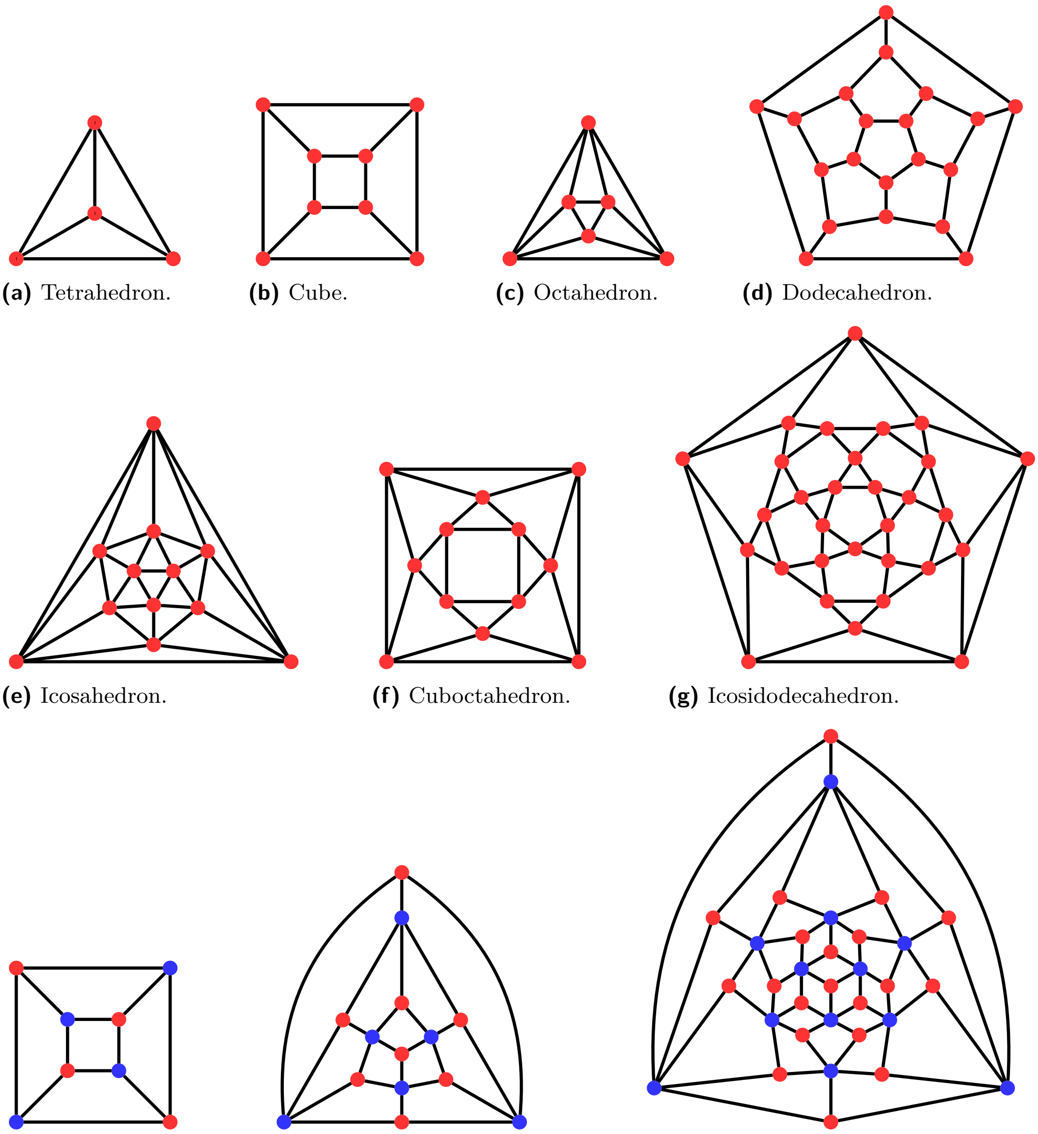
▶ **Theorem 4.** *Let G be a connected planar (directed or undirected) graph of minimum degree at least 3 such that $\chi^2_{\mathsf{WL}}[G](v_1, w_1) = \chi^2_{\mathsf{WL}}[G](v_2, w_2)$ for all $(v_1, w_1), (v_2, w_2) \in E(G)$. Then one of the following holds:*

(A) *G is isomorphic to a tetrahedron (Figure 2a), a cube (Figure 2b), a dodecahedron (Figure 2d), or an icosahedron (Figure 2e),*

(B) *the undirected version $\mathsf{undir}(G)$ is isomorphic to an octahedron (Figure 2c), a cuboctahedron (Figure 2f), or an icosidodecahedron (Figure 2g), or*

(C) *the undirected version $\mathsf{undir}(G)$ is isomorphic to a cube (Figure 2h), a rhombic dodecahedron (Figure 2i), or a rhombic triacontahedron (Figure 2j).*

Note that the classification includes the graphs of all Platonic solids. To prove the theorem, we distinguish two cases. Let $\chi \coloneqq \chi^2_{\mathsf{WL}}[G]$ and let $C_V(G, \chi) \coloneqq \{\chi(v, v) \mid v \in V(G)\}$ denote the set of *vertex colors*. Since $\chi(u, u) = \chi(u', u')$ and $\chi(v, v) = \chi(v', v')$ whenever $\chi(u, v) = \chi(u', v')$, we conclude that $1 \leq |C_V(G, \chi)| \leq 2$. First suppose $|C_V(G, \chi)| = 1$. Then $\mathsf{undir}(G)$ is d-regular for some $d \geq 3$. Since G is planar, $d \leq 5$ and thus, $d \in \{3, 4, 5\}$. A deep analysis of these three cases leads to the graphs listed in Parts A and B. Let us remark at this point that obtaining such a classification is much more challenging than for edge-transitive graphs. Indeed, the proofs for edge-transitive graphs highly exploit that the multiset of sizes of faces incident to an edge (and a vertex, respectively) is always the same. However, we cannot immediately deduce information about the size of faces from considering WL-colors and hence, we cannot directly rely on this type of argument. Instead, our arguments exploit the fact that 2-WL can detect 2-separators [32] as well as the existence of certain short cycles [15].



(a) Tetrahedron. **(b)** Cube. **(c)** Octahedron. **(d)** Dodecahedron.

(e) Icosahedron. **(f)** Cuboctahedron. **(g)** Icosidodecahedron.

(h) Bicolored cube. **(i)** Rhombic dodecahedron. **(j)** Rhombic triacontahedron.

Figure 2 All edge-transitive connected planar graphs of minimum degree 3.

Also, note that the graphs listed in Part A are always undirected since d is odd. On the other hand, every graph listed in Part B also has at least one directed version that is also edge-transitive (we refer the reader to the full version [33] for details).

Finally, for the case $|C_V(G,\chi)| = 2$, it is possible to perform a reduction to the first case by defining an auxiliary graph on one of the two vertex-color classes. This results in the graphs listed in Part C. Here, it is notable that the cube appears for a second time because it is bipartite and directing all edges from one bipartition class to the other one also leads to an edge-transitive graph.

In Theorem 4, we restrict ourselves to graphs that are connected and have minimum degree at least 3. Both of these restrictions can easily be lifted as follows. Let us first consider the restriction on the degree and let G be a connected planar graph such that

$\chi^2_{\mathsf{WL}}[G](v_1,w_1) = \chi^2_{\mathsf{WL}}[G](v_2,w_2)$ holds for all $(v_1,w_1),(v_2,w_2) \in E(G)$. If G has maximum degree 2 or contains a vertex of degree at most 1, then it is easy to see that G is either a cycle or isomorphic to a star $K_{1,h}$ for some $h \geq 0$ ($h = 0$ covers the special case that G consists of a single vertex).

▶ **Definition 5.** *Let H be a graph and $s \geq 1$. The* s-subdivision *of H is the graph $H^{(s)}$ obtained from H by replacing each edge with s parallel paths of length 2. Formally, $H^{(s)}$ is the graph with vertex set $V(H^{(s)}) \coloneqq V(H) \uplus (E(H) \times [s])$ and edge set*

$$E(H^{(s)}) \coloneqq \left\{ v(e,i) \;\middle|\; e \in E(H), v \in e, i \in [s] \right\}.$$

In the remaining case, G has maximum degree at least 3 and minimum degree 2. Then it is easy to see that G is one of the graphs from Theorem 4, or an s-subdivision of one of the graphs from Parts A and B for some $s \geq 1$, a cycle C_ℓ for some $\ell \geq 3$, or the complete graph on two vertices K_2.

Finally, if G is not connected, then it is isomorphic to the disjoint union of ℓ copies of one of its connected components for some $\ell \geq 2$, because all graphs listed above can be distinguished from each other by 2-WL. Actually, it can be checked that all of the graphs are even identified by 2-WL. Overall, this gives the following corollary.

▶ **Corollary 6.** *Let G be a directed planar graph such that $\{\{\chi^2_{\mathsf{WL}}[G](v,w), \chi^2_{\mathsf{WL}}[G](w,v)\}\} = \{\{\chi^2_{\mathsf{WL}}[G](v',w'), \chi^2_{\mathsf{WL}}[G](w',v')\}\}$ holds for all $(v,w),(v',w') \in E(G)$. Then 2-WL determines arc orbits on G. In particular, G is edge-transitive.*

4 Graphs Induced by a Single Edge Color

After considering planar graphs with a single edge color with respect to 2-WL, we now wish to analyze the 2-WL coloring of arbitrary planar graphs. Since, by [32], the algorithm 2-WL implicitly computes the decomposition of a graph into 3-connected components[1], understanding 2-WL on planar graphs essentially amounts to a study of 3-connected planar graphs. Hence, we restrict our attention to those.

4.1 Edge Types

Let G be a 3-connected planar graph and set $\chi \coloneqq \chi^2_{\mathsf{WL}}[G]$. To analyze the coloring χ, we focus on subgraphs induced by a single edge color. Towards this end, let $C_V \coloneqq C_V(G,\chi) = \{\chi(v,v) \mid v \in V(G)\}$ denote the set of *vertex colors*. Similarly, let $C_E \coloneqq C_E(G,\chi) = \{\chi(v,w) \mid vw \in E(G)\}$ be the set of *edge colors*. For $C \subseteq C_E$, we define the graph $G[C]$ with

$$V(G[C]) \coloneqq \{v_1, v_2 \mid \chi(v_1,v_2) \in C\} \quad \text{and} \quad E(G[C]) \coloneqq \{v_1v_2 \mid \chi(v_1,v_2) \in C\}.$$

In case $C = \{c_1,\dots,c_\ell\}$, we also write $G[c_1,\dots,c_\ell]$ instead of $G[\{c_1,\dots,c_\ell\}]$. Observe that $G[C]$ is defined as an undirected graph. However, it may be that $\chi(v_1,v_2) \neq \chi(v_2,v_1)$ holds for some $v_1v_2 \in E(G)$. Since this information turns out to be relevant in some cases, we always assume that $G[C]$ is equipped with an arc coloring where colors are inherited from χ.

As indicated, we are particularly interested in the case $C = \{c\}$ for a single color c. Observe that the ends of c-colored edges have the same vertex color, i.e., if $\chi(v_1,w_1) = \chi(v_2,w_2) = c$, then $\chi(v_1,v_1) = \chi(v_2,v_2)$ and $\chi(w_1,w_1) = \chi(w_2,w_2)$. This implies that $1 \leq |C_V(G[c],\chi)| \leq 2$. We say that $G[c]$ is *unicolored* if $|C_V(G[c],\chi)| = 1$. Otherwise, we say that $G[c]$ is *bicolored*.

[1] For the formal and quite technical definition of this notion, we refer to [32].

To analyze 2-WL on 3-connected planar graphs, we consider the graphs $G[c]$ for suitable edge colors $c \in C_E$. Towards this end, it turns out to be useful to group the graphs $G[c]$ according to the number of faces of each connected component of $G[c]$. Note that 2-WL detects connected components of graphs. More precisely, it holds that

- all connected components in $G[c]$ have the same size because 2-WL detects, for every $w \in V(G[c])$, the set of vertices reachable in the arc-colored graph (G, χ) from w via edges uv of color c (i.e., $\chi(u, v) = c$ or $\chi(v, u) = c$), and
- all connected components in $G[c]$ have the same multiset of vertex degrees in $G[c]$ since otherwise the vertex colors and, thus, also the arc colors would be different.

In combination, all connected components of $G[c]$ have the same number of vertices and edges and hence, by Euler's formula, they also have the same number of faces. We distinguish between three types in C_E.

Type I. For the first category, we consider those graphs $G[c]$ that have only one face. To be more precise, we say that $c \in C_E$ has *Type I* if $(G[c])[A]$ has a single face for every vertex set A of a connected component of $G[c]$. It is not difficult to see that $G[c]$ is isomorphic to a disjoint union of stars $K_{1,h}$ for $h \in [n]$.

Type II. For the second category, we consider those graphs $G[c]$ where every connected component has exactly two faces. Formally, we say that $c \in C_E$ has *Type II* if $(G[c])[A]$ has exactly two faces for every vertex set A of a connected component of $G[c]$. In this case, $G[c]$ is a disjoint union of cycles of the same length. Also, it is not difficult to see that every connected component of $G[c]$ is either a directed cycle (i.e., $\chi(v_1, v_2) \neq \chi(v_2, v_1)$ holds for every edge $v_1v_2 \in E(G[c])$), or an undirected cycle in which all vertices have the same color with respect to 2-WL, or an undirected cycle with two vertex colors that alternate along the cycle.

Type III. Finally, for the last category, we consider those graphs $G[c]$ where each connected component has at least three faces. Again, to be precise, we say that $c \in C_E$ has *Type III* if $(G[c])[A]$ has at least three faces for every vertex set A of a connected component of $G[c]$.

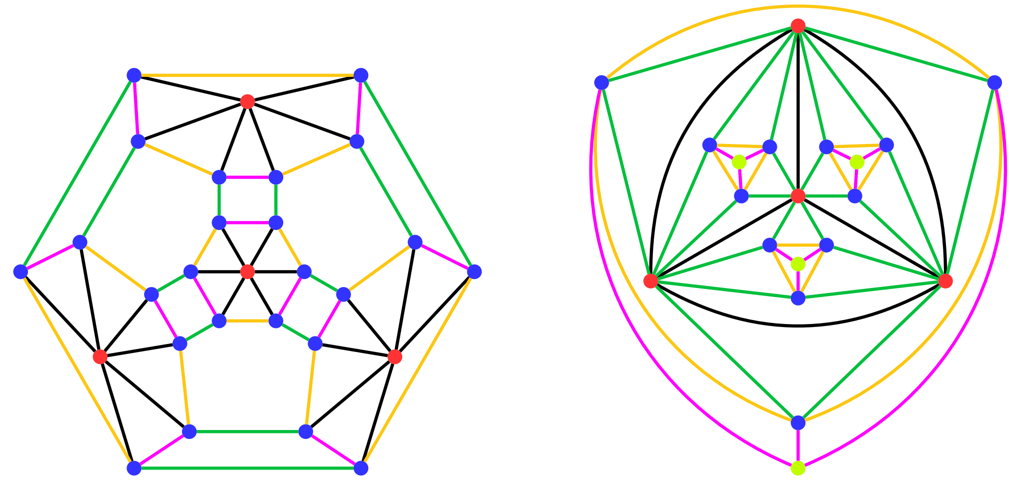
Also, we define the *type* of an edge $v_1v_2 \in E(G)$ as the type of its color $\chi(v_1, v_2)$ (note that the type of $\chi(v_1, v_2)$ is equal to the type of $\chi(v_2, v_1)$).

In the following, we derive several properties of the graphs $G[c]$ depending on the type of c, as well as properties of G depending on which types of edge colors occur. Towards this end, we also define the *type of* G as the maximal type of any edge color $c \in C_E$. So we say that G has Type III if there is some $c \in C_E$ of Type III. The graph G has Type II if there is some $c \in C_E$ of Type II, but there is no $c' \in C_E$ of Type III. Lastly, G has Type I if every $c \in C_E$ has Type I. Two example graphs are displayed in Figure 3.

4.2 Graphs of Fixing Number One

We start by investigating 3-connected planar graphs of Type I (see Figure 3a for an example). It turns out that such graphs have fixing number 1 with respect to 1-WL (after coloring all edges with their 2-WL colors), which in particular implies that 2-WL identifies all graphs of Type I. The proof is based on the following result.

▶ **Theorem 7** ([34, Lemma 23]). *Let G be a 3-connected planar graph and suppose $v_1, v_2, v_3 \in V(G)$ are pairwise distinct vertices lying on a common face of G. Then $\chi^1_{\mathsf{WL}}[G, v_1, v_2, v_3]$ is discrete.*



(a) A graph G of Type I. Each edge color $c \in C_E$ defines a graph $G[c]$ that is isomorphic to a disjoint union of stars. Individualizing an arbitrary blue vertex and performing 1-WL results in a discrete coloring. Hence, the graph is identified by 2-WL.

(b) A graph G of Type III. The edge colors black and green have Type III, yellow has Type II, and pink has Type I. Note that $G[c]$ is connected for every edge color c of Type III whereas the other edge colors induce non-connected subgraphs.

Figure 3 Two 3-connected planar graphs where all vertices and edges are colored by their 2-WL color. For visualization purposes, we only color edges and do not distinguish between potentially different colors of two arcs (v,w) and (w,v).

Here, $\chi^1_{\mathsf{WL}}[G, v_1, v_2, v_3]$ denotes the coloring computed by 1-WL after individualizing v_1, v_2, and v_3. For a vertex coloring $\lambda\colon V \to C$ and $v \in V$, we define $[v]_\lambda := \{w \in V \mid \lambda(v) = \lambda(w)\}$ as the color class of v and $\text{Singles}(\lambda) := \{v \in V \mid |[v]_\lambda| = 1\}$. For a graph G and vertices $v_1, \dots, v_\ell \in V(G)$, we define

$$\text{Singles}_G(v_1, \dots, v_\ell) := \text{Singles}(\chi^1_{\mathsf{WL}}[G, \chi^2_{\mathsf{WL}}[G], v_1, \dots, v_\ell]).$$

In other words, $\text{Singles}_G(v_1, \dots, v_\ell)$ is the set of all vertices appearing in a singleton color class after performing 1-WL on G where every pair is colored with its 2-WL-color, and where $v_1, \dots, v_\ell$ are individualized.

▶ **Lemma 8.** *Let G be a graph and let $v_1, \dots, v_\ell \in V(G)$ such that $\text{Singles}_G(v_1, \dots, v_\ell) = V(G)$. Also define $k := \max\{2, \ell + 1\}$. Then k-WL determines pair orbits in G.*

The following lemma provides a sufficient condition for a 3-connected planar graph to have fixing number 1.

▶ **Lemma 9.** *Let G be a 3-connected planar graph and suppose there is a face F such that every edge $e \in E(F)$ has Type I. Then there is a vertex $v \in V(G)$ such that $\text{Singles}_G(v) = V(G)$.*

Proof. Let H be a directed graph with vertex set $V(H) := V(G)$ and edge set

$$E(H) := \{(v,w) \mid vw \in E(G) \wedge \deg_{G[\chi^2_{\mathsf{WL}}[G](v,w)]}(v) = 1\}.$$

Intuitively speaking, we add a directed edge (v,w) to the graph H if w is the only neighbor of v reachable via an edge of color $\chi^2_{\mathsf{WL}}[G](v,w)$. In particular, if v is individualized, then w is also fixed after performing 1-WL.

For every edge $vw \in E(G)$ of Type I, it holds that $(v, w) \in E(H)$ or $(w, v) \in E(H)$. Hence, there are three vertices $v_1, v_2, v_3 \in V(G)$ lying on the face F of G such that $(v_1, v_2), (v_2, v_3) \in E(H)$ or $(v_1, v_2), (v_1, v_3) \in E(H)$.

Now consider the coloring $\lambda := \chi^1_{\mathsf{WL}}[G, \chi^2_{\mathsf{WL}}[G], v_1]$. Let $c := \chi^2_{\mathsf{WL}}[G](v_1, v_2)$. By definition of the edge set of H, it holds that v_2 is the only neighbor of v_1 which is adjacent via an edge of color c. Hence, $|[v_2]_\lambda| = 1$. By the same argument, $|[v_3]_\lambda| = 1$. Since v_1, v_2, v_3 all lie on the face F, it follows from Theorem 7 that λ is discrete. In other words, $\mathrm{Singles}_G(v_1) = V(G)$. ◀

▶ **Corollary 10.** *Let G be a 3-connected planar graph of Type I. Then there is a vertex $v \in V(G)$ such that $\mathrm{Singles}_G(v) = V(G)$. In particular, 2-WL determines pair orbits of G.*

We also record the following useful lemma, which is another consequence of Theorem 7.

▶ **Lemma 11.** *Let G be a 3-connected planar graph. Suppose $v_1, \ldots, v_\ell \in V(G)$ form a cycle in G, i.e., $v_1v_\ell \in E(G)$ and $v_iv_{i+1} \in E(G)$ holds for all $i \in [\ell-1]$. Let $w \in V(G) \setminus \{v_1, \ldots, v_\ell\}$. Then $\mathrm{Singles}_G(v_1, \ldots, v_\ell, w) = V(G)$.*

The lemma says that in a 3-connected planar graph G, it suffices to fix a cycle and one additional vertex in order to fix the entire graph. For example, this allows us to extract from the presence of certain 2-WL-detectable subgraphs bounds on the fixing number of the entire 3-connected planar graph G. Note that in the case that the fixing number in the subgraph is 1, Lemma 8 yields that 2-WL determines pair orbits in G.

4.3 Three Faces

We now turn to edge colors of Types II and III. For both types, it is not difficult to see that it is impossible to bound the fixing number by 1 in general. Instead, our focus here lies on investigating how edge colors of the corresponding type can appear within a 3-connected planar graph.

We first focus on edge colors of Type III (see Figure 3b for an example). Let G be a 3-connected planar graph and let $c \in C_E(G, \chi)$ be an edge color of Type III, where $\chi := \chi^2_{\mathsf{WL}}[G]$. By Corollary 6, the graph $G[c]$ is edge-transitive, which already puts severe restrictions on $G[c]$. However, as it turns out, due to the planarity and 3-connectedness of G, many edge-transitive graphs can in fact not appear as subgraphs $G[c]$. In the following, we classify the graphs $G[c]$ induced by an edge color c of Type III. The following lemma is a useful tool for the proof of our classification in Theorem 13, but also an interesting insight by itself, since it also yields restrictions on how different colors c, c' of Type III can appear together in one graph G.

▶ **Lemma 12.** *Let G be a 3-connected planar graph and let $c \in C_E(G, \chi^2_{\mathsf{WL}}[G])$ be an edge color of Type III. Then $G[c]$ is connected. Moreover, for every edge color $c' \in C_E(G, \chi^2_{\mathsf{WL}}[G])$ of Type III, it holds that $V(G[c]) \cap V(G[c']) \neq \emptyset$.*

Proof Idea. We focus on the first part of the lemma. The second part can be proved using similar arguments. Let $c \in C_E(G, \chi^2_{\mathsf{WL}}[G])$ be an edge color of Type III and suppose for simplicity that $G[c]$ is unicolored. Also let $A_1, \ldots, A_\ell$ denote the vertex sets of the connected components of $G[c]$ and suppose towards a contradiction that $\ell \geq 2$. Consider the auxiliary graph H with vertex set $V(H) := \{A_1, \ldots, A_\ell\}$ and edges A_iA_j whenever there is a path from a vertex $v_i \in A_i$ to a vertex $v_j \in A_j$ that is internally disjoint from $A_1 \cup \cdots \cup A_\ell$. Note that H is connected because G is connected. Also, we have $\chi^2_{\mathsf{WL}}[H](A_i, A_i) = \chi^2_{\mathsf{WL}}[H](A_j, A_j)$ for all $i, j \in [\ell]$, by exploiting known properties of 2-WL (see, e.g., [9, Theorem 3.1.11]). So H is 2-connected with [32, Theorem 3.15].

Now, consider the graph $G - A_1$. Since H is 2-connected, all sets $A_2, \dots, A_\ell$ are contained in the same connected component of $G - A_1$. Let X denote the vertex set of this component. We claim that $N_G(X) = A_1$. Note that $N_G(X) \neq \emptyset$ since G is connected. Let $v \in N_G(X)$ and let $w \in A_1$. Since $G[c]$ is unicolored, we conclude that $\chi^2_{\mathsf{WL}}[G](v,v) = \chi^2_{\mathsf{WL}}[G](w,w)$. Also, since $v \in N_G(X)$, there is a path from v to another vertex $v' \in A_2 \cup \dots \cup A_\ell$ that is internally disjoint from $A_1 \cup \dots \cup A_\ell$. Since this property can be detected by 2-WL, such a path also exists starting in w. But this is only possible if $w \in N_G(X)$. So overall, $N_G(X) = A_1$.

By contracting the set X to a single vertex, we obtain that adding a universal vertex to $(G[c])[A_1]$ still results in a planar graph. However, by Theorem 4, we have that $(G[c])[A_1]$ is isomorphic to one of the graphs from Figure 2a – 2g. But this gives a contradiction since it is not possible to add a universal vertex to any of those graphs while preserving planarity. So $\ell = 1$, which means that $G[c]$ is connected. ◀

▶ **Theorem 13.** *Let G be a 3-connected planar graph and let $c \in C_E(G, \chi^2_{\mathsf{WL}}[G])$ be of Type III. Then one of the following holds.*

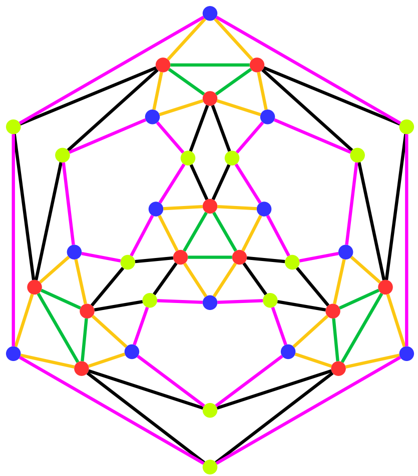
1. *$G[c]$ is bicolored and isomorphic to $K_{2,\ell}$ for some $\ell \geq 3$,*
2. *$G[c]$ is bicolored and isomorphic to a 2-subdivision of a cycle C_ℓ for some $\ell \geq 3$,*
3. *$G[c]$ is bicolored and isomorphic to a graph from Fig. 2h – 2j,*
4. *$G[c]$ is unicolored and isomorphic to a graph from Fig. 2a – 2g,*
5. *$G[c]$ is bicolored and isomorphic to a 1-subdivision of a graph from Fig. 2a – 2g, or*
6. *$G[c]$ is bicolored and isomorphic to a 2-subdivision of a graph from Fig. 2a – 2e.*

Observe that this classification is optimal in the sense that every graph listed in the theorem can actually appear as a graph $G[c]$ for some edge color c within a 3-connected planar graph. An easy way to see this is to take one of the graphs listed in the theorem, embed this graph H in the plane, place a fresh vertex v_F into every face F and connect it to all vertices lying on F. The resulting graph G is 3-connected and planar, and $H = G[c]$ for some edge color $c \in C_E(G, \chi^2_{\mathsf{WL}}[G])$.

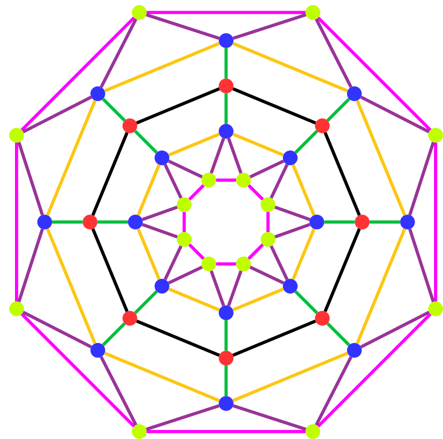
Also note that for a 3-connected planar graph G and an edge color $c \in C_E(G, \chi^2_{\mathsf{WL}}[G])$ of Type III, the automorphism group $\mathrm{Aut}(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(G[c])$. Indeed, every automorphism $\gamma \in \mathrm{Aut}(G)$ naturally restricts to an automorphism $\gamma|_{V(G[c])}$ of $G[c]$ since the coloring computed by 2-WL is invariant. This gives rise to a homomorphism $\varphi\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G[c])\colon \gamma \mapsto \gamma|_{V(G[c])}$. By Theorem 13 and Lemma 11, we obtain that $\mathrm{Singles}_G(w_1, \dots, w_\ell) = V(G)$ where $\{w_1, \dots, w_\ell\} = V(G[c])$. This implies that the kernel of φ is trivial, which implies that $\mathrm{Aut}(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(G[c])$.

5 Disjoint Unions of Cycles

In this section, we consider 3-connected planar graphs of Type II. Let G be a 3-connected planar graph and let $\chi := \chi^2_{\mathsf{WL}}[G]$ be the coloring computed by 2-WL. Suppose that G has Type II, i.e., there is an edge color $c \in C_E(G, \chi)$ of Type II, but there is no edge color of Type III. As before, we wish to understand in which ways edge colors of Type II can occur in G. More precisely, similarly to the case where G has Type III, our goal is to identify and classify connected subgraphs defined by few edge colors. Towards this end, we define three subcategories of edge colors of Type II. Let $c \in C_E(G, \chi)$ be of Type II. If $G[c]$ is unicolored and $\{(v,w) \mid \chi(v,w) = c\} \neq \{(v,w) \mid \chi(w,v) = c\}$ (i.e., $G[c]$ is a disjoint union of directed cycles), then we say that c has Type IIc. If c does not have Type IIc, but $G[c]$ is connected, then we say that c has Type IIb. If c does not have Type IIc, and $G[c]$ is not connected, then we say that c has Type IIa.



(a) A graph G of Type IIa where each edge color has Type IIa. The black and green edges induce a connected subgraph that is isomorphic to a parallel subdivision of a truncated tetrahedron.



(b) A graph G of Type IIb. The color black has Type IIb, yellow, violet and pink have Type IIa, and green has Type I. Note that $\mathrm{Aut}(G)$ is isomorphic to $(\mathrm{Aut}(G[\mathsf{black}])) \times \mathbb{Z}_2$ because, after individualizing all red vertices, we can only swap the "interior" and the "exterior" region of the black cycle.

Figure 4 Two 3-connected planar graphs of Type II. All vertices and edges are colored by their 2-WL color. For visualization purposes, we only color edges and do not distinguish between potentially different colors of two arcs (v, w) and (w, v).

Let us remark that the main point of the subtypes is to distinguish between edge colors c of Type II that induce non-connected subgraphs (Type IIa) and those that induce connected subgraphs (Type IIb). The reason why we additionally single out the directed cycles (Type IIc) is that the existence of an edge color of Type IIc almost always (i.e., with the exception of one graph family) implies that the graph has fixing number 1, because individualizing a single vertex in a directed cycle fixes all other vertices on the cycle as well. In particular, we can show that every graph that contains an edge color of Type IIc is identified by 2-WL.

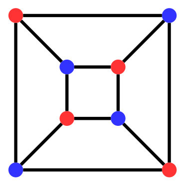
We also point out that the existence of an edge color of Type IIb immediately puts severe restrictions on the structure of G. For example, if $c \in C_E(G, \chi)$ is an edge of Type IIb, it can be shown that $\mathrm{Aut}(G)$ is isomorphic to a subgroup of $(\mathrm{Aut}(G[c])) \times \mathbb{Z}_2$ because, after individualizing all vertices of $G[c]$, an automorphism of G can only swap the "interior" and the "exterior" region of the cycle (see also Figure 4b).

Recall that the type of G is defined as the maximal type of any of its edge colors. We extend this definition to subtypes in the natural way. For example, G has Type IIb if there is an edge color $c \in C_E(G, \chi)$ of Type IIb, but no edge color of Type III or IIc. Two example graphs of Type II are given in Figure 4.

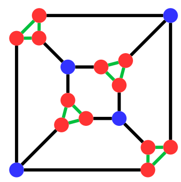
5.1 Directed Cycles

We start by analyzing graphs that contain an edge color c of Type IIc. As indicated above, this is a particularly well-behaved case because we can precisely classify those graphs that do not have fixing number 1. The *bipyramid (of order $m \geq 3$)* is the graph P_m^* with vertex set $V(P_m^*) := \{u_1, u_2\} \cup \{v_i \mid i \in [m]\}$ and edge set $E(P_m^*) = \{u_i v_j \mid i \in [2], j \in [m]\} \cup \{v_i v_{i+1} \mid i \in [\ell - 1]\} \cup \{v_1 v_\ell\}$.

▶ **Lemma 14.** *Let G be a 3-connected planar graph. Also let $c \in C_E(G, \chi_{\mathsf{WL}}^2[G])$ be an edge color of Type IIc. Then there is a vertex $v \in V(G)$ such that* $\mathrm{Singles}_G(v) = V(G)$*, or G is isomorphic to a bipyramid.*



(a) Bicolored cube.



(b) Chamfered tetrahedron.

Figure 5 A chamfered tetrahedron is obtained from a bicolored cube by truncating all red vertices.

▶ **Corollary 15.** *Let G be a 3-connected planar graph and suppose G contains an edge color of Type IIc. Then 2-WL determines pair orbits in G.*

Proof. If there is a vertex $v \in V(G)$ such that $\text{Singles}_G(v) = V(G)$, then 2-WL determines pair orbits in G by Lemma 8. Otherwise, G is a bipyramid and it is easy to check that 2-WL determines pair orbits in G. ◀

5.2 Connected Substructures

In the remainder of this section, we analyze graphs of Type IIa and, similarly as before, aim at finding highly regular connected substructures. (Observe that if G has Type IIb, then a witnessing edge color already provides such an object). Unfortunately, we need to allow for two further possible outcomes. First, we are again satisfied with finding a vertex $v \in V(G)$ such that $\text{Singles}_G(v) = V(G)$, which in particular implies that 2-WL determines pair orbits on G (see Lemma 8). As the other potential outcome, we consider *definable matchings*.

▶ **Definition 16.** *Let G be a graph and let $\chi := \chi^2_{\mathsf{WL}}[G]$. A color $c \in C_E(G,\chi)$* defines a matching *if for every $(v,w) \in \chi^{-1}(c)$, it holds that $\chi(v,v) \neq \chi(w,w)$, $\{v' \in V(G) \mid \chi(v',w) = c\} = \{v\}$, and $\{w' \in V(G) \mid \chi(v,w') = c\} = \{w\}$.*

Suppose there is some $c \in C_E(G,\chi)$ that defines a matching. Such a situation is generally beneficial since we can simply contract all edges of color c and move to a smaller graph. This operation neither changes the 2-WL coloring (see, e.g., [9, Theorem 3.1.11]) nor identification of the graph by 2-WL, as shown in the next lemma (see also [14]).

Let $c \in C_E(G,\chi)$ and let $A_1,\dots,A_\ell$ be the vertex sets of the connected components of $G[c]$. We define G/c as the graph obtained from contracting every set A_i to a single vertex. Formally, $V(G/c) := \{\{v\} \mid v \in V(G) \setminus V(G[c])\} \cup \{A_1,\dots,A_\ell\}$ and $E(G/c) := \{X_1X_2 \mid X_1,X_2 \in V(G/c), \exists v_1 \in X_1, v_2 \in X_2 \colon v_1v_2 \in E(G)\}$. We also define the pair coloring χ/c by setting $(\chi/c)(X_1,X_2) := \{\{\chi(v_1,v_2) \mid v_1 \in X_1, v_2 \in X_2\}\}$ for all $X_1,X_2 \in V(G/c)$.

▶ **Lemma 17.** *Let G be a graph and let $\chi := \chi^2_{\mathsf{WL}}[G]$. Also, let $c \in C_E(G,\chi)$ be an edge color that defines a matching. Suppose that 2-WL determines arc orbits (resp. pair orbits) on the arc-colored graph $(G/c,\lambda)$, where $\lambda(X_1,X_2) := (\chi/c)(X_1,X_2)$ for all $(X_1,X_2) \in A(G/c)$. Then 2-WL determines arc orbits (resp. pair orbits) on G.*

We now provide the main classification result of this section. We start by defining the graphs that appear in it. Let G be a 3-connected planar graph and let $W \subseteq V(G)$ be a set of vertices. We define H to be the graph obtained from G as follows. Let w be a vertex in W. First, subdivide each edge incident to w once. This gives $\ell := \deg_G(w)$ new vertices, which we call $w_1,\dots,w_\ell$ according to the unique cyclic order in any embedding of G. We then remove all edges ww_i and insert edges $w_iw_{(i+1) \bmod \ell}$ for $i \in [\ell]$, turning $w_1,\dots,w_\ell$ into a cycle. Each w_i inherits the color of w. We call these steps the *truncation* of w. One by one, we then truncate each vertex in W. (Note that their order does not matter for the final result.) The obtained graph is H; see Figure 5 for an example. Now, we can define the following graphs.

- For every 3-connected planar graph G, the *truncated* G is obtained from G by truncating all vertices.
- A *chamfered tetrahedron* (Figure 5b) is obtained from a bicolored cube (Figure 5a) by truncating all red vertices[2].
- A *chamfered cube* is obtained from a rhombic dodecahedron (Figure 2i) by truncating all blue vertices.
- A *chamfered octahedron* is obtained from a rhombic dodecahedron (Figure 2i) by truncating all red vertices.
- A *chamfered dodecahedron* is obtained from a rhombic triacontahedron (Figure 2j) by truncating all blue vertices.
- A *chamfered icosahedron* is obtained from a rhombic triacontahedron (Figure 2j) by truncating all red vertices.

Next, let G be a planar graph. We define the *C_4-subdivision of G* to be the graph obtained from G by replacing each edge $vw \in E(G)$ with four vertices $(vw,1)$, $(vw,2)$, $(vw,3)$, $(vw,4)$ and edges $(vw,1)(vw,2)$, $(vw,2)(vw,3)$, $(vw,3)(vw,4)$, $(vw,4)(vw,1)$ and $v(vw,1)$, $w(vw,3)$.

Also, for $m \geq 2$, we define the graph C_m^* with vertex set $V(C_m^*) := [m] \times [4]$ and edge set

$$\begin{aligned} E(C_m^*) := \quad & \{(i,1)(i,2),(i,2)(i,3),(i,3)(i,4),(i,4)(i,1) \mid i \in [m]\} \\ & \cup \{(i,3)(i+1,1) \mid i \in [m-1]\} \cup \{(m,3)(1,1)\}. \end{aligned}$$

Finally, for $h \geq 3$, we define $K_{2,h}^*$ to be the graph with vertex set $V(K_{2,h}^*) := \{u_1,u_2\} \uplus ([h] \times [4])$ and edge set

$$\begin{aligned} E(K_{2,h}^*) := \quad & \{(i,1)(i,2),(i,2)(i,3),(i,3)(i,4),(i,4)(i,1) \mid i \in [h]\} \\ & \cup \{u_1(i,1) \mid i \in [h]\} \cup \{u_2(i,3) \mid i \in [h]\}. \end{aligned}$$

Examples for the last three constructions can be found in Figure 6.

Let H be a graph and $f\colon E(H) \to \mathbb{N}$ be a function. The *f-subdivision of H* is the graph $H^{(f)}$ obtained from H by replacing each edge e with $f(e)$ parallel paths of length 2 (if $f(e)=0$, the edge e remains unaltered). Formally, $H^{(f)}$ is the graph with vertex set $V(H^{(f)}) := V(H) \uplus \{(e,i) \mid e \in E(H), i \in [f(e)]\}$ and edge set $E(H^{(s)}) := \{e \in E(H) \mid f(e)=0\} \cup \{v(e,i) \mid e \in E(H), v \in e, i \in [f(e)]\}$. A graph G is a *parallel subdivision of H* if there is a function $f\colon E(H) \to \mathbb{N}$ such that G is isomorphic to $H^{(f)}$.

▶ **Theorem 18.** *Let G be a 3-connected planar graph of Type IIa and let $\chi := \chi_{\mathsf{WL}}^2[G]$ be the coloring computed by 2-WL. Then one of the following options holds.*

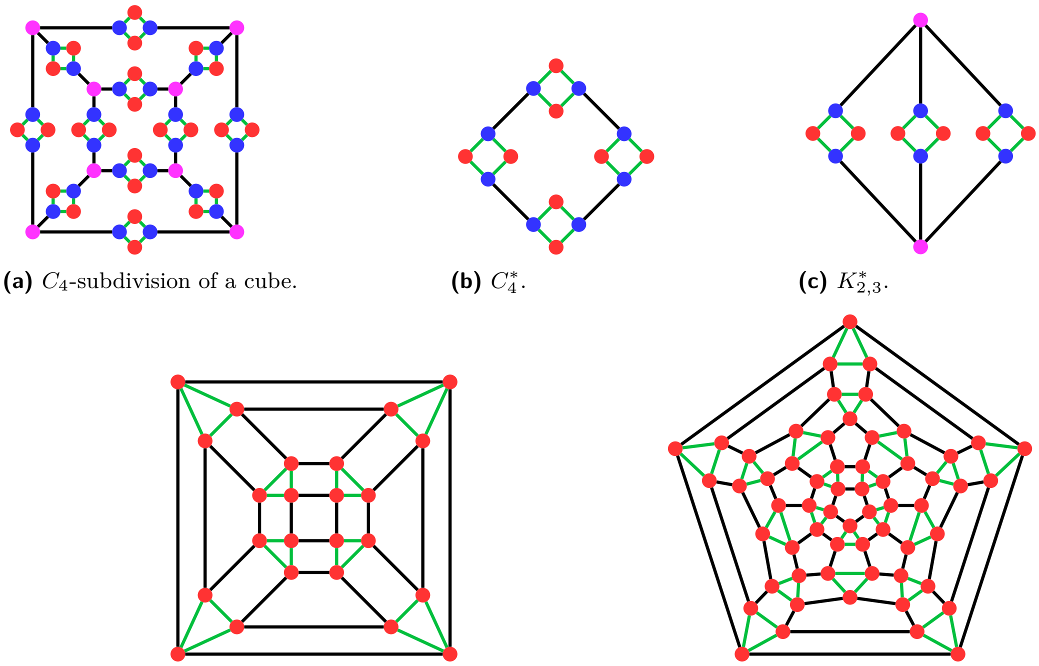
(A) *There is a vertex $v \in V(G)$ such that* $\mathrm{Singles}_G(v) = V(G)$*,*

(B) *there is an edge color $c \in C_E(G,\chi)$ that defines a matching, or*

(C) *there are colors $c,d \in C_E(G,\chi)$ such that $G[c,d]$ is isomorphic to a parallel subdivision of one of the following graphs:*

1. *a truncated tetrahedron, a truncated cube, a truncated octahedron, a truncated dodecahedron, a truncated icosahedron,*
2. *an m-side prism for $m \geq 3$,*

[2] The name *chamfered tetrahedron* comes from an alternative construction that obtains a chamfered tetrahedron by truncation of all edges of a tetrahedron.



(a) C_4-subdivision of a cube. **(b)** C_4^*. **(c)** $K_{2,3}^*$.

(d) Rhombicuboctahedron. **(e)** Rhombicosidodecahedron.

Figure 6 Examples for the constructions from the classification for graphs of Type IIa.

3. *a cuboctahedron (with two edge colors), a rhombicuboctahedron, a rhombicosidodecahedron,*
4. *a C_4-subdivision of one of the graphs from Figure 2a – 2e,*
5. *a C_m^* for $m \geq 2$,*
6. *a $K_{2,h}^*$ for $h \geq 3$,*
7. *a chamfered tetrahedron, a chamfered cube, a chamfered octahedron, a chamfered dodecahedron, or a chamfered icosahedron.*

▶ Remark 19. There are four Archimedean solids that are explicitly listed neither in Theorem 13 nor in Theorem 18. These are the truncated cuboctahedron, the truncated icosidodecahedron, the snub cube, and the snub dodecahedron. The graphs corresponding to these solids have fixing number 1 under 1-WL and hence, they implicitly appear in Theorem 18, Option A.

Note that for Option C from Theorem 18, using the same arguments as for edge colors of Type III, we obtain that $\mathrm{Aut}(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(G[c,d])$.

Overall, by combining Lemmas 9 and 14 and Theorems 13 and 18, we obtain that every 3-connected planar graph G satisfies one of the following options.

(A) There is some $v \in V(G)$ such that $\mathrm{Singles}_G(v) = V(G)$, which implies that 2-WL determines pair orbits of G by Lemma 8,

(B) there is an edge color $c \in C_E(G, \chi^2_{\mathsf{WL}}[G])$ that defines a matching, or

(C) there is a set $C \subseteq C_E(G, \chi^2_{\mathsf{WL}}[G])$ such that $|C| \leq 2$ and $G[C]$ is essentially a Platonic or Archimedean solid, or stems from a small number of infinite families of connected graphs.

Option C contains the graphs listed in Theorems 13 and 18, as well as the class of bipyramids from Lemma 14 and the class of cycles to cover graphs of Type IIb.

It remains an important open question whether 2-WL identifies every planar graph. With the structural insights from this paper, it now suffices to focus on Case C and, as explained above, the classification of the subgraphs $G[C]$ appearing in this case should be a crucial step to determining the WL dimension of planar graphs.

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