

# On Lower Bounds of Approximating Parameterized $k$ -Clique

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## Abstract

Given a simple graph  $G$  and an integer  $k$ , the goal of the  $k$ -CLIQUE problem is to decide if  $G$  contains a complete subgraph of size  $k$ . We say an algorithm approximates  $k$ -CLIQUE within a factor  $g(k)$  if it can find a clique of size at least  $k/g(k)$  when  $G$  is guaranteed to have a  $k$ -clique. Recently, it was shown that approximating  $k$ -CLIQUE within a constant factor is W[1]-hard [20].

We study the approximation of  $k$ -CLIQUE under the *Exponential Time Hypothesis* (ETH). The reduction of [20] already implies an  $n^{\Omega(\sqrt[6]{\log k})}$ -time lower bound under ETH. We improve this lower bound to  $n^{\Omega(\log k)}$ . Using the gap-amplification technique by expander graphs, we also prove that there is no  $k^{o(1)}$  factor FPT-approximation algorithm for  $k$ -CLIQUE under ETH.

We also suggest a new way to prove the *Parameterized Inapproximability Hypothesis* (PIH) under ETH. We show that if there is no  $n^{O(\frac{k}{\log k})}$ -time algorithm to approximate  $k$ -CLIQUE within a constant factor, then PIH is true.

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## 1 Introduction

In this paper, we study the  $k$ -CLIQUE problem: given a simple graph  $G$  and an integer  $k$ , decide whether  $G$  contains a complete subgraph of size  $k$ . As shown in [18],  $k$ -CLIQUE is one of the most classical NP-complete problems. Its inapproximability in the classical complexity regime has also been studied extensively [10, 4, 5, 14, 11, 15, 28]. Along a long line of research, it was proved that even approximating CLIQUE into a ratio of  $n^{1-\varepsilon}$  is NP-hard.

In recent years, the hardness of approximating  $k$ -CLIQUE has received increased attention in the parameterized complexity regime. When guaranteed that the maximum clique is of size  $k$ , people wonder if there is an algorithm which runs in  $f(k)n^{O(1)}$  time, and can find a clique of size at least  $k/g(k)$ , for some computable functions  $f$  and  $g$ . Such an algorithm is called a  $g(k)$ -FPT-approximation for the  $k$ -CLIQUE problem.



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Previously, [6] ruled out all  $g(k)$ -FPT-approximation algorithms of  $k$ -CLIQUE for any  $g(k) = o(k)$  under the *Gap Exponential Time Hypothesis* (Gap-ETH)<sup>1</sup>. They even showed that assuming Gap-ETH, it is impossible to find a clique of size  $\varepsilon(k)$  in  $f(k)n^{o(\varepsilon(k))}$  time. However, as Gap-ETH is such a strong hypothesis that it already gives a gap in hardness of approximation, it is still of great interest to prove the same lower bound under an assumption without an inherent gap. People may further wonder:

*Assuming ETH, does finding a clique of size  $\varepsilon(k)$  in  $k$ -CLIQUE require  $f(k)n^{\Omega(\varepsilon(k))}$  time?*

In a recent work [20], Lin showed that  $k$ -CLIQUE does not admit constant factor FPT-approximation algorithms unless  $W[1] = \text{FPT}$ . This was the first successful attempt to bypass Gap-ETH to prove the hardness of approximating  $k$ -CLIQUE. Unfortunately, [20] reduces a  $k$ -CLIQUE instance to a constant gap  $k'$ -CLIQUE<sup>2</sup> instance with  $k' = 2^{k^6}$ . As there is no  $f(k)n^{o(k)}$  time algorithm for  $k$ -CLIQUE assuming ETH, [20] actually ruled out  $f(k)n^{o(\sqrt[6]{\log k})}$  time constant approximation algorithms for  $k$ -CLIQUE under ETH. Comparing to [6], such a lower bound is still far beyond satisfaction, and it remains open to avoid the huge parameter blow-up in the gap-producing reduction to obtain a better lower bound.

The main result of this paper is

► **Theorem 1.** *Assuming ETH, for any constant  $c > 1$  and any computable function  $f$ , no algorithm can find a clique of size  $k/c$  in the  $k$ -CLIQUE problem in  $f(k)n^{o(\log k)}$  time.*

As an application, we combine our main result with the classical gap-amplification technique to rule out any  $k^{o(1)}$ -ratio FPT-approximation algorithms for  $k$ -CLIQUE under ETH. Let us not fail to mention that [24] recently proved similar lower bound based on a weaker hypothesis  $W[1] \neq \text{FPT}$ . Our result is formally stated as follows.

► **Corollary 2.** *Assuming ETH, for any  $g(k) = k^{o(1)}$ , the  $k$ -CLIQUE problem has no  $g(k)$ -FPT-approximation algorithm.*

We also study the relationship between the constant gap  $k$ -CLIQUE problem and the *parameterized inapproximability hypothesis* (PIH) [21], a central conjecture in parameterized complexity. Roughly speaking, PIH states that it is impossible to approximate a 2-CSP instance over  $k$  variables with alphabet  $[n]$  to a constant factor in FPT time. It is known in [10] that if PIH is true, then there is no FPT algorithm for constant gap  $k$ -CLIQUE. However, the reverse direction is not known yet. Furthermore, although PIH can be deduced from Gap-ETH via standard reductions in [7, 8], proving PIH under gap-free hypotheses (e.g. ETH,  $W[1] \neq \text{FPT}$ ) is still quite open and is believed to require a PCP-like theorem in parameterized complexity. We show that an almost-tight running time lower bound of constant gap  $k$ -CLIQUE could imply PIH. Our theorem suggests a new way to prove PIH under ETH, namely, by using constant gap  $k$ -CLIQUE as an intermediate problem. It is formally stated as follows.

► **Theorem 3.** *If there is no  $f(k)n^{O(\frac{k}{\log k})}$  time algorithm for constant gap  $k$ -CLIQUE, then PIH is true.*

<sup>1</sup> Gap-ETH states that no subexponential time algorithm can distinguish whether a 3SAT formula is satisfiable or every assignment satisfies at most  $1 - \varepsilon$  fraction of clauses for some  $\varepsilon > 0$ .

<sup>2</sup> Given  $k \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$  and a simple graph  $G$ , the constant gap  $k$ -CLIQUE problem is to decide whether  $G$  contains a  $K_k$  subgraph or  $G$  contains no  $K_{\varepsilon k}$  subgraph.

### 1.1 Our Techniques

**From 3SAT to gap  $k$ -CLIQUE.** Recall that the reduction in [20] consists of two steps. First, it reduces  $k$ -CLIQUE to  $k^2$ -VECTORSUM, while introducing a quadratic blow-up of the parameter. Next, it transforms  $k$ -VECTORSUM to CSP on  $k' = 2^{O(k^3)}$  variables  $\{x_{\vec{a}_1, \dots, \vec{a}_k} : \vec{a}_1, \dots, \vec{a}_k \in \mathbb{F}^k\}$ , and then to constant gap  $k'$ -CLIQUE. The two steps together cause the parameter to grow from  $k$  to  $2^{O(k^6)}$ .

To give a tighter lower bound of constant gap  $k$ -CLIQUE under ETH, we deal with the above two steps separately. First, we show a reduction directly from 3SAT to  $k$ -VECTORSUM, resulting in a tighter lower bound of  $k$ -VECTORSUM under ETH. Then, we give a more succinct reduction from  $k$ -VECTORSUM to CSP on  $k' = 2^{O(k)}$  variables  $\{x_{a_1, \dots, a_k} : a_1, \dots, a_k \in \mathbb{F}\}$ , and then to constant gap  $k'$ -CLIQUE. In our new reduction, the parameter blow-up throughout is only  $2^{O(k)}$ , leading to an  $n^{\Omega(\log k)}$  lower bound for constant gap  $k$ -CLIQUE.

Since the second step is more complicated, we will briefly introduce the ideas here. Given an  $k$ -VECTORSUM instance  $(V_1, \dots, V_k, \vec{t})$ , we build a CSP instance on variable set  $X = \{x_{a_1, \dots, a_k} : a_1, \dots, a_k \in \mathbb{F}\}$ . Each variable takes value in  $\mathbb{F}^m$  where  $m$  is the dimension specified by the  $k$ -VECTORSUM problem. In the yes-case, let  $\vec{v}_1 \in V_1, \dots, \vec{v}_k \in V_k$  be a solution that sum up to  $\vec{t}$ , we expect  $x_{a_1, \dots, a_k}$  to take the value  $\sum_{i \in [k]} a_i \vec{v}_i$ . Similar to [20], we want to make the following three types of tests:

- $\forall (a_1, \dots, a_k), (b_1, \dots, b_k) \in \mathbb{F}^k$ , test whether  $x_{a_1, \dots, a_k} + x_{b_1, \dots, b_k} = x_{a_1+b_1, \dots, a_k+b_k}$ .
- $\forall (a_1, \dots, a_k) \in \mathbb{F}^k, a \in \mathbb{F}$ , test whether  $x_{a_1, \dots, a_i+a, \dots, a_k} - x_{a_1, \dots, a_k} \in aV_i$ .
- $\forall (a_1, \dots, a_k) \in \mathbb{F}^k, a \in \mathbb{F}$ , test whether  $x_{a_1+a, \dots, a_k+a} - x_{a_1, \dots, a_k} = a\vec{t}$ .

If an assignment passes most of the linearity tests, then there must be vectors  $\vec{u}_1, \dots, \vec{u}_k \in \mathbb{F}^m$  such that  $x_{a_1, \dots, a_k} = \sum_{i \in [k]} a_i \vec{u}_i$  for most  $(a_1, \dots, a_k) \in \mathbb{F}^k$ . The second step is meant to guarantee that the selected vectors indeed come from the input. Finally we need the third step to check whether they sum up to  $\vec{t}$ .

Note that in our reduction from 3SAT to  $k$ -VECTORSUM, we require the dimension  $m$  to be at least  $\Omega(k \log n)$ . Thus in the CSP instance, we cannot simply leave the alphabet to be  $\mathbb{F}^m = n^{\Omega(k)}$ , which is too large. To reduce the dimension, we pick  $\ell = \Theta(k + \log n)$  matrices  $A_1, \dots, A_\ell \in \mathbb{F}^{k \times m}$  independently at random, and define a new CSP problem on variable set  $Y = \{y_{\vec{\alpha}, \vec{\beta}} : \vec{\alpha}, \vec{\beta} \in \mathbb{F}^k\}$ , where each  $y_{\vec{\alpha}, \vec{\beta}}$  is supposed to take the value

$$\begin{aligned}
 y_{\vec{\alpha}, \vec{\beta}} &= (\vec{\alpha}A_1x_{\vec{\beta}}, \dots, \vec{\alpha}A_\ellx_{\vec{\beta}}) \\
 &= (\vec{\alpha}A_1 \sum_{i \in [k]} \beta_i \vec{v}_i, \dots, \vec{\alpha}A_\ell \sum_{i \in [k]} \beta_i \vec{v}_i) \\
 &= \sum_{i \in [k], j \in [k]} \beta_i \alpha_j (A_1[j] \vec{v}_i, \dots, A_\ell[j] \vec{v}_i) \\
 &\triangleq \sum_{i \in [k], j \in [k]} \beta_i \alpha_j C_{i,j}.
 \end{aligned} \tag{1}$$

Now the alphabet size is only  $\mathbb{F}^\ell = 2^{O(k)} n^{O(1)}$ . With this idea in mind, we add local constraints to enforce that the assignment to  $Y$  is of the above quadratic form (in terms of  $\alpha_1, \dots, \alpha_k$  and  $\beta_1, \dots, \beta_k$ ), and then use locally decodable properties of quadratic polynomials to extract information about vectors  $\vec{v}_1, \dots, \vec{v}_k$ .

In a high level, our construction generalizes that of [20] by replacing the linear code with the Reed–Muller code based on quadratic polynomials.

**Expander graph production.** To obtain an FPT time lower bound for  $k$ -CLIQUE with  $k^{o(1)}$  gap, we apply the standard expander graph product technique. Starting from a constant gap  $k$ -CLIQUE instance, we amplify the gap using an expander graph  $H$  on vertex set  $[k]$ . The new instance contains  $k^t$  groups of vertices. Each group corresponds to a unique path of length- $t$  random walk on  $H$ , and forms an independent set of size  $n^t$ . A vertex in a group represents a length- $t$  sequence of vertices from the original instance. Two (sequences of) vertices are linked if and only if the vertices contained in them form a clique in the original instance. By properties of expander graphs, we get a  $k^t$ -CLIQUE instance with gap  $(\varepsilon')^t$  for some constant  $\varepsilon'$ . Take  $t = o(\log k)$ , we can rule out  $k^{o(1)}$ -ratio FPT-approximation algorithms for  $k$ -CLIQUE under ETH.

**From gap  $k$ -CLIQUE to PIH.** The proof that strong lower bound of constant gap  $k$ -CLIQUE implies PIH goes as follows. First, we reduce constant gap  $k$ -CLIQUE to constant gap  $k$ -BICLIQUE in the canonical way. Next, we use a combinatorial object called disperser to amplify the gap from a constant to  $\frac{k}{\log k}$ . The result then follows from the Kővári-Sós-Turán Theorem which states that every  $2k$ -vertex graph without a  $K_{\log k, \log k}$ -subgraph has at most  $O((2k)^{2 - \frac{1}{\log k}})$  edges.

## 1.2 Organization of the Paper

The paper is organized as follows. In Section 2, we put some preliminaries, including the definitions of problems, hypotheses, and some algebraic and combinatorial tools used in our proofs. In Section 3, we prove the ETH lower bound of constant gap  $k$ -CLIQUE. In Section 4, we show how to amplify the gap to rule out  $k^{o(1)}$ -ratio FPT-approximation algorithms for  $k$ -CLIQUE under ETH. In Section 5, we show how an almost-tight running time lower bound of constant gap  $k$ -CLIQUE implies PIH. Finally, in Section 6, we conclude with a few open questions.

## 2 Preliminaries

### 2.1 Problems

Here we list all the computational problems which are relevant to our paper.

- 3SAT. The input is a 3-CNF formula  $\varphi$  with  $m$  clauses on  $n$  variables. The goal is to decide whether there is a satisfying assignment for  $\varphi$ .
- CSP. The input of a constraint satisfaction problem is a set of variables  $X = \{x_1, \dots, x_n\}$  together with a family of constraints  $\{C_1, C_2, \dots, C_m\}$  and an alphabet  $\Sigma$ . For every  $i \in [m]$ ,  $C_i = (\vec{s}_i, R_i)$ , where  $\vec{s}_i = (x_{j_1}, \dots, x_{j_{\ell_i}})$  is an  $\ell_i$ -tuple of variables for some  $\ell_i \in [n]$ , and  $R_i \subseteq \Sigma^{\ell_i}$  indicates a restriction on valid assignments for those  $\ell_i$  variables. The goal is to find an assignment  $\sigma : X \rightarrow \Sigma$  such that for all  $i \in [m]$ ,  $\sigma(\vec{s}_i) \in R_i$ . We call  $n, m, q$  and  $|\Sigma|$  respectively the number of vertices, the number of clauses, the arity ( $= \max_{i \in [m]} \ell_i$ ), and the alphabet size of this CSP problem.
- $k$ -CLIQUE. The input is an undirected graph  $G = (V_1 \dot{\cup} \dots \dot{\cup} V_k, E)$  with  $n$  vertices divided into  $k$  disjoint groups. The goal is to decide whether we can pick one vertex from each group, such that they form a clique of size  $k$ .
- $k$ -BICLIQUE. The input is an undirected bipartite graph  $G = (V_1 \dot{\cup} \dots \dot{\cup} V_k, U_1 \dot{\cup} \dots \dot{\cup} U_k, E)$ , where  $n$  vertices are divided into  $2k$  disjoint groups. The goal is to decide whether we can pick one vertex from each group, such that they form a biclique  $K_{k,k}$ .

- DENSEST  $k$ -SUBGRAPH. The input is an undirected graph  $G = (V_1 \dot{\cup} \dots \dot{\cup} V_k, E)$  with  $n$  vertices divided into  $k$  disjoint groups. The goal is to pick one vertex from each group, such that they induce maximum number of edges.
- $k$ -VECTORSUM. The input consists of  $k$  groups of vectors  $V_1, \dots, V_k \subseteq \mathbb{F}^d$  together with a target vector  $\vec{t} \in \mathbb{F}^d$ , where  $\mathbb{F}$  is a finite field of constant size. The goal is to decide whether there exists  $\vec{v}_1 \in V_1, \dots, \vec{v}_k \in V_k$  such that  $\sum_{i=1}^k \vec{v}_i = \vec{t}$ . Throughout our paper we only need the version that  $\vec{t}$  equals to  $\vec{0}$ , and will omit it afterwards.

## 2.2 Hypotheses

Now we list some computational complexity hypotheses which are related to our results.

► **Hypothesis 4** (Exponential Time Hypothesis (ETH) [16, 17, 25]). *3SAT with  $n$  variables and  $m = O(n)$  clauses cannot be solved deterministically in  $2^{o(n)}$  time. Moreover, this holds even when restricted to formulae in which each variable appears in at most three clauses.*

Note that the original statement in [16] does not enforce the requirement that each variable appears in at most three clauses. For the restricted version, we first apply the Sparsification Lemma in [17], which implies that without loss of generality we can assume the number of clauses  $m = O(n)$ . Then we apply Tovey's reduction [25], which produces a 3SAT instance with at most  $3m + n = O(n)$  variables and each variable appears in at most three clauses. Thus the restricted version is equivalent to the original statement.

The next hypothesis is *Parameterized Inapproximability Hypothesis (PIH)*, a central conjecture in parameterized complexity. We state it in terms of inapproximability of DENSEST  $k$ -SUBGRAPH as follows.

► **Hypothesis 5** (Parameterized Inapproximability Hypothesis (PIH) [21]). *There exists a constant  $\varepsilon > 0$  such that DENSEST  $k$ -SUBGRAPH has no  $(1 + \varepsilon)$  factor FPT-approximation algorithm. In other words, no algorithm can distinguish the following two cases in  $f(k) \cdot n^{O(1)}$  time, for any computable function  $f$ :*

- (Completeness.) *There exist  $v_1 \in V_1, \dots, v_k \in V_k$  such that they form a clique.*
- (Soundness.) *For any  $v_1 \in V_1, \dots, v_k \in V_k$ , they induce only  $\binom{k}{2} / (1 + \varepsilon)$  edges.*

The factor  $(1 + \varepsilon)$  can be replaced by any constant larger than 1, and the conjecture remains equivalent. Note that the original statement of PIH in [21] says that DENSEST  $k$ -SUBGRAPH is W[1]-hard to approximate, but for our use, we choose a relaxed form which states that it has no constant ratio FPT-approximation algorithm, as in [12].

It is worth noting that the relationship between PIH and gap  $k$ -CLIQUE is not completely known yet. If for a graph the number of edges induced by  $k$  vertices is only  $\approx \varepsilon^2 \binom{k}{2}$ , it cannot have a clique of size  $> \varepsilon k$ . Thus, PIH implies  $k$ -CLIQUE does not admit constant ratio FPT-approximation algorithms. However, the other direction is not necessarily true (forbidding small clique does not imply low edge density), and it remains an important open problem that whether PIH holds if we assume  $k$ -CLIQUE is hard to approximate within any constant factor in FPT time [12].

## 2.3 Low Degree Test

Let  $\mathbb{F}$  be a field of prime cardinality. We say a function  $f$  is  $\delta$ -close to a function class  $\mathcal{F}$  if it is possible to modify at most  $\delta$  fraction of values of  $f$  such that the modified function lies in  $\mathcal{F}$ .

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- The canonical low degree test proposed in [23] can query a function  $f$  at  $d+2$  points, and
- accepts with probability 1 whenever  $f$  is a degree- $d$  polynomial,
  - rejects with probability at least  $\varepsilon > 0$  if  $f$  is not  $\delta$ -close to degree- $d$  polynomials, where  $\varepsilon, \delta$  are two constants.

Throughout our paper we will consider the function class  $\mathcal{F}$  to be vector-valued degree- $d$  polynomials, namely,

$$\mathcal{F} = \{(f_1, f_2, \dots, f_\ell) : \mathbb{F}^m \rightarrow \mathbb{F}^\ell \mid \forall i \in [\ell], f_i \text{ is a degree-}d \text{ multivariate polynomial}\}.$$

By slightly modifying the proof in [23], the low degree test can be easily generalized to vector-valued version, as formally stated below:

- **Lemma 6.** *Let  $d < |\mathbb{F}|/2$  and  $m \in \mathbb{N}$ . There is an algorithm which, by querying the function  $f = (f_1, f_2, \dots, f_\ell) : \mathbb{F}^m \rightarrow \mathbb{F}^\ell$  at  $d+2$  points,*
- *accepts with probability 1 whenever  $f$  lies in  $\mathcal{F}$ ,*
  - *rejects with probability at least  $\min(\delta/2, cd^{-2})$  if  $f$  is not  $\delta$ -close to  $\mathcal{F}$ , where  $c, \delta$  are two constants.*

*Moreover, the queries are generated by selecting  $\vec{x}, \vec{h} \in \mathbb{F}^m$  uniformly at random, and  $f$  is queried at  $\{\vec{x} + i\vec{h} \mid 0 \leq i < d+2\}$ .*

The proof is implicit in literature. We omit it here due to the page limitation. Readers can refer to our arxiv version <sup>3</sup> for a full proof.

### 2.4 Expander Graphs

Given a  $d$ -regular undirected graph  $G$  on  $n$  vertices, define its *normalized adjacency matrix* to be a matrix  $A$  where  $A_{ij}$  equals to the number of edges between  $(i, j)$  divided by  $d$ . Define

$$\lambda(G) = \max_{\|\vec{v}\|=1, \langle \vec{v}, \vec{1} \rangle = 0} \|A\vec{v}\|_2.$$

$G$  is an  $(n, d, \lambda)$ -expander if and only if  $\lambda(G) \leq \lambda$ , and we have the following two Lemmas.

- **Lemma 7** ([2]). *Let  $G$  be an  $(n, d, \lambda)$ -expander,  $\mathcal{B} \subseteq [n]$  be a set of size  $\leq \varepsilon n$  for some  $0 < \varepsilon < 1$  and  $(X_1, X_2, \dots, X_t)$  be a sequence of random variables denoting a length- $t$  random walk where the starting vertex is also picked uniformly at random. Then,*

$$\Pr[\forall 1 \leq i \leq t, X_i \in \mathcal{B}] \leq ((1 - \lambda)\sqrt{\varepsilon} + \lambda)^{t-1}.$$

- **Lemma 8** ([22]). *For some constants  $d \in \mathbb{N}$ ,  $\lambda < 1$  and for sufficiently large  $n$ , an  $(n, \lambda, d)$ -expander can be constructed in  $n^{O(1)}$  time.*

### 2.5 Disperser

- **Definition 9** (Disperser [9, 26, 27]). *For positive integers  $m, k, \ell, r \in \mathbb{N}$  and constant  $\varepsilon \in (0, 1)$ , an  $(m, k, \ell, r, \varepsilon)$ -disperser is a collection  $\mathcal{I}$  of  $k$  subsets  $I_1, \dots, I_k \subseteq [m]$ , each of size  $\ell$ , such that the union of any  $r$  different subsets from the collection has size at least  $(1 - \varepsilon)m$ .*

<sup>3</sup> <https://arxiv.org/abs/2111.14033>

Dispersers could be constructed efficiently by probabilistic methods, as in the following Lemma.

► **Lemma 10.** *For positive integers  $m, \ell, r \in \mathbb{N}$  and constant  $\varepsilon \in (0, 1)$ , let  $\ell = \lceil \frac{3m}{\varepsilon r} \rceil$  and let  $I_1, \dots, I_k$  be random  $\ell$ -subsets of  $[m]$ . If  $\ln k \leq \frac{m}{r}$  then  $\mathcal{I} = \{I_1, \dots, I_k\}$  is an  $(m, k, \ell, r, \varepsilon)$ -disperser with probability at least  $1 - e^{-m}$ .*

Due to the page limitation, we also put the proof into the Appendix of our arxiv version.

### 3 An Improved Lower Bound for Constant Gap $k$ -CLIQUE under ETH

#### 3.1 Reduction from 3SAT to $k$ -VECTORSUM

To prove Theorem 1, we first need an  $f(k) \cdot n^{\Omega(k)}$ -time lower bound for  $k$ -VECTORSUM under ETH. Previously, it is known that  $k$ -CLIQUE has no  $f(k) \cdot n^{o(k)}$ -time algorithms assuming ETH [8]. Combining this with the reduction from  $k$ -CLIQUE to  $\Theta(k^2)$ -VECTORSUM [1], we only have an  $f(k) \cdot n^{\Omega(\sqrt{k})}$ -time lower bound for  $k$ -VECTORSUM under ETH. It is an interesting question whether there is an FPT reduction from  $k$ -CLIQUE to  $k'$ -VECTORSUM with  $k' = O(k)$ . In this section, we give a reduction directly from 3SAT to  $k$ -VECTORSUM, which suits our purpose. Recall that in the ETH statement we can assume without loss of generality that each variable appears in at most 3 clauses, which is a key ingredient in our proof.

► **Theorem 11.** *There is a reduction which, for every integer  $k \in \mathbb{N}$ , and every 3SAT formula  $\varphi$  with  $m$  clauses and  $n$  variables such that each variable appears in at most 3 clauses, outputs a  $k$ -VECTORSUM instance  $\Gamma = (\mathbb{F}, d, V_1, \dots, V_k)$  with the following properties in  $2^{O(n/k)}$  time.*

- $\mathbb{F} = \mathbb{F}_5$ .
- $d = O(n)$ .
- For any  $i \in [k]$ , distinct  $\vec{u}, \vec{v} \in V_i$  and any  $a \in \mathbb{F}_5 \setminus \{0\}$ ,  $\vec{u} \neq a \cdot \vec{v}$ .
- For any  $i \in [k]$ , distinct  $\vec{u}, \vec{v}, \vec{w} \in V_i$  and any  $a \in \mathbb{F}_5 \setminus \{0\}$ ,  $\vec{u} - \vec{w} \neq a \cdot (\vec{v} - \vec{w})$ .
- (Completeness.) If  $\varphi$  is satisfiable, then there exists  $\vec{v}_1 \in V_1, \dots, \vec{v}_k \in V_k$  such that  $\sum_{i=1}^k \vec{v}_i = \vec{0}$ .
- (Soundness.) If  $\varphi$  is not satisfiable, then for any  $\vec{v}_1 \in V_1, \dots, \vec{v}_k \in V_k$ ,  $\sum_{i=1}^k \vec{v}_i \neq \vec{0}$ .

► **Remark 12.** Note if the size of the produced  $k$ -VECTORSUM instance  $N$  appears to be only  $2^{o(n/k)}$ , we can use brute force to solve it in  $N^k = 2^{o(n)}$  time, thus solve 3SAT in  $2^{o(n)}$  time. Therefore, we only need to consider the case  $N = 2^{\Theta(n/k)}$  without loss of generality, and in this case  $d = O(n) = O(k \log N)$ .

**Proof of Theorem 11.** Let  $C = \mathcal{C}_1 \dot{\cup} \dots \dot{\cup} \mathcal{C}_k$  be a partition of the clauses into  $k$  approximately equal-sized parts. We will let vectors in  $V_i$  represent partial satisfying assignments for  $\mathcal{C}_i$ , and use entries of vectors to check consistency of those partial assignments.

Define  $X$  to be the set of variables appearing in exactly two different parts and define  $Y$  to be the set of variables appearing in three different parts. Let  $d = |X| + 2|Y|$ , we associate one entry of vector to each variable  $x \in X$  and two entries to each variable  $y \in Y$ . In the following, we abuse notation a bit and use  $\vec{v}[x]$  to denote the entry in a vector  $\vec{v} \in \mathbb{F}^d$  associated to a variable  $x \in X$ , and use  $\vec{v}[y, 1], \vec{v}[y, 2]$  to denote the two entries associated to a variable  $y \in Y$ .

The construction of vector set  $V_i$  proceeds as follows. Let  $Z_i$  be the set of variables appearing in  $\mathcal{C}_i$ . For an assignment  $\tau : Z_i \rightarrow \{0, 1\}$  which satisfies all clauses in  $\mathcal{C}_i$ , we map it to a vector  $\vec{v} \in \mathbb{F}^d$  in the following way.

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Let  $x \in Z_i \cap X$  be a variable appearing in  $\mathcal{C}_{j_1}$  and  $\mathcal{C}_{j_2}$  ( $j_1 < j_2$ ),

- in case that  $\tau(x) = 0$ , set  $\vec{v}[x] = 0$ .
- in case that  $\tau(x) = 1$ , set  $\vec{v}[x] = 1$  if  $i = j_1$ , and set  $\vec{v}[x] = -1$  if  $i = j_2$ .

Let  $y \in Z_i \cap Y$  be a variable appearing in  $\mathcal{C}_{j_1}, \mathcal{C}_{j_2}$  and  $\mathcal{C}_{j_3}$  ( $j_1 < j_2 < j_3$ ),

- in case that  $\tau(y) = 0$ , set  $\vec{v}[y, 1] = 0$  and  $\vec{v}[y, 2] = 0$ .
- in case that  $\tau(y) = 1$ , set  $\vec{v}[y, 1] = 1$  and  $\vec{v}[y, 2] = 1$  if  $i = j_1$ ; set  $\vec{v}[y, 1] = -1$  and  $\vec{v}[y, 2] = 0$  if  $i = j_2$ ; and set  $\vec{v}[y, 1] = 0$  and  $\vec{v}[y, 2] = -1$  if  $i = j_3$ .

For the remaining entries of  $\vec{v}$  (which are associated to variables in  $(X \cup Y) \setminus Z_i$ ), set them to be 0.

It's easy to see the whole reduction runs in  $2^{O(n/k)}$  time, and the dimension  $d = O(n)$ .

Now we prove the third and the fourth properties.

For two distinct vectors  $\vec{u}, \vec{v} \in V_i$ , suppose  $\vec{u}[j] \neq \vec{v}[j]$ . It must be the case that one of them is 0 and the other is  $\pm 1$ . Thus they still differ after being multiplied by any  $a \in \mathbb{F}_5 \setminus \{0\}$ .

For three distinct vectors  $\vec{u}, \vec{v}, \vec{w} \in V_i$ , suppose  $\vec{u}[j] \neq \vec{w}[j]$ , then either  $\vec{v}[j] = \vec{w}[j]$  or  $\vec{v}[j] = \vec{u}[j]$ . In the former case,  $\vec{u}[j] - \vec{w}[j] \neq 0 = \vec{w}[j] - \vec{v}[j]$ , so they still differ after being multiplied by any  $a \in \mathbb{F} \setminus \{0\}$ . In the latter case, suppose  $\vec{u} - \vec{w} = a \cdot (\vec{w} - \vec{v})$ , then  $\vec{u}[j] - \vec{w}[j] = a \cdot (\vec{w}[j] - \vec{v}[j])$  will lead to  $a = -1$  and thus  $\vec{u} = \vec{v}$ , a contradiction. Therefore,  $\vec{u} - \vec{w} \neq a \cdot (\vec{w} - \vec{v})$  for any  $a \in \mathbb{F}_5 \setminus \{0\}$ .

Next follows the proof of completeness and soundness.

**Completeness.** If the 3SAT formula  $\varphi$  has a satisfying assignment  $\tau$ , we can pick one vector  $\vec{v}_i$  from each  $V_i$  according to the restriction of  $\tau$  on  $Z_i \cap (X \cup Y)$ . Let  $\vec{v} = \sum_{i=1}^k \vec{v}_i$  be the sum of picked vectors.

For a variable  $x \in X$ , let  $\mathcal{C}_{j_1}, \mathcal{C}_{j_2}$  ( $j_1 < j_2$ ) be the two clause parts in which  $x$  appears,

- in case that  $\tau(x) = 0$ ,  $\vec{v}[x] = 0$  since this entry equals to 0 in all vectors.
- in case that  $\tau(x) = 1$ ,  $\vec{v}[x] = 1 + (-1) = 0$  where 1 comes from  $\vec{v}_{j_1}[x]$  and  $-1$  comes from  $\vec{v}_{j_2}[x]$ .

For a variable  $x \in Y$ , let  $\mathcal{C}_{j_1}, \mathcal{C}_{j_2}, \mathcal{C}_{j_3}$  ( $j_1 < j_2 < j_3$ ) be the three clause parts in which  $x$  appears.

- in case that  $\tau(x) = 0$ ,  $\vec{v}[x, 1] = \vec{v}[x, 2] = 0$  since these entries equal to 0 in all vectors.
- in case that  $\tau(x) = 1$ ,  $\vec{v}[x, 1] = 1 + (-1) = 0$  where 1 comes from  $\vec{v}_{j_1}[x, 1]$  and  $-1$  comes from  $\vec{v}_{j_2}[x, 1]$ , and  $\vec{v}[x, 2] = 1 + (-1) = 0$  where 1 comes from  $\vec{v}_{j_1}[x, 1]$  and  $-1$  comes from  $\vec{v}_{j_3}[x, 1]$ .

**Soundness.** If the 3SAT formula  $\varphi$  has no satisfying assignments, any collection of partial assignments satisfying individual clause parts must be inconsistent on some variable in  $X \cup Y$ . For any  $\vec{v}_1 \in V_1, \dots, \vec{v}_k \in V_k$ , let  $\vec{v} = \sum_{i=1}^k \vec{v}_i$ .

Suppose assignments for a variable  $x \in X$  which appears in  $\mathcal{C}_{j_1}$  and  $\mathcal{C}_{j_2}$  are inconsistent, there must be one 0 and one  $\pm 1$  in  $\vec{v}_{j_1}[x]$  and  $\vec{v}_{j_2}[x]$ . Since this entry equals to 0 in all other vectors, it results that  $\vec{v}[x] \neq 0$ .

Suppose assignments for a variable  $x \in Y$  which appears in  $\mathcal{C}_{j_1}, \mathcal{C}_{j_2}$  and  $\mathcal{C}_{j_3}$  ( $j_1 < j_2 < j_3$ ) are inconsistent. If the values for  $x$  specified by  $\vec{v}_{j_1}$  and  $\vec{v}_{j_2}$  are inconsistent, there must be one 0 and one  $\pm 1$  in  $\vec{v}_{j_1}[x, 1]$  and  $\vec{v}_{j_2}[x, 1]$ , while in all other vectors this entry equals to 0, thus  $\vec{v}[x, 1] \neq 0$ . Otherwise the value for  $x$  specified by  $\vec{v}_{j_1}$  and  $\vec{v}_{j_3}$  must be inconsistent, there must be one 0 and one  $\pm 1$  in  $\vec{v}_{j_1}[x, 2]$  and  $\vec{v}_{j_3}[x, 2]$ , while in all other vectors this entry equals to 0, thus  $\vec{v}[x, 2] \neq 0$ .

Therefore,  $\sum_{i=1}^k \vec{v}_i \neq \vec{0}$  as desired. ◀



### 3.2 Reduction from $k$ -VECTORSUM to Constant Gap $k$ -CLIQUE

► **Theorem 13.** *There is an FPT reduction which, given as input a  $k$ -VECTORSUM instance  $\Gamma_0 = (\mathbb{F}, d, V_1, \dots, V_k)$  with the following properties:*

- $\mathbb{F} = \mathbb{F}_5$ ,
  - $d = O(k \log n)$  where  $n = \sum_{i=1}^k |V_i|$  denotes instance size,
  - for any  $i \in [k]$ , distinct  $\vec{u}, \vec{v} \in V_i$  and any  $a \in \mathbb{F}_5 \setminus \{0\}$ ,  $\vec{u} \neq a \cdot \vec{v}$ ,
  - for any  $i \in [k]$ , distinct  $\vec{u}, \vec{v}, \vec{w} \in V_i$  and any  $a \in \mathbb{F}_5 \setminus \{0\}$ ,  $\vec{u} - \vec{w} \neq a \cdot (\vec{v} - \vec{w})$ .
- outputs a  $k'$ -CLIQUE instance  $G = (V, E)$  such that
- $k' \leq c^k$  for some constant  $c$ ,
  - (Completeness.) if  $\Gamma_0$  is a yes-instance of  $k$ -VECTORSUM, then  $G$  contains a clique of size  $k'$ ,
  - (Soundness.) if  $\Gamma_0$  is a no-instance of  $k$ -VECTORSUM, then  $G$  doesn't contain a clique of size  $\varepsilon k'$  for some constant  $\varepsilon < 1$ .

The first step of the reduction involves  $\ell = 2k + 4 \log n$  matrices  $A_1, \dots, A_\ell \in \mathbb{F}^{k \times d}$ . For  $\alpha \in \mathbb{F}^k, v \in \mathbb{F}^d$ , define a bilinear function  $f(\alpha, v) = (\langle \alpha, A_1 v \rangle, \dots, \langle \alpha, A_\ell v \rangle) \in \mathbb{F}^\ell$ , where  $\langle \cdot, \cdot \rangle$  denotes inner product.

► **Lemma 14 ([20]).** *We can find  $\ell = 2k + 4 \log n$  matrices  $A_1, A_2, \dots, A_\ell$  in time polynomial in  $n, k$ , which satisfy the following properties:*

1. for any nonzero vector  $\vec{v} \in \mathbb{F}^d$ , there exists  $i \in [\ell]$  such that  $A_i \vec{v} \neq \vec{0}$ ,
2. for any  $i \in [k]$ , distinct  $\vec{u}, \vec{v} \in V_i$  and nonzero  $\alpha \in \mathbb{F}^k$ ,  $f(\alpha, \vec{u}) \neq f(\alpha, \vec{v})$ ,
3. for any  $i \in [k]$ , distinct  $\vec{u}, \vec{v}, \vec{w} \in V_i$  and  $\alpha, \alpha' \in \mathbb{F}^k$ ,  $f(\alpha, \vec{u}) + f(\alpha', \vec{v}) \neq f(\alpha + \alpha', \vec{w})$ .

The reduction then goes as follows. For every  $\alpha, \beta \in \mathbb{F}^k$ , we introduce a variable  $x_{\alpha, \beta}$  which takes value in  $\mathbb{F}^\ell$ . In the yes-case of  $k$ -VECTORSUM, there exists  $\vec{v}_1 \in V_1, \dots, \vec{v}_k \in V_k$  such that  $\sum_{i=1}^k \vec{v}_i = \vec{0}$ , and we expect  $x_{\alpha, \beta}$  to be  $f(\alpha, \sum_{i=1}^k \beta_i \vec{v}_i)$ , in other words,  $\sum_{i=1}^k \sum_{j=1}^k \alpha_i \beta_j (A_1[i] \vec{v}_j, \dots, A_\ell[i] \vec{v}_j)$  where  $A_w[i]$  indicates the  $i$ -th row of the  $w$ -th matrix. Note that  $f(\alpha, \sum_{i=1}^k \beta_i \vec{v}_i)$  is a degree-2 polynomial of  $\alpha$  and  $\beta$ .

For simplicity of notation, we will use  $e_i \in \mathbb{F}^k$  to denote the  $i$ -th unit vector, and use  $\mathbf{1} \in \mathbb{F}^k$  to denote the all-one vector.

We want to apply four types of tests on those variables:

1. Check whether  $x : \mathbb{F}^{2k} \rightarrow \mathbb{F}^\ell$  is a vector-valued degree-2 polynomial. This can be done by the low-degree test described in Lemma 6. For each  $\vec{\alpha}, \vec{\beta}, \vec{t}_1, \vec{t}_2 \in \mathbb{F}^k$ , check whether  $\{x_{\vec{\alpha} + i \vec{t}_1, \vec{\beta} + i \vec{t}_2} \mid 0 \leq i \leq 3\}$  are point values of a degree-2 polynomial. Each test is applied on 4 variables, and we say the arity of each such test is 4 in shorthand.
2. Check whether  $x$  which maps  $(\alpha, \beta)$  to  $x_{\alpha, \beta}$  is linear in both  $\alpha$  and  $\beta$ , i.e., whether  $x_{\alpha + \alpha', \beta} = x_{\alpha, \beta} + x_{\alpha', \beta}, \forall \alpha, \alpha', \beta \in \mathbb{F}^k$ , and  $x_{\alpha, \beta + \beta'} = x_{\alpha, \beta} + x_{\alpha, \beta'}, \forall \alpha, \beta, \beta' \in \mathbb{F}^k$ . The arity of each such test is 3.
3. For each  $u \in [k], \alpha, \beta \in \mathbb{F}^k$ , check whether  $x_{\alpha, \beta + e_u} - x_{\alpha, \beta} = f(\alpha, \vec{v}) \in \mathbb{F}^\ell$  for some  $\vec{v} \in V_u$ . The arity of each such test is 2.
4. For each  $\alpha, \beta \in \mathbb{F}^k$ , check whether  $x_{\alpha, \beta + \mathbf{1}} - x_{\alpha, \beta} = \vec{0}$ . The arity of each such test is 2.

**Construction of the Graph.** The vertices are divided into three types. Vertices in each type are further partitioned into groups, and each group forms an independent set:

**type-1** There are  $(|\mathbb{F}|^{2k})^2$  groups, each of which indicates a test of type 1, and consists of  $\leq |\mathbb{F}|^{4\ell}$  vertices corresponding to all satisfying assignments of the 4 variables in the test.

**type-2** There are  $2(|\mathbb{F}|^k)^3$  groups, each of which indicates a test of type 2, and consists of  $\leq |\mathbb{F}|^{2\ell}$  vertices corresponding to all satisfying assignments of the 3 variables in the test.

**type-3** There are  $|\mathbb{F}|^{2k}$  groups indexed by  $(\alpha, \beta) \in \mathbb{F}^{2k}$ , each consisting of  $\mathbb{F}^\ell$  vertices which correspond to  $\mathbb{F}^\ell$  possible assignments for variable  $x_{\alpha, \beta}$ .

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We make copies of vertices, so that the numbers of type-1 groups and type-2 groups are the same, and their sum equals to the number of type-3 groups. Specifically, the three types of vertices are made into  $2, |\mathbb{F}|^k, 4\mathbb{F}^{2k}$  copies, respectively. The total number of groups is therefore  $k' = 8|\mathbb{F}|^{4k}$ , while the total number of vertices is at most  $2|\mathbb{F}|^{4k+4\ell} + 2|\mathbb{F}|^{4k+2\ell} + 4|\mathbb{F}|^{4k+\ell} = |\mathbb{F}|^{O(k+\log n)}$ .

The edges are specified as follows:

1. A variable (type-3) vertex and a test (type-1/2) vertex are linked if and only if they specify the same assignment for the variable, or the test is irrelevant of that variable.
2. Two test vertices are linked if and only if they are consistent in all variables which appear in both tests.
3. Two variable vertices are linked if and only if the assignments specified by them can pass the above-mentioned third and fourth tests, or there is no such a test between them.

Two different copies of a same vertex are always linked.

**Proof of Completeness.** If  $\Gamma_0$  is a yes-instance of  $k$ -VECTORSUM, i.e., there exists  $\vec{v}_1 \in V_1, \dots, \vec{v}_k \in V_k$  such that  $\sum_{i=1}^k \vec{v}_i = \vec{0}$ , by letting  $x_{\alpha,\beta}$  take value  $f(\alpha, \sum_{i=1}^k \beta_i \vec{v}_i)$ , it's easy to see that such an assignment can pass all tests. Therefore, by picking a vertex from each group accordingly, one can obtain a clique of size  $k'$ .

**Proof of Soundness.** If  $\Gamma_0$  is a no-instance of  $k$ -VECTORSUM, we will prove that there is no clique of size  $\geq (1 - \varepsilon)k'$  in  $G$  for some small constant  $\varepsilon$ .

Prove by contradiction. If there is a clique of size  $\geq (1 - \varepsilon)k'$ , it must contain vertices from  $\geq (1 - 2\varepsilon)$  fraction of type-3 groups which represent variables, vertices from  $\geq (1 - 4\varepsilon)$  fraction of type-1 groups which represent low-degree tests, vertices from  $\geq (1 - 8\varepsilon)$  fraction of type-2 groups which represent linearity tests  $x_{\alpha+\alpha',\beta} = x_{\alpha,\beta} + x_{\alpha',\beta}$  and vertices from  $\geq (1 - 8\varepsilon)$  fraction of type-2 groups which represent linearity tests  $x_{\alpha,\beta+\beta'} = x_{\alpha,\beta} + x_{\alpha,\beta'}$ .

In the following, we will denote by  $\bar{x}_{\alpha,\beta}$  the assignment for  $x_{\alpha,\beta}$  specified by the clique. If no assignment for  $x_{\alpha,\beta}$  is specified, set  $\bar{x}_{\alpha,\beta}$  arbitrarily as long as it is consistent with all selected test vertices (it is always possible since the selected test vertices are themselves consistent). As almost all low-degree tests and linearity tests are passed, we have:

► **Lemma 15.** *If  $\varepsilon < \frac{c}{16}$  where  $c$  is the constant in Lemma 6, and there is a clique of size  $\geq (1 - \varepsilon)k'$ , then the function  $\pi(\alpha, \beta) = \bar{x}_{\alpha,\beta}$  ( $\alpha, \beta \in \mathbb{F}^k$ ) is  $9\varepsilon$ -close to a function on  $\alpha, \beta$  of the form*

$$\sum_{i=1}^k \sum_{j=1}^k \alpha_i \beta_j C_{i,j}$$

where  $C_{i,j} \in \mathbb{F}^l$  denotes the coefficient of the term  $\alpha_i \beta_j$ .

**Proof.** Plugging  $\delta = 9\varepsilon$  into Theorem 6, if  $\pi(\alpha, \beta)$  is not  $\delta$ -close to any degree-2 polynomial, at least  $\min(\delta/2, c\delta^{-2}) > 4\varepsilon$  fraction of the degree-2 polynomial tests will not be passed. However, when there is a clique of size  $\geq (1 - \varepsilon)k'$ , only  $\leq 4\varepsilon$  fraction of degree-2 polynomial tests may fail. Therefore,  $\pi$  must be  $9\varepsilon$ -close to a function of the form

$$\sum_{i=1}^k \sum_{j=1}^k \alpha_i \alpha_j A_{i,j} + \sum_{i=1}^k \sum_{j=1}^k \beta_i \beta_j B_{i,j} + \sum_{i=1}^k \sum_{j=1}^k \alpha_i \beta_j C_{i,j} + \sum_{i=1}^k D_i \alpha_i + \sum_{i=1}^k E_i \beta_i + F \quad (2)$$

where each coefficient is in  $\mathbb{F}^l$ . We only need to prove that if they also pass most of the linearity tests on  $\alpha$  and on  $\beta$ , all coefficients except  $C_{i,j}$  must be zeros.

Suppose  $A_{i,j} \neq 0$  for some  $i, j \in [k]$ . Regarding  $x_{\alpha+\alpha',\beta} - x_{\alpha,\beta} - x_{\alpha',\beta}$  as a function on  $3k$  variables  $\alpha_1 \dots \alpha_k, \alpha'_1 \dots \alpha'_k, \beta_1 \dots \beta_k$  and expand it by (2). There is a term  $A_{i,j}(\alpha_i \alpha'_j + \alpha'_i \alpha_j)$  which can never be canceled. The function  $x_{\alpha+\alpha',\beta} - x_{\alpha,\beta} - x_{\alpha',\beta}$  is not a zero function and by Schwartz-Zippel Lemma, only  $\frac{2}{|\mathbb{F}|}$  fraction of  $\alpha, \alpha', \beta$  can make it equal to zero. However, by union bound, there are at least  $1 - 8\varepsilon - 3 \cdot (9\varepsilon) > \frac{2}{|\mathbb{F}|}$  fraction of  $(\alpha, \alpha', \beta)$  such that  $\bar{x}_{\alpha,\beta}, \bar{x}_{\alpha',\beta}, \bar{x}_{\alpha+\alpha',\beta}$  are all specified value according to the clique, consistent with equation (2), and satisfying  $\bar{x}_{\alpha,\beta} + \bar{x}_{\alpha',\beta} = \bar{x}_{\alpha+\alpha',\beta}$ , a contradiction. Therefore,  $\forall i, j \in [k], A_{i,j} = 0$ .

Similar arguments can be applied for the other coefficients and are omitted here. The only term remaining is  $\sum_{i=1}^k \sum_{j=1}^k \alpha_i \beta_j C_{i,j}$  as desired. ◀

We call a variable  $x_{\alpha,\beta}$  *good* if the clique consists of a variable vertex of it and it satisfies  $\bar{x}_{\alpha,\beta} = \sum_{i=1}^k \sum_{j=1}^k \alpha_i \beta_j C_{i,j}$ . Let  $\varepsilon' = 2\varepsilon + 9\varepsilon$ , from the above we know at least  $(1 - \varepsilon')$  fraction of variables are good. Recall that from the construction of our graph and the property of clique, all arity-2 constraints between good variables are satisfied.

We call an  $\alpha \in \mathbb{F}^k$  *excellent* if  $\Pr_{\beta \in_R \mathbb{F}^k} [x_{\alpha,\beta} \text{ is good}] \geq \frac{2}{3}$ . By Markov's Inequality, at least  $1 - 3\varepsilon'$  fraction of  $\alpha$ 's are excellent.

► **Lemma 16.** *For each excellent  $\alpha$  and for each  $u \in [k]$ ,  $\sum_{i=1}^k \alpha_i C_{i,u} = f(\alpha, \vec{v})$  for some unique  $\vec{v} \in V_u$ .*

**Proof.** For any fixed  $u \in [k]$  and  $\alpha$ , the set of edges between the vertex of  $x_{\alpha,\beta}$  and the vertex of  $x_{\alpha,\beta+e_u}$  for all  $\beta$  can be partitioned into disjoint length-5 cycles  $\{x_{\alpha,\beta}, x_{\alpha,\beta+e_u}, \dots, x_{\alpha,\beta+4e_u}\}$  since the characteristic of  $\mathbb{F}$  is 5. Observe that if at most two variables in a 5-cycle are not good, there still exist two adjacent vertices that are good.

For an excellent  $\alpha$ , at most  $\frac{1}{3} \leq \frac{2}{5}$  fraction of variables  $x_{\alpha,\beta}$  are not good by definition, so there must exist a  $\beta \in \mathbb{F}^k$  such that  $x_{\alpha,\beta}$  and  $x_{\alpha,\beta+e_u}$  are both good variables. According to the definition of good variables, we have

$$\bar{x}_{\alpha,\beta+e_u} - \bar{x}_{\alpha,\beta} = \sum_{i=1}^k \alpha_i C_{i,u}$$

and from the third type of constraints between them we can infer that

$$\bar{x}_{\alpha,\beta+e_u} - \bar{x}_{\alpha,\beta} = f(\alpha, \vec{v})$$

for some  $\vec{v} \in V_u$ .

Additionally, since for  $\vec{v} \in V_u$ ,  $f(\alpha, \vec{v})$  are all different (the second property in Lemma 14), for an excellent  $\alpha$  and for all  $u \in [k]$ , we can deduce  $\sum_{i=1}^k \alpha_i C_{i,u} = f(\alpha, \vec{v})$  for some unique  $\vec{v} \in V_u$ . ◀

In the following when  $u \in [k]$  is fixed and omitted, we use  $\vec{v}_\alpha$  to denote the unique vector in  $V_u$  specified by an excellent  $\alpha$ . For an  $\alpha$  which is not excellent, we also assign a unique vector  $\vec{v}_\alpha \in V_u$  to it arbitrarily so that  $\vec{v}_\alpha$  is defined for all  $\alpha \in \mathbb{F}^k$ .

► **Lemma 17.** *If there is a clique of size  $\geq (1 - \varepsilon)k'$ , then for each  $u \in [k]$ ,*

$$C_{i,u} = (A_1[i]\vec{v}, \dots, A_\ell[i]\vec{v})$$

for some unique  $\vec{v} \in V_u$ , where  $A_w[i]$  indicates the  $i$ -th row of the  $w$ -th matrix.

**Proof.** Fix any  $u \in [k]$ , we first argue that at least  $\geq \frac{3}{10}$  fraction of  $\alpha$  specify the same  $\vec{v} \in V_u$ . It suffices to prove

$$\Pr_{\alpha, \alpha' \in_R \mathbb{F}^k} [\vec{v}_\alpha = \vec{v}_{\alpha'}] \geq \frac{3}{10}.$$

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If this is not true, by union bound  $3 \cdot \frac{3}{10} + 3(3\varepsilon') < 1$ , there must exist  $\alpha, \alpha'$  such that the following two conditions hold:

1.  $\alpha, \alpha', \alpha + \alpha'$  are all excellent.
2.  $\vec{v}_\alpha, \vec{v}_{\alpha'}, \vec{v}_{\alpha+\alpha'}$  are all different.

Now that they are all excellent, we have

$$\begin{aligned} f(\alpha + \alpha', \vec{v}_{\alpha+\alpha'}) &= \sum_{i=1}^k (\alpha_i + \alpha'_i) C_{i,u} \\ &= f(\alpha, \vec{v}_\alpha) + f(\alpha', \vec{v}_{\alpha'}), \end{aligned}$$

contradicting with the third property in Lemma 14.

Therefore, at least  $\frac{3}{10} - 3\varepsilon' > \frac{1}{|\mathbb{F}|}$  fraction of  $\alpha$  are all excellent and specify the same  $\vec{v} \in V_u$ . Now suppose  $C_{i,u} \neq (A_1[i]\vec{v}, \dots, A_\ell[i]\vec{v})$ , then there are at most  $\frac{1}{|\mathbb{F}|}$  fraction of  $\alpha \in \mathbb{F}^k$  making  $\sum_{i=1}^k \alpha_i C_{i,u} = f(\alpha, \vec{v})$  by Schwartz-Zippel Lemma. However, we have  $> \frac{1}{|\mathbb{F}|}$  fraction of such  $\alpha$ , a contradiction.  $\blacktriangleleft$

► **Lemma 18.** *If there is a clique of size  $\geq (1 - \varepsilon)k'$ , then there exists  $\vec{v}_1 \in V_1, \dots, \vec{v}_k \in V_k$  such that  $\sum_{i=1}^k \vec{v}_i = \vec{0}$ .*

**Proof.** From Lemma 17 we know that for each  $u \in [k]$ ,  $C_{i,u} = (A_1[i]\vec{v}, \dots, A_\ell[i]\vec{v})$  for some unique  $\vec{v} \in V_u$ . Thus  $\bar{x}_{\alpha,\beta}$  indeed equals to  $f(\alpha, \sum_{i=1}^k \beta_i \vec{v}_i)$  for every good variable  $x_{\alpha,\beta}$ .

The remaining proof is very similar to the proof of Lemma 16. Edges between the vertex of  $x_{\alpha,\beta}$  and the vertex of  $x_{\alpha,\beta+1}$  for all  $\beta \in \mathbb{F}^k$  can be divided into disjoint length-5 cycles  $\{x_{\alpha,\beta}, x_{\alpha,\beta+1}, \dots, x_{\alpha,\beta+4 \cdot 1}\}$  since the characteristic of  $\mathbb{F}$  is 5.

For an excellent  $\alpha$ , at most  $\frac{1}{3} \leq \frac{2}{5}$  fraction of variables  $x_{\alpha,\beta}$  are not good, so there must be two variables  $x_{\alpha,\beta}$  and  $x_{\alpha,\beta+1}$  which are both good. According to the definition of good variables, we have

$$\bar{x}_{\alpha,\beta+s} - \bar{x}_{\alpha,\beta} = f\left(\alpha, \sum_{i=1}^k \vec{v}_i\right)$$

and from the fourth type of constraints between them we can infer that

$$\bar{x}_{\alpha,\beta+1} - \bar{x}_{\alpha,\beta} = \vec{0}.$$

Therefore,  $f(\alpha, \sum_{i=1}^k \vec{v}_i) = \vec{0}$  for every excellent  $\alpha$ .

Suppose  $\sum_{i=1}^k \vec{v}_i \neq \vec{0}$ , then there exists  $i \in [\ell]$  such that  $A_i(\sum_{i=1}^k \vec{v}_i) \neq \vec{0}$  by the first property in Lemma 14. There are at most  $\frac{1}{|\mathbb{F}|}$  fraction of  $\alpha$  such that  $f(\alpha, \sum_{i=1}^k \vec{v}_i) = \vec{0}$  by Schwartz-Zippel Lemma, but we have  $\geq 1 - 3\varepsilon'$  fraction of excellent  $\alpha$ , a contradiction.  $\blacktriangleleft$

### 3.3 Putting Things Together

Combing Theorem 11 with Theorem 13, we obtain an improved lower bound for constant gap  $k$ -CLIQUE under ETH as follows.

► **Theorem 1.** *Assuming ETH, for any constant  $c > 1$  and any computable function  $f$ , no algorithm can find a clique of size  $k/c$  in the  $k$ -CLIQUE problem in  $f(k)n^{O(\log k)}$  time.*

**Proof.** Without loss of generality assume  $f$  is non-decreasing and unbounded. Given a 3SAT formula  $\varphi$  with  $n$  variables, each appearing in at most 3 clauses, we first run the reduction in Theorem 11 to produce a  $k$ -VECTORSUM instance of size  $2^{O(n/k)}$ , then run the reduction

in Theorem 13 to produce a constant gap  $c^k$ -CLIQUE instance of size at most  $c^k 2^{O(n/k)}$ . Using the graph product method, we can amplify the gap to any constant, while keeping the parameter  $k' = c^{O(k)}$  and instance size  $n' \leq c^{O(k)} 2^{O(n/k)}$ . Therefore, an  $f(k')n^{o(\log k')}$  time algorithm for constant gap  $k'$ -CLIQUE would lead to an algorithm for 3SAT in

$$f(k')(n')^{o(\log k')} \leq f(c^{O(k)})(c^{O(k)} 2^{O(n/k)})^{o(k)} \leq 2^{o(n)}$$

time, contradicting ETH. The last inequality holds because  $n$  can be sufficiently large compared to  $k$ . ◀

Below we present two remarks about possible extensions of our results on ETH lower bounds of gap  $k$ -CLIQUE.

► **Remark 19 (On higher degree Reed-Muller Codes).** It is natural to extend our idea to obtain a reduction from  $k$ -VECTORSUM to a CSP problem with  $< 2^{o(k)}$  variables using Reed-Muller code with larger degree polynomials. However, the reduction from CSP to  $k$ -CLIQUE has such an important property: when there is a clique of size  $\varepsilon k$  for some constant  $\varepsilon$ , the following two conditions hold:

1. A constant fraction of arity- $d$  constraints ( $d > 2$ ) are satisfied.
2. All arity-2 constraints between a constant fraction of variables are satisfied.

The second condition holds because an arity-2 constraint between two variables can be directly transformed into an edge between two vertices. If there is a large clique, it means all arity-2 constraints between the involved variables are simultaneously satisfied. However, if we use larger degree polynomials, the arity of constraints has to be larger, too. It cannot directly fit into the framework of  $k$ -CLIQUE. If this barrier can be broken, it may be possible to obtain reductions with an even smaller parameter blow-up using larger degree polynomials.

► **Remark 20 (On locally decodable codes).** Our reduction implicitly depends on the property of 2-query locally decodable code, that we could decode  $f(\alpha, v_i)$  for some fixed  $\alpha$  by querying only 2 positions. As pointed out in [13], 2-query locally decodable code has at least an exponential blow up. Hence our method is optimal in this sense. We could also consider how to remove this dependence.

#### 4 $k^{o(1)}$ -Ratio FPT Inapproximability of $k$ -CLIQUE under ETH

In this section, we show how to use expander graphs to amplify the gap efficiently, and how it leads to an improved inapproximability ratio of  $k$ -CLIQUE in FPT time under ETH. The idea comes from the classical technique used to amplify gap in the non-parameterized version of CLIQUE problem, which was proposed by Alon et al. [3].

► **Theorem 21.** *For some constants  $d \in \mathbb{N}, 0 < \lambda < 1$  and for any  $t \in \mathbb{N}$ , there is an algorithm which runs in  $O(k^2 d^{2t} |V|^{2t})$  time, on input an instance  $\Gamma = (V, E)$  of  $k$ -CLIQUE problem, outputs an instance  $\Gamma'$  of  $k'$ -CLIQUE problem such that*

- $k' = kd^{t-1}$ .
- (Completeness.) *If  $\Gamma$  has a  $k$ -clique, then  $\Gamma'$  has a  $k'$ -clique.*
- (Soundness.) *If  $\Gamma$  has no  $\varepsilon k$ -clique, then  $\Gamma'$  has no clique of size  $k'((1-\lambda)\sqrt{\varepsilon} + \lambda)^{t-1}$ .*

**Proof.** Let  $H$  be an  $(k, \lambda, d)$ -expander constructed from Lemma 8. We construct  $\Gamma'$  as follows. Each of the  $k' = kd^{t-1}$  groups in  $\Gamma'$  is associated with a unique path of length- $t$  random walk on  $H$ . We use  $(c_1, \dots, c_t)$  to name a group in  $\Gamma'$ , where each  $c_i \in [k]$  indicates a group in  $\Gamma$ .

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A vertex in  $\Gamma'$  is a length- $t$  sequence of vertices in  $\Gamma$ . Namely, there is a vertex  $(u_1, \dots, u_t)$  in the  $(c_1, \dots, c_t)$ -th group in  $\Gamma'$  if and only if each  $u_i$  belongs to group  $c_i$  in  $\Gamma$ . Therefore, the total number of vertices is at most  $kd^{t-1}|V|^t$  in  $\Gamma'$ .

A vertex  $(u_1, \dots, u_t)$  in group  $(c_1, \dots, c_t)$  is linked to a vertex  $(v_1, \dots, v_t)$  in group  $(d_1, \dots, d_t)$  if and only if

- $(c_1, \dots, c_t) \neq (d_1, \dots, d_t)$ ,
- and the vertices  $\{v_1, \dots, v_t, u_1, \dots, u_t\}$  form a clique in  $\Gamma$ .

The reduction runs in  $O(k^2 d^{2t} |V|^{2t})$  time by simply enumerating every pair of vertices in  $\Gamma'$  and checking if there is an edge between them.

**Completeness.** Let  $\{v_1, \dots, v_k\}$  be an  $k$ -clique in  $\Gamma$ . Then in group  $(c_1, \dots, c_t)$  in  $\Gamma'$ , we can pick the vertex  $(v_{c_1}, \dots, v_{c_t})$ . It's easy to see those vertices form an  $kd^{t-1}$ -clique.

**Soundness.** For any clique  $V$  in  $\Gamma'$ , let  $U$  be the collection of vertices in  $\Gamma$  which appear as part of the name of a vertex in  $V$ . Since  $V$  is a clique in  $\Gamma'$ , it follows by construction that  $U$  is also a clique in  $\Gamma$  and thus  $|U| \leq \varepsilon k$ . Recall that each  $(c_1, \dots, c_t)$  represents a length- $t$  random walk on  $H$ , and all those  $c_i$ 's lie in a set of size  $\leq \varepsilon k$  (which corresponds to the collection of groups that vertices in  $U$  belong to). By plugging  $n = k, |\mathcal{B}| \leq \varepsilon k$  into Lemma 7, the number of different groups that vertices in  $V$  belong to is bounded by  $kd^{t-1}((1 - \lambda)\sqrt{\varepsilon} + \lambda)^{t-1}$ , and so is  $|V|$ . ◀

For any function  $\delta(k) = o(1)$ , by setting  $t$  to be as large as some  $o(\log k)$ , we can make  $\varepsilon' = ((1 - \lambda)\sqrt{\varepsilon} + \lambda)^{t-1}$  smaller than  $k^{-\delta(k)}$  while keeping  $k' = kd^{t-1} \leq k^{O(1)}$ . Thus, by combining Theorem 1 and Theorem 21, we have the following corollary.

► **Corollary 2.** *Assuming ETH, for any  $g(k) = k^{o(1)}$ , the  $k$ -CLIQUE problem has no  $g(k)$ -FPT-approximation algorithm.*

### 5 From Constant Gap $k$ -CLIQUE to PIH

In this section we will show that strong lower bound of constant gap  $k$ -CLIQUE implies PIH. For simplicity of notation, we additionally define problems GAP-CLIQUE( $k, \ell$ ) and GAP-BICLIQUE( $k, \ell$ ) ( $k > \ell$ ), whose definitions are almost the same as  $k$ -CLIQUE and  $k$ -BICLIQUE, except that the soundness parameter is  $\ell$ . We have the following theorem:

► **Theorem 3.** *If GAP-CLIQUE( $k, \varepsilon k$ ) does not admit  $f(k) \cdot n^{O(\frac{k}{\log k})}$ -time algorithms for some  $0 < \varepsilon < 1$ , then PIH is true.*

The proof is relatively elementary and consists of three steps: first reduce constant gap  $k$ -CLIQUE to constant gap  $k$ -BICLIQUE, then use a *dispenser* to compress the soundness parameter to  $\frac{\log k}{k}$ , finally use the Kővári-Sós-Turán Theorem to show that in the soundness case, the density of every  $2k$ -vertex bipartite subgraph is low.

► **Lemma 22.** *If GAP-CLIQUE( $k, \varepsilon k$ ) does not admit  $f(k) \cdot n^{O(\frac{k}{\log k})}$ -time algorithms for some  $0 < \varepsilon < 1$ , then neither does GAP-BICLIQUE( $k, \frac{1+\varepsilon}{2}k$ ).*

**Proof.** We reduce a GAP-CLIQUE( $k, \varepsilon k$ ) instance  $G = (V_1 \dot{\cup} \dots \dot{\cup} V_k, E)$  to a GAP-BICLIQUE( $k, \frac{1+\varepsilon}{2}k$ ) instance  $G' = (U_1 \dot{\cup} \dots \dot{\cup} U_k, W_1 \dot{\cup} \dots \dot{\cup} W_k, E)$  as follows.

The vertex sets in each side are just copies of  $V$ , i.e.,  $U_i = W_i = V_i, \forall i \in [k]$ . Two vertices  $u \in U_i, w \in W_j$  where  $i \neq j$  are linked if and only if their corresponding vertices are linked in  $G$ , while two vertices  $u \in U_i, w \in W_i$  are linked iff they correspond to the same vertex in  $G$ .

The completeness case is obvious. In the soundness case, suppose we can pick  $\frac{1+\varepsilon}{2}k$  vertices from different parts of  $U$  and  $\frac{1+\varepsilon}{2}k$  vertices from different parts of  $W$  such that they form a biclique. Let the collection of picked vertices be  $S$ . Then there must be an index set  $\mathcal{I}$  of size  $\varepsilon k$  such that  $\forall i \in \mathcal{I}, (S \cap U_i \neq \emptyset) \wedge (S \cap W_i \neq \emptyset)$ . Moreover,  $|S \cap U_i| = |S \cap W_i| = 1$  by our construction of edges between  $U_i$  and  $W_i$ . Then consider the set  $\bigcup_{i \in \mathcal{I}} (S \cap U_i)$  which is of size at least  $\varepsilon k$ . The vertices in it must form a clique of size  $|\mathcal{I}|$  in the original graph  $G$ . ◀

► **Lemma 23.** *If  $\text{GAP-BICLIQUE}(k, \varepsilon k)$  does not admit  $f(k) \cdot n^{O(\frac{k}{\log k})}$ -time algorithms for some constant  $0 < \varepsilon < 1$ , then for any constant  $0 < c < 1$ , no algorithm can solve  $\text{GAP-BICLIQUE}(k, c \log k)$  in  $f(k) \cdot n^{O(1)}$  time.*

**Proof.** Given a  $\text{GAP-BICLIQUE}(k, \varepsilon k)$  instance  $G = (U_1 \dot{\cup} \dots \dot{\cup} U_k, W_1 \dot{\cup} \dots \dot{\cup} W_k, E)$ , let  $\ell = \lceil \frac{3k}{\varepsilon c \log k} \rceil$  and let  $\mathcal{I} = (I_1, \dots, I_k)$  be a  $(k, k, \ell, c \log k, 1 - \varepsilon)$ -dispenser. Since the size of  $\mathcal{I}$  is independent of  $n$ , we can deterministically enumerate all possible  $\mathcal{I}$  to find a valid one in  $f(k)$  time. The existence of such a dispenser is guaranteed by Lemma 10. We construct a new  $\text{GAP-BICLIQUE}(k, c \log k)$  instance  $G' = (U'_1 \dot{\cup} \dots \dot{\cup} U'_k, W'_1 \dot{\cup} \dots \dot{\cup} W'_k, E)$  as follows.

The groups of vertices in  $G'$  correspond to the combination of groups in  $G$  according to the dispenser. Specifically, for  $1 \leq i \leq k$ , let  $I_i = \{i_1, \dots, i_\ell\}$ , then each vertex in  $U'_i$  will correspond to a tuple of vertices  $(u_{i_1}, \dots, u_{i_\ell})$  where  $u_{i_j}$  comes from  $U_{i_j}$  in  $G$  for all  $1 \leq j \leq \ell$ . The construction of right vertices  $W'_1, \dots, W'_k$  is similar. The size of the new instance is therefore at most  $n^\ell = n^{O(\frac{k}{\log k})}$ .

An edge between a left vertex  $(u_{i_1}, \dots, u_{i_\ell})$  and a right vertex  $(w_{j_1}, \dots, w_{j_\ell})$  exists if and only if the vertices  $\{u_{i_1}, \dots, u_{i_\ell}, w_{j_1}, \dots, w_{j_\ell}\}$  form a biclique  $K_{\ell, \ell}$  in  $G$ .

The completeness case is still obvious, and we focus on the soundness case. Prove by contradiction, if there exists  $c \log k$  vertices from different groups of  $U'$  and  $c \log k$  vertices from different groups of  $W'$  which form a biclique  $K_{c \log k, c \log k}$ , let  $S$  be the collection of vertices which appear as part of one of the  $2c \log k$  tuples. For  $1 \leq i \leq k$ , arbitrarily pick one vertex from each  $S \cap U_i, S \cap W_i$  if not empty, then we claim that the resulting collection must be a biclique of size  $\geq \varepsilon k$  on both sides. The promise of biclique is from our construction, while the size is guaranteed by properties of the dispenser.

Therefore, an  $f(k) \cdot n^{O(1)}$  time algorithm for the  $\text{GAP-BICLIQUE}(k, c \log k)$  problem would lead to an  $f(k) \cdot \left(n^{O(\frac{k}{\log k})}\right)^{O(1)} = f(k) \cdot n^{O(\frac{k}{\log k})}$  time algorithm for  $\text{GAP-BICLIQUE}(k, \varepsilon k)$  problem. This completes the proof. ◀

► **Theorem 24** (Kővári-Sós-Turán, [19]). *For any graph  $G$  on  $n$  vertices, if  $G$  does not contain  $K_{a,a}$  as a subgraph, then  $G$  has at most  $O(n^{2-1/a})$  edges.*

**Proof of Theorem 3.** By plugging in  $a = c \log k$  for some sufficiently small constant  $0 < c < 1$  such that the  $O(k^{2-1/(c \log k)})$  in Theorem 24 is no more than  $\varepsilon' k^2$  for some constant  $0 < \varepsilon' < 1$ , Theorem 24 and Lemma 23 imply that if  $\text{GAP-BICLIQUE}(k, \varepsilon k)$  does not admit  $f(k) \cdot n^{O(\frac{k}{\log k})}$ -time algorithms, then no FPT algorithm can distinguish the following cases for a  $k$ -BICLIQUE instance  $G = (U, W, E)$  with

- (Completeness.) there exists  $u_1 \in U_1, \dots, u_k \in U_k, w_1 \in W_1, \dots, w_k \in W_k$  such that they form a biclique  $K_{k,k}$ .
- (Soundness.) for all  $u_1 \in U_1, \dots, u_k \in U_k, w_1 \in W_1, \dots, w_k \in W_k$ , the vertex set  $\{u_1, \dots, u_k, v_1, \dots, v_k\}$  induce a subgraph with at most  $O(k^{2-1/(c \log k)}) \leq \varepsilon' k^2$  edges for some constant  $0 < \varepsilon' < 1$ .

We link all pairs of vertices which are on the same side but not in the same group. In the completeness case, we can find  $2k$  vertices from distinct groups such that they form a clique and thus the number of edges induced is  $\binom{2k}{2}$ . In the soundness case, the number of edges induced by  $2k$  vertices from distinct groups is at most  $\varepsilon' k^2 + 2\binom{k}{2} < \varepsilon'' \binom{2k}{2}$  for some  $0 < \varepsilon'' < 1$ .

At last, by Lemma 22, if  $\text{GAP-CLIQUE}(k, \varepsilon k)$  does not admit  $f(k) \cdot n^{O(\frac{k}{\log k})}$ -time algorithms for some  $0 < \varepsilon < 1$ , then neither does  $\text{GAP-BICLIQUE}(k, \frac{1+\varepsilon}{2}k)$ , hence no FPT-algorithm can approximate  $\text{DENSEST } k\text{-SUBGRAPH}$  to an  $\varepsilon''$  factor. ◀

## 6 Conclusion

In this paper, we provide a tighter ETH-lower bound for constant gap  $k$ -CLIQUE by replacing the Hadamard code used in [20] by the Reed-Muller Code with degree-2 polynomials. We use gap amplification techniques by expander graphs to rule out  $k^{o(1)}$ -ratio FPT-approximation algorithms for  $k$ -CLIQUE under ETH. We also study the relationship between the constant gap  $k$ -CLIQUE problem and PIH. We show that almost tight lower bounds for constant gap  $k$ -CLIQUE can imply PIH.

A natural open question is whether we can derive such a lower bound for constant gap  $k$ -CLIQUE under ETH. Formally, it is stated as follows:

**Question 1.** Assuming ETH, does constant gap  $k$ -CLIQUE admit an algorithm in  $f(k) \cdot n^{O(\frac{k}{\log k})}$  time?

It is also worth noting that assuming ETH, there is no  $2^{o(n)}$ -time algorithm for non-parameterized MAX-CLIQUE problem on  $n$ -vertex graphs. Hence, it is natural to ask whether our technique can be analogously applied to non-parameterized MAX-CLIQUE to obtain tight lower bounds:

**Question 2.** Assuming ETH, does constant gap MAX-CLIQUE admit an algorithm in  $2^{o(n)}$  time?

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## References

- 1 Amir Abboud, Kevin Lewi, and Ryan Williams. Losing weight by gaining edges. In *European Symposium on Algorithms*, pages 1–12. Springer, 2014.
- 2 Miklós Ajtai, János Komlós, and Endre Szemerédi. Deterministic simulation in LOGSPACE. In Alfred V. Aho, editor, *Proceedings of the 19th Annual ACM Symposium on Theory of Computing, 1987, New York, New York, USA*, pages 132–140. ACM, 1987. doi:10.1145/28395.28410.
- 3 Noga Alon, Uriel Feige, Avi Wigderson, and David Zuckerman. Derandomized graph products. *Computational Complexity*, 5(1):60–75, 1995.
- 4 Mihir Bellare, Shafi Goldwasser, Carsten Lund, and Alexander Russell. Efficient probabilistically checkable proofs and applications to approximations. In *Proceedings of the twenty-fifth annual ACM symposium on Theory of computing*, pages 294–304, 1993.
- 5 Mihir Bellare and Madhu Sudan. Improved non-approximability results. In *Proceedings of the twenty-sixth annual ACM symposium on Theory of computing*, pages 184–193, 1994.
- 6 Parinya Chalermsook, Marek Cygan, Guy Kortsarz, Bundit Laekhanukit, Pasin Manurangsi, Danupon Nanongkai, and Luca Trevisan. From gap-eth to fpt-inapproximability: Clique, dominating set, and more. In Chris Umans, editor, *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, 2017*, pages 743–754. IEEE Computer Society, 2017. doi:10.1109/FOCS.2017.74.



- 7 Jianer Chen, Xiuzhen Huang, Iyad A. Kanj, and Ge Xia. Linear fpt reductions and computational lower bounds. In *Proceedings of the Thirty-Sixth Annual ACM Symposium on Theory of Computing*, STOC '04, pages 212–221, New York, NY, USA, 2004. Association for Computing Machinery. doi:10.1145/1007352.1007391.
- 8 Jianer Chen, Xiuzhen Huang, Iyad A. Kanj, and Ge Xia. Strong computational lower bounds via parameterized complexity. *J. Comput. Syst. Sci.*, 72(8):1346–1367, 2006. doi:10.1016/j.jcss.2006.04.007.
- 9 Aviad Cohen and Avi Wigderson. Dispersers, deterministic amplification, and weak random sources. In *30th Annual Symposium on Foundations of Computer Science*, pages 14–19. IEEE Computer Society, 1989.
- 10 Uriel Feige, Shafi Goldwasser, Laszlo Lovász, Shmuel Safra, and Mario Szegedy. Interactive proofs and the hardness of approximating cliques. *Journal of the ACM (JACM)*, 43(2):268–292, 1996.
- 11 Uriel Feige and Joe Kilian. Two-prover protocols—low error at affordable rates. *SIAM Journal on Computing*, 30(1):324–346, 2000.
- 12 Andreas Emil Feldmann, Karthik C. S., Euiwoong Lee, and Pasin Manurangsi. A survey on approximation in parameterized complexity: Hardness and algorithms. *Algorithms*, 13(6):146, 2020.
- 13 Oded Goldreich, Howard J. Karloff, Leonard J. Schulman, and Luca Trevisan. Lower bounds for linear locally decodable codes and private information retrieval. *Comput. Complex.*, 15(3):263–296, 2006. doi:10.1007/s00037-006-0216-3.
- 14 Shafi Goldwasser. Introduction to special section on probabilistic proof systems. *SIAM Journal on Computing*, 27(3):737, 1998.
- 15 Johan Hastad. Clique is hard to approximate within  $n^{1-\epsilon}$ . In *Proceedings of 37th Conference on Foundations of Computer Science*, pages 627–636. IEEE, 1996.
- 16 Russell Impagliazzo and Ramamohan Paturi. On the complexity of k-sat. *Journal of Computer and System Sciences*, 62:367–375, 2001.
- 17 Russell Impagliazzo, Ramamohan Paturi, and Francis Zane. Which problems have strongly exponential complexity? *Journal of Computer and System Sciences*, 63(4):512–530, 2001.
- 18 R. M. Karp. Reducibility among combinatorial problems. In *Proceedings of a symposium on the Complexity of Computer Computations, held March 20-22, 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York.*, pages 85–103, 1972.
- 19 Tamás Kővári, Vera T. Sós, and Paul Turán. On a problem of k. zarankiewicz. *Colloquium Mathematicum*, 3:50–57, 1954.
- 20 Bingkai Lin. Constant approximating k-clique is w[1]-hard. In Samir Khuller and Virginia Vassilevska Williams, editors, *STOC '21: 53rd Annual ACM SIGACT Symposium on Theory of Computing, Virtual Event, Italy, June 21-25, 2021*, pages 1749–1756. ACM, 2021. doi:10.1145/3406325.3451016.
- 21 Daniel Lokshantov, M. S. Ramanujan, Saket Saurabh, and Meirav Zehavi. Parameterized complexity and approximability of directed odd cycle transversal. In Shuchi Chawla, editor, *Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, Salt Lake City, UT, USA, January 5-8, 2020*, pages 2181–2200. SIAM, 2020.
- 22 Omer Reingold, Salil P. Vadhan, and Avi Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders and extractors. In *41st Annual Symposium on Foundations of Computer Science, FOCS 2000, 12-14 November 2000, Redondo Beach, California, USA*, pages 3–13. IEEE Computer Society, 2000. doi:10.1109/SFCS.2000.892006.
- 23 Ronitt Rubinfeld and Madhu Sudan. Robust characterizations of polynomials with applications to program testing. *SIAM J. Comput.*, 25(2):252–271, 1996. doi:10.1137/S0097539793255151.
- 24 Karthik C. S. and Subhash Khot. Almost polynomial factor inapproximability for parameterized k-clique. *CoRR*, abs/2112.03983, 2021. arXiv:2112.03983.
- 25 C. Tovey. A simplified np-complete satisfiability problem. *Discret. Appl. Math.*, 8:85–89, 1984.

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- 26 David Zuckerman. On unapproximable versions of np-complete problems. *SIAM J. Comput.*, 25(6):1293–1304, 1996. doi:10.1137/S0097539794266407.
- 27 David Zuckerman. Simulating BPP using a general weak random source. *Algorithmica*, 16(4/5):367–391, 1996. doi:10.1007/BF01940870.
- 28 David Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. *Theory Comput.*, 3(1):103–128, 2007. doi:10.4086/toc.2007.v003a006.