Computability of Finite Simplicial Complexes

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Abstract

The topological properties of a set have a strong impact on its computability properties. A striking illustration of this idea is given by spheres and closed manifolds: if a set X is homeomorphic to a sphere or a closed manifold, then any algorithm that semicomputes X in some sense can be converted into an algorithm that fully computes X. In other words, the topological properties of X enable one to derive full information about X from partial information about X. In that case, we say that X has computable type. Those results have been obtained by Miller, Iljazović, Sušić and others in the recent years. A similar notion of computable type was also defined for pairs (X, A) in order to cover more spaces, such as compact manifolds with boundary and finite graphs with endpoints.

We investigate the higher dimensional analog of graphs, namely the pairs (X, A) where X is a finite simplicial complex and A is a subcomplex of X. We give two topological characterizations of the pairs having computable type. The first one uses a global property of the pair, that we call the ϵ-surjection property. The second one uses a local property of neighborhoods of vertices, called the surjection property. We give a further characterization for 2-dimensional simplicial complexes, by identifying which local neighborhoods have the surjection property.

Using these characterizations, we give non-trivial applications to two famous sets: we prove that the dunce hat does not have computable type whereas Bing's house does. Important concepts from topology, such as absolute neighborhood retracts and topological cones, play a key role in our proofs.

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1 Introduction

Computable analysis is a theory formalizing computations on real numbers using finite but arbitrary precision, and allowing to investigate the theoretical possibility of solving problems on real numbers. The computable aspects of topology are an important research topic in computable analysis. Computability of homology groups was investigated in [10], computability of planar continua in [18], computability of the Brouwer fixed-point theorem was studied in [20] and [3], and computability of Polish spaces is addressed in [11].

A particularly rich topic is the computability of subsets of the plane and of Euclidean spaces. For instance, the computability of Julia sets has thoroughly been studied [4], the computability of the Mandelbrot set is still an open problem [13] and the computability of the set of solutions of a computable equation is generally a non-trivial problem [21].


These studies reveal that many natural definitions of sets induce a *semi-algorithm*, and finding a proper *algorithm* computing the set can be challenging. Informally, a compact subset of the plane is *semicomputable* if there is an algorithm that for each pixel, semidecides whether the pixel is disjoint from the set, i.e. halts exactly in that case. A compact subset of the plane is *computable* if there is an algorithm that decides, for each pixel, whether it intersects the set. This idea can be generalized to higher dimensions, and to subsets of many mathematical spaces.

Although semicomputability of compact sets is strictly weaker than computability in general, it turns out that they are equivalent for many natural sets, and that this phenomenon comes from the topological properties of these sets. For instance, it was prove by Miller [19] that semicomputability and computability are equivalent for spheres, and for every set that is homeomorphic to a sphere. This result leads to the following definition: say that a compact space X has **computable type** if any semicomputable set Y that is homeomorphic to X is actually computable. This property has been intensively studied by Miller [19] and more recently by Iljazović and his co-authors [5, 17, 9, 15, 6, 7] in the recent years. A striking aspect of this property is that it builds a bridge between computability theory and topology. The following results were obtained:

- The n-dimensional sphere $\mathbb{S}_n$ (which is the higher dimensional analog of the circle) has computable type [19],
- Every closed n-manifold (these are compact spaces which are locally homeomorphic to $\mathbb{R}^n$, e.g. the n-dimensional sphere and the n-dimensional torus) has computable type [17].

A line segment or a disk fails to have this property: it is not difficult to build a semicomputable disk which is not computable. However, a similar result can be proved if one requires in addition that the boundary of the set is semicomputable. It leads to the following generalization from compact spaces X to pairs (X, A) where X and $A \subseteq X$ are compact: a pair (X, A) has **computable type** if for any semicomputable pair (Y, B) that is homeomorphic to (X, A), Y is computable. The following results have been obtained for pairs:

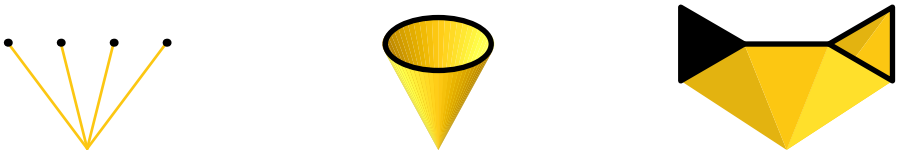
- The n-dimensional ball (which is the higher dimensional analog of the disk) with its bounding sphere $(\mathbb{B}_n, \mathbb{S}_{n-1})$ has computable type [19],
- Every compact manifold with boundary $(M, \partial M)$ has computable type [17],
- Every finite (topological) graph (G, V_1), where V_1 is the set of vertices of degree 1, has computable type [15].

Our goal in this paper is to study the property of having computable type for a broader class of spaces, to characterize the pairs having computable type and to develop a unifying argument for the known examples. Our first observation is that graphs and manifolds have the common property that they are locally topological cones as follows (see Figure 1 for an illustration of this idea):

- A finite graph is locally a cone of a finite set,
- A 2-dimensional manifold is locally a disk, which is the cone of a circle, and more generally an n-dimensional manifold is locally an n-ball, which is the cone of an $(n-1)$-sphere.

In this article, we study the class of finite simplicial complexes which is a large class of spaces that are also locally topological cones, as illustrated in Figure 1c.

Finite simplicial complexes are the higher dimensional analogs of finite graphs. They are made of simplices that are attached together along their faces. This class of compact topological spaces is large enough to include many examples (e.g., most common compact manifolds, geometrical models from computer graphics) and can be easily described using finite combinatorial information, so we can hope to obtain a full characterization of computable type for them. We do not consider infinite simplicial complexes because the usual topologies make them non-compact.



(a) A local cone in a graph.

(b) A local cone in a 2-manifold.

(c) A local cone in a simplicial complex.

Figure 1 Examples of local cones in 3 types of spaces.

Let (X, A) be a pair consisting of a finite simplicial complex X and a subcomplex A. We call such a pair a **simplicial pair**. Our main problem is to understand which simplicial pairs (X, A) have computable type. We give a thorough answer, by giving two topological characterizations of the simplicial pairs (X, A) having computable type. One of them is global whereas the other one is local. The local characterization makes it very easy to check whether a simplicial pair (X, A) has computable type, by inspecting the neighborhoods of each vertex separately. Those neighborhoods are called **local cones**, because they are topological cones with the vertex as the tip (precise definitions will be given in the article). We then use the local characterization to prove or disprove that specific sets, such as **Bing's house** and the **dunce hat**, have computable type. The previous techniques developed in the literature were too specific to be applied to these sets. Our techniques not only make it possible to treat any simplicial complex, but also provide a simple and visual way to settle the question for many sets.

The proofs are non-trivial but the statements are elegant and easy to apply. For instance it is very easy to apply our results to show that the dunce hat does not have computable type. However, the internals of the proofs of the theorems are rather involved and we are not aware of any simpler, more direct argument. Therefore our results provide significant progress in the understanding of the computable type property. Moreover, our approach in this article is new in the sense that the proofs are very different from the arguments developed in the literature on the computable type property.

It turns out that the computability property we are studying is intimately related to topology, so we need to use topology in our investigation. However, we only assume familiarity with basic topology (e.g., continuity and compactness). When we use more advanced topological notions, we give the necessary background (e.g. cones, simplicial complexes).

The results. Let us summarize the main results of this paper. We will be working with pairs (X, A) consisting of a compact metric space X and a compact subset A, to be informally thought as the boundary of X. A typical example is given by the pair $(\mathbb{B}_{n+1}, \mathbb{S}_n)$ consisting of the $(n+1)$-dimensional ball and the n-dimensional sphere:

$$\mathbb{B}_{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\},$$
$$\mathbb{S}_n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\},$$

where $\|\cdot\|$ is the Euclidean norm or any equivalent norm. We introduce two important properties of pairs, given in Definition 3.1 and restated here.

▶ **Definition.** A pair (X, A) has the **surjection property** if every continuous function $f : X \to X$ satisfying $f|_A = \mathrm{id}_A$ is surjective.

Let $\epsilon > 0$. A pair (X, A) has the **ϵ-surjection property** if every continuous function $f : X \to X$ satisfying $f|_A = \mathrm{id}_A$ and $d(f, \mathrm{id}_X) < \epsilon$ is surjective.

For instance, a consequence of Brouwer's fixed-point theorem is that the pair $(\mathbb{B}_{n+1}, \mathbb{S}_n)$ has the surjection property.

The main result of the paper is Theorem 3.4, which relates computable type with these two properties. We restate it here. We recall that a **simplicial pair** (X, A) consists of a finite simplicial complex X and a subcomplex $A \subseteq X$.

▶ **Theorem.** *Let (X, A) be a simplicial pair such that A has empty interior in X. The following conditions are equivalent:*

1. *(X, A) has computable type,*
2. *There exists $\epsilon > 0$ such that (X, A) has the ϵ-surjection property,*
3. *Every local cone pair of (X, A) has the surjection property.*

Condition 2. is the global property mentioned above and condition 3. is the local one. This theorem reduces a computability-theoretic property to purely topological ones. We develop further techniques to determine whether a pair has computable type, by applying this theorem or by analyzing when the topological properties are satisfied. The first one is stability under finite unions (Theorem 4.1 and Corollary 4.2).

▶ **Theorem** (Finite union)**.** *Let (X, A) be a simplicial pair and $(X_i, A_i)_{i \leq n}$ be pairs of subcomplexes such that $X = \bigcup_{i \leq n} X_i$ and $A = \bigcup_{i \leq n} A_i$. If each (X_i, A_i) has computable type, then (X, A) has computable type.*

The second one is a further characterization of the 2-dimensional simplicial pairs having computable type, by reducing the surjection property for local cone pairs to a simple property of graphs (Theorem 4.4). We demonstrate the strength of that result by giving non-trivial applications to two famous sets: the dunce hat (Figure 4a) and Bing's house (Figure 5).

In order to make the paper understandable to a larger audience, we give informal proofs of the main results. The detailed proofs can be found in the full version [14].

The paper is organized as follows. In Section 2, we give the needed background on computability of sets, simplicial complexes and cone spaces. In Section 3, we define the surjection property and the ϵ-surjection property, state and prove our main result. In Section 4, we present techniques to prove or disprove the (ϵ-)surjection property. As an application, we prove that the dunce hat does not have computable type whereas the Bing's house does, by studying the local cones of each of the two sets. In Section 5, we briefly discuss the possible notions of boundary ∂X of a simplicial complex X that make the pair $(X, \partial X)$ have computable type. We finally formulate open questions and discuss a generalization of our results in Section 6.

2 Preliminaries

We give here some necessary preliminaries in computability theory and topology. We start with this central definition.

▶ **Definition 2.1.** *A **pair** (X, A) consists of a compact metrizable space X and a compact subset $A \subseteq X$. A **copy** of a pair (X, A) in a topological space Z is a pair (Y, B) such that $Y \subseteq Z$ is homeomorphic to X and A is sent to B by the homeomorphism.*

2.1 Computability of sets

We recall definitions and results about the Hilbert cube and computable type. We will mainly use the following notion from computability theory: a set $A \subseteq \mathbb{N}$ is **computably enumerable (c.e.)** if there exists a Turing machine that, on input $n \in \mathbb{N}$, halts if and only if $n \in A$. This notion immediately extends to subsets of countable sets, whose elements can be encoded by natural numbers.

Computability in the Hilbert cube. We work in the Hilbert cube because it is universal among the separable metrizable spaces, in particular every compact metrizable space embeds in the Hilbert cube.

▶ **Definition 2.2.** *The **Hilbert cube** is the space $Q = [0,1]^{\mathbb{N}}$ endowed with the metric $d(x,y) = \sum_i 2^{-i}|x_i - y_i|$. We let $(B_i)_{i\in\mathbb{N}}$ be a computable enumeration of the open balls $B(x,r)$ where $x \in Q$ has finitely many non-zero rational coordinates and $r > 0$ is rational; these B_is are called **rational balls**.*

▶ Notation 1. If $X \subseteq Q$ and $f, g : X \to Q$, then let $d_X(f,g) = \sup_{x\in X} d(f(x), g(x))$.

We recall definitions of computability of compact subsets of the Hilbert cube. The reader can find more details about computability of sets in [2, 16].

▶ **Definition 2.3** (Computability of sets). *A compact set $X \subseteq Q$ is:*
- ***Semicomputable** if there exists a c.e. set $E \subseteq \mathbb{N}$ such that $Q \setminus X = \bigcup_{i\in E} B_i$,*
- ***Computable** if it is semicomputable and $\{i \in \mathbb{N} : X \cap B_i \neq \emptyset\}$ is c.e.*

*A pair (X, A) in Q is **semicomputable** if both X and A are semicomputable.*

Intuitively, X is semicomputable if there is an algorithm that takes any rational cube as input (a voxel) and semidecides whether that cube is disjoint from X, i.e. halts exactly in this case. X is computable if there is an algorithm that *decides* whether a cube intersects the set.

For instance, the Mandelbrot set is semicomputable because its definition gives an algorithm that can eventually detect that a point is outside this set; whether it is computable is an open problem, see [13].

▶ **Example 2.4.** The line segment $I = [0,1]$, embedded in the simplest way as $[0,1] \times \{q\} \subseteq Q$ where $q = (0, 0, \ldots)$, is computable. However, if $A \subseteq \mathbb{N}$ is the halting set (a non-computable c.e. set) and $x_A = \sum_{n\in A} 2^{-n}$, then $[x_A, 1] \times \{q\}$ is a copy of I which is semicomputable but not computable.

The Hilbert cube itself is a computable subset of itself. A compact set $X \subseteq Q$ is semicomputable if and only if the set $\{(i_1, \ldots, i_n) \in \mathbb{N}^* : X \subseteq B_{i_1} \cup \ldots \cup B_{i_n}\}$ is c.e., and it is computable if and only if in addition it contains a dense computable sequence. A function $f : Q \to Q$ is **computable** if there exists a c.e. set $E \subseteq \mathbb{N}^2$ such that $f^{-1}(B_i) = \bigcup_{(i,j)\in E} B_j$. The image of a (semi)computable set under a computable function is a (semi)computable set. Semicomputable sets have very useful properties: if $X \subseteq Q$ is semicomputable and $f, g : X \to Q$ are computable, then $\{q \in \mathbb{Q} : d_X(f,g) < q\}$ is c.e.

Computable type. The next definition is the main notion of this article (see [17]).

▶ **Definition 2.5.** *A pair (X, A) has **computable type** if for every semicomputable copy (Y, B) of the pair in the Hilbert cube, Y is computable.*

A compact space X has computable type if the pair $(X, \emptyset)$ has.

▶ Remark 2.6. In fact, in [17] computable type was defined separately for copies in computable metric spaces and computably Haudorff spaces. In a forthcoming article, we show that taking the copies in computably Hausdorff spaces, computable metric spaces or the Hilbert cube are all equivalent using the fact that computable metric spaces embed effectively in the Hilbert cube, as well as Schröder's computable metrization theorem [22].

2.2 Topology

We recall some notions which will be used, like simplicial complexes and cone spaces. We will work with compact metrizable spaces only, and may omit this assumption in the statements.

▶ **Definition 2.7.** *Let (X, A) be a pair. A **retraction** $r : X \to A$ is a continuous function such that $r|_A = \mathrm{id}_A$. If a retraction exists, then we say that A is a **retract** of X.*

Simplicial complex. Let $V = \{0, \dots, n\}$ and $P_+(V)$ be the set of non-empty subsets of V. An **abstract finite simplicial complex** is a set $S \subseteq P_+(V)$ such that if $\sigma \in S$ and $\emptyset \neq \sigma' \subset \sigma$, then $\sigma' \in S$. Its elements $\sigma \in S$ are called the **simplices** of S. If $\sigma \in S$ has $n+1$ elements, then σ is an ***n*-simplex**. The **vertices** of S are the singletons $\{i\} \in S$. $\sigma \in S$ is **free** if there exists exactly one $\sigma' \in S$ with $\sigma \subsetneq \sigma'$. A **subcomplex** of S is an abstract simplicial complex contained in S.

The **support** of a vector $x = (x_0, \dots, x_n) \in [0,1]^{n+1}$ is $\mathrm{supp}(x) = \{i : x_i \neq 0\}$. The **standard realization** of an abstract simplicial complex S is the set

$$|S| = \Big\{x = (x_0, \dots, x_n) \in [0,1]^{n+1} : \sum_i x_i = 1, \mathrm{supp}(x) \in S\Big\}.$$

Any space homeomorphic to the standard realization of an abstract finite simplicial complex is called a **finite simplicial complex**. We often identify an abstract simplicial complex and its standard realization.

A **simplicial pair** (X, A) consists of a finite simplicial complex X and a subcomplex A.

▶ Remark 2.8. For technical reasons, we will implicitly assume that A contains all the free vertices of X (those are the points x having a neighborhood homeomorphic to $[0, 1)$ with a homeomorphism sending x to 0).

In a simplicial complex, each vertex has a neighborhood which is usually called a star and is topologically a cone. Our main result will relate the computable type property with a property of these local cones. Because we are dealing with pairs, we need to define local cone pairs, as follows.

▶ **Definition 2.9.** *Let (X, A) be the standard realization of a simplicial pair and $v_i = (0, \dots, 0, 1, 0, \dots, 0)$ be a vertex. The **local cone pair** at v_i is (K_i, M_i) defined by:*

$$K_i = \{x \in X : x_i \geq 1/2\},$$
$$M_i = \{x \in X : x_i = 1/2\} \cup (K_i \cap A).$$

Note that the coefficient $1/2$ is arbitrary and could be replaced by any number in $(0, 1)$.

▶ Remark 2.10. We call (K_i, M_i) a cone pair because K is a topological cone: let $L_i = \{x \in X : x_i = 1/2\}$, K_i is a copy of the cone of L_i, obtained from $L_i \times [0, 1]$ by identifying all the points $(l, 0)$ together. The point obtained by this identification is the tip of the cone and corresponds to the vertex v_i. If $N_i = \{x \in A : x_i = 1/2\}$, then M_i is the union of L_i and of the cone of N_i.

In the language of simplicial complexes, K_i corresponds to the *star* of v_i and L_i to the *link* of v_i. K_i is homeomorphic to the union of simplices containing v_i. Each such simplex has a face that does not contain v_i, and L_i is the union of these faces.

3 The (ϵ-)surjection property and computable type for simplicial pairs

We now present the main result of this paper, that identifies which simplicial pairs have computable type, using the following topological properties.

▶ **Definition 3.1.** *A pair (X, A) has the **surjection property** if every continuous function $f : X \to X$ satisfying $f|_A = \mathrm{id}_A$ is surjective.*

*A pair (X, A) in Q has the **ϵ-surjection property** for some $\epsilon > 0$, if every continuous function $f : X \to X$ satisfying $f|_A = \mathrm{id}_A$ and $d_X(f, \mathrm{id}_X) < \epsilon$ is surjective.*

▶ **Example 3.2.**

- For every $n \in \mathbb{N}$, the $(n+1)$-dimensional ball and its bounding n-dimensional sphere form a pair $(\mathbb{B}_{n+1}, \mathbb{S}_n)$ that has the surjection property. It is a consequence of an equivalent formulation of Brouwer's fixed-point theorem that $\mathbb{S}_n$ is not a retract of $\mathbb{B}_{n+1}$ (Corollary 2.15 in [12]).
- The pair $(\mathbb{S}_n, \emptyset)$ does not have the surjection property (take a constant function $f : \mathbb{S}_n \to \mathbb{S}_n$), but has the ϵ-surjection property if ϵ is sufficiently small. It can be proved using classical results in topology, or as a consequence of Theorem 3.4 below.
- If $A \subsetneq X$ is a retract of X, then the pair (X, A) does not have the surjection property, as witnessed by the retraction.

Although the ϵ-surjection property depends on the particular copy of a pair (X, A), quantifying over ϵ yields a topological invariant, i.e. a property of the pair that is satisfied either by all copies or by none of them.

▶ **Proposition 3.3.** *Whether there exists $\epsilon > 0$ such that (X, A) has the ϵ-surjection property does not depend on the copy of (X, A) in Q.*

Proof. If (Y, B) is a copy of (X, A), then let $\phi : X \to Y$ be a homeomorphism such that $\phi(A) = B$. By compactness of X, ϕ is uniformly continuous so given $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x, x') < \delta$ then $d(\phi(x), \phi(x')) < \epsilon$. If (Y, B) has the ϵ-surjection property, then we show that (X, A) has the δ-surjection property. Let $f : X \to X$ be continuous, satisfying $f|_A = \mathrm{id}_A$ and $d_X(f, \mathrm{id}_X) < \delta$. Define $g = \phi \circ f \circ \phi^{-1} : Y \to Y$: one has $g|_B = \mathrm{id}_B$ and $d_Y(g, \mathrm{id}_Y) < \epsilon$ by choice of δ so g is surjective, hence f is surjective. ◀

We now state the main result of this paper.

▶ **Theorem 3.4** (The main theorem)**.** *Let (X, A) be a simplicial pair such that A has empty interior in X. The following statements are equivalent:*

1. *(X, A) has computable type,*
2. *(X, A) has the ϵ-surjection property for some $\epsilon > 0$,*
3. *All the local cone pairs (K_i, M_i) have the surjection property.*

We separate the proof into several independent parts.

▶ Remark 3.5. A single topological space X has many different simplicial decompositions, i.e. many abstract simplicial complexes whose realizations are homeomorphic to X. For instance, a triangle can be decomposed into many smaller triangles. At first sight, the third condition in Theorem 3.4 depends on the choice of the decomposition, because the local

cone pairs are taken at the vertices of the decomposition. However, the theorem implies that the choice of the simplicial decomposition is irrelevant, because conditions 1. and 2. do not depend on the decomposition: if all the cone pairs in a simplicial decomposition have the surjection property, then it is still true for all other simplicial decompositions of the space.

For a simplicial pair that is itself homeomorphic to a cone pair, we obtain a further equivalence, which is a consequence of Theorem 3.4.

Let (L, N) be a simplicial pair. The **simplicial cone pair** induced by (L, N) is the pair (K, M) defined by $K = \mathrm{Cone}(L)$ and $N = L \cup \mathrm{Cone}(N)$, as in Remark 2.10.

▶ **Corollary 3.6.** *Let (K, M) be a simplicial cone pair such that M has empty interior in K. The following statements are equivalent:*

1. *(K, M) has computable type,*
2. *(K, M) has the ϵ-surjection property for some $\epsilon > 0$,*
3. *(K, M) has the surjection property.*

Proof. The surjection property implies the ϵ-surjection property for any pair. Conversely, if the pair (K, M) has the ϵ-surjection property then each local cone pair has the surjection property, but (K, M) is itself homeomorphic to one of its local cone pairs. ◀

The rest of this section is devoted to the proof of this result. We will give several applications in the next section.

3.1 The ϵ-surjection property implies computable type

In this section we give an informal idea of the proof of 2. $\Rightarrow$ 1. in Theorem 3.4. The idea of the proof is that if A has empty interior in X and (X, A) has the ϵ-surjection property, then for an open set U the following conditions are equivalent:

- U intersects X,
- There exists a continuous non-surjective function $g : (X \setminus U) \cup A \to X$ such that $g|_A = \mathrm{id}_A$ and $d_X(g, \mathrm{id}_X) < \epsilon$.

This equivalence is straightforward. If U intersects X, then let g be the inclusion map. Conversely, if such a g exists then $(X \setminus U) \cup A$ must differ from X by the ϵ-surjection property for (X, A), so U intersects X.

The finite simplicial complex X has good topological properties because it is a compact Absolute Neighborhood Retract (ANR), which means that any copy of X in Q is a retract of some neighborhood of that copy. The detailed proof consists in showing how to use these properties to prove that the existence of such a function g can be detected by an algorithm if (X, A) is semicomputable. The main idea is that one does not need to search for an arbitrary continuous function g, but for a computable one. Therefore, one can test whether an open set U intersects X, which makes X computable.

3.2 The ϵ-surjection property is equivalent to the local surjection property

In this section we give an informal proof of the equivalence 2. $\Leftrightarrow$ 3. in Theorem 3.4

The ϵ-surjection property implies the local surjection property. It is easy to see that if a local cone pair does not have the surjection property, then for any $\epsilon > 0$, the pair (X, A) does not have the ϵ-surjection property. It relies on the particular property of a cone that it contains arbitrarily small copies of itself, obtained by scaling it down: for any $\lambda \in (0, 1)$,

the set $K_i(\lambda) = \{x \in X : x_i \geq \lambda\}$ is a copy of K_i and it has arbitrarily small diameter as λ approaches 1. Given $\epsilon > 0$, consider λ such that $K_i(\lambda)$ has diameter less than ϵ. Take a non-surjective function f from $K_i(\lambda)$ to itself which is the identity on the corresponding set $M_i(\lambda)$, and extend it to a non-surjective function $g : X \to X$ by simply defining $g(x) = x$ for x outside $K_i(\lambda)$. One has $d(g, \mathrm{id}_X) < \epsilon$, showing that (X, A) does not have the ϵ-surjection property.

The local surjection property implies the ϵ-surjection property. Now, assume that for every $\epsilon > 0$, (X, A) does not have the ϵ-surjection property. We show that some local cone pair does not have the surjection property. The idea is to start from a sufficiently small $\epsilon > 0$, to be defined later, and a non-surjective function $h : X \to X$ such that $h|_A = \mathrm{id}_A$ and $d(h, \mathrm{id}_X) < \epsilon$ and consider its restriction h_0 to a local cone K which is not contained in the image of h. This function h_0 does not immediately disprove the surjection property for the local cone pair (K, M) because $h_0(K)$ may not be contained in K and h_0 may not be the identity on M. However, h_0 almost satisfies these properties: $h_0(K)$ is at distance ϵ from K and h_0 is ϵ-close to the identity on M. Again, using the fact that K is a compact Absolute Neighborhood Retract (ANR) and the properties derived from that, if one takes ϵ sufficiently small, then one can transform h_0 into a continuous function G that sends K to itself, is the identity on M and is still non-surjective. Therefore, (K, M) does not have the surjection property.

3.3 Computable type implies the ϵ-surjection property

We prove 1. $\Rightarrow$ 2. in Theorem 3.4. We show that if a simplicial pair (X, A) does not have the ϵ-surjection property for any $\epsilon > 0$, then it has a semicomputable copy in Q that is not computable. In order to build that semicomputable copy, we show that the pair fails in a computable way to have the ϵ-surjection property, which is expressed by Definition 3.7.

For two non-empty compact sets $A, B \subseteq Q$, their Hausdorff distance is

$$d_H(A, B) = \max(\max_{a \in A} d(a, B), \max_{b \in B} d(b, A)).$$

▶ **Definition 3.7.** *Let $\epsilon > 0$ and $(X, A) \subseteq Q$ fail to have the ϵ-surjection property. Say that $\delta > 0$ is an **ϵ-witness** if there exists a continuous function $f : X \to X$ such that $f|_A = \mathrm{id}_A$, $d_X(f, \mathrm{id}_X) < \epsilon$ and $d_H(f(X), X) > \delta$.*

*Say that (X, A) has **computable witnesses** if there is a computable function $\epsilon \mapsto \delta(\epsilon)$ such that for every $\epsilon > 0$, $\delta(\epsilon)$ is an ϵ-witness.*

For a compact pair (X, A) (not necessarily simplicial), having computable witnesses is sufficient to build a semicomputable copy which is not computable.

▶ **Theorem 3.8.** *Let $(X, A) \subseteq Q$ be a computable pair having computable witnesses. (X, A) does not have computable type.*

Informal proof. In order to give some intuition, let us show precisely another but related result: if we only assume that (X, A) does not have the surjection property, then one can encode the halting problem for *one* program p in a copy of (X, A), in the following sense. Given p, one can produce an algorithm that semicomputes a copy (X_p, A_p) of (X, A); any algorithm computing X_p could be used to decide whether p halts.

Let $(X_0, A_0) \subseteq Q$ be a semicomputable copy of (X, A) and $\delta > 0$ be such that there exists a non-surjective continuous function $f : X_0 \to X_0$ such that $f|_{A_0} = \mathrm{id}_{A_0}$ and $d_H(X_0, f(X_0)) > \delta$.

Given a program p, we define a copy (X_p, A_p). If p does not halt, then $(X_p, A_p) = (X_0, A_0)$. If p halts, then (X_p, A_p) is another copy (X_1, A_1) defined by the following algorithm.

Start enumerating the complements of X_0 and A_0. If p eventually halts then consider a copy (X_1, A_1) of (X_0, A_0) with the following properties:

- (X_1, A_1) is compatible with (i.e. disjoint from) the current enumeration of the complements of X_0 and A_0,
- $d_H(X_1, X_0) > \delta$.

The existence of f implies the existence of (X_1, A_1), which can be effectively found. We then continue enumerating the complements of X_1 and A_1.

We have just given an algorithm that semicomputes a copy (X_p, A_p) of (X, A), be it (X_0, A_0) or (X_1, A_1). Any algorithm that computes X_p could be used to know whether p halts: p halts if and only if $d_H(X_p, X_0) > \delta$, which can be decided from the computable information about X_p.

Now, assuming that (X, A) does not have the ϵ-surjection property for any ϵ, and using the assumption that a witness $\delta(\epsilon)$ can be computed from any ϵ, we apply this strategy against all the programs in parallel and at infinitely many scales. The idea is simple but the details are rather technical. ◀

Note that the standard realization of a simplicial pair is computable. We now show that if it has witnesses, then it always have *computable* witnesses, which together with Theorem 3.8 concludes the proof of 1. ⇒ 2. in Theorem 3.4.

▶ **Proposition 3.9.** *If a simplicial pair (X, A) does not have the ϵ-surjection property for any $\epsilon > 0$, then its standard realization has computable witnesses.*

Proof. By 3. ⇒ 2. in Theorem 3.4, there exists a local cone pair (K_i, M_i) which does not have the surjection property, so there exists a non-surjective function $f_0 : K_i \to K_i$ such that $f_0|_{M_i} = \mathrm{id}_{M_i}$. One can assume w.l.o.g. that $d_X(f_0, \mathrm{id}_X) < 1$. Let $\delta_0 > 0$ be such that $d_H(f_0(X), X) > \delta_0$. Given $\epsilon > 0$, the number $\delta = \delta_0\epsilon$ can be computed from ϵ and is an ϵ-witness. Indeed, the function f obtained by applying f_0 to a version of K_i scaled by a factor ϵ and extended as the identity elsewhere satisfies all the conditions. ◀

4 Techniques for the (ϵ-)surjection property

Theorem 3.4 enables one to reduce the computable type property to topological properties, namely the ϵ-surjection property and the surjection property for local cone pairs. Proving or disproving these properties may not be straightforward, so we develop a few techniques that help in many cases.

4.1 Finite union

The first result is a way to prove that a simplicial pair has the ϵ-surjection property by decomposing it as a finite union of pairs that all have the ϵ-surjection property.

▶ **Theorem 4.1** (Finite union). *Let (X, A) be a finite simplicial pair and $(X_i, A_i)_{i \leq n}$ be pairs of subcomplexes such that $X = \bigcup_{i \leq n} X_i$ and $A = \bigcup_{i \leq n} A_i$. If every pair (X_i, A_i) has the ϵ-surjection property for some $\epsilon > 0$, then (X, A) has the δ-surjection property for some $\delta > 0$.*

Informal proof. We are using good topological properties of finite simplicial complexes. For each i, there exists a neighborhood U_i of X_i and a retraction $r_i : U_i \to X_i$ with a special property: if x belongs to the topological interior of X_i, then the only preimage of x by r_i is x.

Let δ be sufficiently small and assume that (X, A) does not have the δ-surjection property. Let $f : X \to X$ be continuous, non-surjective and satisfy $f|_A = \mathrm{id}_A$ and $d_X(f, \mathrm{id}_X) < \delta$. There must be $i \leq n$ and x in the interior of X_i that is not in the image of f. We can then create a function $f_i : X_i \to X_i$ as follows: f_i is the restriction of $r_i \circ f$ to X_i (it is possible if δ is sufficiently small, so that $f(X_i) \subseteq U_i$).

The special property of r_i implies that x is not in the image of f_i. Moreover, f_i is continuous, is the identity on A_i and is ϵ-close to id_{X_i} if δ is sufficiently small. ◀

▶ **Corollary 4.2.** *Let (X, A) be a simplicial pair and $(X_i, A_i)_{i\leq n}$ be pairs of subcomplexes such that $X = \bigcup_{i\leq n} X_i$ and $A = \bigcup_{i\leq n} A_i$. If every pair (X_i, A_i) has computable type, then (X, A) has computable type.*

For instance, if a finite simplicial complex X is a finite union of subcomplexes that are homeomorphic to spheres, then X has computable type. More generally, if a finite simplicial pair (X, A) is a finite union of pairs of subcomplexes (X_i, A_i) that are homeomorphic to pairs $(\mathbb{S}_n, \emptyset)$ or $(\mathbb{B}_{n+1}, \mathbb{S}_n)$, then (X, A) has computable type.

4.2 Cone of a graph

In a 2-dimensional simplicial pair, the local cones are cones of graphs. We obtain a characterization of the surjection property for such cones. In order to state the result, we need to define the cone pair induced by a pair, already informally discussed in Remark 2.10. Let (L, N) be a pair. We define the cone pair $(K, M) := \mathrm{Cone}(L, N)$ as follows:

- $K = \mathrm{Cone}(L)$ is the quotient of $L \times [0, 1]$ by the equivalence relation $(x, 0) \sim (y, 0)$,
- $M = L \cup \mathrm{Cone}(N)$, where L is embedded in K as $L \times \{1\}$.

The space L is called the **base** of the cone $K = \mathrm{Cone}(L)$, and the equivalence class $L \times \{0\}$ is called the **tip** of K.

▶ **Example 4.3.** Let us illustrate this notion on the usual example of balls and spheres:

- $\mathrm{Cone}(\mathbb{S}_n, \emptyset) = (\mathbb{B}_{n+1}, \mathbb{S}_n)$ with the tip at the center of $\mathbb{B}_{n+1}$,
- $\mathrm{Cone}(\mathbb{B}_n, \mathbb{S}_{n-1}) = (\mathbb{B}_{n+1}, \mathbb{S}_n)$ with the tip in $\mathbb{S}_n$.

Here is the main result of this section.

▶ **Theorem 4.4.** *Let (L, N) be a pair such that L is a finite graph and N is a subset of its vertices. The following statements are equivalent:*

1. $\mathrm{Cone}(L, N)$ *has the surjection property,*
2. *Every edge is in a cycle or a path starting and ending in N.*

We follow the usual convention that in a graph, a path and a cycle do not visit a vertex twice, i.e. they are topologically a line segment and a circle respectively. In particular, a path connects two different points.

▶ **Example 4.5** (Star pair)**.** Fix some $n \geq 1$ and let X be the star with n branches and A be the n endpoints of these branches (see Figure 2), with a special case for $n = 1$: $\mathrm{Cone}(\{v\}, \emptyset) = (\mathbb{B}_1, \mathbb{S}_0)$. The pair (X, A) is precisely $\mathrm{Cone}(A, \emptyset)$. As A has no edge, it satisfies the conditions of Theorem 4.4, therefore (X, A) has the surjection property. One can then obtain Iljazović's result that every finite graph has computable type [15], because the local cones of a finite graph are stars, which have the surjection property.

(a) Star with 5 branches.

(b) Star with 1 branch.

Figure 2 The star pairs (X, A) have the surjection property (Example 4.5) (X in yellow, A in black).

▶ **Example 4.6** (n squares)**.** Fix some $n \geq 2$ and let X be the union of n squares which all meet in one common edge and A be the union of all the other edges (see Figure 3). The pair (X, A) has the surjection property. Indeed, $(X, A) = \mathrm{Cone}(A, \emptyset)$ and A is a graph which is a union of circles (each circle is the boundary of the union of two squares). Therefore, $\mathrm{Cone}(A, \emptyset)$ has the surjection property by Theorem 4.4. Finally, (X, A) has computable type by Corollary 3.6.

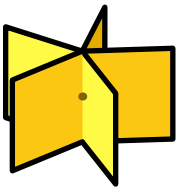


Figure 3 A union of 5 squares is the cone of a graph; the tip is at the center, the graph is in black (Example 4.6).

We expect a generalization of Theorem 4.4 to cones of arbitrary simplicial complexes, by using the notions of n-cycles and relative n-cycles from homology, generalizing cycles and paths respectively [12].

In the next section we apply Theorem 4.4, giving an example of a cone pair of a graph which does not have the surjection property.

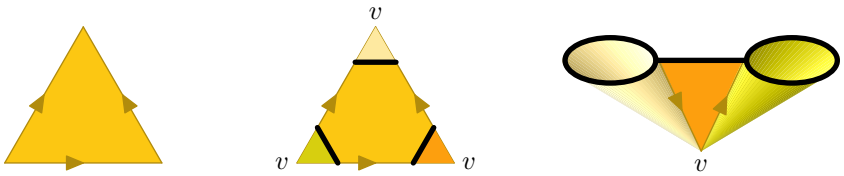
4.3 The dunce hat

The **dunce hat** D is the space obtained from a solid triangle by gluing its three sides together, with the orientation of one side reversed (see Figure 4a). It is a classical example, introduced by Zeeman [23], of a space that is contractible but not intuitively so. It is a 2-dimensional simplicial complex with no free edge, i.e. no edge that belongs to one triangle only.

▶ **Theorem 4.7.** *The dunce hat does not have computable type.*

Proof. First, it is possible to turn the dunce hat into a simplicial complex, so we can apply our results. The vertices of the triangle are identified to a point v, and we show that the local cone pair at that point does not have the surjection property. Indeed, in Figure 4c one can see that the local cone pair at v is $\mathrm{Cone}(L, \emptyset) = (\mathrm{Cone}(L), L)$ where L is the graph consisting of two circles joined by a line segment.

We apply Theorem 4.4: L is a finite graph containing an edge which is neither in a cycle nor in a path from N to N (N is empty), therefore $\mathrm{Cone}(L, N)$ does not have the surjection property. Theorem 3.4 then implies that the dunce hat does not have computable type. ◀



(a) Dunce hat. **(b)** Local cone pair. **(c)** Local cone pair.

Figure 4 (a) The dunce hat is obtained by gluing the edges with the indicated orientations; (b) and (c) a local cone pair $(\mathrm{Cone}(L), L) = \mathrm{Cone}(L, \emptyset)$ with tip at v, with L in black.

As far as we know, there is no simple and visual way of building a semicomputable copy of the dunce hat that is not computable, i.e. the involved construction carried out in the proof of Theorem 3.4 cannot be avoided. The same remark applies to the pair $(\mathrm{Cone}(L), L)$ depicted in Figure 4c.

If A is the identified edges of the triangle, then it can be proved, by analyzing its local cone pairs, that the pair (D, A) has computable type. In particular, the local cone pair at v is $\mathrm{Cone}(L, N)$ where N consists of the two endpoints of the middle interval, so L is the union of two circles and a line segment between two points of N, hence $\mathrm{Cone}(L, N)$ has the surjection property by Theorem 4.4.

▶ Remark 4.8 (Quotient vs pair). It is proved in [8] that for any compact pair (X, A) where A has empty interior, if the quotient space X/A has computable type then the pair (X, A) has computable type. It is also proved that the converse implication fails, the counter-example is given by the circle X and a subset A consisting of a converging sequence together with its limit. The pair (X, A) has computable type, simply because X itself has computable type. However, X/A is homeomorphic to the Hawaiian earring which does not have computable type. This quotient is not a finite simplicial complex.

We give another counter-example of a quotient space which is a finite simplicial complex. Let $L = C_1 \vee I \vee C_2$, X be the cylinder of L and A the two bases of the cylinder. Inspecting the local cones one can show that (X, A) has computable type but X/A does not.

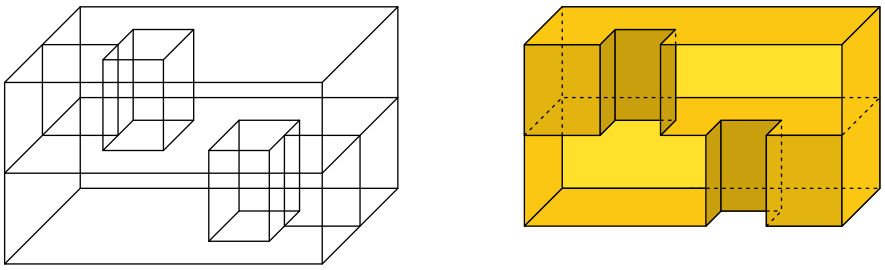
4.4 Bing's house, or the house with two rooms

All the known examples of sets having computable type are non-contractible (note that we are not considering pairs, but single sets), and one might conjecture that no contractible set has computable type. We give a counter-example, which is a famous space that was defined as a counter-example for other properties. It was invented by Bing [1] and is now called **Bing's house**, or the house with two rooms. The set is depicted in Figure 5, together with a half-cut to help visualizing it. It is an example of a space which is contractible but not intuitively so. It can be endowed with a simplicial complex structure (by triangulating each flat surface). It is then a 2-dimensional simplicial complex with no free edge, which means that every edge belongs to at least two triangles.

Using our results we easily show that this set has computable type as a single set, i.e. without adjoining a boundary to it.

▶ **Theorem 4.9.** *Bing's house has computable type.*

It is worth noticing that thanks to our results, it can be proved by looking at pictures only, although the argument can be formalized.



(a) Bing's house. **(b)** Half-cut.

Figure 5 Bing's house with two rooms and a half-cut of it (the full house is obtained by adding the symmetric reflection of the half-cut through the front vertical plane). It consists of two rooms, each of which can be accessed from outside through a tunnel crossing the other room. Each tunnel is linked by an internal wall to a side wall.

Proof. Using Theorem 3.4, it is sufficient to inspect the possible local cones. One easily sees that there are three types of possible cones, depicted in Figure 6. The basis of each cone is a graph which is a union of 1, 2 or 3 cycles, so by Theorem 4.4 each cone pair has the surjection property, therefore Bing's house has computable type by Theorem 3.4.

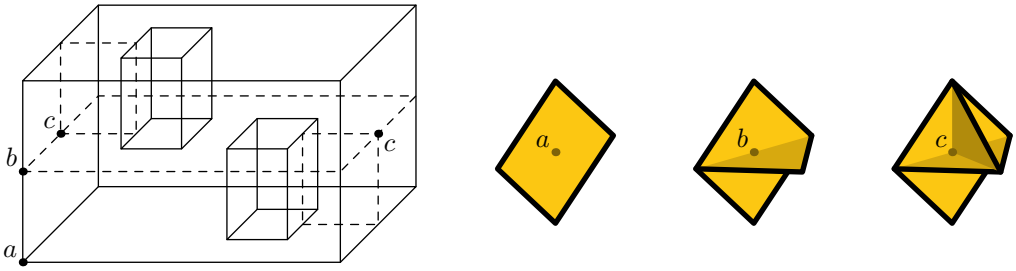


Figure 6 The local cones in Bing's house: their bases (in black) are graphs that are unions of cycles. Each point of Bing's house is the tip of one of these three cones: two points are tips of the third cone, all the other points on the dashed lines are tips of the second cone, all the other points are tips of the first cone. ◀

5 Boundary

Given a simplicial complex X, a natural problem is to understand whether there is a minimal notion of boundary ∂X such that the pair $(X, \partial X)$ has computable type. We make a few observations about three possible candidates. Let

- $\partial_1 X$ be the union of simplices that are contained in *exactly* one simplex of the next dimension, i.e. $\partial_1 X$ is the union of the free simplices of X,
- $\partial_+ X$ be the union of simplices that are contained in *at least* one simplex of the next dimension,
- $\partial_{\text{odd}} X$ be the union of simplices that are contained in *an odd number* of simplices of the next dimension.

In the proofs of the next results, we say that a simplex M in X is **maximal** if it is not contained in a higher-dimensional simplex of X.

▶ **Proposition 5.1.** *Every simplicial pair $(X, \partial_+ X)$ has computable type.*

Proof. Let $(M_i)_{i\leq n}$ be an enumeration of the maximal simplices of X. M_i is a ball, let ∂M_i be its bounding sphere, which is a subcomplex of M_i. One has $X = \bigcup_{i\leq n} M_i$ and $\partial_+ X = \bigcup_{i\leq n} \partial M_i$. Each pair $(M_i, \partial M_i)$ has the surjection property (Example 3.2), so $(X, \partial_+ X)$ has the ϵ-surjection property for some ϵ by Theorem 4.1. As a result, $(X, \partial_+ X)$ has computable type by Theorem 3.4. ◀

▶ **Proposition 5.2.** *Let X be a finite simplicial complex and A a subcomplex. If (X, A) has computable type, then A contains $\partial_1 X$.*

Proof. Assume that some simplex Δ belongs to $\partial_1 X$ but not to A. We show that for every $\epsilon > 0$, (X, A) does not have the ϵ-surjection property, implying that (X, A) does not have computable type by Theorem 3.4. Let $\epsilon > 0$. Let Δ' be the unique maximal simplex having Δ as a face (Δ' has one more vertex than Δ). There is a non-surjective function $f : \Delta' \to \Delta'$ which is ϵ-close to the identity and is the identity on the other faces of Δ': f slightly pushes points of Δ' away from Δ. We extend f as the identity on the rest of X, which gives a continuous function because Δ is free. As Δ is not in A, f is the identity on A. ◀

The following observations can be made:

- Although $(X, \partial_1 X)$ has computable type when X is a 1-dimensional complex (i.e., a graph), it is no more true for 2-dimensional complexes. For the dunce hat D, one has $\partial_1 D = \emptyset$ but we saw in Theorem 4.7 that $(D, \emptyset)$ does not have computable type.
- While $(X, \partial_+ X)$ always has computable type by Proposition 5.1, $\partial_+ X$ is far from optimal. For instance, it is always non-empty (unless X is a single point), but for any sphere $\mathbb{S}_n$, the pair $(\mathbb{S}_n, \emptyset)$ already has computable type.
- In a subsequent paper we prove that $(X, \partial_{\text{odd}} X)$ always has computable type, using homology. Observe that $\partial_{\text{odd}} X$ is in general not optimal, as the example of graphs shows: $(X, \partial_1 X)$ has computable type and $\partial_1 X$ is usually smaller than $\partial_{\text{odd}} X$, which contains all the vertices of odd degrees.

6 Open questions and generalization

We leave two open questions.

▶ **Question 1.** *Is there a canonical notion of boundary ∂X for a simplicial complex X, such that $(X, \partial X)$ always has computable type, and ∂X is minimal in some sense?*

▶ **Question 2.** *For simplicial pairs (L, N), is it possible to characterize the surjection property for $\text{Cone}(L, N)$ in terms of the homology of (L, N)?*

We finally mention that the proof of the main result actually applies to more general spaces. For instance one can prove that if $(M, \partial M)$ is a compact manifold with boundary, then $\text{Cone}(M, \partial M)$ has computable type because it satisfies the surjection property, although it is not always a simplicial complex. These results will appear in a forthcoming article.

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