On the Role of the High-Low Partition in Realizing a Degree Sequence by a Bipartite Graph

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Abstract

We consider the problem of characterizing degree sequences that can be realized by a bipartite graph. If a partition of the sequence into the two sides of the bipartite graph is given as part of the input, then a complete characterization has been established over 60 years ago. However, the general question, in which a partition and a realizing graph need to be determined, is still open. We investigate the role of an important class of special partitions, called *High-Low partitions*, which separate the degrees of a sequence into two groups, the high degrees and the low degrees. We show that when the High-Low partition exists and satisfies some natural properties, analysing the High-Low partition resolves the bigraphic realization problem. For sequences that are known to be not realizable by a bipartite graph or that are undecided, we provide approximate realizations based on the High-Low partition.

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1 Introduction

1.1 Background and Motivation

Graphic degree sequences are among the most well-researched objects in the domain of graph realizations, studied extensively for over 60 years. A sequence $d = (d_1, \ldots, d_n)$ of non-negative integers is said to be a *graphic degree sequence* if there exists an n-vertex simple graph G such that $\deg(G) = d$, where $\deg(G)$ denotes the sequence of vertex degrees of G. The *graphic degree realization (GDR)* problem requires, given a sequence d, to decide whether d is graphic, and if so, to construct a graph G realizing it. Erdös and Gallai [18] gave a complete characterization for graphic degree sequences. However, their method does not provide a realizing graph. Havel and Hakimi [21, 24] gave an algorithm that, given a sequence d, generates a realizing graph, or proves that the sequence is not graphic, in time $\mathcal{O}(\sum_i d_i)$ which is optimal.

In this paper we consider the natural variant of the graphic degree realization problem, referred to as the *bigraphic degree realization (BDR)* problem, where the realizing graph is required to be bipartite. A sequence admitting a bipartite realizing graph is called a *bigraphic degree sequence*. This problem was mentioned in [33] as an open problem over 40 years ago, but we are unaware of any attempt to solve it.

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The literature does contain, however, a sizeable amount of research on the simpler variant, hereafter referred to as the *given partition* version of the bigraphic degree realization problem, BDR^P, where the partition of d is given as part of the input. More explicitly, the input consists of a *partition*, namely, *two* sequences $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_q)$, and the question is to decide whether there exists a bipartite graph $G = (A, B, E)$ such that $|A| = p$, $|B| = q$, and the sequences of degrees of the vertices of A and B are equal to a and b, respectively. We refer to such a pair (a, b) as a *bigraphic degree partition*.

Interestingly, the history of bigraphic degree partitions is as ancient as that of graphic degree sequences. In 1957, Gale and Ryser [19, 35] gave a well-known complete characterization, known as the Gale-Ryser conditions, for a pair of sequences (a, b) to be a bigraphic degree partition. These conditions imply also a polynomial time decision algorithm for BDR^P (by applying a variant of the Havel-Hakimi construction procedure).

Given the Gale-Ryser characterization, the fact that the well-known PARTITION problem is pseudo-polynomial (cf. [9, 16]) would appear to suggest a plausible approach for attacking the (plain) bigraphic degree realization problem BDR, by searching for a bigraphic degree partition for the given sequence d. This approach makes sense because $d_1 < n$ is a necessary condition for a sequence d to be graphic (or bigraphic), and for such d, finding one of the partitions (if one exists) is achievable in polynomial time. However, this approach encounters several immediate obstacles. First, it is possible that some partitions of d are bigraphic while other partitions are not[1]. Second, there may be exponentially many different partitions for a given sequence[2]. Other approaches may be attempted, based on the special structure required by a bigraphic degree partition. So far, however, the bigraphic degree realization remains unresolved: There is no characterization for the class of bigraphic degree sequences, and it is unknown whether the BDR problem is NP-complete.

Towards attacking the bigraphic degree realization problem we study a specific and significant type of partitions that separate the degrees of a sequence by their size, i.e., into two blocks, a block of high degrees and a block of low ones. We refer to such partitions as *High-Low partitions*. These partitions represent an extreme approach, striving to maximize the difference between the degrees on the two sides of the partition. (An opposite extreme approach would be to try to make the two sides as *similar* as possible; we study the role of such partitions, referred to as *equal* partitions, in [8].) High-Low partitions are thus interesting in their own right, and can also be viewed as bipartite counterparts of other partition types considered in the literature, such as *core-periphery* partitions, which commonly occur in social networks, see [3, 4, 5, 11, 34, 46].

1.2 Our Contribution

Our key observation concerning the role of High-Low partitions is that there are special instances where deciding the realizability of the High-Low partition resolves also the BDR problem, i.e., decides whether the given sequence is bigraphic or not (and if so, provides an *exact* realization). Moreover, when the BDR problem is decided in the negative or is unsolved, we are able to generate *approximate* realizations for the given sequence based on its High-Low partition.

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[1] Consider the sequence $(6,6,4,4,2,2,2,2,2,2)$ which has three partitions: *(i.)* $(6,6,4)(4,2,2,2,2,2,2)$, *(ii.)* $(6,4,2,2,2)(6,4,2,2,2)$, and *(iii.)* $(6,6,2,2)(4,4,2,2,2,2)$. However, only the last partition is bigraphic.

[2] Consider the sequence $d = (n, n, n-1, n-1, \ldots, 2, 2, 1, 1)$ of length $2n$, for n divisible by 4. Split d into subsequences $B_j = (x, x, x+1, x+1, x+2, x+2, x+3, x+3)$, for $x = 4(j-1)+1$, noting that each B_j has three partitions: *(i.)* $(x, x+1, x+2, x+3)(x, x+1, x+2, x+3)$, *(ii.)* $(x, x, x+3, x+3)(x+1, x+1, x+2, x+2)$, and *(iii.)* $(x+1, x+1, x+2, x+2)(x, x, x+3, x+3)$. This yields $3^{n/4}$ different partitions for d.

Apart from being easier to generate, approximate realizations are desirable for several other reasons. In many applications it is required to design a network given some partial specifications, and returning that there is no suitable network (even provably) is not satisfactory. Additionally, specifications often rely on imprecise data making exact realizations unattractive. Bar-Noy et al. [6] survey different applications and types of approximate network realizations. Our approximate realizations are either

(i) bipartite *multigraphs* (namely, graphs that allow *parallel edges*) with low *maximum* multiplicity of parallel edges or

(ii) *super-realizations* which are bipartite (plain) graphs where a subset of the vertices adheres to the given degree sequence and with a small number of additional vertices and edges.

The *High-Low partition* of a non-increasing sequence $d = (d_1, \ldots, d_n)$ has the form $H = (d_1, \ldots, d_k)$ and $L = (d_{k+1}, \ldots, d_n)$ for some k. The partition (H, L) is *balanced* if $\sum_{i=1}^{k} d_i = \sum_{i=k+1}^{n} d_i$. Clearly, a bigraphic degree partition is necessarily balanced.

Well-behaved High-Low Partition. It turns out that for High-Low partitions, the first Gale-Ryser conditions are of paramount significance. These conditions stipulate that the largest degree on each side must not exceed the number of vertices on the other side (or formally, $d_1 \leq n - k$ and $d_{k+1} \leq k$). We refer to a balanced High-Low partition (H, L) that satisfies the first Gale-Ryser conditions as a *well-behaved* High-Low partition.

Being well-behaved does not, in itself, ensure that the partition is bigraphic[3]. It does, however, enable us to resolve the BDR problem: as we show in Section 3 that the BDR problem is solvable for a non-increasing sequence d that admits a well-behaved High-Low partition (H, L). Specifically, when d admits a well-behaved High-Low partition (H, L), it suffices to test the entire collection of Gale-Ryser conditions on (H, L). If all the conditions are met, then (H, L) is a bigraphic degree partition, hence d is a bigraphic degree sequence. If, on the other hand, one or more of the Gale-Ryser conditions is violated for (H, L), then *every* partition of d must violate one Gale-Ryser condition and d has *no* bigraphic degree partition. It follows that d itself is not a bigraphic degree sequence. On the positive side, we show in Section 4 that even in case a well-behaved High-Low partition fails to be bigraphic, it is still what we call 2-*bigraphic*, namely, it has a realizing bipartite *multigraph* whose maximum edge multiplicity is 2.

Multigraph Realizations. Next, we look at sequences that have a balanced High-Low partition that is *not* well-behaved, i.e., the first Gale-Ryser conditions are violated. Based on the High-Low partition we provide approximate realizations by bipartite *multigraphs* (without loops) measuring their quality by the maximum multiplicity of edges. We consider this notion for general graphs and bipartite graphs. Let r, t be positive integers. A sequence d of non-negative integers is said to be r-graphic if there exists a multigraph G such that $\deg(G) = d$ and the maximum multiplicity of an edge in G is at most r. Similarly, a partition (a, b) is t-bigraphic if there exists a bipartite multigraph $G(A, B, E)$ such that the maximum multiplicity of an edge in G is at most t and the sequences of degrees of the vertices of A and B are equal to a and b, respectively.

[3] Consider the sequence $((6m)^m, (2m)^{5m+1}, 1^{2m})$ (superscripts denote multiplicities of degrees) which has a well-behaved High-Low partition $H = ((6m)^m, (2m)^{m+1})$, $L = ((2m)^{4m}, 1^{2m})$, but it is not bigraphic.

In Section 4, we show that the balanced High-Low partition (H, L) of an r-graphic sequence is t-bigraphic where $t = \max\{t(d), 2r\}$ and $t(d)$ is a parameter indicating the extent to which (H, L) violates the first Gale-Ryser conditions.

Super-Graph Realizations. In Section 5, we deal with sequences where the High-Low partition is not balanced, and study the *High-Low near-partition* obtained by taking $H = \{d_1, \ldots, d_k\}$ and $L = \{d_{k+1}, \ldots, d_n\}$ for the smallest k such that $\sum_{i=1}^{k} d_i \geq \sum_{i=k+1}^{n} d_i$. For sequences where (H, L) satisfies the first Gale-Ryser conditions (called *quasi-well-behaved*), we come close to resolving its realizability status: Either we provide a bipartite super-realization, i.e., a bipartite graph G where $\deg(G) = d'$, and d is a subsequence of d' such that $|d'| - |d| \leq 2(d_k - 1)$ or we decide that d is not bigraphic.

Finally, in Section 6 we show how to combine the results on the two different types of (single-criterion) approximate realizations to yield bi-criterion approximate realizations, i.e., realizations by bipartite super-multigraphs.

1.3 Related Work

Next to characterizing graphic degree sequences, several related questions were considered: Given a degree sequence, find all the (non-isomorphic) graphs that realize it, count all its (non-isomorphic) realizing graphs, and uniformly sample a random realization. These questions are well-studied, cf. [13, 18, 21, 24, 26, 37, 39, 40, 44, 45], and have important applications in network design, randomized algorithms, social networks [10, 15, 17, 29] and chemical networks [38]. Miller [30] recapitulates reduced criteria for a sequence to be graphic. For surveys on graphic sequences, see [41, 42, 43].

Extensive literature exists on finding realizations having certain properties. A degree sequence is *potentially* P-graphic if it has a realization with property P where P is some graph theoretic property. Rao [33] surveys results (see references therein) on several properties including k-edge/k-vertex connected, hamiltonian and tournament. Characterizing potentially bipartite sequences, i.e., the BDR problem, is mentioned as an open problem.

Additional results include a characterization for trees (cf. [20]). The existing results on *planar* graphs are restricted to k*-sequences*, in which the difference between $\max d_i$ and $\min d_i$ is at most k, for $k = 0, 1, 2$ [1, 36]. Degree sequences of *split* graphs (see [23]), *threshold* graphs (see [22]), *matrogenic* graphs (see [28]) and *difference* graphs (see [22]) are fully characterized. Moreover, Degree sequences of *chordal*, *interval*, and *perfect* graphs were studied in [12].

Realizations by multigraphs were considered by Owens and Trent [32] showing how to realize degree sequences with minimum total number of parallel edges or loops (see [31, 27] for improved algorithms). Interestingly, computing a realization with maximum total number of parallel edges is known to be NP-hard, see [25].

2 Preliminaries and Definitions

Let $G = (V, E)$ be a multigraph without loops. Denote by $E_G(v, w)$ the multiset of edges connecting $v, w \in V$. The *maximum multiplicity* of G is $\mathsf{MaxMult}(G) = \max_{(v,w)\in E}(|E_G(v, w)|)$.

2.1 Degree Sequences of Graphs and Multigraphs

Let $d = (d_1, d_2, \ldots, d_n)$ be a sequence of nonnegative integers in nonincreasing order. The *volume* of d is $\sum d = \sum_{i=1}^{n} d_i$. We call a sequence with even volume a *degree sequence*.

We present the characterization of Erdös and Gallai [18] for graphic degree sequences.

▶ **Theorem 1** (Erdös-Gallai [18]). *A degree sequence $d = (d_1, d_2, \ldots, d_n)$ is graphic if and only if, for $\ell = 1, \ldots, n$,*

$$\sum_{i=1}^{\ell} d_i \leq \ell(\ell-1) + \sum_{i=\ell+1}^{n} \min\{\ell, d_i\}. \tag{1}$$

We refer to Equation (1) as the *ℓ-th Erdös-Gallai inequality* EG_ℓ. The Erdös-Gallai characterization allows us to check if a sequence is graphic or not in polynomial time.

A degree sequence d is *r-graphic*, for a positive integer r, if there is a multigraph G such that $\deg(G) = d$ and $\mathsf{MaxMult}(G) \leq r$. The Erdös-Gallai inequalities were extended to characterize r-graphic sequences.

▶ **Theorem 2** (Chungphaisan [14]). *Let r be a positive integer. The degree sequence $d = (d_1, d_2, \ldots, d_n)$ is r-graphic if and only if, for $\ell = 1, \ldots, n$,*

$$\sum_{i=1}^{\ell} d_i \leq r\ell(\ell-1) + \sum_{i=\ell+1}^{n} \min\{r\ell, d_i\}, \tag{2}$$

Note that, for a given sequence d, the minimum r such that d is r-graphic can be determined in polynomial time.

2.2 Degree Sequences of Bipartite Graphs and Multigraphs

Let d be a degree sequence for which $\sum d = 2m$ for some integer m. A *block* of d is a subsequence a such that $\sum a = m$. Define $B(d) := \{a \subset d \mid \sum a = m\}$ as the set of all blocks of sequence d. For each $a \in B(d)$ there is a disjoint $b \in B(d)$ that completes it to form a partition of d (so that merging them in sorted order yields d). We call such a pair $a, b \in B(d)$ a *(balanced) partition* of d since $\sum a = \sum b$. Denote the set of all degree partitions of d by $\mathsf{BP}(d) = \{\{a, b\} \mid a, b \in B(d),\ d \setminus a = b\}$.

The Gale-Ryser theorem characterizes bigraphic degree partitions.

▶ **Theorem 3** (Gale-Ryser [19, 35]). *Let d be a degree sequence and partition $(a, b) \in \mathsf{BP}(d)$ where $a = (a_1, a_2, \ldots, a_p)$ and $b = (b_1, b_2, \ldots, b_q)$. The partition (a, b) is bigraphic if and only if*

$$\sum_{i=1}^{\ell} a_i \leq \sum_{i=1}^{q} \min\{\ell, b_i\}, \tag{3}$$

for $\ell = 1, \ldots, p$.

We refer to Equation (3) as the *ℓ-th Gale-Ryser inequality* GR^L_ℓ on the left. By symmetry, the partition (a, b) is bigraphic if and only if $\sum_{i=1}^{\ell} b_i \leq \sum_{i=1}^{p} \min\{\ell, a_i\}$, for $\ell = 1, \ldots, q$. We refer to this equation as the *ℓ-th Gale-Ryser inequality* GR^R_ℓ on the right.

Let t be a positive integer. A degree sequence d is t-bigraphic if d has a partition $(a, b) \in \mathsf{BP}(d)$ such that there is a bipartite multigraph $G = (A, B, E)$ such that $\mathsf{MaxMult}(G) \leq t$, $|A| = |a|$, $|B| = |b|$, and the sequences of degrees of the vertices of A and B are equal to a and b, respectively. We also say that partition (a, b) is t-bigraphic. Miller [30] cites the following result of Berge characterizing t-bigraphic partitions.

▶ **Theorem 4** (Berge [30]). *Consider a positive integer t, a degree sequence d and a partition $(a, b) \in BP(d)$ where $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_q)$. The partition (a, b) is t-bigraphic if and only if*

$$\sum_{i=1}^{\ell} a_i \le \sum_{i=1}^{q} \min\{\ell t, b_i\}, \tag{4}$$

for $\ell = 1, \ldots, p$.

2.3 Ferrers Diagrams, Conjugate Sequences and Majorization

Ferrers diagrams (cf. [2]) are instrumental in illustrating integer sequences and partitions graphically (see Figure 1).

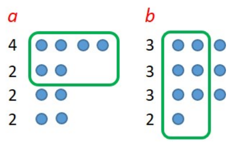

Figure 1 The Ferrers diagram of the partition $a = (4, 2, 2, 2)$ and $b = (3, 3, 3, 1)$.

Conjugate sequences provide us with a convenient alternative way to represent the Gale-Ryser conditions. The *prefix-sum* of a sequence d up to index i is $\sum(d[i]) = \sum_{j=1}^{i} d_j$. Let the sequences a, b have the same length, i.e., $p = q$. (If two sequences are not of the same length, the shorter sequence can be padded with 0's.) Given a degree sequence d, its *conjugate* sequence $\widetilde{d} = (\widetilde{d}_1, \widetilde{d}_2, \ldots, \widetilde{d}_{d_1})$ is defined by $\widetilde{d}_k = |\{i \mid d_i \ge k\}|$.

The duality between $\sum a[i]$ and $\sum \widetilde{b}[i]$ is captured graphically in the Ferrers diagram: a_i is the i-th row on the left, and $\widetilde{b}_i$ is the i-th column on the right. Consequently, $\sum a[i]$ is the sum of the first i rows on the left, whereas $\sum \widetilde{b}[i]$ is the sum of the first i columns on the right. (In the figure, the green ovals capture $\sum a[2]$ and $\sum \widetilde{b}[2]$.)

Majorization is a partial order on degree sequences: a *majorizes* b ($b \trianglelefteq a$) if and only if $\sum(b[i]) \le \sum(a[i])$ for every $i \in [1, q]$.

Observe that if $a \trianglerighteq b$, then $\widetilde{b} \trianglerighteq \widetilde{a}$. The Gale-Ryser theorem can now be reformulated using majorization and conjugates, noting that $\sum_{i=1}^{q} \min\{\ell, b_i\} = \sum_{i=1}^{\ell} \widetilde{b}_i$.

▶ **Theorem 5** (Gale-Ryser [19, 35], conjugate representation). *Let d be a degree sequence and $(a, b) \in BP(d)$. The partition (a, b) is bigraphic if and only if $a \trianglelefteq \widetilde{b}$.*

Furthermore, a graphic description of the ℓ-th Gale-Ryser condition on the left is that the sum of the first ℓ *rows* on the left Ferrers diagram (representing a), must be no greater than the sum of the first ℓ *columns* on the right Ferrers diagram (representing b). (The sequence is bigraphic if and only if these conditions hold for every $\ell \le p$.) As the sequences a and b (or equivalently their Ferrers diagrams) can switch sides, it is clear that the Gale-Ryser conditions are symmetric, i.e., $a \trianglelefteq \widetilde{b}$ if and only if $b \trianglelefteq \widetilde{a}$.

3 Well-Behaved High-Low Partitions

In this section, we study sequences that have a well-behaved High-Low partition. Let $d = (d_1, \ldots, d_n)$ be a degree sequence with a High-Low partition $HL(d) = (H, L)$ where $H = (d_1, \ldots, d_k)$ and $L = (d_{k+1}, \ldots, d_n)$, for some positive integer $k < n$. We assume that $d_1 \le n - k$ and that $d_{k+1} \le k$, i.e., (H, L) is well-behaved.

In the following, we suppose that (H, L) is not bigraphic implying that at least one Gale-Ryser condition on H and L, respectively, is not satisfied. Let x be the index of the first violated Gale-Ryser condition on L, i.e., such that $\sum(L[i]) \leq \sum(\widetilde{H}[i])$, for $i < x$, and

$$\sum(L[x]) > \sum(\widetilde{H}[x]). \tag{5}$$

▶ **Observation 6.** *For a well-behaved $HL(d) = (H, L)$, $d_k < x$.*

Proof. For $i \leq d_k$, we have that $\sum(\widetilde{H}[i]) = i \cdot k$, and $\sum(L[i]) \leq i \cdot d_{k+1}$ (see Figure 2). Since (H, L) is well-behaved, $d_{k+1} \leq k$, and it follows that $\sum(L[i]) \leq \sum(\widetilde{H}[i])$, hence the ith Gale-Ryser condition on L holds, for $i \leq d_k$. Consequently, $x > d_k$. ◀

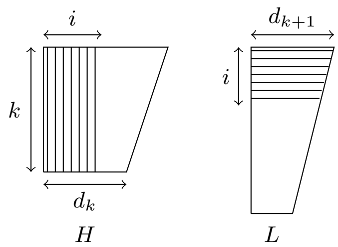

Figure 2 $\sum \tilde{H}[i]$ vs. $\sum L[i]$.

▶ **Observation 7.** *For a well-behaved $HL(d) = (H, L)$, $x > k$.*

Proof. If $d_k \geq k$, then the claim follows from Observation 6. Now suppose $d_k < k$. We need to show that $\sum(\widetilde{H}[j]) \geq \sum(L[j])$ for every $d_k < j \leq k$. This follows because

$$\sum(\widetilde{H}[j]) \geq \sum(\widetilde{H}[d_k]) = k \cdot d_k \geq k \cdot d_{k+1} \geq \sum(L[k]) \geq \sum(L[j]).$$ ◀

Our main goal is to prove the following result.

▶ **Theorem 8.** *Consider a degree sequence d with a well-behaved High-Low partition $HL(d) = (H, L)$. If (H, L) is not bigraphic, then d is not bigraphic (i.e., no partition of d is bigraphic).*

First, we prove a weaker statement. For some other partition $(A, B) \in \mathtt{BP}(d)$, define

$$H^m = H \setminus A, \qquad L^m = L \setminus B, \qquad H^s = H \cap A, \text{ and } \qquad L^s = L \cap B.$$

Moreover, we denote $h = |H^m|$ and $\ell = |L^m|$. Hence, the partition (A, B) can be viewed as obtained from (H, L) by *moving* a subset H^m from the left to the right, and a subset L^m from the right to the left. The subsets H^s and L^s *stay* on their respective sides. Note that, keeping A and B sorted in nonincreasing order, the elements of H^m appear at the beginning of B and the elements of L^m appear at the end of A. Figure 3 illustrates this transformation.

For an index i and an ordered set of integers S, denote the first i elements of S by $S[i]$. Moreover, we use $\sum(S) = \sum_{z \in S} z$ to denote the sum of elements in S. In line with our previous definition, $\sum(S[i])$ is the prefix-sum of S up to index i.

▶ **Definition 9.** *Call the partition (A, B)* benign *if L^m contains at most h of the top x elements of L, i.e., $|L[x] \cap L^m| \leq h$ or equivalently $|L[x] \setminus L^m| \geq x - h$.*

▶ **Lemma 10.** *Consider a degree sequence d with well-behaved High-Low partition $HL(d) = (H, L)$. If (H, L) is not bigraphic, then no benign partition of d is bigraphic.*

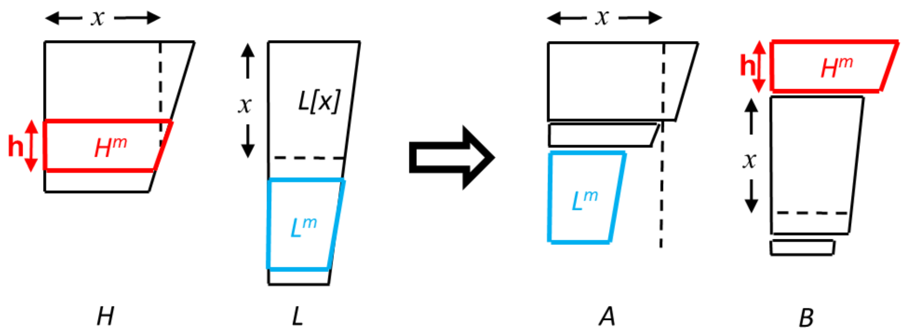

Figure 3 Illustration of the transformation from (H, L) to (A, B). Partition (A, B) is benign since $L[x] \cap L^m$ is empty.

Proof. Let d and $HL(d) = (H, L)$ be as in the lemma, and let (A, B) be some benign partition of d. To show the lemma, we show that $\sum(B[x]) > \sum(\widetilde{A}[x])$ holds, i.e., the partition (A, B) violates GR_x^R. First, verify that

$$\sum(B[x]) = \sum(H^m) + \sum(L^s[x-h]),$$

and that

$$\sum(\widetilde{A}[x]) = \sum(L^m) + \sum(\widetilde{H^s}[x]) \tag{6}$$

since $x > d_{k+1}$ by Observation 6. We need to show that $\sum(L^s[x-h]) > \sum(\widetilde{H^s}[x])$ since $\sum(H^m) = \sum(L^m)$. By Equation (5), it follows that

$$\sum(L[x]) - \sum(L^s[x-h]) + \sum(L^s[x-h]) > \sum(\widetilde{H}[x]) = \sum(\widetilde{H^s}[x]) + \sum(\widetilde{H^m}[x]).$$

To finish the proof, we argue that $\sum(\widetilde{H^m}[x]) \geq \sum(L[x]) - \sum(L^s[x-h])$. Since (A, B) is benign, $L^s[x-h]$ contains at least $x - h$ rows of $L[x]$, or equivalently $L[x] \setminus L^s[x-h]$ are at most h rows of L. It follows that

$$\sum(L[x]) - \sum(L^s[x-h]) \leq h \cdot d_k \leq \sum(\widetilde{H^m}[x])$$

where the last inequality holds due to Observation 6. ◀

Using the symmetry of the Gale-Ryser conditions we prove the following corollary. We introduce another notation, $s = |L[x] \cap L^m|$.

▶ **Corollary 11.** *Consider a degree sequence d with well-behaved High-Low partition $HL(d) = (H, L)$. If (H, L) is not bigraphic, then no partition of d where $x - s \leq k - h$ is bigraphic.*

Proof. The result is obtained from switching the definitions of the moving and staying parts. Partition (B, A) can be viewed as obtained from (H, L) by moving H^s from H to L and L^s from L to H. Note that $|H^s| = k - h$ and $|L^s \cap L[x]| = x - s$. The condition $x - s \leq k - h$ implies that (B, A) is benign, and the corollary follows with Lemma 10. ◀

Another benign case is when the largest element of H^m or H^s is smaller or equal to x. Denote $\lambda = \min\{\max\{H^m\}, \max\{H^s\}\}$.

▶ **Lemma 12.** *Consider a degree sequence d with well-behaved High-Low partition $HL(d) = (H, L)$. If (H, L) is not bigraphic, then no partition of d where $\lambda \leq x$ is bigraphic.*

Proof. Consider d and (H, L) as in the lemma. First, we verify the claim for $\lambda = \max\{H^m\}$. To that end, let (A, B) be a partition of d such that $\max\{H^m\} \le x$. To prove the claim, we show that $\sum(B[x]) > \sum(\widetilde{A}[x])$ holds. With Equations (5) and (6) it follows that

$$\sum(L[x]) > \sum(\widetilde{H}[x]) = \sum(\widetilde{H^s}[x]) + \sum(\widetilde{H^m}[x]) \overset{(\star)}{=} \sum(\widetilde{H^s}[x]) + \sum(H^m) = \sum(\widetilde{A}[x]),$$

where $(\star)$ holds since $\max\{H^m\} \le x$. Because (H, L) is a High-Low partition, B majorizes L and, in particular, $\sum(B[x]) \ge \sum(L[x]) > \sum(\widetilde{A}[x])$ holds.

With the arguments used to prove Corollary 11, the claim also holds in case $\lambda = \max\{H^s\}$, and the lemma follows. ◀

Now we are ready to prove Theorem 8.

Proof of Theorem 8. Consider d and $HL(d) = (H, L)$ as in the lemma, and let (A, B) be some partition of d. Towards a contradiction suppose that (A, B) is bigraphic. It follows that

(a) $\sum(B[x]) \le \sum(\widetilde{A}[x])$,
(b) $\sum(A[x]) \le \sum(\widetilde{B}[x])$,
(c) $\lambda \le \sum(B[1]) \le \sum(\widetilde{A}[1]) = k - h + \ell$,
(d) $\lambda \le \sum(A[1]) \le \sum(\widetilde{B}[1]) = n - k + h - \ell$,
(e) $x < \lambda$,
(f) $s > h$, and
(g) $s < x - k + h$.

Equations (a), (b), (c), (d) are Gale-Ryser conditions (see Theorem 3), and the upper bounds on λ hold by definition. As (A, B) is bigraphic, Equation (e) is implied by Lemma 12. Moreover, Equations (f) and (g) are implied by Lemma 10 and Corollary 11, respectively.

Recall that Equation (5) reads

$$\sum(L[x]) > \sum(\widetilde{H}[x]) = \sum(\widetilde{H^s}[x]) + \sum(\widetilde{H^m}[x]).$$

Note that (d) and (e) imply $x < \lambda \le n - k + h - \ell = |B|$ (i.e., $B[x]$ is a proper subset of B). With Equation (a) we have that

$$\sum(\widetilde{H^s}[x]) + \sum(L^m) = \sum(\widetilde{A}[x]) \ge \sum(B[x]) = \sum(L^s[x-h]) + \sum(H^m).$$

Since $\sum(L^m) = \sum(H^m)$ we get a lower bound on $\sum(\widetilde{H^s}[x])$:

$$\sum(\widetilde{H^s}[x]) \ge \sum(L^s[x-h]). \tag{7}$$

Note that (c) and (e) imply $x < \lambda \le k - h + \ell = |A|$ (i.e., $A[x]$ is a proper subset of A). Since $k - h < x$ and $d_{k+1} < x$, it follows from Observations 6 and 7 that

$$\sum(L^s) + \sum(\widetilde{H^m}[x]) = \sum(\widetilde{B}[x]) \ge \sum(A[x]) = \sum(H^s) + \sum(L^m[x-k+h]).$$

Since $\sum(L^s) = \sum(H^s)$ we get a lower bound on $\sum(\widetilde{H^m}[x])$:

$$\sum(\widetilde{H^m}[x]) \ge \sum(L^m[x-k+h]). \tag{8}$$

Equation (5) together with Equations (7) and (8) yields

$$\sum(L[x]) > \sum(L^s[x-h]) + \sum(L^m[x-k+h]).$$

Observe that $L^s[x-h]$ contains $L[x] \setminus L^m$ as $s > h$. Since $s < x - k + h$, $L^m[x-k+h]$ contains $L[x] \cap L^m$. It follows that $\sum(L^s[x-h]) \cup \sum(L^m[x-k+h])$ contains $L[x]$, and so

$$\sum(L^s[x-h]) + \sum(L^m[x-k+h]) \ge \sum(L[x])$$

holds contradicting the previous equation. Consequently, (A, B) cannot be bigraphic. ◀

Theorem 8 implies that the bigraphic realizability status can be fully resolved for a degree sequence with a well-behaved High-Low partition.

▶ **Theorem 13.** *Let d be a degree sequence with a well-behaved High-Low partition. It can be decided in polynomial time whether d is bigraphic or not. If d happens to be bigraphic, a bipartite graph realizing d can be computed in polynomial time.*

Proof. Let d be as in the theorem. Computing the High-Low partition $HL(d) = (H, L)$ is straight forward. Using the Gale-Ryser theorem (Theorem 3), we decide if (H, L) is bigraphic or not, and due to Theorem 8 if d is bigraphic or not. If (H, L) is bigraphic, a bipartite graph realizing d can be computed by applying the Havel-Hakimi algorithm to one side of the partition (see [7] for details). All steps are performed in polynomial time. ◀

4 Realizations by Bipartite Multigraphs

Our goal is to provide realizations based on multigraphs without loops where the maximum multiplicity of parallel edges is used to measure their quality. We examine degree sequences that have a balanced High-Low partition but are not necessarily bigraphic. Let r be a positive integer. In the following, we consider an r-graphic degree sequence d with High-Low partition $HL(d) = (H, L)$ where $H = (d_1, \dots, d_k)$ and $L = (d_{k+1}, \dots, d_n)$, for some integer $k \in [1, n-1]$. We do not assume that d is well-behaved, and quantify the violation of the first Gale-Ryser conditions with the following definitions. Let

$$t_H(d) = \left\lceil \frac{d_1}{n-k} \right\rceil \qquad \text{and} \qquad t_L(d) = \left\lceil \frac{d_{k+1}}{k} \right\rceil,$$

and define $t(d) = \max\{t_H(d), t_L(d)\}$. (Note that sequence d has a well-behaved High-Low partition if $t(d) = 1$.) First, we observe that $t_H(d)$ is bounded for r-graphic sequences.

▶ **Lemma 14.** *Let d be an r-graphic degree sequence with High-Low partition $HL(d) = (H, L)$. Then, $t_H(d) \le 2r$.*

Proof. Let d be as in the lemma with $H = (d_1, \dots, d_k)$ and $L = (d_{k+1}, \dots, d_n)$. Since (H, L) is a High-Low partition, we have that $k \le n - k$. Moreover, $d_1 \le r(n-1)$ as d is r-graphic. It follows that

$$t_H(d) = \left\lceil \frac{d_1}{n-k} \right\rceil \le \left\lceil \frac{r(n-1)}{n-k} \right\rceil \le \left\lceil \frac{r(k+n-k)}{n-k} \right\rceil \le 2r. \qquad ◀$$

The main result of this section is the next theorem. Its proof is omitted in the conference version of the paper.

▶ **Theorem 15.** *Let d be an r-graphic degree sequence with High-Low partition $HL(d) = (H, L)$ and let $t = \max\{t(d), 2r\}$. Then, (H, L) is t-bigraphic.*

We remark that the conclusion of Theorem 15 does not hold if the degree sequence d is not r-graphic. To see this, consider the non-graphic sequence $d = ((9m)^{m-1}, 6m+1, (3m)^{3m-1}, 1^1)$ for some positive integer m. (We use superscripts to denote the multiplicities of degrees.) Verify that d has a High-Low partition (H, L) where $H = ((9m)^{m-1}, 6m+1)$ and $L = ((3m)^{3m-1}, 1^1)$. We have $t_H(d) = t_L(d) = 3$, but the conditions of Theorem 4 for 3-bigraphic degree sequences are violated. Specifically, the condition for index $m-1$ requires $9m(m-1) \le (3m-1) \cdot 3 \cdot (m-1) + 1$, which is false.

We complement Theorem 15 by providing an existential lower bound.

▶ **Lemma 16.** *There are degree sequences d with High-Low partition $HL(d)$ such that $t(d) > 1$, and d is not t'-bigraphic for any $t' < t(d)$.*

Proof. Let t be a positive integer, and let p be some prime number. Set $x \geq t(p+1)$ such that p is not a prime factor of x, and consider the sequence $d = (x^{p+2}, p^x)$. Verify that sequence d has a High-Low partition $HL(d) = (H, L)$ with blocks $H = (x^{p+1})$ and $L = (x, p^x)$. By the choice of x and p, d does not have other partitions. We have that $t(d) = t_L(d) \geq t$. It follows that d is not t'-bigraphic for any $t' \leq t$. ◀

For graphic degree sequences, we get the following result.

▶ **Corollary 17.** *Let d be a graphic degree sequence with High-Low partition $HL(d) = (H, L)$ and $t = t(d)$.*
(i) *If $t = 1$, then (H, L) is 2-bigraphic.*
(ii) *If $t > 1$, then (H, L) is t-bigraphic.*

If there is a well-behaved High-Low partition, Theorem 8 implies the following result.

▶ **Corollary 18.** *Let d be a graphic degree sequence with well-behaved High-Low partition $HL(d) = (H, L)$. Then, either*
(i) *(H, L) is bigraphic, or*
(ii) *d is not bigraphic and (H, L) is 2-bigraphic.*

Combinatorial Bounds. In the last part of this section, we show bounds on $t_L(d)$ and $t_H(d)$ in case the degree sequence d is r-graphic or bigraphic. The proofs of the following two theorems are omitted in the conference version of the paper.

We start with r-graphic degree sequences. Lemma 14 provides a bound on $t_H(d)$. The next theorem establishes a bound on $t_L(d)$.

▶ **Theorem 19.** *Let d be an r-graphic sequence with High-Low partition $HL(d) = (H, L)$. Then,*

$$t_L(d) \leq \left\lceil \frac{r(k+1)}{2} \right\rceil .$$

We note that the bound of Theorem 19 is tight and that for graphic sequences, $t_L(d) < t_H(d)$ as well as $t_H(d) < t_L(d)$ can occur. To see this, consider the following two examples.

(1) The graphic sequence $d = (6, 3, 3, 3, 3, 3, 3)$ has exactly one (High-Low) partition (H, L) with blocks $H = (6, 3, 3)$ and $L = (3, 3, 3, 3)$. Verify that $t_L(d) = 2$ and $t_H(d) = 1$.

(2) The degree sequence $d' = ((\frac{k(k+1)}{2})^{k+1}, 1^{\frac{k(k+1)}{2}(k-1)})$, for a positive integer k, is graphic (to see this, observe that $\sum d'$ is even, and that the $(k+1)$th-EG inequality holds; for such a block sequence this is sufficient, see, e.g., [30]). The $(k+1)$th-EG inequality reads $(k(k+1)/2) \cdot (k+1) \leq k(k+1) + (k(k+1)/2) \cdot (k-1)$, which trivially holds. Moreover, $HL(d') = (H', L')$ where $H' = ((\frac{k(k+1)}{2})^k)$ and $L' = ((\frac{k(k+1)}{2}), 1^{\frac{k(k+1)}{2}(k-1)})$. Hence, $|H'| = k$ and $d_{k+1} = \frac{k(k+1)}{2}$.

The next result improves the bounds on $t_L(d)$ and $t_H(d)$ for bigraphic degree sequences.

▶ **Theorem 20.** *Let d be a bigraphic sequence with High-Low partition $HL(d)$. Then,*

$$t_H(d) \leq 1, \quad \textit{and} \qquad t_L(d) \leq \left\lceil \frac{k+2+1/k}{4} \right\rceil .$$

5 Realizations by Bipartite Super-Graphs

We next focus on the situation when the given sequence does not admit a balanced High-Low partition (with $\sum H = \sum L$). In this case, we consider the *High-Low near-partition (HLnP)* (H, L) of d, obtained by taking $H = \{d_1, \ldots, d_k\}$ and $L = \{d_{k+1}, \ldots, d_n\}$ for the smallest k such that $\sum H \geq \sum L$ (i.e., $\sum H - d_k < \sum L + d_k$).

Define the *imbalance gap* of the sequence d to be $IG(d) = \sum H - \sum L$. Rearranging the above two inequalities, we get the following.

▶ **Observation 21.** $0 \leq IG(d) < 2d_k$.

In the remainder of this section we assume that $IG(d) > 0$. We refer to such sequences d as *High-Low-imbalanced*. We call the High-Low near-partition (H, L) *quasi-well-behaved* if it satisfies the first Gale-Ryser condition on both sides.

The main result of this section is that when the given sequence d is High-Low-imbalanced but enjoys a quasi-well-behaved High-Low near-partition, it is possible to come close to resolving its realizability status, in the sense that there is a poly-time algorithm that either decides that d is not bigraphic, or constructs a bipartite super-realization for d with a small number of new vertices and edges. We use the operator $\circ$ to merge two sequences.

▶ **Definition 22.** *A bipartite* (n', m') *super-realization of an* n*-integer sequence* d*, for* $n' > n$ *and* $m' > \sum d$*, is a bipartite graph* $G(A, B, E)$ *such that* $|A \cup B| = n + n'$*,* $|E| = \sum d + m'$*, and* $deg(A \cup B) = d \circ d'$ *for some sequence* d' *of* n' *integers.*

Our algorithm hinges on the idea of completing a High-Low near-partition of a given imbalanced sequence d into a (balanced) High-Low partition of a larger sequence in a suitable way. Consider a family D of quasi-well-behaved n-integer nonincreasing sequences. A mapping $\varphi : D \mapsto D'$ is said to be a *valid completion mapping* for D if for every $d \in D$ such that $HLnP(d) = (H, L)$ and $k = |H|$, the generated n'-integer sequence $d' = \varphi(d)$ satisfies the following properties.

(P1) $(H, L \circ d')$ is a well-behaved High-Low partition of $d \circ d'$.
(P2) If d is bigraphic then $d \circ d'$ is bigraphic as well.

The generated sequence $d' = \varphi(d)$ is referred to as the *valid completion* of d.

We now describe a generic Algorithm $A(d, d')$ that, given a High-Low-imbalanced n-integer nonincreasing sequence d with $IG(d) = t$ and a valid completion d' of p integers for d, generates a bipartite (p, t) super-realization for d. The algorithm operates as follows.

1. Construct the High-Low near-partition $HLnP(d) = (H, L)$ for the given sequence d.
2. Let $t \leftarrow IG(d) = \sum H - \sum L$.
3. Set $L' \leftarrow L \circ d'$ (sorted in nonincreasing order).
4. Test all Gale-Ryser conditions on (H, L').
5. If (H, L') is bigraphic, then construct and return a realizing bipartite graph G' for it.
6. Otherwise (* (H, L') is not bigraphic *) return "d is not bigraphic".

▶ **Lemma 23.** *Consider a sequence* d *and let* d' *be a valid completion for* d*. If Algorithm* $A(d, d')$ *returns a graph* G' *(Step 5), then it is a bipartite* (p, t) *super-realization for* d*. If the algorithm returns a negative response (Step 6), then* d *is indeed not bigraphic.*

Proof. The first claim follows immediately by the definition of p and t and the fact that G' is a bipartite realization of (H, L').

To prove the second claim, suppose (H, L') is non-bigraphic. As d' is a valid completion of d, property (P1) implies that (H, L') is a well-behaved High-Low partition of $d \circ d'$. It follows from Theorem 8 that $d \circ d'$ is also non-bigraphic. This, in turn, implies by property (P2) that d is not bigraphic. ◀

▶ **Lemma 24.** *Consider a quasi-well-behaved n-integer nonincreasing sequence d with $t = IG(d)$. The sequence $d' = (1^t)$ is a valid completion for d.*

Proof. To see that (P1) holds, observe that
(1) $(H, L \circ d')$ is a High-Low partition of $d \circ d'$ because the elements of d' are no greater than the elements of H.
(2) it is balanced because $\sum H = \sum(L \circ d')$ by the choice of t.
(3) it satisfies the first Gale-Ryser condition on the left since (H, L) is quasi-well-behaved, so $d_1 \leq n - k \leq n + t - k$.
(4) it satisfies the first Gale-Ryser condition on the right since d_{k+1} satisfies $d_{k+1} \leq k$ by the fact that (H, L) is quasi-well-behaved, and $d'_1 = 1 \leq k$, so $\max(L \circ d') \leq k$.
To see that (P2) holds, suppose d is bigraphic, and let G be a bipartite graph realizing it. Noting that $IG(d)$ must be even (as it is the difference of two integers whose sum is even), let M be a matching consisting of $t/2$ edges. Then $G \cup M$ is a bipartite realization of $d \circ d'$. ◀

We conclude the following.

▶ **Theorem 25.** *Consider a quasi-well-behaved n-integer nonincreasing sequence d and let $d' = (1^{IG(d)})$. Then Algorithm $A(d, d')$, in poly-time, either yields a bipartite $(IG(d), IG(d))$ super-realization for d or decides that d is not bigraphic.*

Our goal is to find a poly-time algorithm that constructs a bipartite super-realization for d with less new vertices. We continue this analysis in the journal version of the paper.

6 Realizations by Super-Multigraphs

We consider super-multigraph realizations based on the High-Low near-partition. Together, Theorems 15 and 25 imply the following.

▶ **Corollary 26.** *Let d be an r-graphic sequence, $d' = d \circ 1^{IG(d)}$, and $t = \max\{2r, t(d')\}$. Then, there is a t-bipartite $(IG(d), IG(d))$ super-realization for d.*

If d is quasi-well-behaved and graphic, we apply Corollary 18 yielding the following.

▶ **Corollary 27.** *Consider a quasi-well-behaved and graphic sequence d and let $d' = (1^{IG(d)})$. Either d is undecided and Algorithm $A(d, d')$ yields in poly-time a bipartite $(IG(d), IG(d))$ super-realization for d, or d is not bigraphic and has a 2-bipartite $(IG(d), IG(d))$ super-realization.*

References

1 P. Adams and Y. Nikolayevsky. Planar bipartite biregular degree sequences. *Discrete Mathematics*, 342:433–440, 2019.
2 G. E Andrews. *The Theory of Partitions*. Cambridge University Press, 1998.
3 C. Avin, M. Borokhovich, Z. Lotker, and D. Peleg. Distributed computing on core–periphery networks: Axiom-based design. *Journal of Parallel and Distributed Computing*, 99:51–67, 2017.

4 C. Avin, Z. Lotker, Y. Nahum, and D. Peleg. Core size and densification in preferential attachment networks. In *International Colloquium on Automata, Languages, and Programming*, pages 492–503. Springer, 2015.
5 C. Avin, Z. Lotker, D. Peleg, Y.-A. Pignolet, and I. Turkel. Elites in social networks: An axiomatic approach to power balance and price's square root law. *PloS one*, 13(10):e0205820, 2018.
6 A. Bar-Noy, T. Böhnlein, D. Peleg, M. Perry, and D. Rawitz. Relaxed and approximate graph realizations. In *International Workshop on Combinatorial Algorithms*, pages 3–19. Springer, 2021.
7 A. Bar-Noy, T. Böhnlein, D. Peleg, and D. Rawitz. On Realizing a Single Degree Sequence by a Bipartite Graph. In *18th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 2022)*, volume 227, pages 1:1–1:17. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022.
8 A. Bar-Noy, T. Böhnlein, D. Peleg, and D. Rawitz. On realizing even degree sequences by bipartite graphs. Unpublished manuscript, 2022.
9 R. Bellman. Notes on the theory of dynamic programming iv-maximization over discrete sets. *Naval Research Logistics Quarterly*, 3(1-2):67–70, 1956.
10 J. K. Blitzstein and P. Diaconis. A sequential importance sampling algorithm for generating random graphs with prescribed degrees. *Internet Mathematics*, 6(4):489–522, 2011.
11 S. P Borgatti and M. G Everett. Models of core/periphery structures. *Social Networks*, 21(4):375–395, 2000.
12 A. A. Chernyak, Z. A. Chernyak, and R. I. Tyshkevich. On forcibly hereditary p-graphical sequences. *Discrete Mathematics*, 64:111–128, 1987.
13 S. A. Choudum. A simple proof of the Erdös-Gallai theorem on graph sequences. *Bull. Austral. Math. Soc.*, 33(1):67–70, 1991.
14 V Chungphaisan. Conditions for sequences to be r-graphic. *Discrete Mathematics*, 7(1-2):31–39, 1974.
15 B. Cloteaux. Fast sequential creation of random realizations of degree sequences. *Internet Mathematics*, 12(3):205–219, 2016.
16 T. H Cormen, C. E Leiserson, R. L Rivest, and C. Stein. *Introduction to algorithms*. MIT press, 2009.
17 Dóra Erdös, R. Gemulla, and E. Terzi. Reconstructing graphs from neighborhood data. *ACM Trans. Knowledge Discovery from Data*, 8(4):23:1–23:22, 2014.
18 P. Erdös and T. Gallai. Graphs with prescribed degrees of vertices [hungarian]. *Matematikai Lapok*, 11:264–274, 1960.
19 D. Gale. A theorem on flows in networks. *Pacific J. Math.*, 7:1073–1082, 1957.
20 G. Gupta, P. Joshi, and A. Tripathi. Graphic sequences of trees and a problem of Frobenius. *Czechoslovak Math. J.*, 57:49–52, 2007.
21 S. Louis Hakimi. On realizability of a set of integers as degrees of the vertices of a linear graph –I. *SIAM J. Appl. Math.*, 10(3):496–506, 1962.
22 P. L. Hammer, T. Ibaraki, and B. Simeone. Threshold sequences. *SIAM J. Algebra. Discr.*, 2(1):39–49, 1981.
23 P. L. Hammer and B. Simeone. The splittance of a graph. *Combinatorica*, 1:275–284, 1981.
24 V. Havel. A remark on the existence of finite graphs [in Czech]. *Casopis Pest. Mat.*, 80:477–480, 1955.
25 H. Hulett, T. G Will, and G. J Woeginger. Multigraph realizations of degree sequences: Maximization is easy, minimization is hard. *Operations Research Letters*, 36(5):594–596, 2008.
26 P.J. Kelly. A congruence theorem for trees. *Pacific J. Math.*, 7:961–968, 1957.
27 D. J. Kleitman. Minimal number of multiple edges in realization of an incidence sequence without loops. *SIAM Journal on Applied Mathematics*, 18(1):25–28, 1970.
28 P. Marchioro, A. Morgana, R. Petreschi, and B. Simeone. Degree sequences of matrogenic graphs. *Discrete Mathematics*, 51(1):47–61, 1984. doi:10.1016/0012-365X(84)90023-2.

29 M. Mihail and N. Vishnoi. On generating graphs with prescribed degree sequences for complex network modeling applications. *3rd ARACNE*, 2002.

30 J. W Miller. Reduced criteria for degree sequences. *Discrete Mathematics*, 313(4):550–562, 2013.

31 A. B Owens. On determining the minimum number of multiple edges for an incidence sequence. *SIAM J. on Applied Math.*, 18(1):238–240, 1970.

32 AB Owens and HM Trent. On determining minimal singularities for the realizations of an incidence sequence. *SIAM J. on Applied Math.*, 15(2):406–418, 1967.

33 S. B. Rao. A survey of the theory of potentially p-graphic and forcibly p-graphic degree sequences. In *Combinatorics and graph theory*, volume 885 of *LNM*, pages 417–440, 1981.

34 M P. Rombach, M. A Porter, J. H Fowler, and P. J Mucha. Core-periphery structure in networks. *SIAM Journal on Applied mathematics*, 74(1):167–190, 2014.

35 H.J. Ryser. Combinatorial properties of matrices of zeros and ones. *Canad. J. Math.*, 9:371–377, 1957.

36 E. F. Schmeichel and S. L. Hakimi. On planar graphical degree sequences. *SIAM J. Applied Math.*, 32:598–609, 1977.

37 G. Sierksma and H. Hoogeveen. Seven criteria for integer sequences being graphic. *J. Graph Theory*, 15(2):223–231, 1991.

38 A. Tatsuya and H. Nagamochi. Comparison and enumeration of chemical graphs. *Computational and structural biotechnology*, 5, 2013.

39 A. Tripathi and H. Tyagi. A simple criterion on degree sequences of graphs. *Discrete Applied Mathematics*, 156(18):3513–3517, 2008.

40 R. Tyshkevich. Decomposition of graphical sequences and unigraphs. *Discrete Mathematics*, 220:201–238, 2000.

41 R. I. Tyshkevich, A. A. Chernyak, and Z. A. Chernyak. Graphs and degree sequences: A survey, I. *Cybernetics*, 23:734–745, 1987.

42 R. I. Tyshkevich, A. A. Chernyak, and Z. A. Chernyak. Graphs and degree sequences: A survey, II. *Cybernetics*, 24:137–152, 1988.

43 R. I. Tyshkevich, A. A. Chernyak, and Z. A. Chernyak. Graphs and degree sequences: A survey, III. *Cybernetics*, 24:539–548, 1988.

44 S.M. Ulam. *A collection of mathematical problems.* Wiley, 1960.

45 N.C. Wormald. Models of random regular graphs. *Surveys in Combin.*, 267:239–298, 1999.

46 X. Zhang, T. Martin, and M. EJ Newman. Identification of core-periphery structure in networks. *Physical Review E*, 91(3):032803, 2015.