Non-Determinism in Lindenmayer Systems and Global Transformations

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Abstract

Global transformations provide a categorical framework for capturing synchronous rewriting systems, generalizing cellular automata to dynamical systems over dynamic spaces. Originally developed for addressing deterministic dynamical systems, the presented work raises the question of non-determinism. While a usual approach is to develop a general non-deterministic setting where deterministic systems can be retrieved as a specific case, we show here that by choosing the right parametrization, global transformations can already be used to handle non-determinism. Context-free Lindenmayer systems, already shown to be captured by global transformation in the deterministic case, are used to illustrate the approach. From this concrete example, the formal obstructions are exhibited, leading to a solution involving a 2-categorical monad and its associated Kleisli construction.

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1 Introduction

Global transformations (GT) has been introduced in [12] as a precise formal description of dynamical systems defined over a space which is also dynamic while being still synchronous. This synchronicity property makes GT apart from the main stream of the literature on graph transformations and graph rewriting systems. They however share with this literature the fact that they are very generically defined using category theory in order not to be tied to a specific kind of space. For instance, GT can be used with directed or undirected graphs, labeled or not, hypergraphs, abstract cell complexes.

In order to present the framework to a public not familiar with category theory, the well known deterministic Lindenmayer systems have been presented in terms of GT in [4]. In this paper, we return to Lindenmayer systems, not as a pedagogical exercise but to explore non-determinism in GT in the simplest possible concrete setting.

A usual approach for studying non-determinism from a deterministic object consists in generalizing the deterministic object into a non-deterministic one and then to show that the original deterministic case is a degenerate case of the new non-deterministic setting. We rather seek to demonstrate that non-determinism is already encompassed as a particular case of the current definition of GT. This is in complete analogy with dynamical systems. Indeed, the general definition of dynamical systems is in terms of sets of states and evolution functions. Non-deterministic dynamical systems are *particular* dynamical systems where

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the sets happen to be powersets of an underlying set of real states. This quest emerged from the necessity to talk about non-deterministic, probabilistic and quantum systems, with the intuition that it should not be very different to what happens for dynamical systems. It is not obvious at first that this is possible, because GT are mainly about the notion of (dynamic) locality, but non-deterministic, probabilistic and quantum systems all exhibit somewhat non-local features of correlation and entanglement.

At this point, it is not possible to say much without properly defining the terms that we use. So we dig into the formal definitions of the main objects in Section 2. This will allow us to show that a naive approach of this quest, based on powersets, does not work as explored in Section 3. In Section 4, we circumvent the obstruction in the most natural way and go to an intuitive solution that does not look as a dynamical system, because the transformation does not go from a set to itself but from a set to a bigger set. Section 5 comes back on the relation between dynamical system and non-deterministic system in terms of monad and Kleisli category. This well-known categorical point of view on dynamical systems leads us to consider a 2-monad and its Kleisli 2-category, linking together the previous attempts with this setting as a coherent whole, and providing a positive answer to our quest.

2 Preliminaries

The following notations are used along the paper. Formal language operations are written as follows. For a given alphabet Σ, Σ^* is the set of finite words on Σ, and ε is the empty word. The length of a word $u \in \Sigma^*$ is written $|u|$ and its ith letter, for $0 \leq i < |u|$, is written u_i. The concatenation of two words $u, v \in \Sigma^*$ is written $u \cdot v$. Concatenation of sets $U, V \subseteq \Sigma^*$ is written $U \cdot V$ and is defined by $\{u \cdot v \mid u \in U, v \in V\}$. For a given set U, the powerset of U is written $\mathrm{P}(U)$. The cartesian product of a family of sets $\{U_i\}_{i \in I}$ is written $\prod_{i \in I} U_i$, and the projection on component i for an element x of that product is written $x(i)$.

The reader is assumed familiar with basic notions of category theory. The colimit of a given diagram D is written $\mathrm{Colim}(D)$. The notation F/x stands for the comma category F over x where x is an object of some category and F is a functor into that category. The first projection is then written $\mathrm{Proj}[F/x]$. The restriction of a category $\mathbf{C}$ to the full subcategory with objects $S \subset \mathbf{C}$ is written $\mathbf{C} \restriction S$. The restriction of a functor F to a subcategory $\mathbf{C}$ of its domain is written $F \restriction \mathbf{C}$ as well.

2.1 Global Transformations

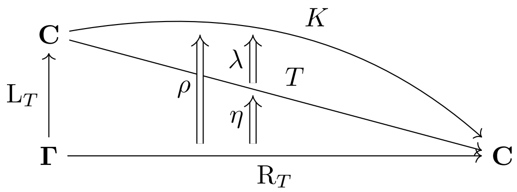
A global transform is a *synchronous* rewriting rule system. This is made possible by considering inclusions between rules in order to make explicit how overlapping applications of rules should be handle, similarly to the notion of amalgamation in classical graph transformation [2, 1]. Asking the coherence of the rule inclusions means exactly to ask them to form categories and functors, leading to the following definitions.

▶ **Definition 1** (rule systems and global transformations). *A* rule system *T on a category $\mathbf{C}$ is a tuple $\langle \mathbf{\Gamma}_T, \mathrm{L}_T, \mathrm{R}_T \rangle$ where $\mathbf{\Gamma}_T$ is a category whose objects and morphism are called* rules *and* rule inclusions*, $\mathrm{L}_T : \mathbf{\Gamma}_T \to \mathbf{C}$ is a full embedding functor called the* l.h.s. functor*, and $\mathrm{R}_T : \mathbf{\Gamma}_T \to \mathbf{C}$ is a functor called the* r.h.s. functor*. A rule system is a* global transformation *when the functor:*

$$T(-) = \mathrm{Colim}(\mathrm{R}_T \circ \mathrm{Proj}[\mathrm{L}_T/-]) \tag{1}$$

abusively also denoted T, is well-defined. The subscript T is omitted when this does not lead to any confusion.

This definition is a simplified version of alternative definitions found in [12, 5] but is enough for the present study. This definition makes GT T into a (pointwise) left Kan extensions of R_T along L_T, *i.e.*, a pair $\langle T, \eta : \mathrm{R}_T \Longrightarrow T \circ \mathrm{L}_T \rangle$ such that any other pair $\langle K, \rho : \mathrm{R}_T \Longrightarrow K \circ \mathrm{L}_T \rangle$ factorizes through η by a unique $\lambda : T \Longrightarrow K$ as in the following diagram. The natural transformation η is the identity, *i.e.*, $T \circ \mathrm{L}_T = \mathrm{R}_T$.



For more information on GT, one can consult [12, 4, 5, 7] but the specific case considered below might be enough to exemplify the relation between synchronous rewriting systems and left Kan extensions as explored by GT.

2.2 Non-Deterministic Lindenmayer Systems

Lindenmayer systems are a variant of formal grammars for specifying languages through a mechanism of parallel string rewriting [13, 10]. The present study focuses on Lindenmayer systems without context, where a word u on an alphabet Σ is synchronously rewritten by mapping each individual letter to a word (deterministic case) or set of words (non-deterministic case).

▶ **Definition 2.** *A* non-deterministic Lindenmayer system *on an alphabet* Σ *is given by a function* $\delta : \Sigma \to \mathrm{P}(\Sigma^*)$ *and produces the function on words* $\Delta : \Sigma^* \to \mathrm{P}(\Sigma^*)$*:*

$$\Delta(u) = \{v_0 \cdot \ldots \cdot v_{|u|-1} \mid (v_0, \ldots, v_{|u|-1}) \in \delta(u_0) \times \ldots \times \delta(u_{|u|-1})\} \tag{2}$$

and the dynamical system $\overline{\Delta} : \mathrm{P}(\Sigma^*) \to \mathrm{P}(\Sigma^*)$*:*

$$\overline{\Delta}(U) = \bigcup_{u \in U} \Delta(u). \tag{3}$$

▶ **Example 3.** Consider the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ with function δ defined by $\delta(\mathtt{a}) = \{\mathtt{a}, \mathtt{ab}\}$ and $\delta(\mathtt{b}) = \{\varepsilon, \mathtt{b}\}$. In this system, each `a` may potentially produce a new `b` on its right, and each `b` remains or vanishes. The behavior on the word `ab` is given by $\Delta(\mathtt{ab}) = \{\mathtt{a}\cdot\varepsilon, \mathtt{ab}\cdot\varepsilon, \mathtt{a}\cdot\mathtt{b}, \mathtt{ab}\cdot\mathtt{b}\} = \{\mathtt{a}, \mathtt{ab}, \mathtt{abb}\}$. Notice that `ab` is produced in two different ways.

In [4], deterministic Lindenmayer systems (without and with context) are presented as GT. This encoding relies on the category $\mathbf{W}$ of finite words that also plays a crucial role in this study. Let us fix the symbol Σ for the alphabet.

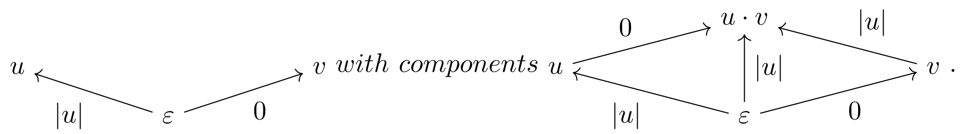
▶ **Definition 4.** *Let* $\mathbf{W}$ *be the category having* Σ^* *as set of objects, and*

$$\mathbf{W}(u, v) := \{\, p \in \{0, \ldots, |v| - |u|\} \mid u_i = v_{p+i} \; \forall i \in \{0, \ldots, |u| - 1\} \,\}$$

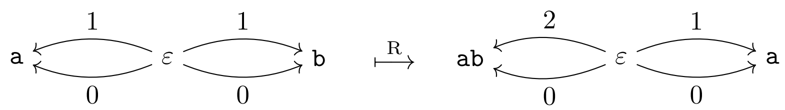
as set of arrows from any $u \in \Sigma^*$ *to any* $v \in \Sigma^*$*. We write* $p : u \to v$ *for* $p \in \mathbf{W}(u, v)$*. The composite* $q \circ p : u \to w$ *of any two arrows* $q : v \to w$ *and* $p : u \to v$ *is given by* $q + p$*,* 0 *being the identity arrow of any* $u \in \Sigma^*$*.*

Category **W** records the many places words appear in each other which is the relevant notion of *locality* for Lindenmayer systems. Indeed, the concatenations involved in Equation (2) correspond to colimits in **W**, as stated by the following theorem, and is the basic ingredient of the expression of deterministic Lindenmayer systems as GT.

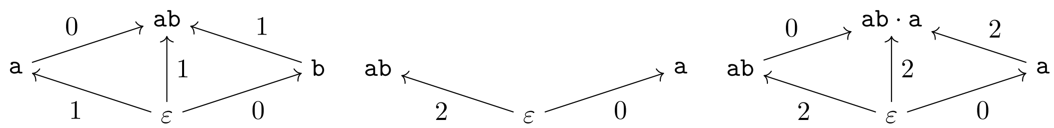
▶ **Theorem 5** (from [4]). *For any two words u, v, the word $u \cdot v$ is the colimit in* **W** *of the diagram:*



▶ **Example 6.** Let us illustrate the construction of [4] with a simple example. Consider the deterministic Lindenmayer system defined on $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ by $\delta(\mathtt{a}) = \{\mathtt{ab}\}$ and $\delta(b) = \{\mathtt{a}\}$. The associated GT $T = \langle \mathbf{\Gamma}, \mathrm{L}, \mathrm{R} \rangle$ is completely determined by the following diagrams presenting respectively the category of rules $\mathbf{\Gamma}$ as a full subcategory of **W** with inclusion functor L, and the image of $\mathbf{\Gamma}$ by R:



The computation of $T(ab)$ as defined by Equation (1) is pictured by the following diagrams:



On the left, the diagram illustrates the information provided by the comma category $\mathrm{L}/\mathtt{ab}$. It corresponds to the pattern matching of the rules l.h.s. in the input word ab. On the middle, the diagram $\mathrm{R} \circ \mathrm{Proj}[\mathrm{L}/\mathtt{ab}]$ is represented. On the right, the application of Theorem 5 for computing the colimit requested by $T(\mathtt{ab})$ constructs the expected result aba.

3 The Challenge of Powersets

Let us recall the goal and make it more precise in light of the formal definitions. We already know from [4] that deterministic Lindenmayer systems are GT, and now we want to establish that non-deterministic Lindenmayer systems are also GT, without any extension of the framework. This means that we want to provide a rule system (Definition 1) based on δ such that $\overline{\Delta}$ (Definition 2) is obtained by the colimit formula of Equation (1). This implies in particular that we need to design the appropriate category, say **PW**, with a calibrated notion of arrows to capture what we can informally call *non-deterministic locality*. The first idea that comes to mind is to take $\mathrm{P}(\Sigma^*)$ as set of objects for **PW** so that an object represents a set of possibilities. It remains to define arrows of **PW**. However, it will cause difficulties as we are now going to see.

To make this more concrete, let us take the simple example of $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, $\delta(\mathtt{a}) = \{\mathtt{a}, \mathtt{b}\}$ and $\delta(\mathtt{b}) = \{\varepsilon\}$. First, notice that $\Delta(\varepsilon) = \{\varepsilon\}$. Second, on the input aa, it produces the behavior $\Delta(\mathtt{aa}) = \{\mathtt{aa}, \mathtt{ab}, \mathtt{ba}, \mathtt{bb}\}$ corresponding to the four possibilities combining the possible evolutions of each $\mathtt{a}$. Using the lessons learned in the deterministic case as illustrated in Example 6, we make the educated guess that the involved diagram should be of the form:

$$\{\mathtt{a},\mathtt{b}\} \xleftarrow{\text{end}} \{\varepsilon\} \xrightarrow{\text{beg}} \{\mathtt{a},\mathtt{b}\} \quad \text{to produce the colimit} \quad \begin{array}{ccccc} & & \{\mathtt{aa},\mathtt{ab},\mathtt{ba},\mathtt{bb}\} & & \\ & \nearrow^{\eta_1} & \uparrow \eta_3 & {}^{\eta_2}\nwarrow & \\ \{\mathtt{a},\mathtt{b}\} & \xleftarrow{\text{end}} & \{\varepsilon\} & \xrightarrow{\text{beg}} & \{\mathtt{a},\mathtt{b}\} \end{array} .$$

The diagram contains $\{\varepsilon\}$ for the matched ε between the two $\mathtt{a}$ in $\mathtt{aa}$, and two times $\{\mathtt{a},\mathtt{b}\}$ for each occurrence of $\mathtt{a}$. The arrows in the diagram indicate that the empty words at the end of the words in the left object need to correspond to the empty words at the beginning of the words in the right object as in Theorem 5. The expected colimit results in the concatenation $\{\mathtt{aa},\mathtt{ab},\mathtt{ba},\mathtt{bb}\}$. Based on these assumptions, we know that:

- the arrow "end" of **PW** should be based on arrows $1 : \varepsilon \to \mathtt{a}$ and $1 : \varepsilon \to \mathtt{b}$ of **W**,
- the arrow "beg" of **PW** should be based on arrows $0 : \varepsilon \to \mathtt{a}$ and $0 : \varepsilon \to \mathtt{b}$ of **W**,
- the arrow η_1 should be based on $0 : \mathtt{a} \to \mathtt{aa}$, $0 : \mathtt{a} \to \mathtt{ab}$, $0 : \mathtt{b} \to \mathtt{ba}$, and $0 : \mathtt{b} \to \mathtt{bb}$,
- the arrow η_2 should be based on $1 : \mathtt{a} \to \mathtt{aa}$, $1 : \mathtt{b} \to \mathtt{ab}$, $1 : \mathtt{a} \to \mathtt{ba}$, and $1 : \mathtt{b} \to \mathtt{bb}$,
- the arrow η_3 should be based on $1 : \varepsilon \to \mathtt{aa}$, $1 : \varepsilon \to \mathtt{ab}$, $1 : \varepsilon \to \mathtt{ba}$, and $1 : \varepsilon \to \mathtt{bb}$.

A natural choice for designing the arrows of **PW** is then to gather of these arrows of **W** into sets of arrows, leading to the following definition.

▶ **Definition 7.** *Let* **PW** *be the category having* $\mathrm{P}(\Sigma^*)$ *as set of objects, and*

$$\mathbf{PW}(U,V) := \mathrm{P}(\{\, p : u \to v \in \mathbf{W} \mid u \in U, v \in V \,\})$$

as set of arrows from any $U \in \mathrm{P}(\Sigma^*)$ *to any* $V \in \mathrm{P}(\Sigma^*)$*. As usual, we write* $P : U \to V$ *for* $P \in \mathbf{PW}(U,V)$*. The composite* $Q \circ P : U \to W$ *of any two arrows* $P : U \to V$ *and* $Q : V \to W$ *is given by* $\{p + q : u \to w \mid p : u \to v \in P, q : v \to w \in Q\}$, $\{0 : u \to u \mid u \in U\}$ *being the identity arrow of any* $U \in \Sigma^*$.

Notice that for any $P \in \mathbf{PW}(U,V)$, P does not contain integers but arrows of **W** with their domain and codomain. To avoid any confusion, we always write $p : u \to v \in P$ for elements of P.

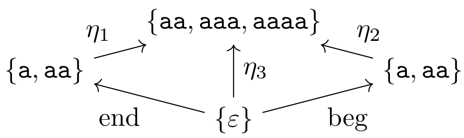
▶ **Example 8.** With this category, the previous example works as expected if we take our previous list of constraints to define end, beg, η_1, η_2, and η_3 as sets of arrows: end $= \{1 : \varepsilon \to \mathtt{a}, 1 : \varepsilon \to \mathtt{b}\}$, beg $= \{0 : \varepsilon \to \mathtt{a}, 0 : \varepsilon \to \mathtt{b}\}$, $\eta_1 = \{0 : \mathtt{a} \to \mathtt{aa}, 0 : \mathtt{a} \to \mathtt{ab}, 0 : \mathtt{b} \to \mathtt{ba}, 0 : \mathtt{b} \to \mathtt{bb}\}$, and so on so forth. To see this, consider another cocone to some object $U \in$ **PW** with components $\rho_1 : \{\mathtt{a},\mathtt{b}\} \to U$, $\rho_2 : \{\mathtt{a},\mathtt{b}\} \to U$, $\rho_3 : \{\varepsilon\} \to U$. We aim at showing that there is a unique arrow $\mu : \{\mathtt{aa},\mathtt{ab},\mathtt{ba},\mathtt{bb}\} \to U$ such that $\rho_i = \mu \circ \eta_i$ for $i \in \{1,2,3\}$.

First notice that ρ_1, ρ_2, and ρ_3 are bijectively related. Indeed, by definition of the composition in **PW** and by commutativity of the cocone, each $f_1 : l_1 \to u \in \rho_1$ compose with the unique appropriate arrow $1 : \varepsilon \to l_1$ of "end" to give a bijectively corresponding arrow in $f_1 + 1 : \varepsilon \to u \in \rho_3$. Surjectivity holds by the definition of the composition. Injectivity stands on the fact that an occurrence of $\mathtt{a}$ and an occurrence of $\mathtt{b}$ in a given word (here u) cannot be at the same position. Therefore, given $f_1 + 1 : \varepsilon \to u \in \rho_3$, there is a unique letter l_1 such that $f_1 : l_1 \to u \in \rho_1$. Similarly, there is also such a bijection mapping each $f_2 : l_2 \to u \in \rho_2$ to $f_2 : \varepsilon \to u \in \rho_3$. Consequently, the mediating μ has to contain exactly arrows with domain the concatenation of source letters l_i of two bijectively related arrows $f_1 : l_1 \to u \in \rho_1$ and $f_2 : l_2 \to u \in \rho_2$, and with codomain u. Formally μ has to be $\{f_1 : l_1 \cdot l_2 \to u \mid f_1 : l_1 \to u \in \rho_1$ and corresponding $f_1 + 1 : l_2 \to u \in \rho_2\}$.

The fact that this mediating commutes appropriately comes directly from the fact that $\mu \circ \eta_1 = \{f_1 : l_1 \to u \mid 0 : l_1 \to l_1 \cdot l_2 \in \eta_1, f_1 : l_1 \cdot l_2 \to u \in \mu\}$ which turns to be ρ_1. Commutations for ρ_2 and ρ_3 hold as well. Finally, μ is unique by construction.

While the previous example works, the proof uses explicitly the properties specific to the example. In the general case, the construction fails as shown by the following example.

▶ **Example 9.** We consider a different non-deterministic Lindenmayer system where $\Sigma = \{\mathtt{a}\}$ and $\delta(\mathtt{a}) = \{\mathtt{a}, \mathtt{aa}\}$. This produces the behavior $\Delta(\mathtt{aa}) = \{\mathtt{aa}, \mathtt{aaa}, \mathtt{aaaa}\}$. Notice in particular that the concatenations $\mathtt{a} \cdot \mathtt{aa}$ and $\mathtt{aa} \cdot \mathtt{a}$ give the same word $\mathtt{aaa}$ in $\Delta(\mathtt{aa})$. Following the same construction as presented in Example 8, we consider the following colimit:



where
- $\eta_1 = \{0 : \mathtt{a} \to \mathtt{aa}, 0 : \mathtt{a} \to \mathtt{aaa}, 0 : \mathtt{aa} \to \mathtt{aaa}, 0 : \mathtt{aa} \to \mathtt{aaaa}\}$,
- $\eta_2 = \{1 : \mathtt{a} \to \mathtt{aa}, 1 : \mathtt{aa} \to \mathtt{aaa}, 2 : \mathtt{a} \to \mathtt{aaa}, 2 : \mathtt{aa} \to \mathtt{aaaa}\}$, and
- $\eta_3 = \{1 : \varepsilon \to \mathtt{aa}, 1 : \varepsilon \to \mathtt{aaa}, 2 : \varepsilon \to \mathtt{aaa}, 2 : \varepsilon \to \mathtt{aaaa}\}$.

To see that this cocone is *not* a colimit, consider the cocone to $\{\mathtt{aaa}\}$ having components $\rho_1 = \{0 : \mathtt{a} \to \mathtt{aaa}\}$, $\rho_2 = \{1 : \mathtt{aa} \to \mathtt{aaa}\}$, and $\rho_3 = \{1 : \varepsilon \to \mathtt{aaa}\}$. The only possible mediating is $\mu = \{0 : \mathtt{aaa}, \mathtt{aaa}\}$ but it fails to respect the required commutation property. Indeed, $\mu \circ \eta_1 = \{0 : \mathtt{a} \to \mathtt{aaa}, 0 : \mathtt{aa} \to \mathtt{aaa}\}$ which definitively differs from ρ_1.

So the first and simplest intuitive idea does not work and we have not designed the appropriate category. In particular, the construction fails since it is not able to distinguish the different ways for generating a given output (case $\mathtt{aaa}$ in Example 9). We then deduce that an appropriate category, if it exists, needs to keep track of this information. Moreover, notice that in Equation (1), the arrows of the category are not only used for constructing the result as a colimit, but also to decompose an input U into a coma category L_T/U and produce the diagram by the formula $\mathrm{R} \circ \mathrm{Proj}[\mathrm{L}_T/U]$. So the previous discussion has only addressed one side of the problem.

4 From Sets to Indexed Families

There are plenty of other possible definitions for designing arrows of **PW** and the research space to get the right one is pretty large. However, having considered several attempts of definitions, we come to the following working hypothesis.

▶ **Conjecture 10.** *There is no category with set of objects* $\mathrm{P}(\Sigma^*)$ *and an appropriate choice of arrows* $\mathrm{beg}_U, \mathrm{end}_U : \{\varepsilon\} \to U$ *producing concatenation as a colimit.*

Irrespectively of the validity of that conjecture, we choose to circumvent directly the problem that occurred in the previous example and to jump to other aspects of the program. As previously evoked, the obstruction was that $\mathtt{a} \cdot \mathtt{aa}$ and $\mathtt{aa} \cdot \mathtt{a}$ merged into a single word $\mathtt{aaa}$, so that the mediating arrow could not specify whether it needed this word as a result of the first concatenation or of the second one. To keep track of that information, we simply allow for a word to appear many times and we take families of words instead of sets of words. Taking this path of least resistance, we obtain the following category where an arrow from a source family of words to a target family of words, picks a unique word from the source for each word in the target. This additional "unique source word" property is respected by all the arrows considered up to now. Taking the view that an arrow $p : u \to v$ in $\mathbf{W}$ selects a subword of v, this constraint says that an arrow to a family of words similarly selects a subword for each member of that family.

▶ **Definition 11.** *For any* $\mathbf{C} \in \mathbf{Cat}$*,* $\mathbb{O}\mathbf{C} \in \mathbf{Cat}$ *has pairs* $(I : \mathrm{Set}, U \in \mathbf{C}^I)$ *as objects and*

$$\mathbb{O}\mathbf{C}((I,U),(J,V)) := \{\, (P : J \to I, P' : \prod_{j \in J} \mathbf{C}(U_{P(j)}, V_j)) \,\}$$

as set of arrows from any $(I,U) \in \mathbb{O}\mathbf{C}$ *to any* $(J,V) \in \mathbb{O}\mathbf{C}$*. As usual, we write* $(P,P') : (I,U) \to (J,V)$ *for* $(P,P') \in \mathbb{O}\mathbf{C}((I,U),(J,V))$*. The composite* $(Q,Q') \circ (P,P') : (I,U) \to (K,W)$ *of any two arrows* $(P,P') : (I,U) \to (J,V)$ *and* $(Q,Q') : (J,V) \to (K,W)$ *is given by the pair* $(R : K \to I, R' : \prod_{k \in K} \mathbf{C}(U_{R(k)}, W_k))$ *where :*

$$R(k) = P(Q(k)) \text{ and } R'(k) = Q'(k) \circ P'(Q(k)) : U_{P(Q(k))} \to V_{Q(k)} \to W_k \in \mathbf{C}.$$

The identity arrow of any (I,U) *is* (P,P') *where* $P(i) = i$ *and* $P'(i) = \mathrm{id}_{P(i)} : P(i) \to P(i)$*.*

One may have recognised in this definition the construction for non-determinism of [11]. It happens to be the free cartesian completion of $\mathbf{C}$, the dual of $\mathrm{Fam}(\mathbf{C})$ construction for free coproduct completion [9, 3, 14], i.e. $\mathbb{O}\mathbf{C} = \mathrm{Fam}(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}}$.

It is not hard to see that each object of this category has many isomorphic objects of bijective index set, so that the particular index set used is irrelevant. The real information contained in an equivalence class of isomorphic objects is the number of times each word occurs. Here, we allow this cardinality to be arbitrary. The issue of cardinality is a detail at this point, and we do not bother commenting on this issue before the conclusion. In the meantime, one can freely add the word *finite* anywhere one feels it is needed.

At this time, some notations are required to handle families and some relevant elements of $\mathbb{O}\mathbf{W}$. Given an arbitrary set U, we write $\overline{U}$ for the corresponding family containing each elements of U exactly once and given by the pair (U, id_U). Also, for any $(I,U) \in \mathbb{O}\mathbf{W}$, we consider the appropriate arrows $\mathrm{beg}_{(I,U)}, \mathrm{end}_{(I,U)} : \overline{\{\varepsilon\}} \to (I,U)$ identifying the occurrences of the empty words respectively at the beginning and at the end of the words of the family (I,U), and which are given by $\mathrm{beg}_{(I,U)} = ([i \mapsto \varepsilon], [i \mapsto (0 : \varepsilon \to U_i)])$ and $\mathrm{end}_{(I,U)} = ([i \mapsto \varepsilon], [i \mapsto (|U_i| : \varepsilon \to U_i)])$. We make use here of the notation $[x \mapsto f(x)]$ to specify succinctly an anonymous function; domains and codomains can always be retrieved from the context.

With the category $\mathbb{O}\mathbf{W}$ and these beginning and ending arrows, we obtain concatenation as wanted and in a very similar way to concatenation in $\mathbf{W}$ with Theorem 5.

▶ **Proposition 12.** *For any two families* $(I,U), (J,V) \in \mathbb{O}\mathbf{W}$*, a colimit of the diagram*

$$(I,U) \xleftarrow{\mathrm{end}_{(I,U)}} \overline{\{\varepsilon\}} \xrightarrow{\mathrm{beg}_{(I,U)}} (J,V)$$

is given by the cocone

$$(I,U) \xrightarrow{\eta_1} (I \times J, (i,j) \mapsto U_i \cdot V_j) \xleftarrow{\eta_2} (J,V), \quad \overline{\{\varepsilon\}} \xrightarrow{\eta_3} (I \times J, (i,j) \mapsto U_i \cdot V_j), \quad (I,U) \xleftarrow{\mathrm{end}_{(I,U)}} \overline{\{\varepsilon\}} \xrightarrow{\mathrm{beg}_{(I,U)}} (J,V)$$

where

- $\eta_1 = ([(i,j) \mapsto i], [(i,j) \mapsto (0 : U_i \to U_i \cdot V_j)])$,
- $\eta_2 = ([(i,j) \mapsto j], [(i,j) \mapsto (|U_i| : V_j \to U_i \cdot V_j)])$*, and*
- $\eta_3 = ([(i,j) \mapsto \varepsilon], [(i,j) \mapsto (|U_i| : \varepsilon \to U_i \cdot V_j)])$.

Proof. Indeed, consider another cocone to some (K, W) with components, $(P, P') : (I, U) \to (K, W)$, $(Q, Q') : (J, V) \to (K, W)$, and $(R, R') : \overline{\{\varepsilon\}} \to (K, W)$. The mediating arrow is given by $([k \mapsto (P(k), Q(k))], [k \mapsto P'(k) : U_{P(k)} \cdot V_{Q(k)} \to W_k])$. Notice that this mediating follows the exact same definition as the one exhibited in Example 8. ◀

It is now time to summarize what we have just achieved. We took a diagram shape that allowed to obtain *deterministic* Lindenmayer systems as GT by encoding concatenation as colimit. Then, we changed the objects and arrows in this diagram to obtain what we can informally call *non-deterministic concatenation*. But the diagram shape itself arises from the *deterministic* decomposition of a single input word (see [4]). In other words, the pattern matching is not considered in $\mathbb{O}\mathbf{W}$ but in $\mathbf{W}$. So for now, we only have the following.

▶ **Definition 13.** *For any non-deterministic Lindenmayer system* $(\Sigma, \delta : \Sigma \to \mathrm{P}(\Sigma^*))$*, we write* $\mathbf{D} : \mathbf{W} \to \mathbb{O}\mathbf{W}$ *for the functor mapping words* $u \in \mathbf{W}$ *to* $\mathbf{D}(u) = (I, V)$ *where*

$$I := \delta(u_0) \times \ldots \times \delta(u_{|u|-1}) \text{ and } V_{(v_0,\ldots,v_{|u|-1})} := v_0 \cdot \ldots \cdot v_{|u|-1}$$

and arrows $p : u' \to u$ *to* $\mathbf{D}(p) = (P, P')$ *where*

$$P((v_0, \ldots, v_{|u|-1})) := (v_p, \ldots, v_{p+|u'|-1}) \text{ and}$$

$$P'((v_0, \ldots, v_{|u|-1})) := |v_0| + \ldots + |v_{p-1}| : v_p \cdot \ldots \cdot v_{p+|u'|-1} \to v_0 \cdot \ldots \cdot v_{|u|-1}.$$

The functor $\mathbf{D}$ is a categorical counterpart of $\Delta : \Sigma^* \to \mathrm{P}(\Sigma^*)$ of Definition 2 but with families making sure that we keep the multiple instances of each word. Indeed, for $\mathbf{D}(u) = (I, U)$, each index $(v_0, \ldots, v_{|u|-1}) \in I$ corresponds to a choice of a word v_i among the possibilities provided by $\delta(u_i)$, for each letter u_i of u. For such a choice, the associated resulting word $V_{(v_0,\ldots,v_{|u|-1})}$ is simply given by the concatenation of the v_i. The definition of $\mathbf{D}(p)$ expresses the monotony of Δ. The monotony can be illustrated as follows. Consider $u = \alpha_1 \cdot u' \cdot \alpha_2$ with $|\alpha_1| = p$. Taking $v' \in \Delta(u')$, $\gamma_1 \in \Delta(\alpha_1)$, and $\gamma_2 \in \Delta(\alpha_2)$, we have $v = \gamma_1 \cdot v' \cdot \gamma_2 \in \Delta(u)$. So we have an arrow $|\gamma_1| : v' \to v$. As a family of arrows, $\mathbf{D}(p)$ gathers all of these arrows. In the formula of Definition 13, we have $\alpha_1 = u_0 \ldots u_{p-1}$, $u' = u_p \ldots u_{p+|u'|-1}$, $\gamma_1 = v_0 \cdot \ldots \cdot v_{p-1}$, and $v' = v_p \cdot \ldots \cdot v_{p+|u'|-1}$.

Exactly as Δ is generated from its sole behavior on letters given by δ as stated by Definition 2, we will see that the functor $\mathbf{D}$ is generated from its restriction to the letters and ε. We start by defining the categorical counterpart $\mathbf{d}$ of δ.

▶ **Definition 14.** *For any non-deterministic Lindenmayer system* $(\Sigma, \delta : \Sigma \to \mathrm{P}\Sigma^*)$*, we write* $\mathbf{d} : \mathbf{W} \restriction \Sigma \cup \{\varepsilon\} \to \mathbb{O}\mathbf{W}$ *for the functor from the full subcategory* $\mathbf{W} \restriction \Sigma \cup \{\varepsilon\}$ *of* $\mathbf{W}$ *to* $\mathbb{O}\mathbf{W}$ *defined as* $\mathbf{d} = \mathbf{D} \restriction (\mathbf{W} \restriction \Sigma \cup \{\varepsilon\})$.

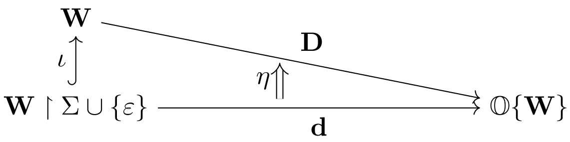
The functor $\mathbf{d}$ is entirely characterized in terms of arrows beg and end.

▶ **Lemma 15.** *For any* $a \in \Sigma$*, we have* $\mathbf{d}(0 : \varepsilon \to a) = \mathrm{beg}_{\mathbf{d}(a)}$ *and* $\mathbf{d}(1 : \varepsilon \to a) = \mathrm{end}_{\mathbf{d}(a)}$.

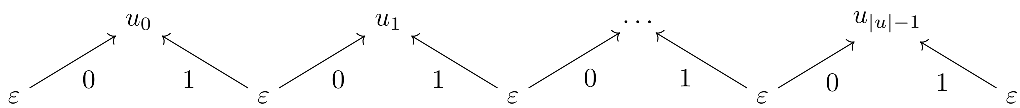
Proof. Consider $0 : \varepsilon \to a$. By Defs 14 and 13, $\mathbf{d}(\varepsilon) = \mathbf{D}(\varepsilon) = (\{\varepsilon\}, [\varepsilon \mapsto \varepsilon]) = \overline{\{\varepsilon\}}$, and $\mathbf{d}(a) = \mathbf{D}(a) = (\delta(a), [i \mapsto i])$. By the same definitions, $\mathbf{d}(0 : \varepsilon \to a) = \mathbf{D}(0 : \varepsilon \to a) = (P, P')$ with $P(i) = \varepsilon$ and $P'(i) = |\varepsilon| : \varepsilon \to i$ where i ranges over $\delta(a)$. Clearly, $\mathbf{d}(0 : \varepsilon \to a) = \mathrm{beg}_{(\delta(a),[i \mapsto i])} = \mathrm{beg}_{\mathbf{d}(a)}$ as expected. We get $\mathbf{d}(1 : \varepsilon \to a) = \mathrm{end}_{\mathbf{d}(a)}$ similarly. ◀

We can now establish that $\mathbf{D}$ is obtained as an extension of $\mathbf{d}$ thereby providing a categorical counterpart of Definition 2.

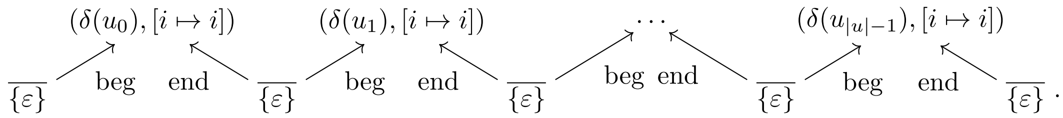
▶ **Proposition 16.** $\mathbf{D}$ *is a pointwise left Kan extension of* $\mathbf{d}$ *along the inclusion functor* $\iota : \mathbf{W} \upharpoonright \Sigma \cup \{\varepsilon\} \to \mathbf{W}$ *as in the following diagram where* η *is the identity.*



Proof. Using the explicit definition of pointwise left Kan extensions in terms of colimit, we are left to show that $\mathbf{D}(-) = \mathrm{Colim}(\mathbf{d} \circ \mathrm{Proj}[\iota/-])$. In [4], it is already proved that the diagram $\mathrm{Proj}[\iota/u]$ has the following zigzag shape:



Using Lemma 15, the diagram $\mathbf{d} \circ \mathrm{Proj}[\iota/u]$ is:



Iteratively using Prop. 12 on this finite sequence, the colimit of this diagram is clearly the non-deterministic concatenation of the $(\delta(u_k), [i \mapsto i])$, $0 \leq k < |u|$, which is also the definition of $\mathbf{D}(u)$ as given in Def. 13. To prove that η is the identity, it is enough to consider the particular case of the previous reasoning with $|u| \leq 1$ that shows that $(\mathbf{D} \circ \iota)(u) = \mathbf{d}(u)$.

Given some $p : u' \to u$, for proving that $\mathbf{D}(p) = \mathrm{Colim}(\mathbf{d} \circ \mathrm{Proj}[\iota/p])$ we remark that $\mathbf{D}(p)$ has to be a mediating arrow. By unicity of the mediating arrow, it remains to show that $\mathbf{D}(p)$ obeys the requested commutations of mediating arrows, which is straightforward. ◀

So far, for a non-deterministic Lindenmayer system (Σ, δ), we have $\mathbf{d}$ as a categorical counterpart of δ, which gives rise by left Kan extension to $\mathbf{D}$, the categorical counterpart of Δ. However, we still do not have a dynamical system, since the domain and codomain of $\mathbf{D} : \mathbf{W} \to \mathbb{O}\mathbf{W}$ are not strictly the same. In other words, we now want a left Kan extension counterpart of $\overline{\Delta} : \mathrm{P}(\Sigma^*) \to \mathrm{P}(\Sigma^*)$ of Definition 2, say $\overline{\mathbf{D}} : \mathbb{O}\mathbf{W} \to \mathbb{O}\mathbf{W}$. Clearly, we already know the expected definition of $\overline{\mathbf{D}}$ since we want to apply independently $\mathbf{D}$ on each element of a family (I, U) and to flatten the results altogether.

▶ **Definition 17.** *Let* $\overline{\mathbf{D}} : \mathbb{O}\mathbf{W} \to \mathbb{O}\mathbf{W}$ *be the functor defined as*

$$\overline{\mathbf{D}}((I, U)) = \left(\bigcup_{i \in I} (\{i\} \times J_i)\,,\, [(i, j) \mapsto (V_i)_j] \right) \text{ where } (J_i, V_i) = \mathbf{D}(U_i),$$

and $\overline{\mathbf{D}}((P, P') : (I', U') \to (I, U)) = (Q, Q')$ *such that, for each* $(i, j) \in \bigcup_{i \in I} (\{i\} \times J_i)$*:*

$$Q((i, j)) = (P(i), R_i(j)) \text{ and } Q'((i, j)) = R'_i(j) \text{ where } (R_i, R'_i) = \mathbf{D}(P'(i)).$$

Obtaining $\overline{\mathbf{D}}$ as a Kan extension consists in embedding $\mathbf{W}$ into $\mathbb{O}\mathbf{W}$, then extending along this embedding. The notation $\overline{U}$ that we have introduced earlier can be turned into a *singleton functor* for defining this embedding.

▶ **Definition 18.** *For any category* $\mathbf{C}$*, the singleton functor* $\mathrm{sing}_{\mathbf{C}} : \mathbf{C} \to \mathbb{O}\mathbf{C}$ *is defined as*

$$\mathrm{sing}_{\mathbf{C}}(x) = \overline{\{x\}} \text{ and } \mathrm{sing}_{\mathbf{C}}(f : x \to y) = ([y \to x], [y \mapsto (f : x \to y)]).$$

Unfortunately, the program stops here since $\overline{\mathbf{D}}$ fails to be the extension of $\mathbf{d}$ along $\mathrm{sing}_{\mathbf{W}} \circ \iota : \mathbf{W} \upharpoonright \Sigma \cup \{\varepsilon\} \to \mathbf{W} \hookrightarrow \mathbb{O}\mathbf{W}$. In fact, the arrows of $\mathbb{O}\mathbf{W}$ are not well-suited for decomposing a family of words in the appropriate way for the expected concatenation. For instance, consider the family in inputs $\overline{\{\mathtt{aa}, \mathtt{bb}\}}$. The comma category $\mathrm{sing}_{\mathbf{W}} \circ \iota / \overline{\{\mathtt{aa}, \mathtt{bb}\}}$ fails to identify the occurrences of `a` in this family. Indeed, an arrow from $\overline{\{\mathtt{a}\}}$ to $\overline{\{\mathtt{aa}, \mathtt{bb}\}}$ requires to identify an occurrence of `a` in `bb` but there is none. So there is no arrow between those two families and the diagram $\mathbf{d} \circ \mathrm{Proj}[\mathrm{sing}_{\mathbf{W}} \circ \iota / \overline{\{\mathtt{a}, \mathtt{b}\}}]$ of Equation (1) contains only ε's and does not exhibit the expected zigzag shape.

5 The Kleisli 2-Category of the 2-Monad of Families

As explained in Section 1, we propose to solve the last issue by placing oneself in a 2-categorical context. Since this solution can seem more elaborate than necessary, let us make precise why this transition to 2-categories is conceptually natural with respect to our goal.

Let us develop the relation between dynamical systems and their non-deterministic counterparts. At a general level of description, dynamical systems can be defined once we have a collection of objects to model the states, and a way to specify endo-functions on these objects to model the dynamics. For instance, in the category of sets and functions, the states are modeled as a set and the dynamics as a function; the usual case of (deterministic) dynamical systems is captured. But in the category of topological spaces and continuous functions, states are modeled as a topological space and the dynamics by a continuous function, allowing to handle the so-called topological dynamical systems. Similarly, in the category of sets and relations, states are modeled as a set and the dynamics by a relation. This last case is particularly interesting for our concern since it is the place to deal with non-deterministic dynamical systems. Formally this latter category is equivalently described as the *Kleisli category* of the so-called *powerset monad*. This is based on the fact that $R \subseteq X \times Y$ is equivalently a function $f : X \to \mathrm{P}(Y)$, that singletons allow any set X to be seen as included in $\mathrm{P}(X)$, and that unions allow any sets of sets in $\mathrm{P}(\mathrm{P}(X))$ to be simplified in a simple set of $\mathrm{P}(X)$. The two lessons we learn here are that (1) dynamical systems are parametrized by the nature of the objects and the arrows they rely on, and that (2) the parametrization for the non-determinitic counterpart is based on the powerset monad.

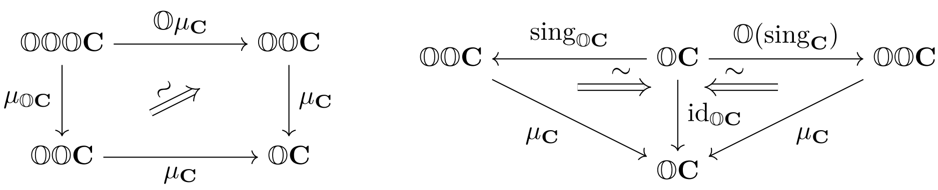
We now proceed to apply the same scheme for the GT. The difference with dynamical systems is that GT are not defined with two layers (an object for the states and an arrow for the dynamics) but with three layers: categories, functors and natural transformations as it can be seen in the pointwise left Kan extension diagram of Section 2.1. So they are parametrized by a 2-category. For instance, the simple GT as defined in Definition 1 are parametrized by **Cat**, the 2-category of categories. Following the second lesson on non-deterministic dynamical systems, for the particular case of non-deterministic GT, we propose this 2-category parameter to be set to the Kleisli 2-category induced by the 2-monad of families.

We already have all the ingredients of a 2-monad on **Cat** as we now proceed to show. Firstly, the construction $\mathbb{O}\mathbf{C}$ of Definition 11 can be extended to act on functors and natural transformations and yields a 2-functor $\mathbb{O} : \mathbf{Cat} \to \mathbf{Cat}$.

▶ **Definition 19.** *For any functor* $F : \mathbf{C} \to \mathbf{C}'$*, the functor* $\mathbb{O}F : \mathbb{O}\mathbf{C} \to \mathbb{O}\mathbf{C}'$ *is defined as* $\mathbb{O}F((I, U)) = (I, F \circ U)$*, and* $\mathbb{O}F((P, P') : (I, U) \to (J, V)) = (P, F \circ P')$*. For any natural transformation* $\alpha : F \Longrightarrow G : \mathbf{C} \to \mathbf{C}'$*, the natural transformation* $\mathbb{O}\alpha : \mathbb{O}F \Longrightarrow \mathbb{O}G : \mathbb{O}\mathbf{C} \to \mathbb{O}\mathbf{C}'$ *has components* $(\mathbb{O}\alpha)_{(I,U)} = ([i \mapsto i], [i \mapsto \alpha_{U_i}]) : (I, F \circ U) \to (I, G \circ U)$*.*

To make the 2-functor $\mathbb{O}$ into a 2-monad, we need to consider the obvious pairs of operations, the first one, $\text{sing}_{\mathbf{C}} : \mathbf{C} \to \mathbb{O}\mathbf{C}$, lifting an object of a category $\mathbf{C}$ to a singleton family of $\mathbb{O}\mathbf{C}$, and the second one, $\mu_{\mathbf{C}} : \mathbb{O}\mathbb{O}\mathbf{C} \to \mathbb{O}\mathbf{C}$, flattening a family of families of objects into a simple family of objects. Notice that this last construction has already been encountered in Definition 17 of $\overline{\mathbf{D}}$, whose role of flattening has also been underlined above. In particular, the function $\overline{\mathbf{D}}$ is in fact obtained as $\mu_{\mathbf{W}} \circ \mathbb{O}\mathbf{D}$ from Definition 13. The two operations form indeed a 2-monad.

▶ **Proposition 20.** *Operations* $\text{sing}\_$ *and* $\mu\_$ *make* $\mathbb{O} : \mathbf{Cat} \to \mathbf{Cat}$ *into a 2-monad, i.e., all instances of the following diagrams weakly commute.*



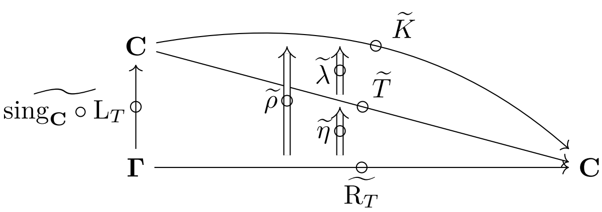
Proof. For the first square, and given an object $(I, U) \in \mathbb{O}\mathbb{O}\mathbb{O}\mathbf{C}$, the top-right path leads to index set of the form $(i, (j, k))$ for the object in $\mathbb{O}\mathbf{C}$ while the left-bottom path leads to the form $((i, j), k)$, hence the weak commutation. For the triangles on the right, an object $(I, U) \in \mathbb{O}\mathbf{C}$ sees each index $i \in I$ transformed into $((I, U), i) \in \{(I, U)\} \times I$ by the left path and into $(i, U_i) \in \{(i, U_i) \mid i \in I\}$ by the right path. ◀

In order to ease the reading of elements of the Kleisli weak 2-category, let us introduce some notations. We write $\widetilde{F} : \mathbf{C} \multimap \mathbf{D}$ to represent a functor $F : \mathbf{C} \to \mathbb{O}\mathbf{D}$ of the Kleisli weak 2-category. A 2-arrow $\widetilde{\eta} : \widetilde{F} \Rightarrow \widetilde{G}$ stands simply for a natural transformation $\eta : F \Longrightarrow G$. The composition of arrows in the Kleisli weak 2-category, written $\widetilde{G} \circ \widetilde{F} : \mathbf{C} \multimap \mathbf{D} \multimap \mathbf{E}$, is the functor $\mu_{\mathbf{E}} \circ \mathbb{O}G \circ F : \mathbf{C} \to \mathbb{O}\mathbf{D} \to \mathbb{O}\mathbb{O}\mathbf{E} \to \mathbb{O}\mathbf{E}$.

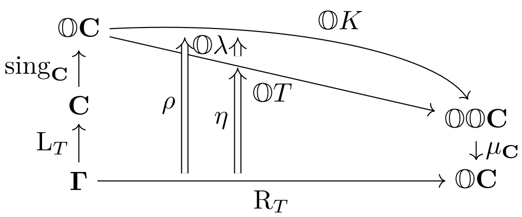
We finally reach our initial goal as we are now able to show that the diagram of Prop. 16 is in fact a summary of a GT in the Kleisli weak 2-category induced by the 2-monad $\mathbb{O}$. More accurately, considering the GT diagram of the rule system $\langle \mathbf{W} \restriction \Sigma \cup \{\varepsilon\}, \widetilde{\text{sing}_{\mathbf{W}} \circ \iota}, \widetilde{\mathbf{d}} \rangle$ is completely equivalent to considering the diagram of Prop. 16, achieving the fact that the initial non-determinitic Lindenmayer system is indeed a GT. Moreover, this works for any non-deterministic rule system $\langle \mathbf{\Gamma}, \widetilde{\text{sing}_{\mathbf{C}} \circ \mathrm{L}}, \widetilde{\mathrm{R}} \rangle$ on any category $\mathbf{C}$. Notice the particular form of the l.h.s. functor defined using $\mathrm{L} : \Gamma \to \mathbf{C}$ which is still required to be a full embedding.

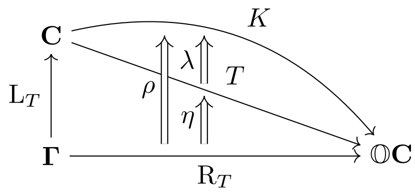
▶ **Theorem 21.** *Let* $\widetilde{T} = \langle \mathbf{\Gamma}_T, \widetilde{\text{sing}_{\mathbf{C}} \circ \mathrm{L}_T}, \widetilde{\mathrm{R}_T} \rangle$ *be a rule system in the Kleisli weak 2-category induced by the 2-monad* $\mathbb{O}$*.* $\widetilde{T}$ *is a GT iff* T *is the left Kan extension of* R_T *along* L_T *in the 2-category* $\mathbf{Cat}$*.*

Proof. The rule system being a GT, we have a pair $\langle \widetilde{T}, \widetilde{\eta} : \widetilde{\mathrm{R}_T} \Longrightarrow \widetilde{T} \circ \widetilde{\text{sing}_{\mathbf{C}} \circ \mathrm{L}_T} \rangle$ which is a left Kan extension in the Kleisli 2-category and takes the following diagrammatic form for any other pair $\langle \widetilde{K}, \widetilde{\rho} : \widetilde{\mathrm{R}_T} \Longrightarrow \widetilde{K} \circ \widetilde{\text{sing}_{\mathbf{C}} \circ \mathrm{L}_T} \rangle$:



This diagram corresponds by definition to the diagram in **Cat** on the left below. By naturality of sing and Prop. 20, it (weakly) simplifies into the expected diagram, on the right below.





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6 Final Discussion

This journey started with the general goal of going toward non-deterministic, probabilistic and quantum GT. Starting with the first concrete step of capturing non-deterministic Lindenmayer systems as GT, we guessed some constructions based on the deterministic case. The result of these guesses did not have the precise form of a GT in the 2-category of categories, functors and natural transformations. But we showed they correspond in fact to a GT in another (weak) 2-category. The latter is induced by Kleisli's construction on a particular (weak) 2-monad that we made explicit. This is to be expected since the same architecture happens for non-deterministic dynamical systems. Indeed, they are also dynamical systems in another category, with the latter being induced by a monad. All in all, we ended with a general solution for mixing non-determinism with locality as described in the global transformation framework.

Along the way, we mentioned a few technicalities on which we now come back. The first one is the Conjecture 10 on which we want to add a comment. If it is wrong, then families can be simplified into sets. But in this case, there should be a relation between the family-based solution just presented and this new set-based solution. But thinking of this relation as a functor from families to sets reinforce us in the belief that the conjecture is true.

The second technicality is about the size of the families considered, which is related to the size issues for the "2-category of categories". For most practical purpose, it is possible to restrict oneself to finite families. In this case, a small category $\mathbf{C}$ leads to a small category $\mathbb{O}\mathbf{C}$. In this case, the 2-functor $\mathbb{O}$ is indeed an endomorphism of the 2-category of *small* categories. Dropping the finiteness constraint though, one then considers a 2-functor $\mathbb{O}\mathbf{C}$ from *small* categories to *large* categories. This is however perfectly fine, since this describes a so-called *relative pseudomonad* with an associate Kleisli's construction, as defined in [8].

In [6], one can find a direct account of the 2-category described here in terms of 2-monad. In particular, the *open* functors and *open* natural transformations are introduced using presheaves and proved to form a weak 2-category. More precisely, an open functor F from a category $\mathbf{C}$ to a category $\mathbf{D}$ is the data of a presheaf on $\mathbf{C}$ together with a functor from the category of elements of that presheaf to $\mathbf{D}$. An interesting feature of this presheaf presentation is that it allows to speak directly about special properties arising from the association of locality and non-determinism. For instance, correlations and intrications correspond to obstructions of the presheaf to be a sheaf. The formal definition in [6] can be made easier to manipulate by the use of discrete fibrations instead of categories of elements through the so-called Grothendieck construction, and doing so presents this bicategory as a particular bicategory of spans. Moreover, this bicategory is a sub-bicategory of the bicategory of profunctors (*a.k.a.* distributors). Notice that the many presentations of this bicategory are strongly related to the many possible presentations one can have of the notion of "relation": powerset monad (as in this paper), spans, and characteristic functions of the relation.

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