

The Hamilton Compression of Highly Symmetric Graphs

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Abstract

We say that a Hamilton cycle $C = (x_1, \dots, x_n)$ in a graph G is k -symmetric, if the mapping $x_i \mapsto x_{i+n/k}$ for all $i = 1, \dots, n$, where indices are considered modulo n , is an automorphism of G . In other words, if we lay out the vertices x_1, \dots, x_n equidistantly on a circle and draw the edges of G as straight lines, then the drawing of G has k -fold rotational symmetry, i.e., all information about the graph is compressed into a $360^\circ/k$ wedge of the drawing. We refer to the maximum k for which there exists a k -symmetric Hamilton cycle in G as the *Hamilton compression of G* . We investigate the Hamilton compression of four different families of vertex-transitive graphs, namely hypercubes, Johnson graphs, permutahedra and Cayley graphs of abelian groups. In several cases we determine their Hamilton compression exactly, and in other cases we provide close lower and upper bounds. The cycles we construct have a much higher compression than several classical Gray codes known from the literature. Our constructions also yield Gray codes for bitstrings, combinations and permutations that have few tracks and/or that are balanced.

2012 ACM Subject Classification Mathematics of computing \rightarrow Graph theory; Mathematics of computing \rightarrow Combinatorics

Keywords and phrases Hamilton cycle, Gray code, hypercube, permutahedron, Johnson graph, Cayley graph, abelian group, vertex-transitive

Digital Object Identifier 10.4230/LIPIcs.MFCS.2022.54

Related Version Full proofs are provided in the preprint version of this extended abstract:
Full Version: <https://arxiv.org/abs/2205.08126> [15]

Funding This work was supported by Czech Science Foundation grant GA 22-15272S.

Petr Gregor: Partially supported by ARRS project P1-0383.

Arturo Merino: German Science Foundation grant 413902284 and ANID Becas Chile 2019-72200522

Torsten Mütze: German Science Foundation grant 413902284

Acknowledgements We thank Fedor Petrov and Michal Koucký for ideas that simplified some proofs in the full version. We also thank the reviewers for their helpful comments.

1 Introduction

A *Hamilton cycle* in a graph is a cycle that visits every vertex of the graph exactly once. This concept is named after the Irish mathematician and astronomer Sir William Rowan Hamilton (1805–1865), who invented the Icosian game, in which the objective is to find a Hamilton cycle along the edges of the dodecahedron. Figure 1 shows the dodecahedron with a Hamilton cycle on the circumference. Hamilton cycles have been studied intensively from



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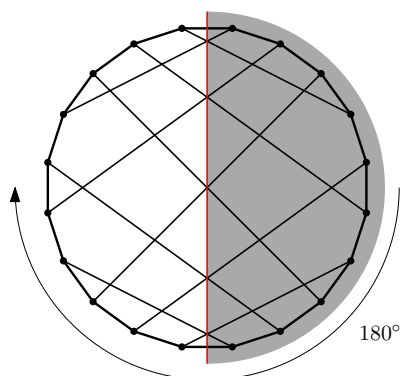
47th International Symposium on Mathematical Foundations of Computer Science (MFCS 2022).

Editors: Stefan Szeider, Robert Ganian, and Alexandra Silva; Article No. 54; pp. 54:1–54:14

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



■ **Figure 1** The dodecahedron with a 2-symmetric Hamilton cycle.

various different angles, such as graph theory (necessary/sufficient conditions, packing and covering etc. [11, 12, 13, 18]), optimization (shortest tours, approximation [2]), algorithms (complexity [10], exhaustive generation [22, 25]) and algebra (Cayley graphs [5, 19, 24, 28]). In this work we introduce a new graph parameter that quantifies how symmetric a Hamilton cycle in a graph can be. For example, the cycle in the dodecahedron shown in Figure 1 is 2-symmetric, as the drawing has 2-fold (i.e., $360^\circ/2 = 180^\circ$) rotational symmetry.

1.1 Hamilton cycles with rotational symmetry

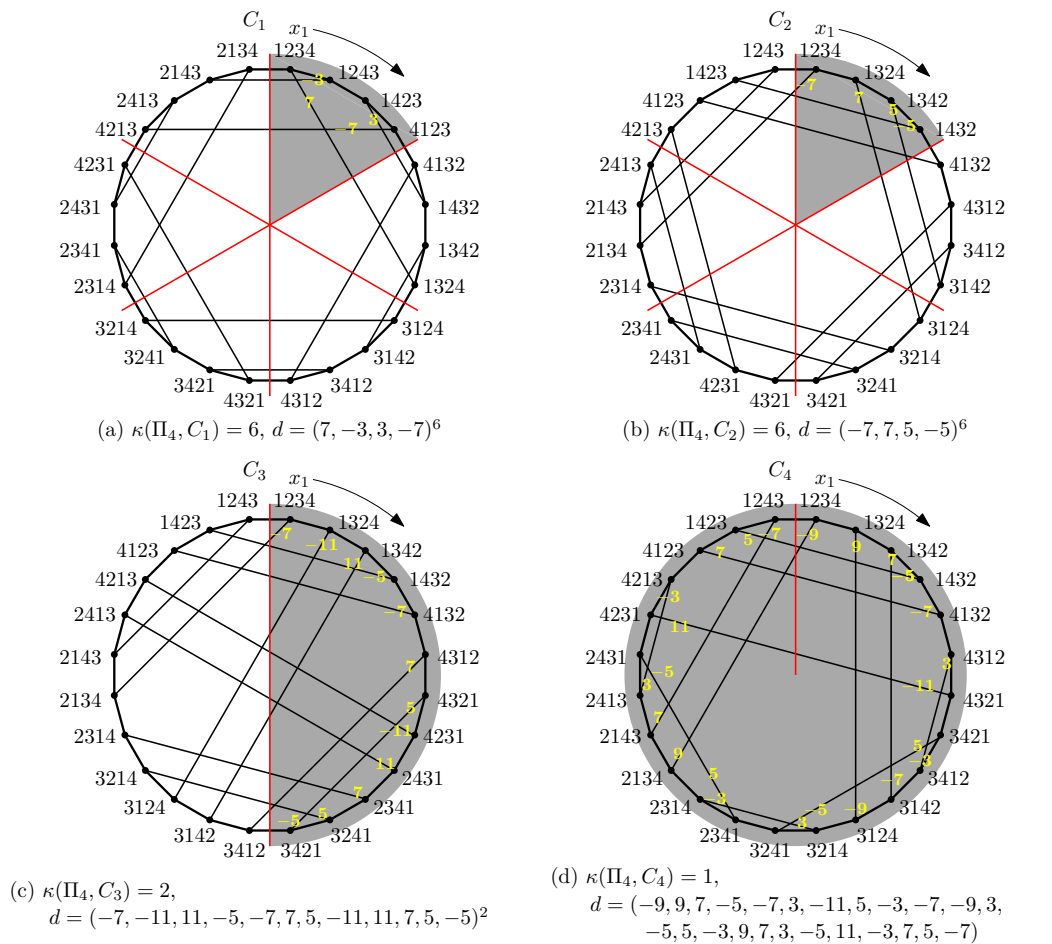
Formally, let $G = (V, E)$ be a graph with n vertices. We say that a Hamilton cycle $C = (x_1, \dots, x_n)$ is k -symmetric if the mapping $f : V \rightarrow V$ defined by $x_i \mapsto x_{i+n/k}$ for all $i = 1, \dots, n$, where indices are considered modulo n , is an automorphism of G . I.e., we have

$$C = P, f(P), f^2(P), \dots, f^{k-1}(P) \quad \text{for the path } P := (x_1, \dots, x_{n/k}). \quad (1)$$

The idea is that the entire cycle C can be reconstructed from the path P , which contains only a $1/k$ -fraction of all vertices, by repeatedly applying the automorphism f to it. In other words, if we lay out the vertices x_1, \dots, x_n equidistantly on a circle, and draw edges of G as straight lines, then we obtain a drawing of G with k -fold rotational symmetry, i.e., f is a rotation by $360^\circ/k$; see Figure 2. We refer to the maximum k for which the Hamilton cycle C of G is k -symmetric as the *compression factor* of C , and we denote it by $\kappa(G, C)$.

1.2 Connection to LCF notation

There is yet another interesting interpretation of the compression factor in terms of the LCF notation of a graph, named after its inventors Lederberg, Coxeter and Frucht (see [9]). The idea is to describe a 3-regular Hamiltonian graph (such as the dodecahedron) concisely by considering one of its Hamilton cycles $C = (x_1, \dots, x_n)$. Each vertex x_i has the neighbors x_{i-1} and x_{i+1} (modulo n) in the graph, plus a third neighbor x_j , which is $d_i := j - i$ (modulo n) steps away from x_i along the cycle. The *LCF sequence* of G is the sequence $d = (d_1, \dots, d_n)$, where each d_i is chosen so that $-n/2 < d_i \leq n/2$. Clearly, we also have $d_i \notin \{-1, 0, +1\}$. Note that if C is k -symmetric, then the LCF sequence d of G is k -periodic, i.e., it has the form $d = (d_1, \dots, d_{n/k})^k$, where the k in the exponent denotes k -fold repetition; see Figure 2. While LCF notation is only defined for 3-regular graphs, we can easily extend it to arbitrary graphs with a Hamilton cycle $C = (x_1, \dots, x_n)$, by considering a sequence of sets $D = (D_1, \dots, D_n)$, where D_i is the set of distances to all neighbors of x_i on the cycle except x_{i-1} and x_{i+1} ; see Figure 3 (a)+(d). As before, if C is k -symmetric, then the corresponding sequence D is k -periodic, i.e., it has the form $D = (D_1, \dots, D_{n/k})^k$. Frucht [9] writes:



■ **Figure 2** Hamilton cycles C_1, \dots, C_4 in the 4-permutahedron Π_4 with different LCF sequences and compression factors.

“What happens with the LCF notation if we replace one hamiltonian circuit by another one? The answer is: nearly everything can happen! Indeed the LCF notation for a graph can remain unaltered or it can change completely [...] In such cases we should choose of course the shortest of the existing LCF notations.”

This observation is illustrated in Figure 2, which shows four different Hamilton cycles of the same graph G that have different LCF sequences and compression factors.

1.3 Hamilton compression

Frucht’s suggestion is to search for a Hamilton cycle C in G whose compression factor $\kappa(G, C)$ is as large as possible. Formally, for any graph G we define

$$\kappa(G) := \max \{ \kappa(G, C) \mid C \text{ is a Hamilton cycle in } G \}, \tag{2}$$

and we refer to this quantity as the *Hamilton compression of G* . If G has no Hamilton cycle, then we define $\kappa(G) := 0$. While the maximization in (2) is simply over all Hamilton cycles in G , and the automorphisms arise as possible rotations of those cycles, this definition is somewhat impractical to work with. In our arguments, we rather consider all automorphisms

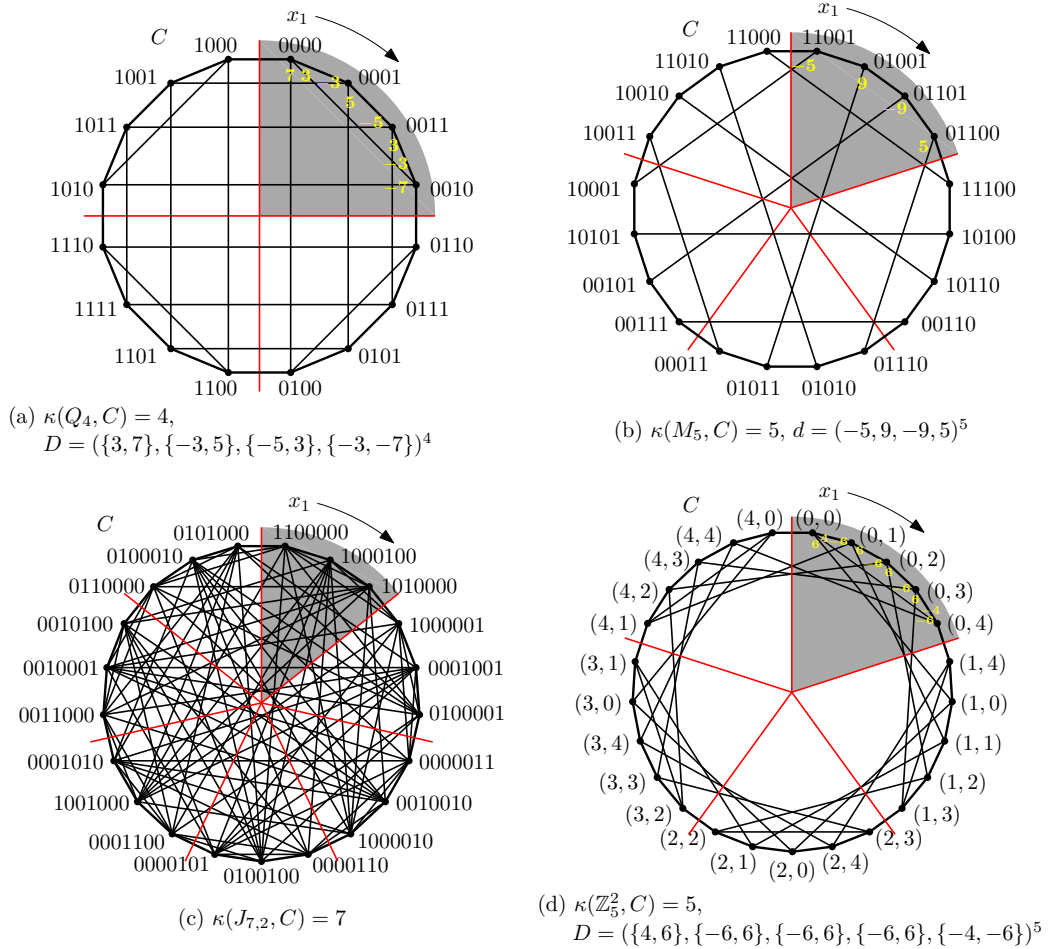


Figure 3 Symmetric Hamilton cycles in the (a) 4-cube; (b) middle levels of the 5-cube; (c) Johnson graph $J_{7,2}$; (d) abelian Cayley graph $(\mathbb{Z}_5^2, \{(0, 1), (1, 0)\})$.

of G , and then search for a Hamilton cycle that is k -symmetric under the chosen automorphism. Specifically, proving a lower bound of $\kappa(G) \geq k$ amounts to finding an automorphism f of G and a k -symmetric Hamilton cycle under f . To prove an upper bound of $\kappa(G) < k$, we need to argue that there is no k -symmetric Hamilton cycle in G , for any choice of f .

By what we said in the beginning, the quantity $\kappa(G)$ can be seen as a measure for the nicest (i.e., most symmetric) way of drawing the graph G on a circle. Thus, our paper contains many illustrations that convey the aesthetic appeal of this problem.

1.4 Easy observations and bounds

We collect a few basic observations about the quantity $\kappa(G)$. Trivially, we have $0 \leq \kappa(G) \leq n$, where n is the number of vertices of G . The upper bound n can be improved to

$$\kappa(G) \leq \max_{f \in \text{Aut}(G)} \text{ord}(f), \tag{3}$$

where $\text{Aut}(G)$ is the automorphism group of G , and $\text{ord}(f)$ is the order of f . An immediate consequence of (1) is that all orbits of the automorphism f must have the same size n/k , and the path $P = (x_1, \dots, x_{n/k})$ visits every orbit exactly once. This can be used to improve (3) further by restricting the maximization to automorphisms from $\text{Aut}(G)$ whose orbits all have the same size. Furthermore, as k must divide n , we obtain that $\kappa(G) \in \{0, 1, n\}$ for prime n .

Clearly, every Hamilton cycle of a graph G is 1-symmetric, by taking the identity mapping $f = \text{id}$ as automorphism. Consequently, we have $\kappa(G) \geq 1$ for any Hamiltonian graph. On the other hand, if G is Hamiltonian and highly symmetric, i.e., if it has a rich automorphism group, then intuitively G should have a large value of $\kappa(G)$, i.e., it should admit highly symmetric Hamilton cycles. For example, for the cycle C_n on n vertices and the complete graph K_n on n vertices we have $\kappa(C_n) = \kappa(K_n) = n$. More generally, note that $\kappa(G) = n$ if and only if G is a special circulant graph, namely the vertices of G can be labeled with $1, \dots, n$ such that vertex i is adjacent to all vertices $j = i + d$ (modulo n) with $d \in L$, where L is a fixed list with $1 \in L$. Note that general circulant graphs do not require that $1 \in L$, but the aforementioned characterization requires this assumption.

2 Our results

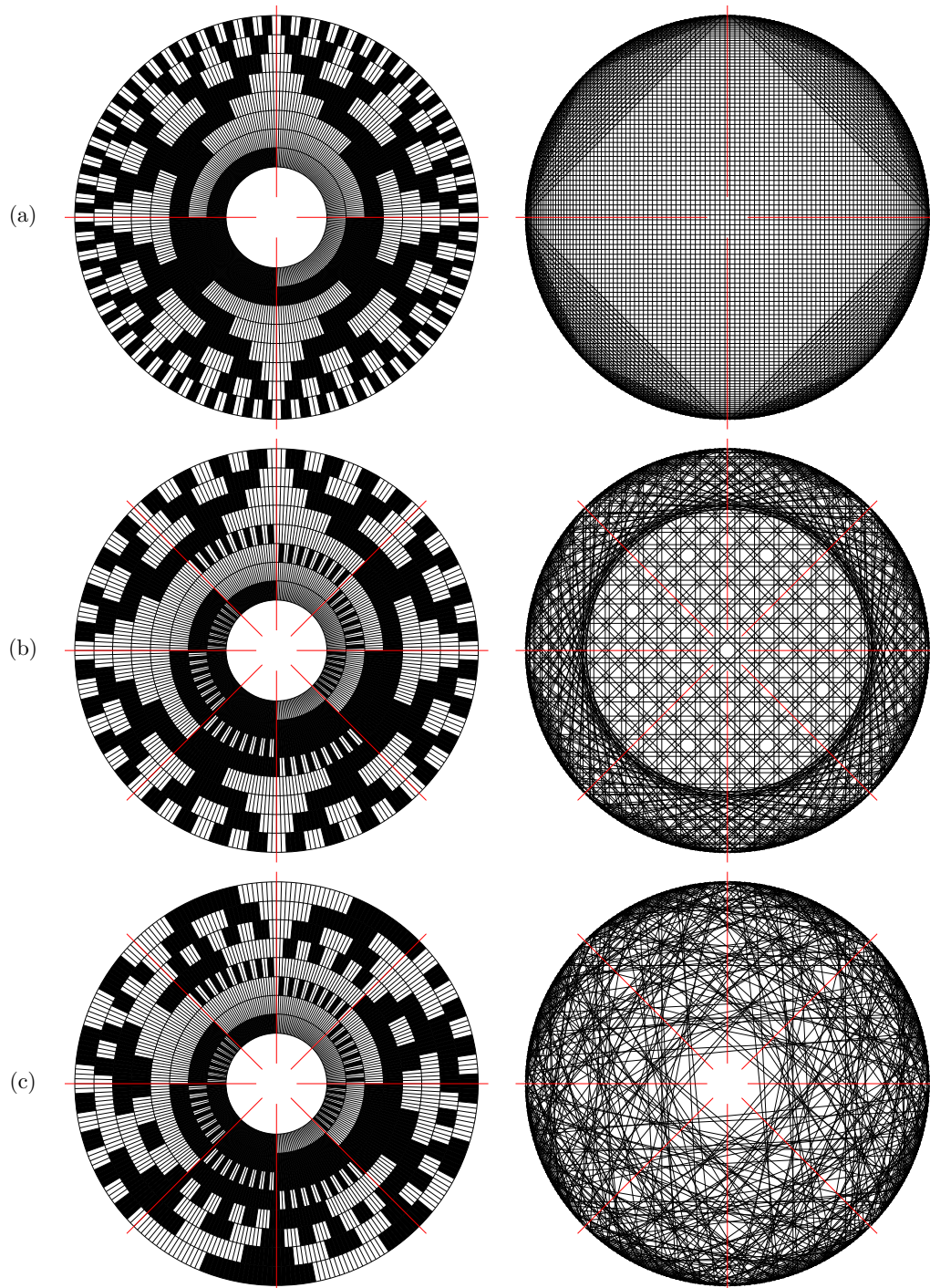
Vertex-transitive graphs are a prime example of highly symmetric graphs. A graph is *vertex-transitive* if for any two vertices there is an automorphism that maps the first vertex to the second one. In other words, the automorphism group of the graph acts transitively on the vertices. In this paper we investigate the Hamilton compression $\kappa(G)$ of four families of vertex-transitive graphs G , namely hypercubes, Johnson graphs, permutahedra, and Cayley graphs of abelian groups. In the following definitions the letter n denotes a graph parameter and not the number of vertices of the graph as in Section 1. The *n -dimensional hypercube* Q_n , or *n -cube* for short, has as vertices all bitstrings of length n , and an edge between any two strings that differ in a single bit; see Figure 3 (a). The *Johnson graph* $J_{n,m}$ has as vertices all bitstrings of length n with fixed Hamming weight m , and an edge between any two strings that differ in a transposition of a 0 and 1; see Figure 3 (c). The *n -permutahedron* Π_n , has as vertices all permutations of $[n] := \{1, \dots, n\}$, and an edge between any two permutations that differ in an adjacent transposition, i.e., a swap of two neighboring entries of the permutations in one-line notation; see Figure 2. For a group Γ and generating set $S \subseteq \Gamma$, the Cayley graph $G(\Gamma, S)$ has Γ as its vertex set and undirected edges $\{x, y\}$ for all $x, y \in \Gamma$ and $s \in S$ with $y = xs$; see Figure 3 (d). Note that the hypercube is isomorphic to a Cayley graph of the abelian group \mathbb{Z}_2^n .

Hamilton cycles with various additional properties in the aforementioned families of graphs have been the subject of a long line of previous research under the name of *combinatorial Gray codes* [22, 25]. We will see that some classical constructions of such cycles have a non-trivial small compression factor, and we construct cycles with much higher compression factor that we show to be optimal or near-optimal. Along the way, many interesting number-theoretic and algebraic phenomena arise. Due to space constraints, in this extended abstract we only mention our main results, while all proofs can be found in the preprint [15].

2.1 Hypercubes

One of the classical constructions of a Hamilton cycle in Q_n is the well-known *binary reflected Gray code (BRGC)* [14]. This cycle in Q_n is defined inductively by $\Gamma_0 := \varepsilon$ and $\Gamma_n := 0\Gamma_{n-1}, 1\overleftarrow{\Gamma_{n-1}}$ for all $n \geq 1$, where ε is the empty sequence and $\overleftarrow{\Gamma_{n-1}}$ denotes the reversal of the sequence Γ_{n-1} . In words, the cycle Γ_n is obtained by concatenating the vertices of Γ_{n-1} prefixed by 0 with the vertices of Γ_{n-1} in reverse order prefixed by 1. The cycle Γ_n is shown in Figure 3 (a) and Figure 4 (a) for $n = 4$ and $n = 8$, and these drawings have 4-fold rotational symmetry.

► **Proposition 1.** *The BRGC Γ_n has compression $\kappa(Q_n, \Gamma_n) = 4$ for $n \geq 2$.*



■ **Figure 4** Symmetric Hamilton cycles in Q_8 . Cycles are on the left (0=white, 1=black), with the first and last bit on the inner and outer track, respectively. The full graph Q_8 is on the right, with vertices arranged in cycle order and edges drawn as straight lines. (a) Binary reflected Gray code Γ_8 with compression 4; (b) Hamilton cycle with compression 8 from Theorem 2; (c) 2-track Hamilton cycle with compression 8 from Theorem 8.

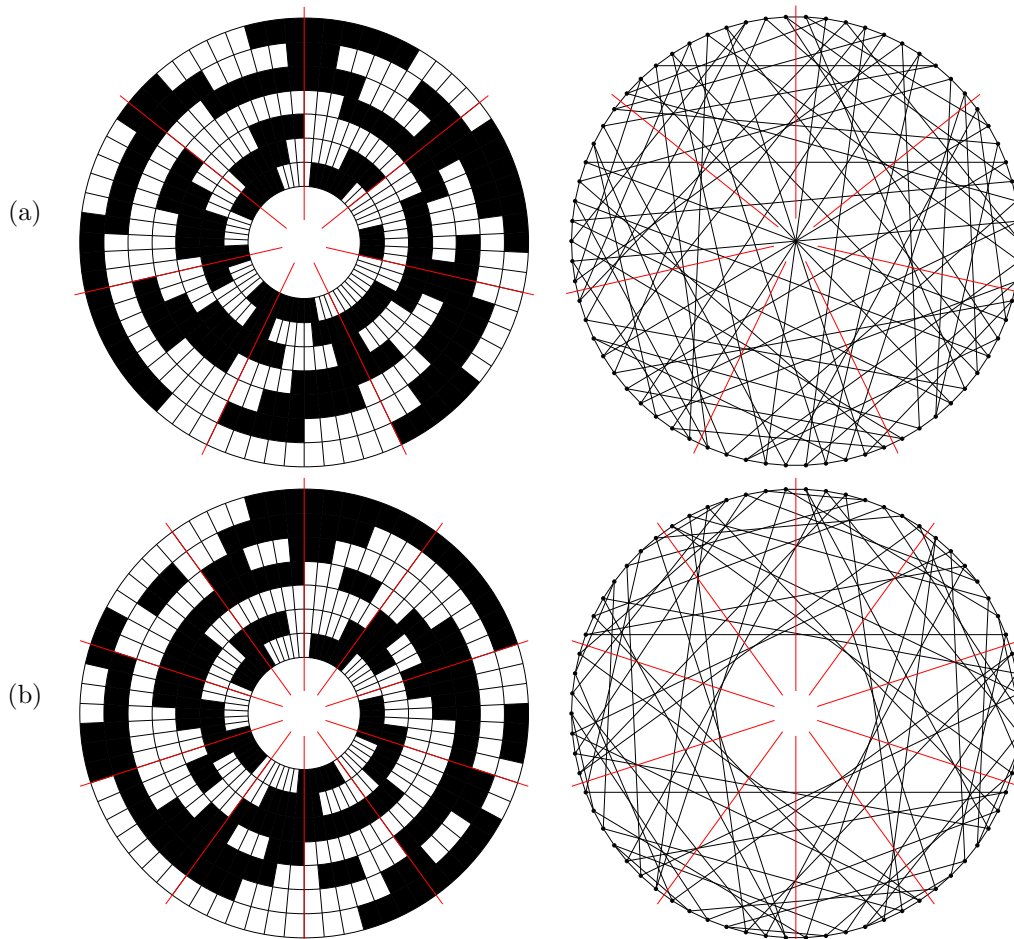
We improve upon this by constructing new Hamilton cycles in Q_n that have optimal linear Hamilton compression; see Figure 4 (b).

► **Theorem 2.** *We have $\kappa(Q_2) = 4$ and $\kappa(Q_n) = 2^{\lceil \log_2 n \rceil}$ for all $n \geq 3$.*

Note that $n \leq \kappa(Q_n) < 2n$ for $n \geq 2$, in particular $\kappa(Q_n) = \Theta(n)$, i.e., the optimal compression grows linearly with n .

2.2 Johnson graphs and relatives

Our definition of Hamilton compression is inspired by a variant of the well-known middle levels problem raised by Knuth in Problem 56 in Section 7.2.1.3 of his book [17]. Let M_{2n+1} denote the subgraph of Q_{2n+1} induced by all bitstrings with Hamming weight n or $n+1$. In other words, M_{2n+1} is the subgraph of the cover graph of the Boolean lattice of dimension $2n+1$ induced by the middle two levels. There is a natural automorphism of M_{2n+1} all of whose orbits have the same size, namely cyclic left-shift of the bitstrings by one position. Knuth asked whether M_{2n+1} admits a $(2n+1)$ -symmetric Hamilton cycle



■ **Figure 5** Symmetric Hamilton cycles in the middle levels graph M_7 : (a) A solution to Knuth's problem with compression 7 for f being cyclic left-shift; (b) Hamilton cycle with compression 10 for f being left-shift of the last 5 bits and complementation of all bits.

under this automorphism, and he rated this the hardest open problem in his book, with a difficulty rating of 49/50. Such cycles are shown in Figure 3 (b) and Figure 5 (a) for the graphs M_5 and M_7 , respectively. Knuth’s problem was answered affirmatively in full generality in [21], which establishes the lower bound $\kappa(M_{2n+1}) \geq 2n + 1$. We show that this is at most a factor of 2 away from optimality.

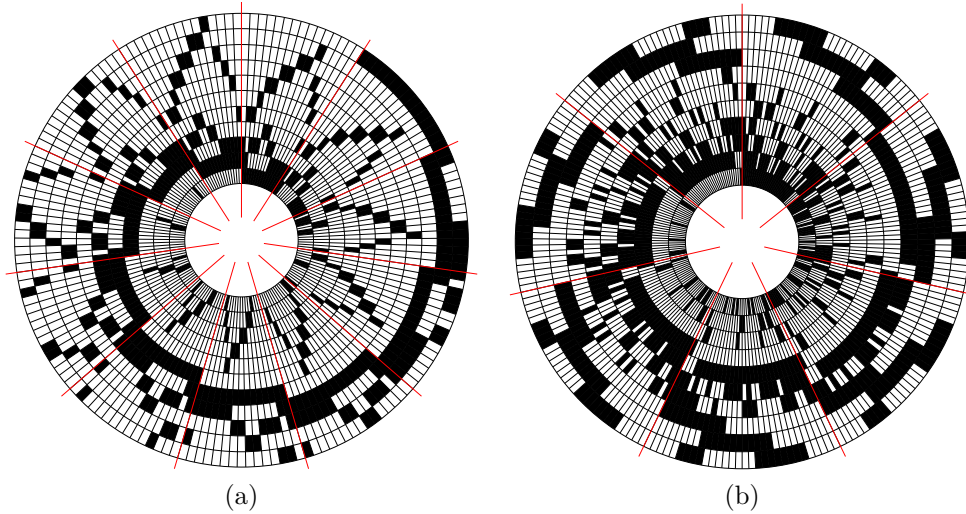
► **Theorem 3.** *For all $n \geq 1$ we have $2n + 1 \leq \kappa(M_{2n+1}) \leq 2(2n + 1)$.*

Interestingly, it seems that both bounds in Theorem 3 can be improved. For example, for $n = 3$ we can take the automorphism f of M_7 defined by $x_1 \cdots x_7 \mapsto \overline{x_1 x_2 x_4 x_5 x_6 x_7 x_3}$, which fixes the first two bits, cyclically left-shifts the remaining five bits by one position, and then complements all bits. A 10-symmetric Hamilton cycle under this f is shown in Figure 5 (b), whereas the lower and upper bounds are 7 and 14, respectively. In fact, computer experiments show that $\kappa(M_7) = 10$.

For the Johnson graphs $J_{n,m}$, we obtain the following exact results and bounds. Part (i) and (ii) of the theorem are illustrated in Figure 6 (a) and (b), respectively.

► **Theorem 4.** *The Hamilton compression of the Johnson graph $J_{n,m}$, where $n > m > 0$, has the following properties:*

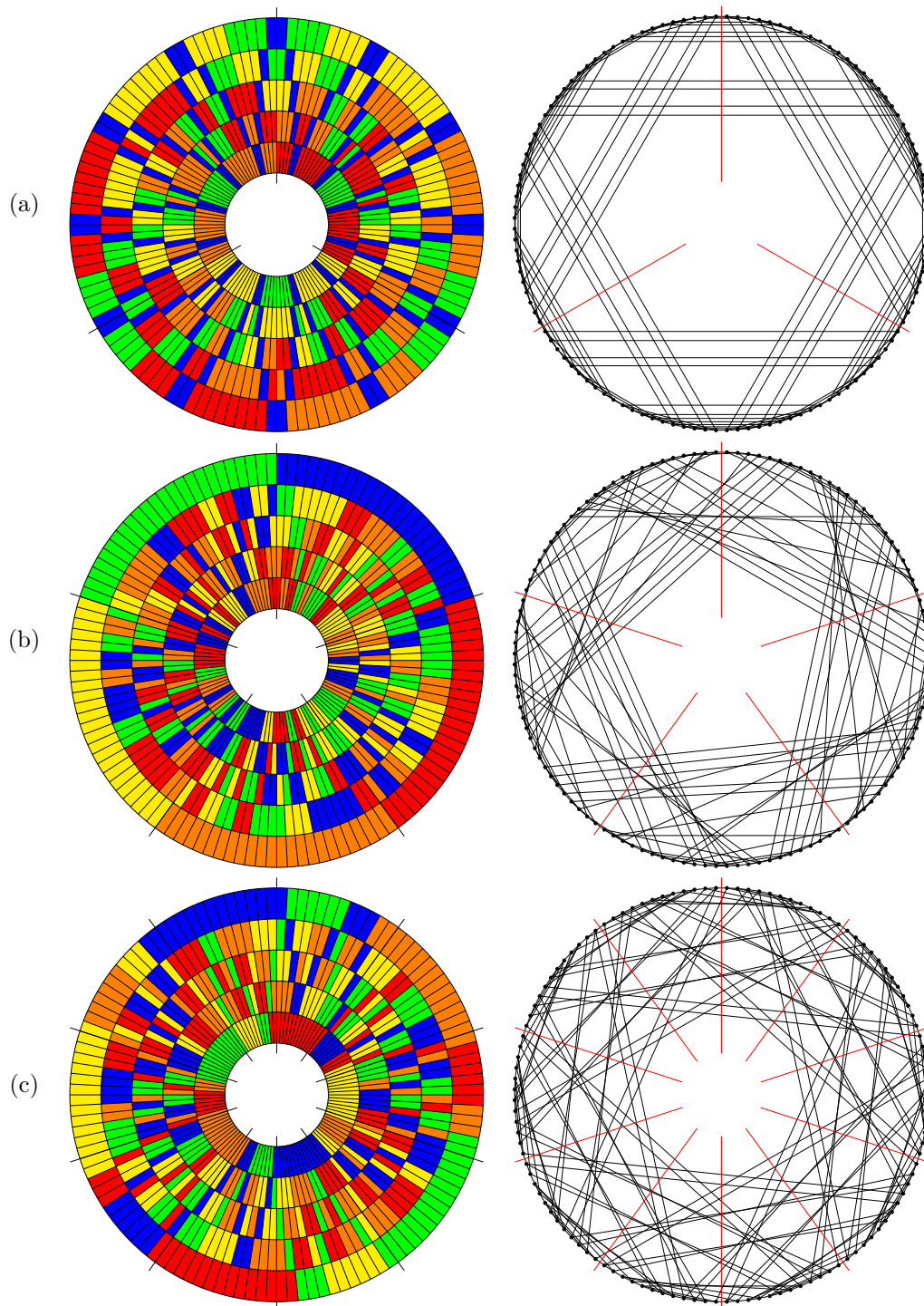
- (i) *If n and m are coprime, we have $\kappa(J_{n,m}) = n$.*
- (ii) *If n and m are not coprime and $n \neq 2m$, we have $n/2 < \max\{m, n - m\} < \kappa(J_{n,m}) \leq n$.*
- (iii) *If n and m are not coprime and $n = 2m$, we have $n/2 < \kappa(J_{n,m}) \leq 2n$.*
- (iv) *For any $\varepsilon > 0$ there is an n_0 such that for all $n > n_0$ with $n \neq 2m$ we have $(1 - \varepsilon)n \leq \kappa(J_{n,m}) \leq n$. In particular, we have $\kappa(J_{n,m}) = (1 - o(1))n$ for $n \neq 2m$.*



■ **Figure 6** Symmetric Hamilton cycles in Johnson graphs: (a) Balanced 1-track Hamilton cycle in $J_{11,3}$ with compression $n = 11$; the automorphism left-shifts all n bits; (b) 4-track Hamilton cycle in $J_{10,4}$ with compression $q = 7$; the automorphism left-shifts the first q bits.

2.3 Permutahedra

Another classical Gray code is produced by the *Steinhaus-Johnson-Trotter (SJT) algorithm*, which generates permutations by adjacent transpositions. This algorithm computes a Hamilton cycle in Π_n , which can be described inductively as follows: $\Lambda_1 := 1$ and for all



■ **Figure 7** Symmetric Hamilton cycles in Π_5 (1=red, 2=orange, 3=yellow, 4=green, 5=blue): (a) Steinhaus-Johnson-Trotter cycle Λ_5 with compression 3; (b) Cycle with compression 5; (c) Cycle with optimal compression 10.

$n \geq 2$ the cycle Λ_n is obtained from Λ_{n-1} by replacing each permutation of length $n-1$ by the n permutations given by inserting n in every possible position, alternatingly from right to left or vice versa. The cycle Λ_n is shown in Figure 2 (a) and Figure 7 (a) for $n = 4$ and $n = 5$, respectively, and these drawings have 6-fold or 3-fold rotational symmetry.

► **Proposition 5.** *The SJT cycle Λ_n has compression $\kappa(\Pi_n, \Lambda_n) = 6$ for $n = 3, 4$ and $\kappa(\Pi_n, \Lambda_n) = 3$ for $n \geq 5$.*

We improve upon this by constructing new Hamilton cycles in Π_n that have mildly exponential Hamilton compression; see Figure 7 (b)+(c). Specifically, the growth of the optimum compression is determined by *Landau's function* $\lambda(n)$, which is defined as the maximum order of an element in the symmetric group S_n .

► **Theorem 6.** *We have $\kappa(\Pi_n) = \Theta(\lambda(n)) = e^{(1+o(1))\sqrt{n \ln n}}$.*

The lower and upper bounds in the proof of Theorem 6 differ at most by a factor of 2 for every $n \geq 3$. Moreover, we achieve the optimal compression in infinitely many cases, in particular for the following values of $n \leq 100$: $n = 3, 4, 5, 15, 22, 46, 49, 51, 52, 53, 55, 68, 69, 72, 73, 74, 75, 80, 82, 85, 87, 88, 89, 91, 92, 93, 96, 97, 99, 100$.

2.4 Abelian Cayley graphs

A classical folklore result asserts that every Cayley graph of an abelian group has a Hamilton cycle. The Chen-Quimpo theorem [4] asserts that in fact much stronger Hamiltonicity properties hold. It is thus natural to ask whether Cayley graphs of abelian groups have highly symmetric Hamilton cycles.

► **Theorem 7.** *Let Γ be an abelian group.*

- (i) *If $|\Gamma|$ is a product of distinct odd primes, then for the canonical generating set $S \subseteq \Gamma$, the Cayley graph $G = G(\Gamma, S)$ has compression $\kappa(G) = 1$.*
- (ii) *If $|\Gamma| \geq 3$ is even or divisible by a square greater than 1, then for any generating set $S \subseteq \Gamma$, the Cayley graph $G = G(\Gamma, S)$ has compression $\kappa(G) \geq 2$.*

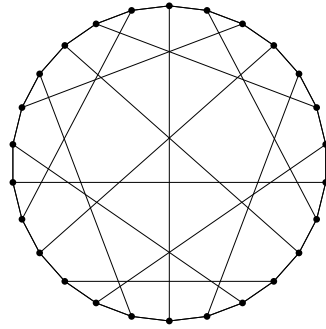
In particular, toroidal grids $\mathbb{Z}_p \times \mathbb{Z}_q$ for distinct odd primes p, q have only compression 1. The canonical generating set of a finite abelian group Γ is the set of unit vectors in the decomposition of Γ as a product of cyclic groups given by the structure theorem of finite abelian groups.

3 Related problems

We proceed to discuss some applications of our results to closely related problems.

3.1 Lovász' conjecture

A well-known question of Lovász' [20] asks whether there are infinitely many vertex-transitive graphs that do not admit a Hamilton cycle. So far only five such graphs are known, namely K_2 , the Petersen graph, the Coxeter graph, and the graphs obtained from the latter two by replacing every vertex by a triangle. Vertex-transitive graphs have a lot of automorphisms, and we may take the quantity $\kappa(G)$ as a measure of how strongly G is Hamiltonian. In particular, Lovász' question may be rephrased as 'Are there infinitely many vertex-transitive graphs G with $\kappa(G) = 0$?' More generally, we may ask: 'Are there infinitely many vertex-transitive graphs G with $\kappa(G) = k$, for each fixed integer k ?' A particularly relevant subclass



■ **Figure 8** One of the smallest vertex-transitive non-Cayley graphs G with $\kappa(G) = 1$.

of vertex-transitive graphs are Cayley graphs, so we may ask the same question about Cayley graphs. From our results mentioned in Section 2.4 we obtain an infinite family of Cayley graphs G with $\kappa(G) = 1$. Computer experiments show that the smallest vertex-transitive non-Cayley graphs G with $\kappa(G) = 1$ have 26 vertices, and one of them is shown in Figure 8.

The path P in (1) is a Hamilton path in the quotient graph G/f obtained by collapsing each orbit of f into a single vertex. The idea of constructing a Hamilton cycle in G by constructing a Hamilton cycle in the much smaller graph G/f that is then “lifted” to the full graph is well known in the literature, and has been used to solve some special cases of Lovász’ problem affirmatively; see e.g. [1, 6, 19, 21, 27]. It is particularly useful for computer searches, as it reduces the search space dramatically.

3.2 t -track and balanced Gray codes

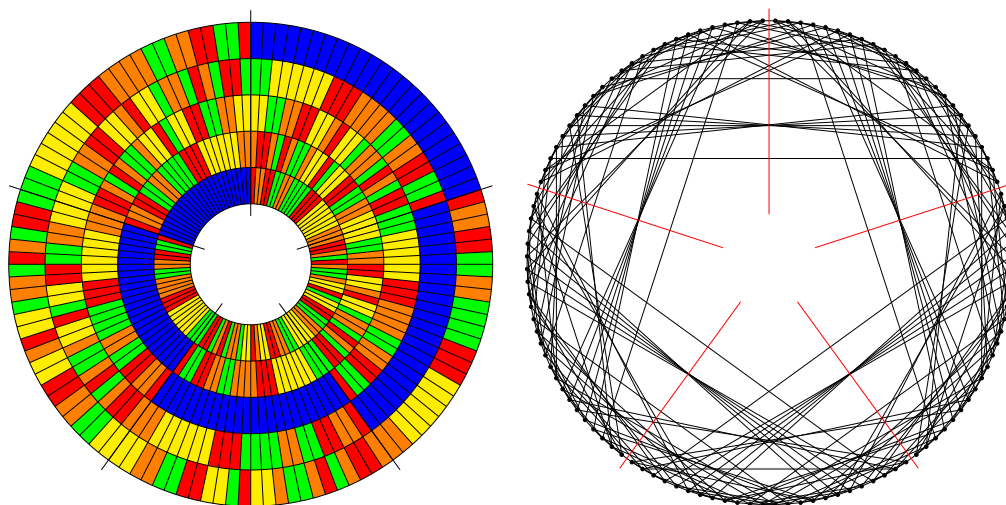
We say that a sequence C of strings of length n consists of t tracks if in the $|C| \times n$ matrix corresponding to C there are t columns such that every other column is a cyclically shifted copy of one of these columns. This property is relevant for applications, as it saves hardware when implementing Gray-coded rotary encoders. Instead of using n tracks and n reading heads aligned at the same angle (each reading one track), one can use only t tracks, and place some of the n reading heads at appropriately rotated positions.

Hiltgen, Paterson, and Brandestini [16] showed that the length of any 1-track cycle in Q_n must be a multiple of $2n$. In particular, such a cycle cannot be a Hamilton cycle unless n is a power of 2. For the case $n = 2^r$, $r \geq 3$, Etzion and Paterson [7] showed that there is 1-track cycle of length $2^n - 2n$, and Schwartz and Etzion [26] subsequently showed that the length $2^n - 2n$ is best possible. Taken together, these results show that there is no 1-track Hamilton cycle in Q_n for any $n \geq 3$. We complement this negative result by constructing a 2-track Hamilton cycle in Q_n , for every n that is a sum of two powers of 2; see Figure 4 (c).

► **Theorem 8.** *For every $n = 2^r$ and $m = 2^s$, where $r \geq 2$ and $r \geq s \geq 0$, there is a $2n$ -symmetric Hamilton cycle in Q_{n+m} that has 2 tracks.*

More generally, we obtain t -track Hamilton cycles in Q_n for every n that is a sum of $t \geq 2$ powers of 2. In particular, every dimension n admits a Hamilton cycle with at most logarithmically many tracks.

► **Theorem 9.** *For every $n = 2^r$ and $(m_1, \dots, m_{t-1}) = (2^{s_1}, \dots, 2^{s_{t-1}})$, where $r, t \geq 2$ and $r \geq s_1 \geq \dots \geq s_{t-1} \geq 0$, there is a $2n$ -symmetric Hamilton cycle in $Q_{n+m_1+\dots+m_{t-1}}$ that has t tracks.*



■ **Figure 9** Balanced 1-track Hamilton cycle in Π_5^+ with compression 5 from Theorem 11 (cyclically adjacent transpositions).

From our construction in the Johnson graph $J_{n,m}$ when n and m are coprime, we obtain 1-track Hamilton cycles that are also *balanced*, i.e., each bit is flipped equally often (cf. [3, 8]); see Figure 6 (a).

► **Theorem 10.** *Let $n > m > 0$ be such that n and m are coprime. Then $J_{n,m}$ has an n -symmetric Hamilton cycle that has 1 track and is balanced, i.e., each bit is flipped equally often ($\binom{n}{m}/n$ many times).*

We write Π_n^+ for the graph obtained from the permutahedron Π_n by adding edges that correspond to transpositions of the first and last entry of a permutation, i.e., we allow cyclically adjacent transpositions. The next theorem is illustrated in Figure 9.

► **Theorem 11.** *For every odd $n \geq 3$ there is an n -symmetric Hamilton cycle in Π_n^+ that has 1 track and is balanced, i.e., each of the n transpositions is used equally often ($(n-1)!$ many times).*

4 Open questions

The Hamilton compression $\kappa(G)$ is a newly introduced graph parameter, so many natural follow-up questions arise. We conclude this paper by listing several of these problems.

- Can the Gray codes constructed in this paper be computed efficiently? While our proofs translate straightforwardly into algorithms whose running time is polynomial in the size of the graph, a more ambitious goal are algorithms whose running time per generated vertex is polynomial in the length of the vertex labels (bitstrings, permutations, etc.).
- What is the Hamilton compression of the middle levels graph (recall Theorem 3)?
- For any integer $n \geq 1$, the odd graph O_n has as vertices all bitstrings of length $2n + 1$ with Hamming weight n , and an edge between any two strings that have no 1s in common. Odd graphs O_n , $n \geq 3$, were shown to have a Hamilton cycle in [23], so $\kappa(O_n) \geq 1$. We can use cyclic shifts as the automorphism, and it is easy to see that $\kappa(O_n) \leq 2n + 1$. We conjecture that $\kappa(O_n) = 2n + 1$ for all $n \geq 4$, which we confirmed for $n = 4$.

- In view of Section 2.4, the main open question here is whether the Cayley graph $G = G(\Gamma, S)$, where Γ is an abelian group such that $|\Gamma|$ is a product of distinct odd primes and S is a non-canonical set of generators, has $\kappa(G)$ equal to 1 or exceeding 1.
- What is the Hamilton compression of the associahedron, which has as automorphism group the dihedral group of a regular n -gon? For $n = 5, 6, 7, 8$ we determined the values 5, 2, 7, 2 by computer, and we suspect that the primality of n plays a role.
- Instead of asking about the largest number $k = \kappa(G)$ such that $\text{Aut}(G, C)$ (automorphisms of G that preserve C) contains the cyclic subgroup of order k for some Hamilton cycle C in G , we may ask for the dihedral subgroup of the largest order, which would allow not only for rotations of the drawings but also reflections.
- Is there a 1-track Hamilton cycle in Π_n (recall Theorem 11)? Equivalently, can all $n!$ permutations be listed by adjacent transpositions so that every column is a cyclic shift of every other column?
- Is there a balanced Hamilton cycle in Π_n ? Equivalently, can all $n!$ permutations be listed using each of the $n - 1$ adjacent transpositions equally often? Alternatively, what about using each of the $\binom{n}{2}$ transpositions equally often (see [8])? For $n = 5$, we found orderings satisfying the constraints of both questions.

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