A Discrete-Continuous Algorithm for Globally Optimal Free Flight Trajectory Optimization

Ralf Borndörfer
Zuse Institute, Berlin, Germany
Freie Universität Berlin, Germany

Fabian Danecker[1]
Zuse Institute, Berlin, Germany

Martin Weiser
Zuse Institute, Berlin, Germany

Abstract

We present an efficient algorithm that finds a globally optimal solution to the 2D Free Flight Trajectory Optimization Problem (aka Zermelo Navigation Problem) up to arbitrary precision in finite time. The algorithm combines a discrete and a continuous optimization phase. In the discrete phase, a set of candidate paths that densely covers the trajectory space is created on a directed auxiliary graph. Then Yen's algorithm provides a promising set of discrete candidate paths which subsequently undergo a locally convergent refinement stage. Provided that the auxiliary graph is sufficiently dense, the method finds a path that lies within the convex domain around the global minimizer. From this starting point, the second stage will converge rapidly to the optimum. The density of the auxiliary graph depends solely on the wind field, and not on the accuracy of the solution, such that the method inherits the superior asymptotic convergence properties of the optimal control stage.

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1 Introduction

Flight planning deals with finding the shortest flight path between two airports for an aircraft subject to a number of constraints, in particular, to wind conditions. The problem can be addressed from a discrete and from a continuous point of view and both approaches have received significant attention in the literature. Today's flight planning system follow the discrete approach, which treats the problem as a time-dependent shortest path problem in a world-wide 3D Airway Network, see [19] for a comprehensive survey, and a number of algorithms have been developed that address different aspects of the problem. For the most basic version, [11] and [28] suggested dynamic programming methods, [29] discusses graph

[1] corresponding author

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preprocessing, and [5] and [21] present A*-type algorithms. [4] integrates overflight costs and [22] traffic restrictions. [16] investigates the free route case, in which the Airway Network can be enriched by additional, artificial waypoints and segments. This setting blends into the Free Flight Trajectory Optimization Problem, aka Zermelo Navigation problem, to find the (globally) time-optimal route from A to B with respect to wind conditions. This classic of continuous optimization is usually solved using direct or indirect methods from Optimal Control [7]. These are highly efficient, but suffer from one key drawback, namely, they only converge locally. Such methods therefore depend on a sufficiently good starting point, which makes them, used as a standalone tool, incapable of meeting airlines' high expectations regarding the global optimality of routes. In other words, what is called an "optimal solution" in Control theory is only locally optimal, and not globally optimal in the sense of Discrete optimization.

As far as we know, Global Optimization has received little attention in this context so far, but inspiration can be drawn from related fields such as interstellar space mission design [10], robot motion planning [18, 26, 30], or even molecular structure optimization [15]. In all these cases, the central challenge is always to find the right balance between sufficient exploration of the search space on the one hand and accurate exploitation of promising regions on the other hand [20]. Two main types of approaches have been used to provide this balance, namely, *stochastic* and *deterministic* algorithms. In both cases, finding solutions takes at least exponential time, the runtime increasing with the required accuracy.

Stochastic methods scan the search space with some sort of Multistart approach, i.e., a set of starting points is chosen from the search space more or less at random, and these are explored. The exploration may be enhanced by allowing the candidates (then called *agents*) to wander around with a certain (decreasing) probability (e.g. Simulated Annealing [25, 10]). The deeper investigation of promising areas can be implemented as a local optimization step (e.g. Monotonic Basin Hopping [1]) or via interaction of the candidates attracting each other to the best known solution (e.g., Particle Swarm Optimization [6]). Even though these methods have received a lot of attention over the last decades and show promising results in a variety of applications, they are generally not able to guarantee global optimality in finite time. At best, they will asymptotically converge to a global optimizer (e.g., PRM* or RRT* [18]).

Deterministic approaches are usually based on the principle of Branch and Bound and converge to the global optimizer up to arbitrary precision in finite time [3, 14, 12, 17]. The complexity is generally exponential in the number of problem dimensions and the actual performance depends strongly on the quality of the lower bound.

We propose in this paper a efficient deterministic algorithm that finds the global optimizer of the Free Flight Trajectory Optimization Problem in finite time. It is not based on the Branch-and-Bound paradigm. Instead, a two-stage approach combines discrete and continuous optimization methods in a refinement of the concept of the hybrid algorithm DisCOptER [7]. In the first stage, the search space is sampled by calculating discrete paths on a sufficiently dense artificial digraph. In the second stage, the candidate solutions are refined using efficient techniques from optimal control. Under mild assumptions, namely, the existence of an isolated global minimizer and bounded wind speeds and wind derivatives, the problem is convex in a certain neighborhood of the minimizer. A sufficiently dense graph then contains a path within this neighborhood. This path can be determined by means of Yen's algorithm, and standard nonlinear programming methods will then efficiently produce the global optimizer up to any requested accuracy. In this way, our approach focuses on the exploration of the relevant parts of the search space. Moreover, the density of the auxiliary graph depends solely on the convexity properties of the problem, i.e., on the wind field, and *not* on the required accuracy. Hence, the method inherits, on the one hand, the superior asymptotic convergence properties of the second stage, which, in turn, is the key to its efficiency. Typically, only a handful of paths have to be enumerated and investigated. On the other hand, the method also benefits from all advancements in the area of Discrete Flight Planning, e.g. [5, 29].

2 The Free Flight Trajectory Optimization Problem

As the Free Flight Trajectory Optimization Problem is ultimately looking for a smooth trajectory, we start our discussion from the Optimal Control point of View.

2.1 Continuous Point of View: Optimal Control

The Free Flight Trajectory Optimization Problem can be formally described as follows. Let a spatially heterogeneous, twice continuously differentiable wind field $w : \mathbb{R}^2 \to \mathbb{R}^2$ with a bounded magnitude $\|w\|_{L^\infty(\mathbb{R}^2)} < \overline{v}$ be given. A valid trajectory is any Lipschitz-continuous path $x : [0, T] \to \mathbb{R}^2$ with $\|x_t - w\| = \overline{v}$ almost everywhere, connecting the origin x_O and the destination x_D. Among those, we want to find one of minimal flight duration $T \in \mathbb{R}$ (flight duration is essentially proportional to fuel consumption [31]). This classic of optimal control is also known as Zermelo's navigation problem [33].

It can easily be shown that in case of bounded wind speed, the optimal trajectory cannot be arbitrarily longer than the straight connection of origin and destination. Hence every global minimizer is contained in an ellipse $\Omega \subset \mathbb{R}^2$ with focal points x_O and x_D.

Assume the flight trajectory $x \in H^1([0, 1]) : [0, T] \to \mathbb{R}^2$ is given by a strictly monotonuously increasing parametrization $t(\tau)$ on $[0, 1]$ as $x(t(\tau)) = \xi(\tau)$, such that $\xi : [0, 1] \to \mathbb{R}^2$ is a Lipschitz continuous path. Due to Rademacher's theorem, its derivative with respect to the time ξ_τ exists almost everywhere, and we assume it not to vanish. Then, $t(\tau)$ is defined by the state equation $x_t = v + w \neq 0$ and the airspeed constraint $\|v\| = \overline{v}$, with $v \in L^2([0, 1])$ being the airspeed vector. Indeed,

$$\overline{v} = \|x_t - w\| \quad \text{and} \quad x_t t_\tau = \xi_\tau \neq 0$$

imply

$$\begin{aligned}
& (t_\tau^{-1}\xi_\tau - w)^T (t_\tau^{-1}\xi_\tau - w) = \overline{v}^2 \\
\Leftrightarrow\quad & t_\tau^{-2}\xi_\tau^T\xi_\tau - 2t_\tau^{-1}\xi_\tau^T w + w^T w - \overline{v}^2 = 0 \\
\Leftrightarrow\quad & (\overline{v}^2 - w^T w)t_\tau^2 + 2\xi_\tau^T w t_\tau - \xi_\tau^T\xi_\tau = 0 \\
\Leftrightarrow\quad & t_\tau = \frac{-\xi_\tau^T w + \sqrt{(\xi_\tau^T w)^2 + (\overline{v}^2 - w^T w)(\xi_\tau^T \xi_\tau)}}{\overline{v}^2 - w^T w} =: f(t, \xi, \xi_\tau)
\end{aligned} \tag{1}$$

due to $t_\tau > 0$. The flight duration T is then given by integrating the ODE (1) from 0 to 1 as $T = t(1)$. Let us assume for ease of presentation that the wind w is stationary, i.e., independent of t, and thus $f(t, \xi, \xi_\tau) = f(\xi, \xi_\tau)$. Doing so, we obtain the simple integral

$$T(\xi) = \int_0^1 f\big(\xi(\tau), \xi_\tau(\tau)\big)\, d\tau. \tag{2}$$

Since the flight duration T as defined in (2) is based on a reparametrization $x(t) = \xi(\tau(t))$ of the path such that $\|x_t(t) - w(x(t))\| = \overline{v}$, the actual parametrization of ξ is irrelevant for the value of T. Calling two paths $\xi, \tilde{\xi}$ equivalent if there exists a Lipschitz-continuous bijection $r : [0,1] \to [0,1]$ such that $\xi(r(\tau)) = \tilde{\xi}(\tau)$, we can restrict the optimization to equivalence classes. Every equivalence class contains a representative with constant ground speed $\|\xi_\tau(\tau)\| = L$, that can be obtained from any $\tilde{\xi}$ with $\|\tilde{\xi}_\tau(\tau)\| \neq 0\,\forall\tau$ via

$$\xi(\tau) := L \int_0^\tau \frac{\tilde{\xi}_\tau(t)}{\|\tilde{\xi}_\tau(t)\|} dt, \tag{3}$$

where $L := \int_0^1 \|\tilde{\xi}_\tau(\tau)\| d\tau$. Hence we will subsequently consider the following equivalent constrained minimization problem:

$$\min_{\xi \in X,\, L \in \mathbb{R}} T(\xi), \quad \text{s.t.} \quad \|\xi_\tau(\tau)\|^2 = L^2 \quad \forall \tau \in [0,1]; \tag{4}$$

here, the admissible set is the affine space

$$X = \{\xi \in H^1([0,1], \mathbb{R}^2) \;\mid\; \xi(0) = x_O,\ \xi(1) = x_D\}. \tag{5}$$

Note that L also represents the path length of a solution, since

$$\int_0^1 \|\xi_\tau\| d\tau = L. \tag{6}$$

We finally express the constant ground speed requirement by means of a constraint $h(z) = 0$, where $z := (L, \xi) \in Z := \mathbb{R} \times X$ and

$$h : Z \to \Lambda := L^2(]0,1[), \quad z \mapsto \xi_\tau^T \xi_\tau - L^2 \tag{7}$$

for $L \leq L_{\max}$, with an arbitrary continuation for $L > L_{\max}$ that is linear in $\|\xi_\tau\|$. In order to take the boundary constraints $\xi(0) = x_O$, $\xi(1) = x_D$ into account, we restrict deviations $\delta\xi$ from the trajectory ξ to the space

$$\delta X := \{H^1([0,1], \mathbb{R}^2) \mid \delta\xi(0) = \delta\xi(1) = 0\}. \tag{8}$$

The goal of the present paper is to find a isolated globally optimal solution $\xi^{\star\star}$ to (4) that satisfies $T(\xi^{\star\star}) < T(\xi)\ \forall \xi \in X$, contrary to a local optimizer $\xi^\star$ that is only superior to trajectories in a certain neighborhood, $T(\xi^\star) \leq T(\xi)\ \forall \xi \in \mathcal{N}(\xi^\star) \subseteq X$. A isolated global minimizer satisfies the necessary Karush-Kuhn-Tucker (KKT) optimality conditions [23] given that it is a regular point, which is always the case since

$$h'(z) : \delta Z \mapsto \Lambda\ \forall z \in Z, \quad \delta z \mapsto \xi_\tau^T \delta\xi_\tau - L\delta L. \tag{9}$$

The KKT-conditions result from the variation of the Lagrangian

$$\mathcal{L}(z, \lambda) := T(\xi) + \langle \lambda, h(z) \rangle \tag{10}$$

with respect to z and λ:

$$0 = T'(\xi^{\star\star})[\delta\xi, \delta\xi_\tau] + \int_0^1 \lambda^{\star\star}({\xi_\tau}^{\star\star T} \delta\xi_\tau) d\tau - L^{\star\star}\delta L \int_0^1 \lambda^{\star\star} d\tau \qquad \forall\ \delta z \in \delta Z, \tag{11a}$$

$$0 = \int_0^1 \delta\lambda({\xi_\tau}^{\star\star T} {\xi_\tau}^{\star\star} - L^{\star\star 2})\, d\tau \qquad \forall\ \delta\lambda \in \Lambda, \tag{11b}$$

where $\delta z := (\delta L, \delta\xi)$ and $\delta Z := \mathbb{R} \times \delta X$. Consider the unconstrained problem $\min_{\xi \in X} T(\xi)$ and a global minimizer $\tilde{\xi}^{\star\star}$. As discussed before, there is an equivalent route $\xi^{\star\star}$ that satisfies the constraint and hence – together with L from (6) – is a global minimizer of the constrained problem.

▶ **Lemma 1.** *Let* $(z^{\star\star}, L^{\star\star})$ *be a global minimizer of* (4)*. Then, this solution together with*

$$\lambda^{\star\star} = 0 \tag{12}$$

satisfies the necessary conditions (11)*.*

Proof. Since $\xi^{\star\star}$ is also a global minimizer of the unconstrained problem, the necessary condition states that $T'(\xi^{\star\star})[\delta\xi, \delta\xi_\tau] = 0$. The terms $\int_0^1 \lambda^{\star\star}({\xi_\tau}^{\star\star T}\delta\xi_\tau)\, d\tau$ and $\int_0^1 \lambda^{\star\star}\, d\tau$ of (11a) both vanish for $\lambda^{\star\star} = 0$. (11b) is satisfied because $\|{\xi_\tau}^{\star\star}\| = L^{\star\star}\ \forall \tau \in [0, 1]$. ◀

Now we turn to the second order sufficient conditions for optimality. In general, a stationary point $(z^\star, \lambda^\star)$ is a minimizer, iff the well known Ladyzhenskaya–Babuška–Brezzi (LBB) conditions (e.g. [9]) are satisfied, which comprise a) the so called *inf-sup*-condition

$$\inf_{\substack{\delta\lambda\in\Lambda\\ \delta\lambda\neq 0}} \sup_{\substack{\delta z\in\delta Z\\ \delta z\neq 0}} \frac{\langle \delta\lambda, h'(z)[\delta z]\rangle}{\|\delta z\|_{H^1}\|\delta\lambda\|_\Lambda} \geq C > 0 \tag{13}$$

and b) the requirement that the Lagrangian's Hessian regarding z, $\mathcal{L}_{zz}$, need be positive definite on the kernel of h'. Formally speaking, there must be a $\underline{\mathcal{B}} > 0$ such that

$$\mathcal{L}_{zz}(z^\star)[\delta z]^2 \geq \underline{\mathcal{B}}\, \|\delta z\|^2_{L^2}$$

for any $\delta z \in \delta Z$ that satisfies

$$\langle \delta\lambda, h'(z^\star)[\delta z]\rangle = 0 \quad \forall\ \delta\lambda \in \Lambda.$$

In our case, the second order sufficient condition is

$$T''(\xi^\star)[\delta\xi, \delta\xi_\tau]^2 + 2\int_0^1 \lambda^\star(\delta\xi_\tau^T\delta\xi_\tau - \delta L^2)d\tau \geq \underline{\mathcal{B}}(\delta L^2 + \|\delta\xi\|^2_{L^2} + \|\delta\xi_\tau\|^2_{L^2})$$

for any $(\delta L, \delta\xi) \in \mathbb{R} \times \delta X$ such that

$$\int_0^1 \delta\lambda({\xi_\tau}^{\star T}\delta\xi_\tau - L^\star\delta L)d\tau = 0 \quad \forall\ \delta\lambda \in L^2([0, 1]).$$

In case of a global minimizer $z^{\star\star}$, this can be simplified using $\langle\lambda^{\star\star}, h''\rangle = 0$ from Lemma 1. Moreover, the constraint is equivalent to requiring that ${\xi_\tau}^{\star\star T}\delta\xi_\tau = L^{\star\star}\delta L \quad \forall\ \tau \in [0, 1]$. With this, we conclude that for any isolated global minimizer $(z^{\star\star}, L^\star)$ of (4) there exists a $\underline{\mathcal{B}} > 0$ such that

$$T''(\xi^{\star\star})[\delta\xi, \delta\xi_\tau]^2 \geq \underline{\mathcal{B}}\left(\delta L^2 + \|\delta\xi\|^2_{L^2} + \|\delta\xi_\tau\|^2_{L^2}\right) \tag{14}$$

for any $\delta z \in \delta Z$ such that ${\xi_\tau}^{\star\star T}\delta\xi_\tau = L^{\star\star}\delta L \quad \forall\ \tau \in [0, 1]$.

2.2 Discrete Point of View: Shortest Paths

If flight trajectories are restricted to airway segments connecting given waypoints, flight planning is a special kind of shortest path problem on a graph. It can be described as follows. Let $V \subset \mathbb{R}^2$ be a finite set of waypoints including x_O and x_D, and $A \subset V \times V$ a set of segments such that $G = (V, A)$ is a connected directed graph. A discrete flight path is a finite sequence $(x_i)_{0\leq i\leq n}$ of waypoints with $(x_{i-1}, x_i) \in E$ for $i = 1, \ldots, n$, connecting $x_0 = x_O$ with $x_n = x_D$. Shortest path problems on static graphs with non-negative weights are usually solved with the A^* variant of Dijkstra's algorithm [27].

Define a mapping $\Xi : (x_i)_{0\le i\le n} \mapsto \xi \in X$ of discrete flight paths to continuous paths by piecewise linear interpolation

$$\xi(\tau) = x_{\lfloor n\tau\rfloor} + (n\tau - \lfloor n\tau\rfloor)(x_{\lceil n\tau\rceil} - x_{\lfloor n\tau\rfloor}), \tag{15}$$

resulting in polygonal chains, which are Lipschitz-continuous with piecewise constant derivative. Denote the image by $X_G := \operatorname{im}\Xi \subset X$, i.e., X_G is the set of flight trajectories with constant ground speed in the Euclidean plane that can be realized by adhering to the airway network. The discrete flight planning problem then reads

$$\min_{\xi\in X_G} T(\xi). \tag{16}$$

With any $\xi \in X_G$ satisfying the constraint for constant ground speed, this differs from its continuous counterpart (4) essentially by the admissible set, which effectively acts as a particular discretization. The class of (h,l)-dense graphs used in this work was introduced in [7] and is defined as follows.

▶ **Definition 2.** *A digraph $G = (V, A)$ is said to be (h,l)-dense in a convex set $\Omega \subset \mathbb{R}^2$ for $h, l \ge 0$, if it satisfies the following conditions:*

1. containment: $V \subset \Omega$,
2. vertex density: $\forall p \in \Omega : \exists v \in V : \|p - v\| \le h$,
3. local connectivity: $\forall u, v \in V, \|u - v\| \le l + 2h : (u, v) \in E$.

▶ **Definition 3.** *We call an (h,l)-dense digraph rectangular, if the vertex positions can be described by,*

$$x_{ij} = x_0 + \sqrt{2}h[i,j]^T \quad \textit{for} \quad i \in M \subseteq \mathbb{Z},\ j \in N \subseteq \mathbb{Z} \tag{17}$$

with $x_{ij} \in \Omega$ and M, N being connected subsets of the integers.

An example for such a rectangular (h,l)-dense airway digraph is shown in Figure 1 a). Note that, even for $l \to 0$, the minimum local connectivity length of $2h$ guarantees that a vertex is connected to all its direct neighbors. It is easy to show that any (h,l)-dense digraph is connected, such that a path from origin to destination exists.

2.3 Discrete-Continuous Point of View: Hybrid Algorithm DisCOptER

In [7] a hybrid algorithm was proposed that combines the strengths of the discrete and the continuous approach to flight planning. In a nutshell, it works as follows: First, an artificial locally connected digraph of defined density is created, as in Definition 2 (blue dots in Figure 1 b), arcs omitted). The shortest path on this graph (red) serves as an initial guess for a subsequent refinement stage in which a suitable nonlinear programming formulation of the same problem is solved, leading to a continuous locally optimal solution (green). As follows from this paper, this solution is also globally optimal, provided that the graph is sufficiently dense.

In numerical experiments, we observed that even for scenarios that are far more challenging than any real world situation, a very sparse graph is already sufficient to find the globally optimal solution, rendering the hybrid approach highly efficient. In case of the example illustrated in Figure 1 b), the global optimum was found using any graph with node spacing $h \le \frac{1}{15\sqrt{2}}$, which corresponds to 16 or more nodes between origin and destination. Note that in similar scenarios with n vortices one can expect $\mathcal{O}(2^n)$ local minima.

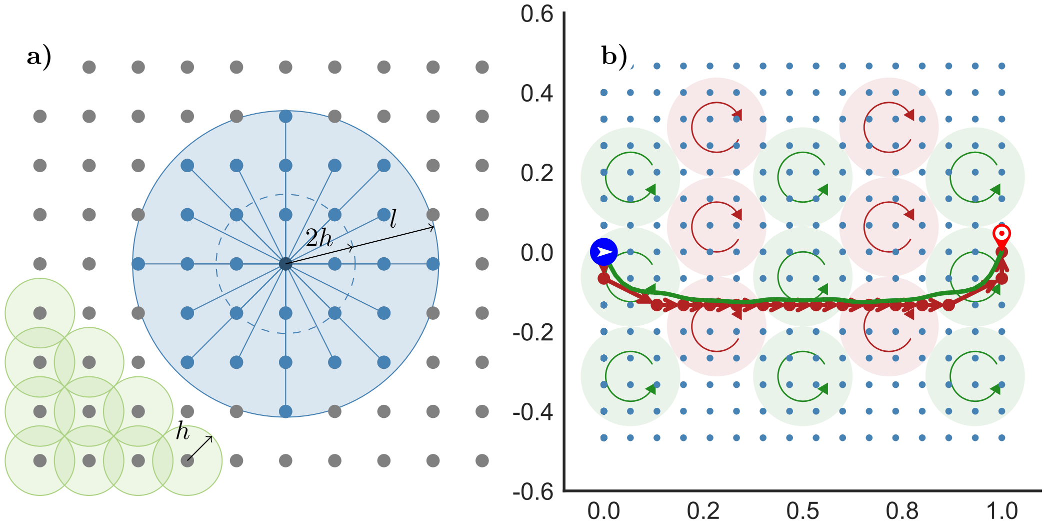

Figure 1 a) A rectangular (h, l)-dense digraph. The center vertex (dark blue) is connected to all vertices in a circular neighborhood of radius $2h + l$ (light blue) with edges in both directions. b) Illustration of the classical hybrid algorithm DisCOptER. The planar wind field consists of 15 regularly aligned vortices indicated by the green and red discs. Blue dots: locally connected vertices of the (h, l)-dense graph, see a). Red: Shortest path on the graph, Green: Continuous solution obtained via refinement.

We quickly recap the complexity analysis from [7]. The novel algorithm DisCOptER was compared against the traditional, purely graph-based approach in terms of accuracy of the provided solution compared to the continuous optimum. Trajectories of the desired accuracy can in principle be obtained by solving the shortest path problem on a sufficiently dense, locally complete digraph, that can be characterized by its vertex density h and local connectivity radius l, see Definition 2. An optimized combination of these properties is $h = \overline{\sigma} l^2/L^2$, where $\overline{\sigma}$ is an upper bound for the curvature of the optimal trajectory and L denotes its path length [8, Theorem 4]. Hence, l^{-1} may serve as a suitable measure for the solution accuracy. The number of vertices $|V|$ in such a digraph is in $\mathcal{O}(l^{-4})$ and the number of arcs $|A|$ is in $\mathcal{O}(l^{-6})$. The complexity of solving the shortest path problem with Dijkstra's algorithm is $\mathcal{O}(|A| + |V| \log |V|)$ and so the overall time complexity is in

$$\mathcal{O}(l^{-6}). \tag{18}$$

Since the required graph density is dictated exclusively by the wind conditions, the complexity of the hybrid algorithm approach is asymptotically inherited from the Optimal Control stage. Using a direct collocation method, the problem is discretized over the time domain with quasi equidistant steps $\delta\tau$. A comparable accuracy measure is then defined as $l := L\delta\tau$. Solving the first order necessary conditions – well known as Karush-Kuhn-Tucker (KKT) conditions – for the discretized problem via Newton's method rapidly yields a solution, provided that the starting point was already sufficiently close. Due to the problem structure each iteration step essentially involves a linear system of equations with an arrow-shaped matrix, which can be solved efficiently by specialized band-solvers. The overall time complexity of the hybrid algorithm is determined by the number of iterations and the cost of each step, which is in

$$\mathcal{O}(l^{-1}). \tag{19}$$

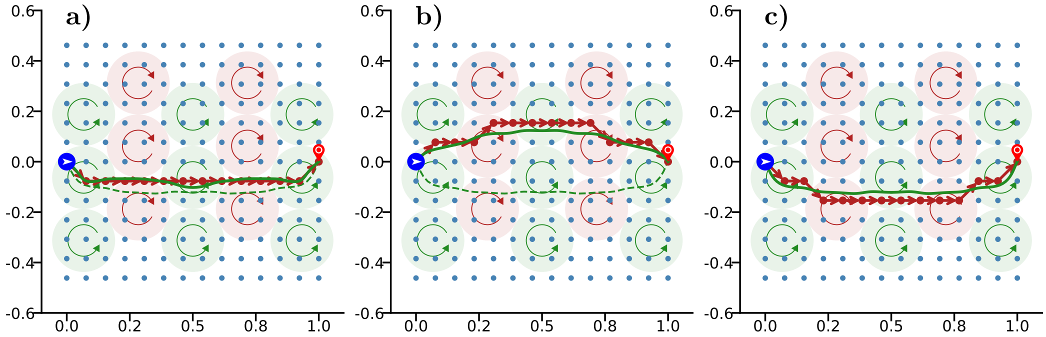

Figure 2 Illustration of the hybrid algorithm DisCOptER. The planar wind field consists of 15 regularly aligned vortices indicated by the green and red discs. Blue dots: locally connected vertices of the (h, l)-dense graph, see Figure 1 a). Red: k^{th} shortest path on the graph, Green: Continuous solution obtained via refinement. a) Starting from the very shortest path the refinement stage does not converge. b) The 5^{th} shortest path on the graph leads to a local optimum. c) The 14^{th} shortest path on the graph finally leads to the global optimum.

3 Towards Global Optimality

In terms of runtime the hybrid algorithm DisCOptER appears to be clearly superior to the traditional graph-based approach. One key question, however, remains: What is the right graph density? This sections answers this question and presents a variant of the algorithm which is guaranteed to find a global minimizer in finite time by calculating not only one but multiple shortest paths. We exploit the fact that, by continuity, there is a sufficiently large neighborhood around the minimizer over which the objective function is convex, see Theorem 4. If started within this neighborhood, optimal control methods will quickly converge up to arbitrary precision. Using a sufficiently dense graph, as described in Lemma 5, we guarantee that there is a path that lies in this neighborhood of the global minimizer.

This path can be found by computing paths by Yen's algorithm [32], which computes shortest simple paths in the order of increasing travel time. A suitable stopping criterion is technically not necessary, but anyway provided in Theorem 6. The required graph density is dictated by the wind conditions. Adverse scenarios will require dense graphs leading to a large number of feasible paths that is, e.g., exponential in the number of vortices, cf. again the example in Fig. 2. The number will, however, always be finite and – most importantly – independent of the desired solution accuracy.

▶ **Theorem 4.** *Let* $\|w(p)\| \leq \bar{c}_0 < \overline{v}/\sqrt{5}$, $\|w_x(p)\| \leq \bar{c}_1$, $\|w_{xx}(p)\| \leq \bar{c}_2$, *and* $\|w_{xxx}(p)\| \leq \bar{c}_3$ *for every* $p \in \Omega$. *Moreover, let* $z^{\star\star} := (\xi^{\star\star}, L^{\star\star}) \in Z$ *be a global minimizer of problem* (4), *that satisfies the necessary and sufficient conditions* (11), (13), *and* (14) *with* $C > 0$ *and* $\underline{\mathcal{B}} > 0$. *Then the problem* (4) *is convex in a neighborhood of* $z^{\star\star}$, *i.e., there is a* $R_C > 0$ *exclusively depending on the wind conditions such that the LBB-conditions are satisfied for any* $z \in Z$ *with*

$$\|\Delta z\|_{H^1([0,1])} := \|z - z^{\star\star}\|_{H^1([0,1])} \leq R_C. \tag{20}$$

Proof. According to (13), there is a $C > 0$ such that

$$\inf_{\substack{\delta\lambda \in \Lambda \\ \delta\lambda \neq 0}} \sup_{\substack{\delta z \in \delta Z \\ \delta z \neq 0}} \frac{\langle \delta\lambda, h'(z^{\star\star})[\delta z] \rangle}{\|\delta z\|_{H^1} \|\delta\lambda\|_\Lambda} \geq C$$

with h as defined in (7). Moreover, it holds that

$$T''(\xi^{\star\star})[\delta\xi, \delta\xi_\tau]^2 \geq \underline{\mathcal{B}}\left(\delta L^2 + \|\delta\xi\|^2_{L^2} + \|\delta\xi_\tau\|^2_{L^2}\right)$$

for any $\delta z \in \delta Z$ such that ${\xi_\tau}^{\star\star T}\delta\xi_\tau = L^{\star\star}\delta L \quad \forall \tau \in [0,1]$, see (14). Due to the continuity of the bilinear form, the inf-sup-condition is satisfied for any z with $\|\Delta z\| \leq R_{C1}$, such that

$$\inf_{\substack{\delta\lambda\in\Lambda \\ \delta\lambda\neq 0}} \sup_{\substack{\delta z\in\delta Z \\ \delta z\neq 0}} \frac{\langle \delta\lambda, h'(z)[\delta z]\rangle}{\|\delta z\|_{H^1}\|\delta\lambda\|_\Lambda} \geq \frac{C}{2} > 0.$$

Similarly, the continuity of T as given in (2), guarantees that there is a $R_{C2} > 0$ such that

$$T''(\xi)[\delta\xi, \delta\xi_\tau]^2 \geq \frac{\underline{\mathcal{B}}}{2}\left(\delta L^2 + \|\delta\xi\|^2_{L^2} + \|\delta\xi_\tau\|^2_{L^2}\right)$$

for any $z \in Z$ such that $\|z - z^{\star\star}\|_{H^1([0,1])} \leq R_{C2}$ and any $\delta z \in \delta Z$ such that $\xi_\tau^T\delta\xi_\tau = L\delta L \quad \forall\tau \in [0,1]$. Consequently, the sufficient conditions are satisfied for any z with $\|\Delta z\| \leq R_C := \min(R_{C1}, R_{C2})$. ◀

Providing a sufficiently (h,l)-dense graph, we can guarantee that there is a discrete path within the convex neighborhood of the global minimizer $B_{R_C}(\xi^{\star\star})$. The following Lemma involves a result from [8, Theorem 3] stating that the curvature of a global minimizer of (4) is bounded by

$$\|\xi^{\star\star}_{\tau\tau}\| \leq \overline{\sigma} := \frac{\overline{c}_1 L^{\star\star 2}}{\underline{v} - \overline{c}_0}\left(\sqrt{2}\overline{v} + \frac{\overline{v} + \overline{c}_0}{\underline{v} - \overline{c}_0}\left((1+\sqrt{2})\overline{v} + \overline{c}_0\right)\right). \tag{21}$$

▶ **Lemma 5.** *Let $(L^{\star\star}, \xi^{\star\star})$ be a minimizer of (4). For any $R_C > 0$ there is a h small enough such that the corresponding (h,l)-dense digraph contains a valid path ξ_R with $\|\xi^{\star\star} - \xi_R\|_{H^1([0,1])} \leq R_C$. The connectivity length l shall here be given as $l = L^{\star\star}\sqrt{h/\overline{\sigma}}$, which is an optimized choice as derived in [8, Theorem 4].*

Proof. In [8, Theorem 3], it was proved that for every $\xi \in X$ with $\|\xi_\tau\| = L$, there is a trajectory $\xi_R(\xi)$ on an (h,l)-dense digraph with

$$\|\xi_R(\xi) - \xi\|_{H^1([0,1])} \leq 2\overline{\sigma}\frac{l}{L} + 2h\frac{L}{l} + 3h.$$

Since $\|{\xi_\tau}^{\star\star}\| = L^{\star\star}$, this bound holds for a global optimizer $(L^{\star\star}, \xi^{\star\star})$ of (4). Together with $l = L^{\star\star}\sqrt{h/\overline{\sigma}}$ this reads

$$\|\xi_R(\xi) - \xi^{\star\star}\|_{H^1([0,1])} \leq 4\sqrt{\overline{\sigma}h} + 3h,$$

which directly proves that $\|\xi_R(\xi) - \xi^{\star\star}\|_{H^1([0,1])} \leq R_C$ for sufficiently small h. ◀

Having defined a spatially bounded (h,l)-dense digraph, we use Yen's algorithm [32] to enumerate paths in order of increasing travel time. Each generated discrete path $\xi_{G,i}$ undergoes a locally convergent refinement stage. If $\xi_{G,i}$ is the path on the graph that is closest to the minimizer $\xi^{\star\star}$, then Theorem 4 and Lemma 5 guarantee that it lies in the convex domain. For this reason we do not require the solver to incorporate any globalization strategies. Instead, the KKT system (11) can be solved via Newton's method, which either converges quadratically or is terminated in case of non-convexity.
Since any other local minimizer may be found as well, the preliminary solution shall be denoted as $\xi^\star(\xi_{G,i})$ in Algorithm 1 and may replace the current best solution ξ_C if $T(\xi^\star(\xi_{G,i})) < T(\xi_C)$. A suitable stopping criterion builds on the following local error bound.

▶ **Theorem 6.** *Let $(L^{\star\star}, \xi^{\star\star})$ be a global minimizer of* (4) *and define $\Delta\xi := \xi - \xi^{\star\star}$. Then there are constants $\overline{\mathcal{B}} > 0$ and $R_E > 0$ exclusively depending on the wind conditions, such that for any $\xi \in X$ with $\|\Delta\xi\|_{H^1} \leq R_E$, the error in the objective function T as defined in* (2) *is bounded by*

$$T(\xi) - T(\xi^{\star\star}) \leq \frac{1}{2}\overline{\mathcal{B}}\|\Delta\xi\|^2_{H^1([0,1])}. \tag{22}$$

Proof. As shown in the proof of [8, Theorem 2] the second directional derivative of T is bounded from above at a global minimizer. Let this bound be compactly given as

$$|T''(\xi)[\delta\xi, \delta\xi_\tau]^2| \leq 2\overline{\mathcal{B}}\|\delta\xi_\tau\|^2_{H^1([0,1])}$$

with some $\overline{\mathcal{B}} > 0$ that only depends on the wind conditions. Due to the continuity of T there is a $R_E > 0$ such that for any $\xi \in X$ with $\|\Delta\xi\|_{H^1} \leq R_E$, the second directional derivative of T is bounded by

$$|T''(\xi)[\delta\xi, \delta\xi_\tau]^2| \leq \overline{\mathcal{B}}\|\delta\xi_\tau\|^2_{H^1([0,1])}.$$

We use this bound, the optimality of $\xi^{\star\star}$, and Taylor's Theorem to validate that

$$\begin{aligned} T(\xi) &= T(\xi^{\star\star}) + \underbrace{T'(\xi^{\star\star})[\Delta\xi, \Delta\xi_\tau]}_{=0} + \int_0^1 (1-\nu)T''(\xi^{\star\star} + \nu\Delta\xi)[\Delta\xi, \Delta\xi_\tau]^2 d\nu \\ &\leq T(\xi^{\star\star}) + \frac{1}{2}\overline{\mathcal{B}}\|\Delta\xi\|^2_{H^1([0,1])}. \end{aligned}$$ ◀

Since we are only interested in discrete paths within the convex domain of the global minimizer $B_R(\xi^{\star\star})$, the generation of new paths is terminated if the extra cost of the next discrete path cannot be compensated by convergence to a nearby local minimizer anymore, i.e., if

$$T(\xi_{G,i}) - T(\xi_C) \geq \frac{1}{2}\overline{\mathcal{B}}R^2 =: \epsilon, \tag{23}$$

where $\xi_{G,i}$ denotes the ith shortest path, ξ_C the current best guess and $R := \min(R_C, R_E)$.

▶ Remark. We finally want to point out that the required graph density is exclusively dictated by the wind conditions and *independent of the requested solution accuracy*. Therefore, even though the enumeration of multiple discrete paths is certainly more expensive than finding the single shortest path as in the original DisCOpter concept, this difference vanishes asymptotically such that the proposed algorithm for global optimality inherits the superior convergence properties of the optimal control method given in Equation (19).

4 Conclusion

We presented a novel discrete-continuous algorithm that computes globally optimal solutions of the Free Flight Trajectory Optimization Problem in finite time to any desired accuracy. The main advantage of the method, and the key to its efficiency, is that the density of the discretization in the first graph-search stage of the algorithm depends only the problem data, and not on the desired accuracy. In this way, the algorithm inherits the superior asymptotic convergence properties of the second optimal control stage. A next step is a demonstration of computational efficiency. This requires improvements in the discrete part, in particular, an adaptive graph construction and the use of k-shortest path or k-dissimilar path algorithms that are, at least in practice, faster than Yen's algorithm, such as [13, 24] or [2], respectively.

Algorithm 1 This algorithm provides a globally optimal solution to the Free Flight trajectory optimization problem (4) in finite time.

Data: $x_O, x_D, \Omega, \overline{v}, w, \overline{c}_0, \overline{c}_1, \overline{c}_2, \overline{c}_3, C, \underline{\mathcal{B}}, \overline{\mathcal{B}}, R_E, R_C, TOL$
Result: (L_C, ξ_C) with $T(\xi_C) - T(\xi^{\star\star}) \leq TOL$ and $\|\xi_\tau{}^{\star\star}\| - L_C \leq TOL$
$(L_C, \xi_C) \leftarrow$ None; $T_C \leftarrow \infty$; $i \leftarrow 0$; $R \leftarrow \min(R_C, R_E)$;
$\epsilon \leftarrow$ Calculate the error bound for $\|\delta\xi\|_{H^1} \leq R$ from Theorem 6;
$(h, l) \leftarrow$ Calculate $h(R)$ and $l(h)$ as in Lemma 5;
Define a rectangular, spatially bounded (h, l)-dense digraph covering Ω;
do
  Calculate the ith shortest path $\xi_{G,i}$;
  if $T(\xi_{G,i}) - T_C \geq \epsilon$ **then**
    return (L_C, ξ_C);
  end
  `/* Optimal Control stage */`
  $(converged, L^\star, \xi^\star) \leftarrow$ (Try to) Calculate a local minimizer starting from $(L(\xi_{G,i}), \xi_{G,i})$ up to tolerance TOL;
  `/* Update */`
  if *converged and* $T(\xi^\star) < T_C$ **then**
    $(L_C, \xi_C) \leftarrow (L^\star, \xi^\star)$;
    $T_C \leftarrow T(\xi^\star)$;
  end
  $i \leftarrow i + 1$;
while *true*;

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