

# Approximating CSPs with Outliers

Suprovat Ghoshal ✉

University of Michigan, Ann Arbor, MI, USA

Anand Louis ✉

Indian Institute of Science, Bangalore, India

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## Abstract

Constraint satisfaction problems (CSPs) are ubiquitous in theoretical computer science. We study the problem of STRONG-CSPs, i.e. instances where a large induced sub-instance has a satisfying assignment. More formally, given a CSP instance  $\mathcal{G}(V, E, [k], \{\Pi_{ij}\}_{(i,j) \in E})$  consisting of a set of vertices  $V$ , a set of edges  $E$ , alphabet  $[k]$ , a constraint  $\Pi_{ij} \subseteq [k] \times [k]$  for each  $(i, j) \in E$ , the goal of this problem is to compute the largest subset  $S \subseteq V$  such that the instance induced on  $S$  has an assignment that satisfies all the constraints.

In this paper, we study approximation algorithms for UNIQUEGAMES and related problems under the STRONG-CSP framework when the underlying constraint graph satisfies mild expansion properties. In particular, we show that given a STRONGUNIQUEGAMES instance whose optimal solution  $S^*$  is supported on a regular low threshold rank graph, there exists an algorithm that runs in time exponential in the threshold rank, and recovers a large satisfiable sub-instance whose size is independent on the label set size and maximum degree of the graph. Our algorithm combines the techniques of Barak-Raghavendra-Steurer (FOCS'11), Guruswami-Sinop (FOCS'11) with several new ideas and runs in time exponential in the threshold rank of the optimal set. A key component of our algorithm is a new threshold rank based spectral decomposition, which is used to compute a “large” induced subgraph of “small” threshold rank; our techniques build on the work of Oveis Gharan and Rezaei (SODA'17), and could be of independent interest.

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## 1 Introduction

An instance of a 2-Constraint Satisfaction Problem (2-CSP)  $\mathcal{G}(V, E, [k], \{\Pi_{ij}\}_{(i,j) \in E})$  consists of a set of vertices  $V$ , a set of edges  $E$ , alphabet  $[k]$ , and a constraint  $\Pi_{ij} \subseteq [k] \times [k]$  for each  $(i, j) \in E$ . The goal of this problem is to compute an assignment  $f : V \rightarrow [k]$  such that the fraction of constraints satisfied is maximized; this optimal fraction is also called the *value* of this instance, and is formally denoted by  $\text{Val}(\mathcal{G})$ . Many common optimization problems such as Max Cut, Unique Games, Graph Coloring, 2-SAT, etc. are 2-CSPs. Designing approximation algorithms for specific CSPs are central problems in the study of algorithms and have been studied extensively, for e.g., Max-Cut [20], Unique Games [11, 12], etc. There is also a long line of work which deal with algorithms for general CSPs (see [35, 36, 8, 21]).



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A particular parameter regime of interest is when the CSP instance is “almost” fully satisfiable. There are several ways for quantifying this, one of which is by asking the value of the CSP instance be close to 1. This can also be viewed as the setting where deleting a small number of edges from the instance results in an instance that is fully satisfiable. There has been extensive work on designing algorithms for CSPs in this regime; we give a brief survey in Section 1.2. Another way a CSP can be almost satisfiable is if a small number of *outlier* vertices can be deleted (all the edges incident on these vertices would also be deleted) to obtain an instance which is fully satisfiable. The main focus of our work is to study algorithms for CSPs in this model; we define it below formally.

► **Problem 1** (STRONG-CSP). *Given an instance  $\mathcal{G}(V, E, [k], \{\Pi_{ij}\}_{(i,j) \in E})$  consisting of a set of vertices  $V$ , a set of edges  $E$ , alphabet  $[k]$ , and a constraint  $\Pi_{ij} \subseteq [k] \times [k]$  for each  $(i, j) \in E$ , compute the largest subset  $S \subseteq V$  such that the instance induced on  $S$  has value 1.*

We refer to an optimal set of vertices for Problem 1 as *good vertices*<sup>1</sup>, and denote them by  $V_{\text{good}}$ . A naturally arising such instantiation of STRONG-CSP’s is the ODDCYCLETRANSVERSAL problem. Here, given a graph  $G = (V, E)$  as input, the objective is to delete the smallest fraction of vertices so that the graph induced on the remaining vertices is bipartite. This is easily seen as an instance of a STRONG-CSP – here the predicate on the edges is the “Not Equals” predicate on the label set  $\{0, 1\}$ . ODDCYCLETRANSVERSAL is a well studied problem. In general, it is known to be constant factor inapproximable [7] (assuming the Unique Games Conjecture), and the best known upper bounds (in terms of fraction of vertices deleted) are  $O(\delta \sqrt{\log |V|})$  [1] and  $O(\sqrt{\delta \log d})$  [18] – where  $\delta$  is the optimal fraction of vertices to be deleted and  $d$  is the maximum degree of the graph – the latter bound is also tight upto constant factors assuming the Unique Games Conjecture [18]. Given these worst case bounds, one might ask if there are natural classes of instances under which ODDCYCLETRANSVERSAL admits better approximation?

For the specific setting of ODDCYCLETRANSVERSAL, there are several such classes which exhibit improved approximation guarantees. For instance, for the setting of planar graphs, the natural linear programming relaxation is known to be exact [14], and therefore admits an exact polynomial time algorithm. Furthermore, for  $K_r$ -minor closed graphs, Alev and Lau [2] gave an  $O(r)$ -approximation algorithm. On the other hand, since ODDCYCLETRANSVERSAL is fixed parameter tractable with respect to treewidth [26], it admits exact polynomial time algorithms for graphs with bounded treewidth. Note that these also happen to be characterizations which end up implying easy instances for Max-CSPs. Motivated by this connection, we investigate whether there are spectral characterizations under which ODDCYCLETRANSVERSAL (and more generally, STRONG-CSP’s) admit improved approximation. In particular, we study instances which are expanding, or more generally, have low threshold rank. Formally, the threshold rank of a graph is defined as follows.

► **Definition 2** (Threshold rank). *Given an undirected graph  $G = (V, E)$ , let  $A$  denote its weighted adjacency matrix  $G$  and let  $D$  denote the diagonal matrix where  $D(i, i)$  is the weighted degree of vertex  $i$ . The  $(1 - \varepsilon)$  threshold rank of  $G$ , denoted by  $\text{rank}_{\geq 1 - \varepsilon}(G)$  is defined as the number of eigenvalues of  $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$  that are greater than or equal to  $1 - \varepsilon$ .*

In the setting of CSPs, low threshold rank instances have been studied extensively – the study of such instances was instrumental in the development of sub-exponential time algorithms for UNIQUEGAMES and SMALLSETEDGEEXPANSION [24, 3, 8]. In particular, for

<sup>1</sup> Note that such a set of vertices may not be unique, in which case, we will fix such a collection of vertices, and call it the set of good vertices.

the edge deletion analogue of ODDCYCLETRANSVERSAL i.e., MAX-CUT, [8] gave a  $(1/\lambda_t)$ -approximation algorithm running in time  $n^{\text{poly}(t)}$ , where  $\lambda_t$  is the  $t^{\text{th}}$  largest eigenvalue of the normalized Laplacian. Surprisingly, to the best of our knowledge, no such analogous results are known for ODDCYCLETRANSVERSAL. Furthermore, random instances of CSPs are expanding, and naturally have low threshold rank. This motivates us to explore the approximability of ODDCYCLETRANSVERSAL and other STRONG-CSP's in low threshold instances. In fact, we study them under the more stringent setting where only the graph induced on good vertices (constituting the fully satisfiable sub-instance) is assumed to have low threshold rank, as opposed to the full graph having low threshold rank.

**Max-CSPs vs. StrongCSPs.** This choice of the setting, in addition to making the task more challenging, is also motivated by our wish to exhibit a separation between the approximability of edge deletion and vertex deletion problems, i.e., namely MAX-CSPs and STRONG-CSP's. We point out that under an identical setting (where only a  $(1 - \delta)$ -sized subset has low threshold rank), MAX-CSPs can be arbitrarily hard to approximate. Indeed, consider a MAX-CSP instance where the  $(1 - \delta)$ -sized subset  $V_{\text{good}}$  induces a constant degree expander with trivially satisfiable constraints, and the edges going across  $V_{\text{good}}, V_{\text{good}}^c$  encode a denser hard to approximate UNIQUE GAME instance with large gap and larger vertex degrees. It is easy to see that such instances do not admit efficient constant factor approximation guarantees with respect to the edge satisfaction objective i.e., that of finding an assignment that satisfies the maximum fraction of constraints. On the other hand, our results in the current work show that the same instances when interpreted as STRONG-CSP's are easy (i.e, with respect to the vertex deletion objective, see Definition 1). Therefore, it is not immediately obvious if the conditions under which MAX-CSPs are easy also translate to conditions under which STRONG-CSP's are easy, and vice versa. Consequently, the broader agenda of identifying clean characterizations under which there is a separation in the approximability of the two classes of problems might yield useful insights towards understanding the limitations of the approximation techniques for problems from either class.

**Connection to Fortification.** A final motivation for studying STRONG-CSP's in the above setting is that the problem of finding slightly smaller sub-instances with better “local” approximation guarantees is closely related to notion of *fortification*. Informally, a MAX-CSP instance is said to be fortified if every large sub-instance of the CSP has (relative) optimal value no larger than the global optimal. Fortification is widely studied in the context of parallel repetition [32, 9, 33], and in particular, recent works [33] show that fortified UNIQUE GAME instances with hypercontractive small set expansion profiles can be used to bypass bottlenecks towards establishing *strong parallel repetition* for UNIQUE GAME instances. Consequently, this reduces the task of establishing UGC to that of showing that a family of fortified Boolean CSPs on small-set-expanders are hard. Given that STRONG-CSP's can be re-interpreted as the task of deciding whether an instance is fortified (in the perfect completeness regime), and the tight connections between small-set-expansion and threshold rank (e.g., [3, 25, 28]), these considerations further motivate the study of STRONG-CSP's even in the simpler setting where the full underlying constraint graph has low threshold rank. Motivated by the above considerations, we study the STRONGUNIQUEGAMES and related problems in this setting:

► **Problem 3** (STRONGUNIQUEGAMES). *Given an instance  $\mathcal{G}(V, E, [k], \{\Pi_{ij}\}_{(i,j) \in E})$  consisting of a set of vertices  $V$ , a set of edges  $E$ , alphabet  $[k]$ , and a bijection  $\pi_{ij} \subseteq [k] \times [k]$  for each  $(i, j) \in E$ , the goal of this problem is to compute the largest  $S \subseteq V$  such that the instance induced on  $S$  has value 1.*

The STRONGUNIQUEGAMES problem is a natural variant of UNIQUEGAMES, it and its variants express several well studied problems such as ODDCYCLETRANSVERSAL, among others. There have been extensive work on the above problems, see Section 1.2 for a detailed review. Our main results in this paper are improved approximation algorithms for these problems in the setting where the induced graph on the good vertices has low threshold rank.

## 1.1 Our Results

Our main result is a new approximation algorithm for the STRONGUNIQUEGAMES problem where the induced sub-graph on the satisfiable set has low threshold rank. In order to make the theorem statements concise, we will define the notion of a subset being  $\lambda^*$ -good.

► **Definition 4** ( $\lambda^*$ -good). *Given a CSP constraint graph  $G = (V, E)$ , a subset  $V^* \subseteq V$  is said to be  $\lambda^*$ -good if the following conditions hold.*

1.  $\text{rank}_{\geq 1-\lambda^*}(\mathcal{G}[V^*]) \leq (1/\lambda^*)^{10}$ .<sup>2</sup>
2.  $G[V^*]$  is regular.

The above is a quantitative characterization of induced low-rank instances studied in this paper – all of our results are based on the above setting. Our first result is for STRONGUNIQUEGAMES instances with small vertex induced low threshold rank, as stated in the following theorem.

► **Theorem 5.** *Let  $\delta, \lambda^* \in (0, 1)$  be such that  $\delta \leq (\lambda^*)^{100}$ . Let  $\mathcal{G}(V, E, [k], \{\pi_e\}_{e \in E})$  be a STRONGUNIQUEGAMES instance such that there exists<sup>3</sup> a  $\lambda^*$ -good subset  $V_{\text{good}}$  of size at least  $(1 - \delta)n$  such that  $\text{Val}(\mathcal{G}[V_{\text{good}}]) = 1$ . Then there exists a randomized algorithm that runs in time  $n^{\text{poly}(k/\delta)}$  and outputs a subset  $\tilde{V} \subseteq V$  of size at least  $(1 - \delta^{1/12})n$  and a partial labeling  $\sigma : \tilde{V} \rightarrow [k]$  such that  $\sigma$  satisfies all induced constraints in  $\mathcal{G}[\tilde{V}]$ .*

The above theorem illustrates the tractability of the STRONGUNIQUEGAMES problem in the setting where just the instance induced on the satisfiable set has low threshold rank. To put the above result in perspective, [18] showed that given a STRONGUNIQUEGAMES instance with value  $(1 - \delta)$ , it is Unique Games hard to output a satisfiable subset of relative size  $(1 - \Omega(\sqrt{\delta \log d \log k}))$ , where  $d$  is the maximum degree of the graph and  $k$  is the label set size. We remark that the exponent in the fraction of vertices deleted (i.e.,  $\delta^{1/12}$ ) might be improvable and we have not made further attempts towards optimizing it. Theorem 5 almost directly leads to quantitatively similar results for the ODDCYCLETRANSVERSAL and BALANCEDVERTEXSEPARATOR problems, stated as corollaries.

► **Corollary 6.** *Let  $\delta, \lambda^* \in (0, 1)$  be such that  $\delta \leq (\lambda^*)^{100}$ . Let  $\mathcal{G} = (V, E)$  be a graph for which there exists a  $\lambda^*$ -good subset  $V_{\text{good}} \subseteq V$  of size at least  $(1 - \delta)n$  such that  $G[V_{\text{good}}]$  is bipartite. Then there exists an algorithm which runs in times  $n^{\text{poly}(1/\delta)}$  which outputs a set  $V' \subseteq V$  of size at least  $(1 - \delta^{1/12})n$  such that  $G[V']$  is bipartite.*

► **Theorem 7.** *Let  $\delta, \lambda^* \in (0, 1)$  be such that  $\delta \leq (\lambda^*)^{100}$ . Let  $\mathcal{G} = (V, E)$  be a graph for which there exists a  $\lambda^*$ -good subset  $V_{\text{good}} \subseteq V$  of size at least  $(1 - \delta)n$  such that the following holds. There exists a partition  $V_{\text{good}} = A \uplus B$  such that  $E_{G[V_{\text{good}}]}(A, B) = \emptyset$  i.e,  $A$  is disconnected from  $B$  in  $G[V_{\text{good}}]$ . Then there exists a randomized algorithm which runs in time  $n^{\text{poly}(1/\delta)}$  and outputs a set  $S$  of size at most  $O(\delta^{1/12}n)$  and a partition  $A', B'$  of  $V \setminus S$  such that (a)  $E_{G[V \setminus S]}(A, B) = \emptyset$  and (b)  $(\gamma - \delta^{1/12})n \leq \min(|A'|, |B'|) \leq (\gamma + \delta^{1/12})n$  where  $\gamma = \min(|A|, |B|)/n$ .*

<sup>2</sup> The constant 10 in the exponent is arbitrary, and can be chosen to any large constant  $C$ , at the cost of loss in  $\text{poly}(C)$ -multiplicative factors in the fraction of vertices deleted by the algorithm. We instantiate it to be 10 for ease of notation.

<sup>3</sup> We do not assume that such a set is unique, we just need the existence at least one such subset.

For both `ODDCYCLETRANSVERSAL` as well as `BALANCEDVERTEXSEPARATOR`, the best known approximation algorithm for general instances have an approximation guarantee of  $O(\sqrt{\log |V|})$  [13, 1]. Furthermore, [18] showed that given a  $(1 - \delta)$ -satisfiable instance of `ODDCYCLETRANSVERSAL`, assuming `UGC`, it is `NP-Hard` to find set of size  $(1 - \Omega(\sqrt{\delta \log d}))$  which induces a bipartite graph. *It is important to note that our results hold for more restrictive setting where we assume the low threshold rank guarantee on the good set.* In particular, the technical core of our results is a spectral decomposition theorem which can be used to find a large subset that induces a sub-graph with relatively small threshold rank. We state an informal version of it here for reference.

► **Theorem 8** (Informal version of Theorem 4.2 [19]). *The following holds for every  $0 < \delta \leq 0.1$ . Let  $G = (V, E)$  be a  $d$ -regular graph on  $n$ -vertices such that there exists a set  $V_{\text{good}} \subseteq V$  of size at least  $(1 - \delta)n$  such that  $\text{rank}_{\geq 1 - \delta^{0.1}}(G[V_{\text{good}}]) \leq K$ . Furthermore, suppose  $K \leq 1/\delta^{100}$ . Then there exists an efficient algorithm outputs a set  $V'' \subseteq V$  of size at least  $(1 - O(\delta^{1/10}))n$  such that  $\text{rank}_{\geq 1 - \delta^{0.1}}(G[V'']) \leq \text{poly}(1/\delta)$ . Moreover, the subset  $V''$  itself is a disjoint union of constant number of  $\Omega(n)$ -sized subsets, each of which induces an expander.*

The above decomposition result adds to the already extensive literature on spectral decomposition – however, the above decomposition result is incomparable in terms of its setting and guarantees to the ones existing in the literature. For comparison, we describe the two previous such results which are closest to this work in terms of the setting and the guarantees:

- In [3], Arora, Barak and Steurer show that any  $n$ -vertex graph can be decomposed into non-expanding subsets which induce sub-graphs of  $(1 - \varepsilon^5)$ -threshold rank at most  $n^\varepsilon$ . While their result does not require the graph to contain a large low threshold rank sub-graph, their decomposition result can only guarantee a substantially weaker threshold rank bound of  $n^\varepsilon$  (as opposed to the constant bounds guaranteed in Theorem 8). We clarify that their  $n$ -dependent bound on the threshold rank is indeed unavoidable, since they make no assumptions on the threshold rank structure of the graph [34].
- In [15], Oveis Gharan and Rezaei show that given a regular graph which contains a  $\kappa n$ -sized spectral expander, one can efficiently find subset of size at least  $3\kappa n/8$  with spectral gap multiplicatively comparable to that of the optimal induced expander. Again, their result is not directly comparable to ours since even in graphs which contain a  $(1 - \delta)n$ -sized induced expander, their algorithm is only guaranteed to output a  $3(1 - \delta)n/8$ -sized subset which induces an expander. In comparison, for similar instances, Theorem 8 guarantees a  $(1 - \delta^{0.1})$ -sized subset which induces a “low threshold rank graph” – which itself is guaranteed to be a union of linear sized expanders. On the other hand, our result only applies in the setting  $\kappa \rightarrow 1$ , whereas their result holds for any constant  $\kappa \in (0, 1)$ .

We point out that our actual spectral decomposition theorem (see Theorem 4.2 of [19]) differs from the informal version stated above (i.e., Theorem 8) in a couple of crucial ways. Firstly, we only assume that only the underlying good graph  $G[V_{\text{good}}]$  is regular (as opposed to the full graph being regular) and make no assumptions on the degree distribution of the set of outlier vertices  $V \setminus V_{\text{good}}$  – indeed, these assumptions allow us to include instances which show a separation between the approximability of the `MAX-CSP` and `STRONG-CSP` objectives. Secondly, our actual guarantee is slightly more robust in the following sense: given any  $(1 - \delta^{O(1)})$ -sized subset  $V' \subseteq V$  (where  $V' \not\subseteq V_{\text{good}}$ ), one can find another subset  $V'' \subseteq V'$  of size  $(1 - \delta^{O(1)})$  such that  $\text{rank}_{1 - \delta^{O(1)}}(G[V'']) \leq \text{poly}(1/\delta)$ . The structural fact that we can still recover a large low threshold rank subgraph within any large subset  $V'$  is interesting on its own, we are not aware of similar results in the previous literature on spectral decomposition.



► **Remark 9** (On the regularity assumption). We point out that our threshold rank decomposition result, and more generally the approximation guarantees from Theorem 5 and its corollaries also hold as long as  $V_{\text{good}}$  is  $\lambda^*$ -good (Definition 4) and is contained in any subset  $\tilde{V}$  (where  $\tilde{V}$  may strictly contain  $V_{\text{good}}$ ) for which  $G[\tilde{V}]$  induces a regular subgraph – this naturally subsumes the more commonly studied setting where the full graph has low threshold rank and is regular [3, 8]. As in these works, our results will also hold for the setting where the graph is non-regular; in that setting, the guarantees of the threshold decomposition result and our algorithm will involve bounds on the volume of the subset deleted by the algorithm (as opposed to bounds on the size of the subset).

### Hardness of STRONG-CSPs

Given our algorithmic results hold for structured instances i.e., the subgraph induced by the good set has low threshold rank, an immediate question is if it is possible to obtain quantitatively similar approximation guarantees without making any assumptions. Towards that, our first observation is that arbitrary Strong 2-CSPs can be almost polynomially hard to approximate, as stated by the following fact.

► **Observation 10** (Hardness of General Strong 2-CSPs). *The following holds for any small  $\varepsilon > 0$ . Given a 2-CSP  $\Psi(V, E, \{\psi\}_{e \in E})$  over label set  $\{0, 1\}$ , it is NP-Hard to find a subset  $V' \subseteq V$  of size  $|V'| \geq n^{1-\varepsilon}|V^*|$  such that all induced constraints on  $V'$  are satisfiable. Here  $V^*$  is a set of largest cardinality for which there exists a labeling which satisfies all the induced constraints on  $V^*$ .*

The fact follows simply by using the observation that the Maximum Independent Set problem can be modeled as STRONG-CSP on label set  $\{0, 1\}$  with arity 2 (see Appendix B of the full version [19] for a formal explanation). On the other hand, it is known that all general 2-CSPs admit constant factor approximation (when the label set size is a constant). For e.g., for any 2-CSP on  $\{0, 1\}$  just a random assignment itself satisfies at least 1/4-fraction of constraints in expectation. This shows that STRONG-CSP's can be strictly harder than MAX-CSPs. Clearly, one can expect general STRONG-CSP's to only get harder for larger arities, so we choose to relax the requirements of STRONG-CSP's and ask the following question. Consider a MAX-CSP which is known to be hard to approximate to a factor of  $\alpha$ . Then it is natural to ask, if given such an instance, can we delete a few vertices, and then output a labeling on the remaining instance which has approximation factor strictly better than  $\alpha$ . The following theorem answers the question in the negative for the specific setting where the CSP is Max-4-Lin.

► **Theorem 11.** *The following holds for any constants  $\alpha, \eta, \nu \in (0, 1)$ . Given a system of equations  $\Psi$  of arity 4, on variables  $X_1, X_2, \dots, X_n$  taking values in  $\mathbb{F}_2$ , it is NP-Hard to distinguish between the following cases:*

- *There exists an assignment to the variables which satisfies at least  $(1 - \eta)$ -fraction of constraints in  $\Psi$ .*
- *No subset  $S \subseteq V$  of size at least  $\alpha n$  induces a system of equations for which there exists an assignment which satisfies at least  $(1/2 + \nu)$  fraction of the induced constraints.*

The above can be thought of as an instance of approximation resistance in a STRONG-CSP sense; it is a strengthening of  $(1/2 + \nu)$ -inapproximability for Max-3-Lin shown by Håstad in the seminal work [23]. We prove the above hardness result by combining the techniques from [23] with novel application of expansion properties of the inner and outer verifiers. In particular, Theorem 11 says that one cannot hope to do slightly better than its inapproximability factor (which is matched by the naive random guessing algorithm) on any smaller sub-instance for approximation resistant predicates.

## 1.2 Related Work

### Strong Unique Games

Ghoshal and Louis [18] gave an algorithm that takes as input an instance of STRONGUNIQUEGAMES having a satisfiable set of size  $(1 - \varepsilon)n$ , and outputs a satisfiable subset of size at least  $(1 - \tilde{O}(k^2) \varepsilon \sqrt{\log n}) n$ . In a similar setting, they also gave another algorithm which outputs satisfiable subsets of size  $(1 - \tilde{O}(k^2) \sqrt{\varepsilon \log d}) n$ , where  $d$  is the largest vertex degree of the instance. Complementing these upper bounds, they also showed that it is Unique Games hard (in certain regimes of parameters) to compute a set of size larger than  $1 - O(\sqrt{\varepsilon \log d \log k})$  such that the induced instance on this set is satisfiable. These results were obtained via a connection between STRONGUNIQUEGAMES and small-set vertex expansion in graphs, and used the machinery (hypergraph orthogonal separators) developed in the context of approximation algorithms for small-set vertex expansion in graphs and hypergraph small-set expansion [27] in obtaining their approximation algorithms.

### General CSPs

There have been several works which give approximation algorithms for 2-CSPs. [5] were the first to study UNIQUEGAMES in the setting where the underlying constraint graph is an expander; they gave an algorithm with the approximation factor depending on only the second largest eigenvalue of the normalized Laplacian matrix of the instance. Subsequent works by Barak, Raghavendra and Steurer [8] and Guruswami and Sinop [21] extended this framework to general 2-CSPs when the underlying constraint graph and the label extended graph have low threshold rank respectively, with the algorithms running time exponential in threshold rank. On the other hand, Kolla [24] gave spectral approximation algorithms for UNIQUEGAMES and SMALLSETEDGEEXPANSION. Building on this, Arora, Barak and Steurer [3] gave sub-exponential time algorithms for UNIQUEGAMES and SMALLSETEDGEEXPANSION. In a recent work, [6] give efficient algorithms for UNIQUEGAMES based on the Sum Of Squares (SoS) hierarchy, when the underlying constraint graph is an SoS certifiable small set expander.

### Graph Partitioning and CSPs with Cardinality Constraints

Graph partitioning with vertex/edge expansion objectives has been extensively studied under the lens of approximation algorithms. Feige, Lee and Hajhiyaghayi [13] and Louis, Raghavendra and Vempala [29] give approximation algorithms for finding small size balanced vertex separators and minimizing vertex expansion respectively. Guruswami and Sinop [21, 22] gave improved approximation algorithms for several graph partitioning problems dealing with edge expansion for low threshold rank instances. [30] studied a planted model of instances where the graph induced on either side of the planted cut satisfies a lower bound requirement on its spectral gap in addition to satisfying some other properties; they gave exact and constant factor bi-criteria approximation algorithms for balanced vertex expansion for various ranges of parameters. They also gave a constant factor bi-criteria approximation algorithm for balanced vertex expansion for instances where one side of the optimal cut has a subgraph on  $\Omega(n)$  vertices satisfying a lower bound requirement on its spectral gap. [31] gave some similar results for  $k$ -way edge expansion and  $k$ -way vertex expansion.

The problem of decomposing a graph into expanders is also a well studied problem and has several applications to approximation algorithms. In [38], Trevisan gave a decomposition of a graph into non-expanding set which induce expanders. There have been several subsequent works [3, 16, 17] which deal with the problem of partitioning a graph into expanding/low

threshold rank graphs. Oveis Gharan and Rezaei [15] study the problem of finding a large subset of vertices such that the graph induced on them is an expander; we discuss this more in Section 2.

## 2 Overview and Techniques

We begin by reviewing the by now standard *Propagation Rounding* based framework which was introduced informally in [5] and then later developed in [8, 21]. For simplicity, we shall restrict our discussion to the setting of **UNIQUEGAMES**. Consider the following convex program which is the  $R$ -level Sum-of-Squares (SoS) lifting of SDP relaxation for **UNIQUEGAMES**:

$$\min_{\substack{\mu \text{ is a degree-}R \\ \text{pseudo-distribution}^4}} \mathbb{E}_{(i,j)=e \sim E} \Pr_{(X_i, X_j) \sim \mu} [X_i \neq \pi_{j \rightarrow i}(X_j)]. \quad (1)$$

The above convex program is intended to minimize the number of unsatisfied edges by the (pseudo)-distribution. The algorithm proceeds along the following steps.

1. Solve the  $R$ -round Lasserre relaxation for the SDP where  $R$  is chosen large enough as a function of the error to be tolerated, and the threshold-rank of the instance. Let  $\mu := \{\mu_{S,\alpha}\}$  be the degree- $R$  pseudo-distribution corresponding to the optimal value of the relaxation.
2. Choose a subset  $S$  appropriately, sample an assignment  $x_S$  to the variables in  $S$  from the local distribution  $\mu_S$ .
3. Label the remaining vertices  $i \in V \setminus S$  by sampling from their respective conditional distributions  $\mu_{i|x_S}$  independently.

The main idea used in the aforementioned works for relating the expected value of the rounded solution to the SDP objective is the so called *local-to-global* correlation property [8, 21], which has the following key consequence. If the underlying constraint graph has constant threshold rank, then conditioning on constant levels of the SoS solution should result in pseudo-distributions that have small average local correlation i.e.,

$$\mathbb{E}_{(i,j) \sim E} [\text{Corr}_{\mu_{i|x_S}}(X_i, X_j)] \leq o(1).$$

Consequently, independent sampling from the marginals of conditional pseudo-distribution  $\mu_{i|x_S}$  will result in labelings that which have value close to optimal of the lifted SDP. While this recipe and its variants has been remarkably successful in dealing with **MAX-CSPs** [8, 21, 6], it is easy to see that this framework does not translate well to the framework of **STRONG-CSP**'s studied in this paper, as we briefly describe below.

Firstly, note that in the setting of **STRONG-CSP**'s, the emphasis is on deleting vertices to ensure that all surviving constraints are simultaneously satisfiable. This is in direct contrast to the aforementioned results where the algorithms are allowed to output labelings which satisfy “almost all”, but not necessarily, “all”, constraints. A naive approach towards extending the above to our setting would be to first find a good labeling that satisfies almost all edges, and then delete the vertices corresponding to the violated edges. However, doing so might result in approximation guarantees that are worse by a factor of the max-degree. Furthermore, this

<sup>4</sup> Informally, a degree- $R$  pseudo-distribution is a collection of local distributions  $\{\mu_S\}_S$  for every subset  $S \subseteq V$  of size at most  $R$ , which are pairwise consistent up to all variables sets of size at most  $R$  (see Section 3.2 of the full version [19] for more details).



approach can fail badly in instances where the induced sub-instance on the good vertices  $\mathcal{G}[V_{\text{good}}]$  is sparse (i.e, constant degree), but the full graph is relatively dense and almost non-satisfiable, since these algorithms are designed to compete against the global optimum for the edge-satisfaction version of the problem. A final hurdle is that the local-to-global correlation guarantee, which was the key property used to guarantee the goodness of the rounding algorithm, might not hold for the full constraint graph of  $\mathcal{G}$  since in our setting, the constant threshold-rank guarantee may only hold for the constraint graph induced on  $V_{\text{good}}$ . In fact, the threshold rank of the full graph can be as large as  $\Omega(|V \setminus V_{\text{good}}|)$  which implies that conditioning on constant levels of the SoS solution might result in pseudo-distributions that don't guarantee any local-to-global correlation like property. These issues taken together guide our approach to the design of our algorithm (described informally in Figure 1); we describe and motivate the various steps of the algorithm details in the remainder of this section.

**Input:** A UNIQUEGAMES instance  $\mathcal{G}(V_G, E_G, [k], \{\pi_e\}_{e \in E})$  satisfying the conditions of Theorem 5.

**Algorithm:**

- ▶ **Threshold Rank Decomposition.** As a first step, we compute a  $(1 - O(\delta^c))$ -sized subset  $V''$  of  $V_G$  with low threshold-rank and bounded degree using our spectral decomposition algorithm.
- ▶ **SDP with Slack Variables.** We solve the  $R$ -level SoS relaxation of a modified SDP for UNIQUEGAMES instance induced on  $V''$  with the extended label set  $[k] \cup \{*\}$ , where the label  $*$  is meant to indicate vertices which are to be deleted.
- ▶ **Low Variance Rounding.** We sample an assignment  $\alpha$  for an appropriately chosen subset  $S$ , and then label the vertex  $i \in V$  with the label with the largest probability in the conditional marginal  $\mu_i|_{X_S=\alpha}$ .

■ **Figure 1** StrongUG-Informal.

### Finding a large bounded-degree low threshold-rank graph

Since the full instance in our setting can have arbitrarily large threshold rank (due to the edges incident on the set of outlier vertices), a natural way to overcome this issue would be to zoom into a large (i.e,  $(1 - o_\delta(1))$ -sized) subset of vertices which induces a subgraph with (comparably) low threshold rank. We do this by using a new threshold rank based spectral decomposition algorithm with the following guarantee: given a graph  $G = (V, E)$  for which there exists a  $(1 - \delta)|V|$  sized subset that induces a regular low threshold rank sub-graph, the algorithm returns a  $(1 - \delta^{O(1)})$ -sized subset with threshold<sup>5</sup> rank at most  $\text{poly}(1/\delta)$ . This algorithm is the main technical contribution of this paper; in particular, it combines a classical approximation algorithm for the partial vertex cover problem and extensions of spectral partitioning primitives from Oveis Gharan and Rezaei [15] to the setting of low threshold rank graphs. We defer a more detailed discussion of this step to Section 2.1 for now and proceed with our discussion of the subsequent steps of the full algorithm.

<sup>5</sup> Here the threshold parameter is dependent on  $\delta$  and the optimal value of the STRONGUNIQUEGAMES instance.

### Solve SoS relaxation with Slack Variables

In the next step, we consider the SDP for UNIQUEGAMES modified with slack variables. Specifically, let  $\mathcal{G}(V, E, [k], \{\pi_e\}_{e \in E})$  be the UNIQUEGAMES instance. Due to the above step, we can directly assume that the full graph has low threshold-rank and bounded vertex degrees. Furthermore, since the previous step only removes a tiny fraction of vertices, we can assume that there exists a subset  $V_{\text{sat}} \subseteq V$  such that  $|V_{\text{sat}}| \geq (1 - 2\delta)|V|$  and  $\mathcal{G}[V_{\text{sat}}]$  is fully satisfiable<sup>6</sup>. Now given  $\mathcal{G}$ , we consider a partial<sup>7</sup> Unique Game  $\mathcal{G}'(V, E, [k] \cup \{*\}, \{\Pi_e\}_{e \in E})$  with the global constraint that the fraction of vertices that can be labeled  $*$  is at most  $2\delta$ . Here the label  $*$  is meant to indicate vertices that are supposed to be deleted. Consequently, for any edge  $e \in E$ , we define the extended constraint set  $\Pi_e = \pi_e \cup (\{*\} \times \Sigma) \cup (\Sigma \times \{*\})$ . Note that this constraint is no longer a “unique game” constraint. The final SoS relaxation used is almost identical to Eq. 1, along with the following modifications:

$C_1$ : The pseudo-distribution is now over assignments to variables from the extended label set  $[k] \cup \{*\}$ .

$C_2$ : We add the global cardinality constraint  $\Pr_{i \sim V} \Pr_{X_i \sim \mu}[X_i = *] \leq 2\delta$ .

$C_3$ : We also add the constraint

$$\Pr_{(X_i, X_j) \sim \mu} [\pi_{i \rightarrow j}(X_i) \neq X_j] \leq \Pr_{X_i \sim \mu} [X_i = *] + \Pr_{X_j \sim \mu} [X_j = *],$$

for every edge  $(i, j) \in E$

The cardinality constraint ( $C_2$ ) is intended to ensure that conditioned on any assignment that is assigned a non-zero probability mass by the SDP solution, the fraction of vertices that are labeled  $*$  under the resulting conditional distribution is at most  $\delta$ . The edge violation constraints ( $C_3$ ) are intended to ensure that an edge constraint is allowed to be violated only when one of the end points is labeled  $*$ . It is easy to verify that this SDP is feasible for  $\mathcal{G}'$ . Furthermore, since the previous step guarantees that the max-degree of the surviving graph is at most a constant times the average degree, this implies that the optimal value of the SoS relaxation is at most  $\delta^{O(1)}$ .

### Low Variance Rounding

In the final step, we have to round the SDP solution to output a large set with the corresponding labeling which satisfies all induced constraints. As mentioned above, the *local-to-global* correlation argument in itself is not sufficient for this purpose, as it can only guarantee that a labeling which violates a small fraction of edges. However, it is well known that for certain kinds of CSPs e.g., UNIQUEGAMES, 3-COLORING, the low threshold-rank guarantee implies the stronger property of “*conditioning reduces variance*” [8, 21, 4]<sup>8</sup>, which says that for an appropriately chosen subset  $S \subseteq V$  we have

$$\mathbb{E}_{X_S \sim \mu_S} \left[ \mathbb{E}_{i \sim V} \text{Var} \left[ X_i | X_S \right] \right] \leq \frac{\text{SDP}}{\lambda_m}, \quad (2)$$

whenever  $\text{rank}_{\geq 1 - \lambda_m}(G) \leq m$ . Note that this is a strictly stronger property than local-to-global correlation (see Appendix D of [19] for an example which separates the two properties). To see why this property is useful in constructing labelings which satisfy all induced constraints,

<sup>6</sup> Note that  $V_{\text{sat}}$  may be a strict subset of  $V_{\text{good}}$  since the previous step may remove a few vertices from  $V_{\text{good}}$ .

<sup>7</sup> The nomenclature “partial” Unique Game was introduced in [37] and refers to a Unique Game with the additional property that a fixed fraction of vertices are allowed to be left as unlabeled.

<sup>8</sup> [4] actually showed a variant of this statement tailored towards finding large independent sets.

consider a constraint  $(i, j) \in E_G$  such that for some partial assignment  $X_S \leftarrow \alpha$ , the conditional marginals of vertices  $i$  and  $j$  have low variance i.e.,  $\text{Var}[X_i|X_S = \alpha] \leq 0.1$  and  $\text{Var}[X_j|X_S = \alpha] \leq 0.1$ . Therefore, it follows that there exists labels  $a, b \in [k] \cup \{*\}$  for which  $\mu_{i=a|X_S=\alpha} \geq 0.9$  and  $\mu_{j=b|X_S=\alpha} \geq 0.9$ . Furthermore, suppose we assume that  $a, b \in [k]$ . Then we claim that the SDP constraints imply that  $\pi_{i \rightarrow j}(a) = b$ . This is because for any  $(a', b') \in [k] \times [k]$  which violates the edges  $(i, j)$ , using the edges violation constraints  $(C_3)$  and a union bound we get that

$$\Pr_{(X_i, X_j) \sim \mu|X_S=\alpha} [X_i = a', X_j = b'] \leq 0.2.$$

On the other hand, our choice of labels  $a$  and  $b$  for vertices  $i$  and  $j$  (respectively) imply that

$$\Pr_{(X_i, X_j) \sim \mu|X_S=\alpha} [X_i = a, X_j = b] \geq 1 - \Pr_{X_i \sim \mu|X_S=\alpha} [X_i \neq a] - \Pr_{X_j \sim \mu|X_S=\alpha} [X_j \neq b] \geq 0.8.$$

Therefore, it must be that  $\pi_{j \rightarrow i}(a) = b$  i.e., the labeling  $(a, b)$  satisfies the edge  $(i, j)$ . In summary, low variance vertices whose leading labels are not  $'*'$  induce a satisfiable instance. We point out that a similar observation was also made by Arora and Ge [4] who used it to find large independent sets in low threshold-rank graphs. The above discussion naturally suggests the following rounding process:

1. Let  $S$  be the subset for which Eq. 2 holds. Sample an assignment  $\alpha \sim \mu_S$  for  $X_S$
2. Delete the vertices for which  $\text{Var}_{\mu|X_S=\alpha}[X_i] > 0.1$ .
3. For the remaining vertices  $i \in V$ , assign the maximum likelihood labeling

$$\sigma(i) = \operatorname{argmax}_{a \in [k] \cup \{*\}} \Pr_{X_i \sim \mu|X_S=\alpha} [X_i = a].$$

4. Delete the vertices labeled as  $*$  and output the surviving vertices with the corresponding labeling.

The above discussion ensures that the set output by the rounding scheme is satisfiable. Combining (2) with the SDP bound and the threshold-rank bound established in the previous steps imply that  $O(\delta^{O(1)})$  vertices get deleted in step 3. Furthermore, the global cardinality constraint ensures that the fraction of vertices labeled  $'*'$  (and hence deleted in the round step) is  $O(\delta)$ . This with the bound on the vertices deleted in the previous steps imply that the total fraction of vertices deleted is  $\delta^{O(1)}$ , which concludes the analysis of the algorithm.

## 2.1 Threshold Rank based Spectral Partitioning

As mentioned above, the first step of our algorithm (i.e., the threshold rank decomposition step) is the key technical contribution of this work. Formally, our objective here is the following: given a graph  $G = (V, E)$  which contains a  $(1 - \delta)$ -sized subset  $V_{\text{good}}$  that induces a regular subgraph with low threshold rank, the objective is to recover a  $(1 - o_\delta(1))$ -sized subset that has relatively small threshold rank (say  $\text{poly}(1/\delta)$ ). This in itself is a well motivated question and various versions of it have been studied in the design of approximation algorithms for UNIQUEGAMES and SMALLSETEDGEEXPANSION (see [3] and references therein). However, we point out that the techniques from these earlier works do not immediately apply to our setting as in these works, the emphasis there is rather on finding sub-linear sized sets which induce graphs with threshold rank growing with the number of vertices (with of course, no assumption on the spectrum of the full graph). In our setting, we instead want to design algorithms that exploit the “almost low threshold” structure of the instance and output

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sets that satisfy the stronger guarantees of being  $\approx (1 - o(1))$ -sized and having constant threshold rank. In the remainder of this section, we motivate and describe the design of such an algorithm.

**Finding a Linear Sized Low Rank Set.** To begin with, let us first consider the simpler setting where we assume that the max-degree of the graph is at most a constant times the degree of the induced good graph  $G[V_{\text{good}}]$  (we shall later discuss how to achieve this condition at the cost of deleting a few additional vertices). Furthermore, let us first address the even simpler goal of finding a linear (say  $n/1000$ ) sized subset which induces a low threshold rank graph. Again, this in itself is a well motivated problem, and several previous works [38, 15] study the related question when the induced subgraph has to be an expander<sup>9</sup>. Most of these works build on the following basic spectral partitioning primitive that also forms the basis of our algorithm:

- ▷ *There exists an efficient algorithm that given a graph  $G = (V, E)$ , outputs a partition  $\mathcal{P} := (S, T)$  of  $V$  such that either  $|S| \geq 3n/4$  and  $G[S]$  is an expander, or  $(S, T)$  is balanced<sup>10</sup> and has small expansion.* (3)

The above algorithm is a simple recursive application of the spectral partitioning algorithm from Cheeger’s inequality (see Lemmas 3.1, 5.1 from the full version [19] for more details). Note that the above algorithm may either output a large set which induces an expander (in which case we are done), or a balanced partition (say  $\mathcal{P}_0$ ) with small expansion. How do we proceed if the latter is the case? Following an idea from [15], we again apply the spectral partitioning (i.e, (3)) to each set in the partition in partition  $\mathcal{P}_0$  to construct a refinement of the partition, say  $\mathcal{P}_1$ . Again, if  $\mathcal{P}_1$  contains a linear sized subset which induces an expander, then we are done – otherwise, we again keep repeating the above process. We iteratively continue constructing a sequence of refinements  $\mathcal{P}_0 \subseteq \mathcal{P}_1 \subseteq \dots \subseteq \mathcal{P}_t$  until one of the partitions contains a linear sized set which induces an expander. But then one can ask that how can we guarantee that the process terminates? This is where the *higher order Cheeger’s inequality* [28, 25] comes to the rescue i.e., we show that if the process continues beyond some iteration  $t = t(\delta)$ , then  $\mathcal{P}_t$  is a balanced  $K := 2^{t-1}$ - partition of the vertex set. In particular, using the higher order Cheeger inequality and the fact that  $G[V_{\text{good}}]$  as low threshold rank, we can show that at least one of the  $K$ -sets in the partition must have large edge boundary i.e.,

$$\max_{i \in [K]} |\partial_G(S_i)| \geq \Omega(\varepsilon dn) \quad (\text{by an appropriate instantiation of parameters for (3).}) \quad (4)$$

On the other hand, note that since the algorithm proceeds beyond iteration  $t$ , it follows that for each application of (3) in each of the  $t$  iterations, the spectral partitioning algorithm returns a non-expanding partition (using the “or” guarantee from (3)), and hence the fraction of edges crossing the various sets in the partition  $\mathcal{P}_t$  must be small i.e.,

$$\sum_{i \in [K]} |\partial_G(S_i)| \leq O(\varepsilon^2 dn), \quad (5)$$

<sup>9</sup> Here we refer to any graph whose spectral gap is at least a constant as an expander.

<sup>10</sup> Here we say a partition  $V = S \sqcup T$  is *balanced* if  $|S|, |T| \in [|V|/4, 3|V|/4]$ .

which contradicts the upper bound on the expansion from (4). In summary, the above arguments taken together imply that the above process must terminate during some iteration  $t' \leq t$ , resulting in a subset of size at least  $2^{-t'} \cdot n = \tilde{\Omega}(n)$ <sup>11</sup> which induces an expander.

**Finding Many Low Rank Sets.** Now that we have an algorithm that finds a  $\Omega(n)$ -sized set (say  $S_1$ ) which induces an expander, the next step is to find many such vertex disjoint sets in the graph. This is easily achieved by deleting the first such subset  $S_1$  recovered by the above algorithm, and then again running the above algorithm on the graph  $G[V \setminus S_1]$  to recover a linear sized subset  $S_2 \subseteq V \setminus S_1$  which again induces an expander in  $G$ . However, note that the induced sub-graph  $G[V \setminus S_1]$  does not automatically inherit the structural properties of  $G[V_{\text{good}}]$  and hence, additional care is need to ensure that the above algorithm will still succeed on the smaller induced sub-graph  $G[V \setminus S_1]$ . In particular, we shall again need to establish an analogue of (4) where we show that any balanced  $K$ -way partition  $\mathcal{P}$  of the smaller set  $V \setminus S_1$  will still have one expanding set. This is done by showing that any balanced  $K$ -way partition of  $V \setminus S_1$  can be carefully extended to a balanced  $K$ -way partition  $\mathcal{P}'$  of  $V$  such that

$$\max_{S \in \mathcal{P}} |\partial_{G[V \setminus S_1]}(S)| \gtrsim \max_{S' \in \mathcal{P}'} |\partial_{G[V]}(S')|, \quad (6)$$

Note that the above immediately implies the desired  $K$ -way expansion bound for  $\mathcal{P}$  since the latter term can again be lower bounded by combining the higher order Cheeger's inequality with the threshold rank guarantee of the full graph  $G[V]$ . We point out that establishing (6) is precisely where the bounded degree assumption on the graph comes in handy. The above (i.e., (6)), along with an appropriately tailored version of (5) will allow us to establish that the algorithm will again find a linear sized subset  $S_2 \subset V_1 \setminus S_1$  which induces an expander. Overall, we keep iteratively finding and removing linear sized subsets  $S_2, S_3, \dots$ , – each of which induces an expander – until only  $o_\delta(1)$ -vertices remain; this results in an almost<sup>12</sup> partition  $\mathcal{P} := \{S_i\}_{i \in [N]}$  of the vertex set where each of the subsets in partition has small edge boundary, is linear sized, and induces an expander in the full graph  $G$ .

**Stitching the sets together.** Recall that our final objective is not to find an almost partition consisting of induced low threshold rank subgraphs, but to find one large  $(1 - o_\delta(1))$ -subset  $V'$  that induces a low threshold rank subgraph. To that end, we just show that the set  $V' := \cup_{i \in [N]} S_i$  is itself such a set. To see this, observe that the adjacency matrix  $A[V']$  of induced subgraph  $G[V']$  is almost block diagonal (since the above step guarantees that only few edges cross the partition  $\{S_i\}_{i \in [N]}$ ). Hence with some additional work we can conclude that the number of large eigenvalues in  $A[V']$  must be at most the sum of number of large eigenvalues in each of the blocks  $A[S_1], \dots, A[S_N]$ , each of which is again small on account of  $G[S_i]$ 's being expanders i.e., we can conclude  $\text{rank}_{1-\delta o(1)}(G[V']) \leq O(N)$ . Furthermore, since each of the sets in the partition  $\mathcal{P}$  is linear sized, this establishes that  $N$  is at most a constant (possibly depending on  $\delta$ ), which implies that the threshold rank of  $G[V']$  is at most  $O_\delta(1)$ .

**Reducing to the Bounded Degree Setting.** Lastly, we address the issue that in general the max degree of the underlying constraint graph can be arbitrarily large compared to the degree  $d$  of the underlying good graph  $G[V_{\text{good}}]$ . Towards addressing this, we introduce

<sup>11</sup> Here  $\tilde{\Omega}$  hides poly-logarithmic in  $\delta$  factors.

<sup>12</sup> An almost partition of a set  $[n]$  is a collection of disjoint sets whose union contain  $(1 - o(1))$ -fraction of the elements.

an additional pre-processing step which reduces the average degree of the remaining graph to  $O(d)$  by deleting a small number of vertices, and then additionally deletes the vertices in the remaining subgraph which have degree larger than  $d/\delta^{O(1)}$ . For the first part, we use a 2-factor approximation algorithm for the Partial Vertex Cover problem [10] that can be used to identify a small number of vertices that hits  $\approx (d_{\text{avg}}(G) - d)n/2$  edges (where  $d_{\text{avg}}$  denotes the average degree). The subsequent deletion step again just removes a small number of vertices; this follows from a simple application of Markov's inequality. Finally, we remark that this again perturbs the spectral structure of the graph used that is used in the subsequent steps, and hence additional care is needed to make all of the above arguments go through.

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