Algorithms for Coloring Reconfiguration Under Recolorability Digraphs

Soichiro Fujii
Research Institute for Mathematical Sciences, Kyoto University, Japan
School of Mathematical and Physical Sciences, Macquarie University, Sydney, Australia

Yuni Iwamasa
Graduate School of Informatics, Kyoto University, Japan

Kei Kimura
Faculty of Information Science and Electrical Engineering, Kyushu University, Fukuoka, Japan

Akira Suzuki
Graduate School of Information Sciences, Tohoku University, Sendai, Japan

Abstract

In the k-Recoloring problem, we are given two (vertex-)colorings of a graph using k colors, and asked to transform one into the other by recoloring only one vertex at a time, while at all times maintaining a proper k-coloring. This problem is known to be solvable in polynomial time if $k \leq 3$, and is PSPACE-complete if $k \geq 4$. In this paper, we consider a (directed) recolorability constraint on the k colors, which forbids some pairs of colors to be recolored directly. The recolorability constraint is given in terms of a digraph $\overrightarrow{R}$, whose vertices correspond to the colors and whose arcs represent the pairs of colors that can be recolored directly. We provide algorithms for the problem based on the structure of recolorability constraints $\overrightarrow{R}$, showing that the problem is solvable in linear time when $\overrightarrow{R}$ is a directed cycle or is in a class of multitrees.

2012 ACM Subject Classification Mathematics of computing → Graph algorithms

Keywords and phrases combinatorial reconfiguration, graph coloring, recolorability, recoloring

Digital Object Identifier 10.4230/LIPIcs.ISAAC.2022.4

Funding *Soichiro Fujii*: Partially supported by JST, ERATO Grant Number JPMJER1603 (HASUO Metamathematics for Systems Design Project) and JSPS Overseas Research Fellowships.
Yuni Iwamasa: Partially supported by JSPS KAKENHI Grant Numbers JP20K23323, JP20H05795, 22K17854, Japan.
Kei Kimura: Partially supported by JSPS KAKENHI Grant Numbers JP19K22841, JP21K17700, Japan.
Akira Suzuki: Partially supported by JSPS KAKENHI Grant Numbers JP18H04091, JP20K11666, JP20H05794, Japan.

1 Introduction

In a *combinatorial reconfiguration problem*, we are given two feasible solutions of a search problem, and asked whether there exists a step-by-step transformation (satisfying a given *reconfiguration rule*) between them, such that all intermediate solutions are also feasible. In other words, our task is to check the reachability between two given feasible solutions in a *reconfiguration (di)graph*, a (di)graph whose vertices correspond to the feasible solutions and whose arcs to the one-step transformations. (From this perspective, the traditional search problem is about deciding whether the reconfiguration (di)graph is empty or not.) Since the framework of combinatorial reconfiguration was developed by Ito et al. in 2008 [11, 12], combinatorial reconfiguration problems have been actively studied in theoretical computer science. We refer the reader to the surveys by van den Heuvel [10] and Nishimura [17] for more information on combinatorial reconfiguration in general.

One of the most fundamental reconfiguration problems is k-Recoloring. In this problem, we are given two k-colorings α and β of a graph G, and asked to determine whether there exists a sequence $(\alpha = \gamma_0, \gamma_1, \dots, \gamma_\ell = \beta)$ of k-colorings of G such that for each $i \in \{1, 2, \dots, \ell\}$, γ_i is obtained from γ_{i-1} by recoloring a single vertex in G.

The computational complexity of k-Recoloring has been studied with respect to various parameters such as the number k of colors [3, 4, 14] and the input graph classes [1, 9, 23]. In particular, it is known that k-Recoloring is polynomial-time solvable if $k \leq 3$ [4, 14] and is PSPACE-complete if $k \geq 4$ [3]. (On the other hand, k-Coloring, which asks to determine if a given graph admits a k-coloring, is polynomial-time solvable if $k \leq 2$ and is NP-complete if $k \geq 3$ [8].) While some of these parameters also appear in traditional search problems, others do not. In order to analyze the problem from a viewpoint specific to reconfiguration, Osawa et al. [19] introduced an undirected recolorability constraint for k-Recoloring, and Osawa [18] further considered its directed variant. More precisely, let $C = \{0, 1, \dots, k-1\}$ be the color set for k-Recoloring. We define a **recolorability digraph** on C as a digraph $\overrightarrow{R} = (V(\overrightarrow{R}), A(\overrightarrow{R}))$ with $V(\overrightarrow{R}) = C$; the idea is that each arc in $A(\overrightarrow{R})$ represents a pair of colors allowed to be recolored directly. Then k-Recoloring under the recolorability constraint $\overrightarrow{R}$ is formulated as follows.

k-Recoloring Under $\overrightarrow{R}$-Recolorability

Given A graph G and k-colorings α and β of G.

Problem Determine if there exists a sequence $\mathcal{W} = (\alpha = \gamma_0, \gamma_1, \dots, \gamma_\ell = \beta)$ of k-colorings of G such that for each $i \in \{1, 2, \dots, \ell\}$, there exists $v \in V(G)$ with $(\gamma_{i-1}(v), \gamma_i(v)) \in A(\overrightarrow{R})$ and for all $w \in V(G) \setminus \{v\}$, $\gamma_{i-1}(w) = \gamma_i(w)$ (such a sequence $\mathcal{W}$ is called an **(α, β)-reconfiguration sequence**).

Throughout this paper, we assume for simplicity that the input graph G is connected; otherwise, we can solve the problem in each connected component of G separately. For an undirected graph $R = (V(R), E(R))$ with $V(R) = C$ (which we call a **recolorability graph**), we similarly define k-Recoloring Under R-Recolorability by replacing "$(\gamma_{i-1}(v), \gamma_i(v)) \in A(\overrightarrow{R})$" in the above definition by "$\{\gamma_{i-1}(v), \gamma_i(v)\} \in E(R)$." Notice that k-Recoloring is equivalent to k-Recoloring Under K_k-Recolorability, where K_k denotes the undirected complete graph with k vertices.

Several (in)tractability results in terms of the recolorability constraint have been obtained in [19, 20, 18]. In [19], Osawa et al. showed that k-Recoloring Under R-Recolorability is PSPACE-complete if the undirected recolorability graph R contains a connected component having more than one cycle or a vertex of degree at least four. The same authors showed in [20] that this problem can be solved in polynomial time if the maximum degree of R is at most two. Since k-Recoloring is equivalent to k-Recoloring Under K_k-Recolorability, these results generalize the classification of the computational complexity of k-Recoloring in terms of the number k of colors; PSPACE-complete if $k \geq 4$ (in this case K_k has more than one cycle) and solvable in polynomial time if $k \leq 3$ (in this case the maximum degree of K_k is at most two).

For a directed recolorability constraint, Osawa [18] showed that k-Recoloring Under $\overrightarrow{R}$-Recolorability can be NP-hard even when $\overrightarrow{R}$ is a polytree (i.e., a directed graph whose underlying undirected graph is a tree), and solvable in polynomial time when $\overrightarrow{R}$ is an arborescence (i.e., a directed rooted tree). However, most of the previous results are for undirected recolorability R, and it is fair to say that the corresponding problems for directed recolorability $\overrightarrow{R}$ are not well-investigated.

Our contribution

We provide two linear-time algorithms for k-Recoloring Under $\overrightarrow{R}$-Recolorability in the cases where

1. $\overrightarrow{R}$ is a directed cycle; and
2. $\overrightarrow{R}$ is a multitree (i.e., a directed acyclic graph (DAG) in which there exists at most one directed path between any two vertices) satisfying the condition (S) stated in Section 4.

The former result is the directed analogue of the tractability of k-Recoloring Under R-Recolorability when R is an undirected cycle [20], and the latter is a generalization of the tractability of k-Recoloring Under $\overrightarrow{R}$-Recolorability when $\overrightarrow{R}$ is an arborescence [18].

When $\overrightarrow{R}$ is a directed cycle, the key to our algorithm is the notion of *potential function* on the set of vertices induced by a reconfiguration sequence. We rephrase the existence of an (α, β)-reconfiguration sequence as the existence of a suitable potential function (Theorem 3), which in turn is characterized as a nonnegative integer solution of a system of linear equations, whose coefficient matrix is the incidence matrix of the input graph G (endowed with edge orientation). This implies that k-Recoloring Under $\overrightarrow{R}$-Recolorability is solvable in linear time. We also introduce a similar potential function in the case where $\overrightarrow{R}$ is an *undirected* cycle, and characterize the existence of an (α, β)-reconfiguration sequence as the existence of an integer solution to the same system of linear equations as in the directed case (Theorem 8). These characterizations clarify the relationship of the problems when $\overrightarrow{R}$ is a directed cycle and those when $\overrightarrow{R}$ is an undirected cycle. Any yes-instance (G, α, β) for the former is also a yes instance for the latter, and the difference is precisely the additional requirement of nonnegativity (for the corresponding potential function) in the former. We also shed new light on a criterion of the existence of an (α, β)-reconfiguration sequence given in [20, Section 4.2] in terms of the system of linear equations. We can compute a shortest reconfiguration sequence in linear time when $\overrightarrow{R}$ is a directed cycle, as in the case of undirected cycles.

We then show that k-Recoloring Under $\overrightarrow{R}$-Recolorability is also solvable in linear time when $\overrightarrow{R}$ is a multitree satisfying condition (S). This generalizes the tractability of k-Recoloring Under $\overrightarrow{R}$-Recolorability when $\overrightarrow{R}$ is an arborescence shown in [18]. It might be worth remarking that a certain aspect of the algorithm in [18] is preserved in this more general setting: for any yes-instance, we can always find a reconfiguration sequence in which once we change the color of some vertex in G, we keep changing the color of that vertex until we reach the target color (see Remark 14). In this case, any reconfiguration sequence has the same length and a shortest reconfiguration sequence can be obtained trivially.

Related work

One of the features of this study is that the reconfiguration rule is asymmetric, that is, even when a solution can be reached from another solution in one step, the reverse transformation is not available in general. The computational complexity of reconfiguration problems with asymmetric reconfiguration rules (that is, with a directed reconfiguration graph) has been studied since at least late 1990s. For example, Sokoban [7] is a well-known example of such a reconfiguration problem. In recent years, research on this kind of reconfiguration problems has grown rapidly; in particular, Demaine et al. [5, 6] introduced a useful tool to analyze the computational complexity of such problems in 2022. Ito et al. [13] considered an asymmetric version of the independent set reconfiguration under the token sliding rule. We note that an auxiliary digraph used in Section 4 is similar to the one introduced in [13].

k-Recoloring Under $\overrightarrow{R}$-Recolorability is related to the multi-agent path finding problem studied in motion planning and multi-agent systems (see, e.g., [22] for a survey on the problem). In a variant of the multi-agent path finding problem, we are given a digraph $\overrightarrow{R}$ and k agents, where each agent has a start vertex and a goal vertex in $\overrightarrow{R}$. Then we are asked to assign a directed walk in $\overrightarrow{R}$ to each agent that does not cause collisions on the vertices among the agents, in such a way that the sum of the time steps required for every agent to reach its goal vertex is minimized. (It is assumed that passing through an arc of $\overrightarrow{R}$ takes a unit time, and that agents can stay at vertices to avoid collision.) This problem is similar to k-Recoloring Under $\overrightarrow{R}$-Recolorability with the input graph K_k, where each vertex of K_k corresponds to an agent and a coloring $\alpha\colon V(K_k) \to V(\overrightarrow{R})$ corresponds to a configuration of the agents on $\overrightarrow{R}$. The coloring constraint (i.e., adjacent vertices must have different colors) corresponds to the requirement that the agents do not collide on the vertices. While each agent can move simultaneously in the multi-agent path finding problem, agents can only move one by one in k-Recoloring Under $\overrightarrow{R}$-Recolorability. Therefore, any reconfiguration sequence in k-Recoloring Under $\overrightarrow{R}$-Recolorability gives an upper bound on the above variant of the multi-agent path finding problem.

Preliminaries

For a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. An edge incident to vertices v and w is written as vw or $\{v, w\}$. For a digraph $\overrightarrow{H}$, we denote its vertex set by $V(\overrightarrow{H})$ and its arc set by $A(\overrightarrow{H})$. An arc from v to w is written as (v, w). All graphs and digraphs considered in this paper are finite and simple.

Let C be a finite set of colors, and $\overrightarrow{R}$ a recolorability digraph on C (defined above). For $\overrightarrow{R}$ and a graph G, we define the **$\overrightarrow{R}$-reconfiguration digraph** on G, denoted by $\mathcal{C}_{\overrightarrow{R}}(G)$, as follows. A vertex of $\mathcal{C}_{\overrightarrow{R}}(G)$ is a coloring of G with respect to the color set C, i.e., a function $\alpha\colon V(G) \to C$ such that $\alpha(v) \neq \alpha(w)$ holds whenever $vw \in E(G)$. Given two colorings $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$, there is an arc (α, β) in $\mathcal{C}_{\overrightarrow{R}}(G)$ if and only if there exists $v \in V(G)$ such that $(\alpha(v), \beta(v)) \in A(\overrightarrow{R})$ and $\alpha(w) = \beta(w)$ for each $w \in V(G) \setminus \{v\}$. See Figure 1 for an example of $\mathcal{C}_{\overrightarrow{R}}(G)$. A directed walk in $\mathcal{C}_{\overrightarrow{R}}(G)$ from α to β is called an **(α, β)-reconfiguration sequence**. We denote by $\mathsf{dist}(\alpha, \beta)$ the length of a shortest (α, β)-reconfiguration sequence; when there is no (α, β)-reconfiguration sequence, we define $\mathsf{dist}(\alpha, \beta) = +\infty$.

Organization

The remainder of this paper is organized as follows. Section 2 considers the case where $\overrightarrow{R}$ is a directed cycle. Section 3 provides a novel view to the case where $\overrightarrow{R}$ is an undirected cycle based on the results in Section 2. Section 4 deals with the case where $\overrightarrow{R}$ is a multitree satisfying condition (S).

Due to space limitation, the proofs of the statements marked with ($\star$) are deferred to the appendix.

2 Directed cycle

In this section, we consider k-Recoloring Under $\overrightarrow{R}$-Recolorability when $\overrightarrow{R}$ is a directed cycle, and show that it is solvable in linear time. Throughout this section, we fix a natural number $k \geq 3$, the k-element set $C = \{0, 1, \ldots, k-1\}$ and the recolorability digraph

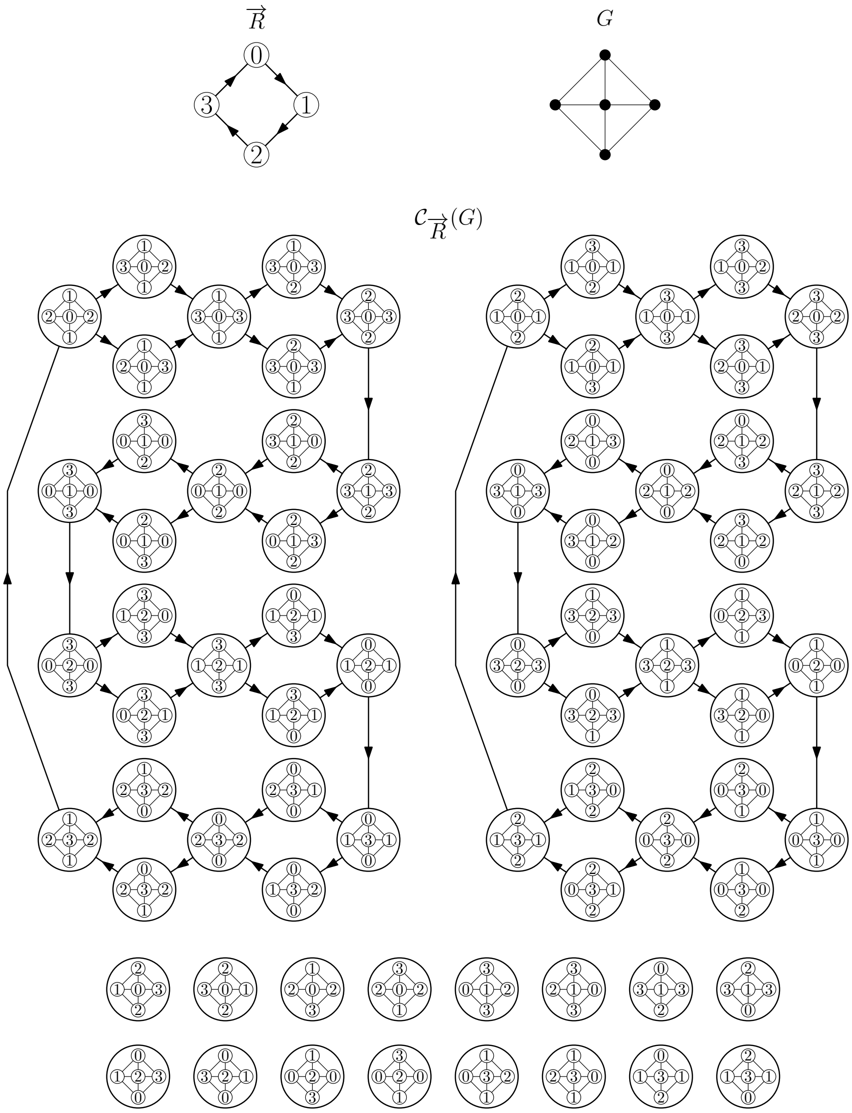

Figure 1 Examples of $\overrightarrow{R}$, G, and $\mathcal{C}_{\overrightarrow{R}}(G)$.

$\overrightarrow{R}$ on C whose arc set is given by $A(\overrightarrow{R}) = \{(0,1),(1,2),\ldots,(k-1,0)\}$. For each $c, d \in C$, we denote by $\Delta(c,d)$ the length of the shortest (directed) (c,d)-path in $\overrightarrow{R}$; in other words, $\Delta(c,d)$ is the unique natural number in $\{0,1,\ldots,k-1\}$ satisfying

$$c + \Delta(c,d) = d \pmod{k}. \tag{1}$$

Notice that whenever $c \neq d$, we have $\Delta(c,d) = k - \Delta(d,c)$. For each $c \in C$, define $c^+ \in C$ to be the (unique) color satisfying $\Delta(c,c^+) = 1$.

Given a coloring $\alpha \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ of a graph G, we say that a vertex $v \in V(G)$ is

- **blocked** with respect to α if for any $\beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ with $(\alpha,\beta) \in A(\mathcal{C}_{\overrightarrow{R}}(G))$, we have $\alpha(v) = \beta(v)$; and
- **frozen** with respect to α if for any $\beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ with $\mathsf{dist}(\alpha,\beta) < +\infty$, we have $\alpha(v) = \beta(v)$.

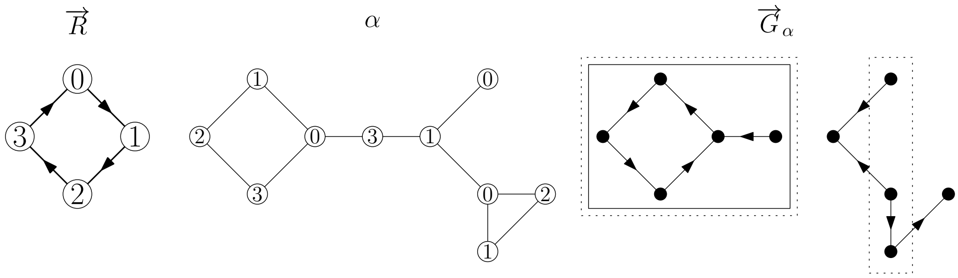

Figure 2 Examples of $\overrightarrow{R}$, α, and $\overrightarrow{G}_\alpha$. The vertices in the dotted rectangles are the blocked vertices, and those in the solid rectangle are the frozen ones.

Intuitively, v is blocked with respect to α if v cannot be recolored *directly* from α, whereas v is frozen with respect to α if v cannot be recolored by any successive recoloring starting from α.

Let G be a graph and $\alpha \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ be a coloring, and suppose that a vertex $v \in V(G)$ is blocked with respect to α. Then, since the only color to which v can be recolored is $\alpha(v)^+$, it follows that there exists $w \in V(G)$ such that $vw \in E(G)$ and $\alpha(w) = \alpha(v)^+$. In order to represent this blocking relation, we define the **blocking digraph** $\overrightarrow{G}_\alpha$ on α as $V(\overrightarrow{G}_\alpha) = V(G)$ and for each $v, w \in V(G)$, $(v, w) \in A(\overrightarrow{G}_\alpha)$ if and only if $vw \in E(G)$ and $\alpha(w) = \alpha(v)^+$.[1] Clearly, a vertex $v \in V(G)$ is blocked with respect to α if and only if the outdegree of v in $\overrightarrow{G}_\alpha$ is positive. Notice that any vertex of G which is in a directed cycle in $\overrightarrow{G}_\alpha$, is frozen with respect to α. (More precisely, a vertex $v \in V(G)$ is frozen with respect to α if and only if there exists a directed path from v to a directed cycle in $\overrightarrow{G}_\alpha$.) See Figure 2 for an example of $\overrightarrow{G}_\alpha$.

Any directed walk $\mathcal{W} = (\gamma_0, \gamma_1, \ldots, \gamma_\ell)$ in $\mathcal{C}_{\overrightarrow{R}}(G)$ induces the **potential function**[2] $p^{\mathcal{W}} \colon V(G) \to \mathbb{N}$ associated with $\mathcal{W}$, mapping each vertex $v \in V(G)$ to

$$p^{\mathcal{W}}(v) = |\{\, i \in \{1, 2, \ldots, \ell\} \mid \gamma_{i-1}(v) \neq \gamma_i(v) \,\}|.$$

So $p^{\mathcal{W}}(v)$ is the number of arcs in $\mathcal{W}$ where the vertex v is recolored. Note that we have $\sum_{v \in V(G)} p^{\mathcal{W}}(v) = \ell$.

The following lemma establishes a crucial property of potential functions; cf. [24, Lemma 4.1].

▶ **Lemma 1.** *Let G be a graph and $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ be colorings. Given any (α, β)-reconfiguration sequence $\mathcal{W} = (\alpha = \gamma_0, \gamma_1, \ldots, \gamma_\ell = \beta)$ and any pair of vertices $v, w \in V(G)$ such that $vw \in E(G)$, we have*

$$\Delta(\alpha(v), \alpha(w)) + p^{\mathcal{W}}(w) = p^{\mathcal{W}}(v) + \Delta(\beta(v), \beta(w)).$$

Proof. For each $i \in \{0, 1, \ldots, \ell\}$, let $\mathcal{W}_i = (\gamma_0, \gamma_1, \ldots, \gamma_i)$. We show that

$$\Delta(\gamma_0(v), \gamma_0(w)) + p^{\mathcal{W}_i}(w) = p^{\mathcal{W}_i}(v) + \Delta(\gamma_i(v), \gamma_i(w)) \quad (\forall vw \in E(G)) \tag{2}$$

holds for each $i \in \{0, 1, \ldots, \ell\}$, by induction on i.

---

[1] The digraph $\overrightarrow{G}_\alpha$ is the same as $\overrightarrow{H}_\alpha$ in [20, Section 4.1], although $\overrightarrow{H}_\alpha$ is defined for the case where the recolorability graph is an undirected cycle.

[2] We remark that this notion is different from the notion of *potential of an (oriented) edge in G with respect to a coloring* introduced in [20, Section 4.2]. We have decided to adopt the current terminology since it seems to be more consistent with the usage of the term “potential” in the literature (see, e.g., [2, Chapter 20] and [16, Chapter 9]).

When $i = 0$, $p^{\mathcal{W}_0}(v) = p^{\mathcal{W}_0}(w) = 0$ and hence both sides of (2) are equal.

Let $j \in \{1, 2, \dots, \ell\}$ and assume that (2) holds for $i = j - 1$. Let $u \in V(G)$ be the vertex such that $\gamma_{j-1}(u) \neq \gamma_j(u)$. If $u \notin \{v, w\}$, then clearly (2) continues to hold for $i = j$. If $u = v$, then $p^{\mathcal{W}_j}(v) = p^{\mathcal{W}_{j-1}}(v) + 1$ and $\Delta(\gamma_j(v), \gamma_j(w)) = \Delta(\gamma_{j-1}(v), \gamma_{j-1}(w)) - 1$, while $p^{\mathcal{W}_j}(w) = p^{\mathcal{W}_{j-1}}(w)$; hence (2) holds for $i = j$. The remaining case $u = w$ is similar. ◀

▶ Remark 2. The equation in Lemma 1 can be rewritten as

$$p^{\mathcal{W}}(v) - p^{\mathcal{W}}(w) = \Delta(\alpha(v), \alpha(w)) - \Delta(\beta(v), \beta(w)) \quad (\forall vw \in E(G)). \tag{3}$$

We can interpret this equation using the notion of incidence matrix of a digraph. Let us arbitrarily choose an orientation of each edge of G and denote the resulting digraph by $\overrightarrow{G}$. Let $\mathbf{M} \in \mathbb{R}^{V(\overrightarrow{G}) \times A(\overrightarrow{G})}$ be the incidence matrix of $\overrightarrow{G}$, i.e.,

$$\mathbf{M}(v, e) = \begin{cases} 1 & \text{if } e = (v, w) \text{ for some } w \in V(G); \\ -1 & \text{if } e = (w, v) \text{ for some } w \in V(G); \\ 0 & \text{otherwise.} \end{cases}$$

If we regard the potential function $p^{\mathcal{W}}$ as the row vector $\mathbf{p}^{\mathcal{W}} \in \mathbb{R}^{V(\overrightarrow{G})}$ and define another row vector $\boldsymbol{\Delta}^{\alpha,\beta} \in \mathbb{R}^{A(\overrightarrow{G})}$ by $\boldsymbol{\Delta}^{\alpha,\beta}((v, w)) = \Delta(\alpha(v), \alpha(w)) - \Delta(\beta(v), \beta(w))$, then (3) can be written as $\mathbf{p}^{\mathcal{W}}\mathbf{M} = \boldsymbol{\Delta}^{\alpha,\beta}$. This implies that for the existence of an (α, β)-reconfiguration sequence, it is necessary that the vector $\boldsymbol{\Delta}^{\alpha,\beta}$ be in the row space of $\mathbf{M}$, also known as the **tension space** of $\overrightarrow{G}$ (see, e.g., [2, Chapter 20]). It turns out that if there are no frozen vertices with respect to α, then the latter condition is equivalent to the existence of an (α, β)-reconfiguration sequence; see Remark 6. ⌟

The following theorem is the key to our algorithm. It allows us to reduce the task of deciding the existence of an (α, β)-reconfiguration sequence to that of deciding the existence of a suitable function $p \colon V(G) \to \mathbb{N}$, which is straightforward.

▶ **Theorem 3.** *Let G be a graph. For colorings $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ and a function $p \colon V(G) \to \mathbb{N}$, there exists an (α, β)-reconfiguration sequence $\mathcal{W}$ such that $p = p^{\mathcal{W}}$ if and only if the following conditions hold:*

(C1) *for each $v \in V(G)$ in a directed cycle in $\overrightarrow{G}_\alpha$, we have $p(v) = 0$;*

(C2) *for each $v \in V(G)$, we have $\alpha(v) + p(v) = \beta(v) \pmod k$; and*

(C3) *for each $vw \in E(G)$, we have $\Delta(\alpha(v), \alpha(w)) + p(w) = p(v) + \Delta(\beta(v), \beta(w))$.*

Proof. Condition (C1) is necessary since a vertex in a directed cycle in $\overrightarrow{G}_\alpha$ is frozen with respect to α, (C2) is clearly necessary, and (C3) is necessary by Lemma 1.

The sufficiency of conditions (C1)–(C3) is established by induction on $\sum_{v \in V(G)} p(v)$. The base case (i.e., when $p(v) = 0$ for all $v \in V(G)$) is clear by (C2). Assume that a function $p \colon V(G) \to \mathbb{N}$ with $\sum_{v \in V(G)} p(v) > 0$ and colorings α, β satisfying conditions (C1)–(C3) are given. We claim that there exists a vertex $w \in V(G)$ such that $p(w) > 0$ and that w is not blocked with respect to α, i.e., w has outdegree 0 in the blocking digraph $\overrightarrow{G}_\alpha$. To show this, let $V' \subseteq V(G)$ be the set consisting of all $v \in V(G)$ with $p(v) > 0$. Then there is no arc in $\overrightarrow{G}_\alpha$ from V' to $V(G) \setminus V'$. Indeed, $(v, w) \in A(\overrightarrow{G}_\alpha)$ implies $p(v) \leq p(w)$: by (C3) we have $\Delta(\alpha(v), \alpha(w)) + p(w) = p(v) + \Delta(\beta(v), \beta(w))$, whereas $\Delta(\alpha(v), \alpha(w)) = 1 \leq \Delta(\beta(v), \beta(w))$. So if there is no $w \in V(G)$ with $p(w) > 0$ such that w is not blocked with respect to α, then it follows that each vertex in the induced subgraph $\overrightarrow{G}_\alpha|_{V'}$ has positive outdegree. Therefore the digraph $\overrightarrow{G}_\alpha|_{V'}$ (which is nonempty by $\sum_{v \in V(G)} p(v) > 0$) contains a directed cycle, contradicting (C1).

Now that we know the existence of $w \in V(G)$ such that $p(w) > 0$ and that w is not blocked with respect to α, let us define $\alpha' \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ to be the coloring obtained from α by recoloring w (necessarily to $\alpha(w)^+$), and $p' : V(G) \to \mathbb{N}$ to be the function given by

$$p'(v) = \begin{cases} p(v) - 1 & \text{if } v = w; \\ p(v) & \text{otherwise.} \end{cases}$$

Since we have $(\alpha, \alpha') \in A(\mathcal{C}_{\overrightarrow{R}}(G))$, it now suffices to show that there exists an (α', β)-reconfiguration sequence $\mathcal{W}'$ such that $p' = p^{\mathcal{W}'}$. For this we can use the induction hypothesis, since the triple (α', β, p') again satisfies (C1)–(C3). (To see that (α', β, p') satisfies (C1), observe that the directed cycles in $\overrightarrow{G}_\alpha$ and $\overrightarrow{G}_{\alpha'}$ remain unchanged.) ◀

▶ Remark 4. Let $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ be colorings of a (nonempty and connected) graph G and $p : V(G) \to \mathbb{N}$ be a function satisfying (C3). Then the triple (α, β, p) satisfies (C2) if and only if there exists *some* $v \in V(G)$ satisfying $\alpha(v) + p(v) = \beta(v) \pmod k$. Indeed, using (C3) and (1), it follows that the set of all vertices $v \in V(G)$ for which $\alpha(v) + p(v) = \beta(v) \pmod k$ holds is a union of connected components of G, i.e., is either $\emptyset$ or $V(G)$. ⌟

Theorem 3 suggests Algorithm 1 for k-Recoloring Under $\overrightarrow{R}$-Recolorability. This algorithm checks the existence of a function $p : V(G) \to \mathbb{N}$ satisfying (C1)–(C3). If the output of Algorithm 1 is "yes", then the length $\mathsf{dist}(\alpha, \beta)$ of a shortest (α, β)-reconfiguration sequence is given by $\sum_{v \in V(G)} \overline{p}(v)$, and such a sequence can be obtained by repeatedly recoloring any non-blocked vertex $w \in V(G)$ with $\overline{p}(w) > 0$ (and then updating $\overline{p}$ as in the proof of Theorem 3) in a greedy manner.

Algorithm 1 Algorithm for k-Recoloring Under $\overrightarrow{R}$-Recolorability with $\overrightarrow{R}$ a directed cycle.

Input : A (connected) graph G and two colorings $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$.
Output : "Yes" if there exists an (α, β)-reconfiguration sequence, and "no" otherwise.

Step 1. Choose a vertex $v_0 \in V(G)$ and define $\tilde{p}(v_0) = \Delta(\alpha(v_0), \beta(v_0))$.
Step 2. Extend $\tilde{p}$ to a function $\tilde{p} : V(G) \to \mathbb{Z}$ so that for each edge $vw \in E(G)$, $\Delta(\alpha(v), \alpha(w)) + \tilde{p}(w) = \tilde{p}(v) + \Delta(\beta(v), \beta(w))$ holds. (For example, one can first choose a spanning tree T of G and then extend the value of $\tilde{p}$ along edges in T using the equation. Then one checks if the equation holds for each edge in $E(G) \setminus E(T)$.) Output "no" if this is impossible.
Step 3. Let $a = \min\{j \in \mathbb{N} \mid \forall v \in V(G).\ 0 \le \tilde{p}(v) + jk\}$ and define $\overline{p} : V(G) \to \mathbb{N}$ by $\overline{p}(v) = \tilde{p}(v) + ak$ for each $v \in V(G)$.
Step 4. Compute $\overrightarrow{G}_\alpha$. Output "yes" if for each $v \in V(G)$ which is in a directed cycle in $\overrightarrow{G}_\alpha$, we have $\overline{p}(v) = 0$; output "no" otherwise.

▶ **Theorem 5.** *For any directed cycle $\overrightarrow{R}$, k-Recoloring Under $\overrightarrow{R}$-Recolorability is solvable in* $\mathrm{O}(|V(G)| + |E(G)|)$ *time.*

Proof. Since the correctness of Algorithm 1 follows from Theorem 3, we only estimate its running time. It is clear that Steps 1, 2, and 3 in Algorithm 1 can be done in $\mathrm{O}(1)$, $\mathrm{O}(|E(G)|)$, and $\mathrm{O}(|V(G)|)$ time, respectively. In Step 4, $\overrightarrow{G}_\alpha$ can be computed in $\mathrm{O}(|E(G)|)$ time by definition and the subsequent procedure can be done in $\mathrm{O}(|V(\overrightarrow{G}_\alpha)| + |A(\overrightarrow{G}_\alpha)|) = \mathrm{O}(|V(G)| + |E(G)|)$ time by computing the strongly connected components of $\overrightarrow{G}_\alpha$. Hence, the running time of Algorithm 1 is $\mathrm{O}(|V(G)| + |E(G)|)$. ◀

▶ Remark 6. This remark is a continuation of Remark 2, and we use the notations introduced there. Given colorings $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$, we shall give a necessary and sufficient condition for the existence of an (α, β)-reconfiguration sequence in terms of the vector $\mathbf{\Delta}^{\alpha,\beta}$.

First we consider the case where there is no directed cycle in $\overrightarrow{G}_\alpha$, i.e., no vertex is frozen with respect to α. In this case, *there exists an (α, β)-reconfiguration sequence if and only if $\mathbf{\Delta}^{\alpha,\beta}$ belongs to the tension space of $\overrightarrow{G}$.* We have already seen the necessity of the latter condition. To show its sufficiency, suppose that $\mathbf{\Delta}^{\alpha,\beta}$ is in the tension space of $\overrightarrow{G}$, i.e., the row space of the incidence matrix $\mathbf{M}$ of $\overrightarrow{G}$. Since $\mathbf{M}$ is totally unimodular (see, e.g., [15, Theorem 5.27]), it follows that there exists an integer vector $\tilde{\mathbf{p}} \in \mathbb{Z}^{V(\overrightarrow{G})}$ satisfying $\tilde{\mathbf{p}}\mathbf{M} = \mathbf{\Delta}^{\alpha,\beta}$ (see, e.g., [21, Section 19.1]). Now notice that the all-one (row) vector $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^{V(\overrightarrow{G})}$ is in the left kernel of $\mathbf{M}$ (i.e., $\mathbf{1M} = \mathbf{0}$), and in fact spans the left kernel of $\mathbf{M}$ since G is assumed to be connected. Therefore, we can add a suitable natural number multiple of $\mathbf{1}$ to $\tilde{\mathbf{p}}$ to obtain $\mathbf{p} \in \mathbb{N}^{V(\overrightarrow{G})}$ which moreover satisfies (C2) (cf. Remark 4). Since (C1) is vacuous by our assumption on α, Theorem 3 guarantees the existence of an (α, β)-reconfiguration sequence.

When there is a directed cycle in $\overrightarrow{G}_\alpha$, choose a vertex $v \in V(G)$ that is in such a directed cycle. If $\mathbf{\Delta}^{\alpha,\beta}$ is in the tension space of $\overrightarrow{G}$, then it follows from the above discussion that there exists a unique (integer) vector $\mathbf{p} \in \mathbb{Z}^{V(\overrightarrow{G})}$ satisfying $\mathbf{pM} = \mathbf{\Delta}^{\alpha,\beta}$ and $\mathbf{p}(v) = 0$. Then an (α, β)-reconfiguration sequence exists if and only if all components of $\mathbf{p}$ are nonnegative and $\mathbf{p}$ satisfies (C1) and (C2).

The tension space of $\overrightarrow{G}$ is the orthogonal complement of the **circulation space** of $\overrightarrow{G}$, the vector subspace of $\mathbb{R}^{A(\overrightarrow{R})}$ spanned by the (signed) characteristic vectors of the cycles in G (see, e.g., [2, Proposition 20.1]). A basis of the circulation space of $\overrightarrow{G}$ can be obtained by choosing a spanning tree T of G and taking the set of all characteristic vectors of the fundamental cycles with respect to T (see, e.g., [15, Theorem 2.11]).[3] The role of Step 2 of Algorithm 1 can be understood from this perspective; it decides whether $\mathbf{\Delta}^{\alpha,\beta}$ is in the tension space or not, by checking its orthogonality to the circulation space. ⌟

▶ Remark 7. Here, we estimate the diameter of $V(\mathcal{C}_{\overrightarrow{R}}(G))$, i.e., the value

$$\max\{\, \mathsf{dist}(\alpha, \beta) \mid \alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G)), \mathsf{dist}(\alpha, \beta) < +\infty \,\}.$$

Given colorings $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$, the length of a shortest (α, β)-reconfiguration sequence, if it exists, is given by $\sum_{v \in V(G)} \overline{p}(v)$ from Algorithm 1. We have

$$\begin{aligned} \sum_{v \in V(G)} \overline{p}(v) &\le (k-1)n + \sum_{i=1}^{n-1} (k-2)i \\ &= \frac{((k-2)n + k)\, n}{2}, \end{aligned}$$

since the minimum value of $\{\overline{p}(v) \mid v \in V(G)\}$ is at most $k-1$ and $|\overline{p}(v) - \overline{p}(w)| \le k-2$ for any adjacent vertices v and w. The value $((k-2)n + k)\, n/2$ is attained by the n-vertex path $P = (v_1, v_2, \dots, v_n)$ with $\alpha = (0, 1, 2, \dots, k-1, 0, 1, \dots, r(n,k) - 1)$ and $\beta = (k-1, k-2, \dots, 1, 0, k-1, \dots, k - r(n,k))$, where $r(n,k)$ is the remainder of n divided by k. Hence, the diameter of $V(\mathcal{C}_{\overrightarrow{R}}(P))$ is $\Theta(kn^2)$ which matches that in the case where the recolorability graph is an undirected cycle.[4] ⌟

[3] The tension space and the circulation space of $\overrightarrow{G}$ are called the cocycle space and the cycle space of $\overrightarrow{G}$ in [15], respectively.

[4] The construction of graphs with the diameter of the reconfiguration graph $\Omega(n^2)$ when R is the undirected cycle with three vertices in [4] can be easily generalized to that with $\Omega(kn^2)$ when R is the undirected cycle with k vertices.

3 Undirected cycle

A linear-time algorithm for k-Recoloring Under R-Recolorability where R is an undirected cycle has been given in [20]. Here we shall establish an analogue of Theorem 3 and shed new light on results of [20]. Throughout this section, we fix a natural number $k \geq 3$, the k-element set $C = \{0, 1, \ldots, k-1\}$ and the (undirected) recolorability graph R on C whose set of edges is given by $E(R) = \{01, 12, \ldots, (k-1)0\}$. Notions for reconfiguration problems with undirected recolorability graph are explained in [20]. We may also regard R as a digraph with $A(R) = \{(0,1), (1,2), \ldots, (k-1, 0)\} \cup \{(0, k-1), (k-1, k-2), \ldots, (1, 0)\}$. Note that since R is undirected, the R-reconfiguration "digraph" $\mathcal{C}_R(G)$ is also undirected for any graph G: for $\alpha, \beta \in V(\mathcal{C}_R(G))$, we have $(\alpha, \beta) \in A(\mathcal{C}_R(G))$ if and only if $(\beta, \alpha) \in A(\mathcal{C}_R(G))$.

For each $c, d \in C$, we denote by $\Delta(c, d)$ the unique natural number $\Delta(c, d) \in \{0, 1, \ldots, k-1\}$ satisfying $c + \Delta(c, d) = d \pmod k$. Given $c \in C$, define $c^+, c^- \in C$ to be the (unique) colors satisfying $\Delta(c, c^+) = 1$ and $\Delta(c^-, c) = 1$. Notice that given a coloring $\alpha \in V(\mathcal{C}_R(G))$, a vertex $v \in V(G)$ can only be recolored (directly) to $\alpha(v)^+$ or $\alpha(v)^-$. It is convenient to refine the notion of blocked vertices as follows. We say that a vertex $v \in V(G)$ is

- **forward-blocked** with respect to α if for any $\beta \in V(\mathcal{C}_R(G))$ with $(\alpha, \beta) \in A(\mathcal{C}_R(G))$, we have $\alpha(v) = \beta(v)$ or $\alpha(v)^- = \beta(v)$; and
- **backward-blocked** with respect to α if for any $\beta \in V(\mathcal{C}_R(G))$ with $(\alpha, \beta) \in A(\mathcal{C}_R(G))$, we have $\alpha(v) = \beta(v)$ or $\alpha(v)^+ = \beta(v)$.

The **blocking digraph** $\overrightarrow{G}_\alpha$ on a coloring $\alpha \in V(\mathcal{C}_R(G))$ is defined as in Section 2: $V(\overrightarrow{G}_\alpha) = V(G)$ and for each $v, w \in V(G)$, $(v, w) \in A(\overrightarrow{G}_\alpha)$ if and only if $vw \in E(G)$ and $\alpha(w) = \alpha(v)^+$ (or equivalently $\alpha(v) = \alpha(w)^-$). Notice that $v \in V(G)$ is forward-blocked with respect to α if and only if the outdegree of v in $\overrightarrow{G}_\alpha$ is positive, whereas v is backward-blocked with respect to α if and only if the indegree of v in $\overrightarrow{G}_\alpha$ is positive.

Let G be a graph. A walk $\mathcal{W} = (\gamma_0, \gamma_1, \ldots, \gamma_\ell)$ in $\mathcal{C}_R(G)$ induces the **potential function** $p^{\mathcal{W}} \colon V(G) \to \mathbb{Z}$ associated with $\mathcal{W}$, defined for each vertex v as

$$p^{\mathcal{W}}(v) = |\{\, i \in \{1, 2, \ldots, \ell\} \mid \gamma_{i-1}(v)^+ = \gamma_i(v) \,\}| - |\{\, i \in \{1, 2, \ldots, \ell\} \mid \gamma_{i-1}(v)^- = \gamma_i(v) \,\}|.$$

Note that we have $\sum_{v \in V(G)} |p^{\mathcal{W}}(v)| \leq \ell$.

The following is the analogue of Theorem 3. The only difference is that now the function p is allowed to take negative integer values.

▶ **Theorem 8.** *Let G be a graph. For colorings $\alpha, \beta \in V(\mathcal{C}_R(G))$ and a function $p \colon V(G) \to \mathbb{Z}$, there exists an (α, β)-reconfiguration sequence $\mathcal{W}$ such that $p = p^{\mathcal{W}}$ if and only if conditions (C1)–(C3) of Theorem 3 hold. Moreover, if these conditions are satisfied, then there exists an (α, β)-reconfiguration sequence $\mathcal{W} = (\alpha = \gamma_0, \gamma_1, \ldots, \gamma_\ell = \beta)$ with $p = p^{\mathcal{W}}$ which moreover satisfies $\sum_{v \in V(G)} |p^{\mathcal{W}}(v)| = \ell$.*

Proof. This can be proved by an argument analogous to the proof of Theorem 3. One can show the sufficiency of conditions (C1)–(C3) by induction on $\sum_{v \in V(G)} |p(v)|$. In the induction step, we may assume that either

1. there exists $v \in V(G)$ with $p(v) > 0$; or
2. there exists $v \in V(G)$ with $p(v) < 0$.

Accordingly, we can show (using $(v, w) \in A(\overrightarrow{G}_\alpha) \implies p(v) \leq p(w)$) that either

1. there exists $w \in V(G)$ with $p(w) > 0$ and w is not forward-blocked with respect to α; or
2. there exists $w \in V(G)$ with $p(w) < 0$ and w is not backward-blocked with respect to α.

The rest of the proof is straightforward. ◀

We can modify Algorithm 1 to obtain Algorithm 2. If the output of Algorithm 2 is "yes" and there exists a directed cycle in $\overrightarrow{G}_\alpha$, then the length $\mathsf{dist}(\alpha, \beta)$ of a shortest (α, β)-reconfiguration path is given by $\sum_{v \in V(G)} |\tilde{p}(v)|$, and such a path can be obtained by repeatedly recoloring any non-blocked vertex $w \in V(G)$ with $\tilde{p}(w) \neq 0$. If the output of Algorithm 2 is "yes" but there is no directed cycle in $\overrightarrow{G}_\alpha$, then $\mathsf{dist}(\alpha, \beta)$ might be strictly smaller than $\sum_{v \in V(G)} |\tilde{p}(v)|$. To find the precise value of $\mathsf{dist}(\alpha, \beta)$, consider $\tilde{p}[j] \colon V(G) \to \mathbb{Z}$ defined by $\tilde{p}[j](v) = \tilde{p}(v) + jk$ for each $j \in \mathbb{Z}$. Then we have

$$\mathsf{dist}(\alpha, \beta) = \min_{j \in \mathbb{Z}} \Bigl(\sum_{v \in V} |\tilde{p}[j](v)| \Bigr).$$

This value can be easily computed using the convexity of the function $j \mapsto \sum_{v \in V} |\tilde{p}[j](v)|$.

Algorithm 2 Algorithm for k-Recoloring Under R-Recolorability with R an undirected cycle.

Input : A (connected) graph G and two colorings $\alpha, \beta \in V(\mathcal{C}_R(G))$.
Output : "Yes" if there exists an (α, β)-reconfiguration sequence, and "no" otherwise.

Step 1. Compute $\overrightarrow{G}_\alpha$. If $\overrightarrow{G}_\alpha$ contains a directed cycle, choose a vertex $v_0 \in V(G)$ in such a directed cycle; otherwise, choose $v_0 \in V(G)$ arbitrarily. Define $\tilde{p}(v_0) = \Delta(\alpha(v_0), \beta(v_0))$.
Step 2. Extend $\tilde{p}$ to a function $\tilde{p} \colon V(G) \to \mathbb{Z}$ so that for each edge $vw \in E(G)$, $\Delta(\alpha(v), \alpha(w)) + \tilde{p}(w) = \tilde{p}(v) + \Delta(\beta(v), \beta(w))$ holds. (For example, one can first choose a spanning tree T of G and then extend the value of $\tilde{p}$ along edges in T using the equation. Then one checks if the equation holds for each edge in $E(G) \setminus E(T)$.) Output "no" if this is impossible.
Step 3. Output "yes" if for each $v \in V(G)$ which is in a directed cycle in $\overrightarrow{G}_\alpha$, we have $\tilde{p}(v) = 0$; output "no" otherwise.

From Algorithm 2, we obtain the following theorem, which was first proved in [20].

▶ **Theorem 9.** *For any undirected cycle R, k-Recoloring Under R-Recolorability is solvable in* $\mathrm{O}(|V(G)| + |E(G)|)$ *time.*

Proof. This can be proved in a similar way as Theorem 5. ◀

▶ Remark 10. As mentioned in Remarks 2 and 6, we can interpret condition (C3) via the incidence matrix $\mathbf{M} \in \mathbb{R}^{V(\overrightarrow{G}) \times A(\overrightarrow{G})}$ of a digraph $\overrightarrow{G}$ obtained from G by arbitrarily choosing an orientation of each edge. We can define the row vector $\mathbf{\Delta}^{\alpha,\beta} \in \mathbb{R}^{A(\overrightarrow{G})}$ from any pair of colorings $\alpha, \beta \in V(\mathcal{C}_R(G))$ as in Remark 2.

Let $\alpha, \beta \in V(\mathcal{C}_R(G))$ be colorings. If there is no directed cycle in $\overrightarrow{G}_\alpha$, then *there exists an (α, β)-reconfiguration sequence if and only if $\mathbf{\Delta}^{\alpha,\beta}$ belongs to the tension space of $\overrightarrow{G}$*. So in this case k-Recoloring Under R-Recolorability can be solved by checking whether $\mathbf{\Delta}^{\alpha,\beta}$ is orthogonal to the circulation space of $\overrightarrow{G}$. This explains the criterion for the existence of an (α, β)-reconfiguration sequence given in [20, Theorem 13] (or more precisely its special case where no frozen vertices are involved).[5] We can turn this observation into a linear-time

[5] Note that, as mentioned in footnote 2, the meaning of the term "potential" is different in [20] and in this paper.

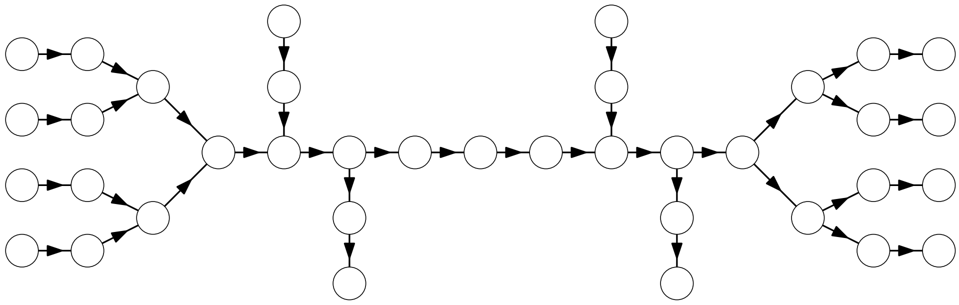

Figure 3 A polytree $\overrightarrow{R}$ such that k-Recoloring Under $\overrightarrow{R}$-Recolorability is NP-complete [18].

algorithm by using the fact that the basis of the circulation space is given by the set of all (signed) characteristic vectors of the fundamental cycles in G with respect to any spanning tree T of G; cf. [20, Lemma 17].

The case where $\overrightarrow{G}_\alpha$ contains directed cycles can be treated as in Remark 6. ⌟

▶ Remark 11. Let us examine how the length of a shortest reconfiguration sequence changes when the recolorability graph is changed from an undirected cycle R to a directed cycle $\overrightarrow{R}$ of the same size k. Namely, for any graph G and colorings $\alpha, \beta \in V(\mathcal{C}_R(G))$, we compare $d = \mathsf{dist}(\alpha, \beta)$ in $V(\mathcal{C}_R(G))$ with $d' = \mathsf{dist}(\alpha, \beta)$ in $V(\mathcal{C}_{\overrightarrow{R}}(G))$, provided that d' is finite. Assume that d satisfies $d = \sum_{v \in V} |\tilde{p}(v)|$ for some potential function $\tilde{p}$ of G. Let $M = -\min_{v \in V} \tilde{p}(v)$ and $j \in \mathbb{N}$ be such that $(j-1)k < M \leq jk$. Then from Algorithm 1 we have $d' = \sum_{v \in V} \tilde{p}'(v)$, where $\tilde{p}'(v) = \tilde{p}(v) + jk$ for each $v \in V$. By definition, we have $M \leq d$ and $jk < M + k$. Hence, we have

$$\begin{aligned} d' &\leq d + njk \\ &< d + n(M+k) \\ &\leq d + (d+k) \\ &= (n+1)d + nk. \end{aligned}$$

Using this inequality we can also obtain a multiplicative bound $d' < knd$, which is tight as seen from the following instance. Consider the n-vertex path $P = (v_1, v_2, \ldots, v_n)$, and its colorings $\alpha = (0, 1, 2, \ldots, k-1, 0, 1, \ldots, r(n,k)-1)$ and $\beta = (k-1, 1, 2, \ldots, k-1, 0, 1, \ldots, r(n,k)-1)$ (thus, β is obtained by changing only the color of v_1 from α), where $r(n,k)$ is the remainder of n divided by k. Then $d = 1$ and $d' = kn - 1$. Hence, the bound $d' < knd$ is tight. ⌟

4 A class of multitrees

In this section, we show that k-Recoloring Under $\overrightarrow{R}$-Recolorability is linear-time solvable whenever $\overrightarrow{R}$ is a multitree satisfying the condition (S) introduced below, where a **multitree** is a directed acyclic graph (DAG) $\overrightarrow{R}$ such that for each $c, d \in V(\overrightarrow{R})$, there exists at most one directed (c,d)-path in $\overrightarrow{R}$.

First observe that k-Recoloring Under $\overrightarrow{R}$-Recolorability is in NP whenever $\overrightarrow{R}$ is a DAG [18], since in this case the lengths of reconfiguration sequences in $\mathcal{C}_{\overrightarrow{R}}(G)$ are bounded by $|V(G)| \times |V(\overrightarrow{R})|$. For a DAG $\overrightarrow{R}$ and $c, d \in V(\overrightarrow{R})$, we write $c \leq d$ if there exists a (c,d)-path in $\overrightarrow{R}$. Then $\leq$ is a partial order on $V(\overrightarrow{R})$, and given any graph G and colorings $\alpha, \beta \in \mathcal{C}_{\overrightarrow{R}}(G)$, an obvious necessary condition for the existence of an (α,β)-reconfiguration sequence is *pointwise reachability*:

(PR) for each $v \in V(G)$, we have $\alpha(v) \leq \beta(v)$.

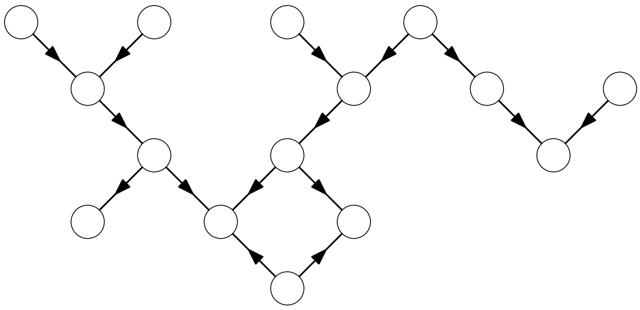

Figure 4 An example of a recolorability digraph satisfying (S) but not (S').

When $\overrightarrow{R}$ is moreover a multitree, for every $c, d \in V(\overrightarrow{R})$ such that $c \leq d$, the set of all vertices in the unique directed (c, d)-path in $\overrightarrow{R}$ is given by $[c, d] = \{e \in V(\overrightarrow{R}) \mid c \leq e \leq d\}$. To see its implication to the reconfiguration problem, let $\alpha, \beta \in \mathcal{C}_{\overrightarrow{R}}(G)$ be colorings satisfying (PR). Then we can tell in advance the sequence of colors on individual vertices $v \in V(G)$ in an (α, β)-reconfiguration sequence (if any): for each (α, β)-reconfiguration sequence $\mathcal{W} = (\alpha = \gamma_0, \gamma_1, \ldots, \gamma_\ell = \beta)$ and $v \in V(G)$, we have

$$\{\, c \in V(\overrightarrow{R}) \mid \exists i \in \{0, 1, \ldots, \ell\}.\, \gamma_i(v) = c \,\} = [\alpha(v), \beta(v)].$$

In particular, the lengths of all (α, β)-reconfiguration sequences are the same, and is given by

$$\sum_{v \in V(G)} \bigl(\bigl|[\alpha(v), \beta(v)]\bigr| - 1\bigr).$$

Although the above discussion simplifies the situation greatly, Osawa [18] constructed a multitree $\overrightarrow{R}$ for which k-Recoloring Under $\overrightarrow{R}$-Recolorability is NP-complete; see Figure 3. (In fact this $\overrightarrow{R}$ is a *polytree*, i.e., a multitree whose underlying (undirected) graph is a tree.) Roughly speaking, the source of difficulty lies in the fact that it might be necessary to change the color of some vertex v of G a bit, but then "wait" in the middle of a reconfiguration sequence in order to "let colors on adjacent vertices pass" before reaching the final color of v; see the proof of [18, Theorem 14] for details.

Then a natural question is whether k-Recoloring Under $\overrightarrow{R}$-Recolorability is solvable in polynomial time if the multitree $\overrightarrow{R}$ prohibits such a phenomenon (cf. Remark 14). In our first attempt to find a sufficient condition on such multitrees $\overrightarrow{R}$, we were lead to the following:

(S') Let $(c_0, c_1, \ldots, c_\ell)$ be a directed path in $\overrightarrow{R}$ and $i, j \in \{0, 1, \ldots, \ell\}$. If c_i has indegree > 1 and c_j has outdegree > 1 in $\overrightarrow{R}$, then $i \leq j$.

Intuitively, this says that no "merging" occurs after "splitting" in $\overrightarrow{R}$.

It then turned out that we can improve our result, and show (using condition (S') in the proof) that k-Recoloring Under $\overrightarrow{R}$-Recolorability is linear-time solvable whenever $\overrightarrow{R}$ is a multitree satisfying the following slightly more general condition:

(S) Let $(c_0, c_1, \ldots, c_\ell)$ be any directed path in $\overrightarrow{R}$ and $i, j \in \{0, 1, \ldots, \ell\}$. If c_i has indegree > 1 and c_j has outdegree > 1, then either $i \leq j$, $i = \ell$, or $j = 0$.

See Figure 4 for an example of a recolorability digraph satisfying (S) but not (S'). (A multitree $\overrightarrow{R}$ satisfies (S) if and only if it does not contain a subgraph $\overrightarrow{R}'$ as in Figure 6 in the appendix.)

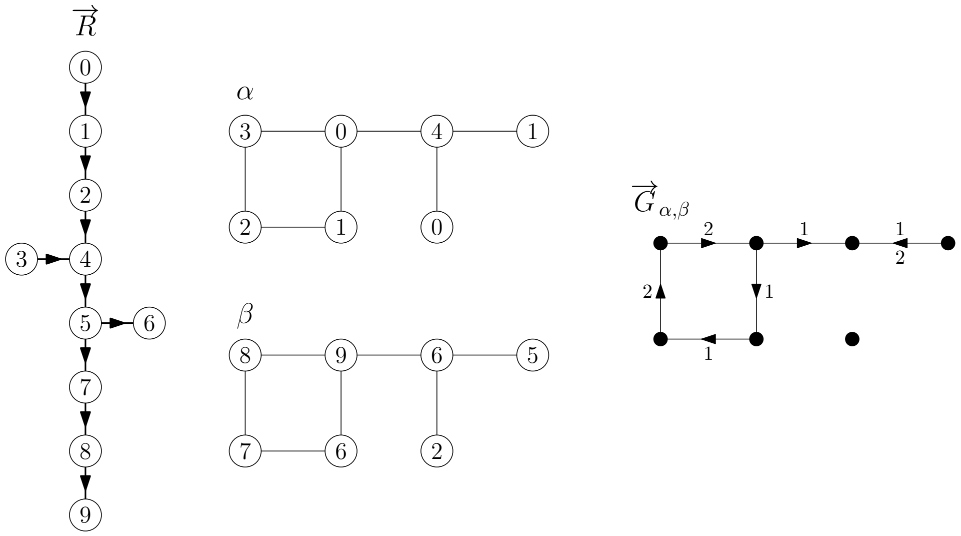

Figure 5 Examples of $\overrightarrow{R}$, α, β, and $\overrightarrow{G}_{\alpha,\beta}$. In $\overrightarrow{G}_{\alpha,\beta}$, the arcs from $A(\overrightarrow{G}^1_{\alpha,\beta})$ (resp. $A(\overrightarrow{G}^2_{\alpha,\beta})$) are labeled by 1 (resp. 2).

Throughout the rest of this section, let $\overrightarrow{R}$ be a multitree satisfying (S) unless otherwise stated. The outline of our linear-time algorithm for k-Recoloring Under $\overrightarrow{R}$-Recolorability is as follows. It first checks whether (PR) holds for the input data (G, α, β). If (G, α, β) satisfies (PR), then we construct an auxiliary digraph $\overrightarrow{G}_{\alpha,\beta}$. Now Theorem 12 below guarantees that we can obtain the answer for the instance (G, α, β) by checking whether $\overrightarrow{G}_{\alpha,\beta}$ contains a directed cycle or not.

We proceed to the definition of the auxiliary digraph $\overrightarrow{G}_{\alpha,\beta}$; see also Figure 5. Let G be a graph and $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ be colorings satisfying (PR). First define a digraph $\overrightarrow{G}^1_{\alpha,\beta}$ by $V(\overrightarrow{G}^1_{\alpha,\beta}) = V(G)$ and for each $v, w \in V(G)$, $(v, w) \in A(\overrightarrow{G}^1_{\alpha,\beta})$ if and only if $vw \in E(G)$ and $\alpha(w) \in [\alpha(v), \beta(v)]$. Similarly, define a digraph $\overrightarrow{G}^2_{\alpha,\beta}$ by $V(\overrightarrow{G}^2_{\alpha,\beta}) = V(G)$ and for each $v, w \in V(G)$, $(v, w) \in A(\overrightarrow{G}^2_{\alpha,\beta})$ if and only if $vw \in E(G)$ and $\beta(v) \in [\alpha(w), \beta(w)]$. Finally we define the digraph $\overrightarrow{G}_{\alpha,\beta}$ as the union of $\overrightarrow{G}^1_{\alpha,\beta}$ and $\overrightarrow{G}^2_{\alpha,\beta}$: $V(\overrightarrow{G}_{\alpha,\beta}) = V(G)$ and $(v, w) \in A(\overrightarrow{G}_{\alpha,\beta})$ if and only if $(v, w) \in A(\overrightarrow{G}^1_{\alpha,\beta})$ or $(v, w) \in A(\overrightarrow{G}^2_{\alpha,\beta})$. Intuitively, that (v, w) is an arc in $\overrightarrow{G}_{\alpha,\beta}$ means that w is an obstacle for recoloring v from $\alpha(v)$ to $\beta(v)$. We remark that similar auxiliary digraphs are also used in [13].

▶ **Theorem 12** ($\star$). *Let G be a graph and $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ be colorings. There exists an (α, β)-reconfiguration sequence if and only if (PR) and the following condition hold:*
(DAG) *the digraph $\overrightarrow{G}_{\alpha,\beta}$ does not contain a directed cycle.*

We show the "if" part of Theorem 12; the proof of the "only if" part is deferred to the appendix.

▶ **Lemma 13.** *Let G be a graph and $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ be colorings. There exists an (α, β)-reconfiguration sequence if conditions (PR) and (DAG) hold.*

Proof. We prove this by induction on the sum of the lengths of the paths $[\alpha(v), \beta(v)]$ over $v \in V(G)$. Assume that $\alpha \neq \beta$. Every vertex $v \in V(G)$ incident to an arc of $\overrightarrow{G}_{\alpha,\beta}$ satisfies $\alpha(v) \neq \beta(v)$, because otherwise $\overrightarrow{G}_{\alpha,\beta}$ would contain a directed cycle of length 2, contradicting (DAG). It follows from (DAG) that there exists a vertex $w \in V(G)$ with $\alpha(w) \neq \beta(w)$ and whose outdegree is 0 in $\overrightarrow{G}_{\alpha,\beta}$.

Take any such $w \in V(G)$, and define $\alpha' \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ by

$$\alpha'(v) = \begin{cases} \beta(w) & \text{if } v = w; \\ \alpha(v) & \text{otherwise.} \end{cases}$$

There exists an (α, α')-reconfiguration sequence since w has outdegree 0 in $\overrightarrow{G}^1_{\alpha,\beta}$. It suffices to show that the pair (α', β) satisfies (DAG). By definition, $A(\overrightarrow{G}^2_{\alpha',\beta}) \subseteq A(\overrightarrow{G}^2_{\alpha,\beta})$ holds. We also have $A(\overrightarrow{G}^1_{\alpha',\beta}) \subseteq A(\overrightarrow{G}^1_{\alpha,\beta})$, since w has outdegree 0 in $\overrightarrow{G}^2_{\alpha,\beta}$. So $A(\overrightarrow{G}_{\alpha',\beta}) \subseteq A(\overrightarrow{G}_{\alpha,\beta})$ and (DAG) holds for (α', β). ◀

▶ Remark 14. Note that the above proof reveals the following striking property of k-Recoloring Under $\overrightarrow{R}$-Recolorability with $\overrightarrow{R}$ a multitree satisfying (S): for any graph G and colorings $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$, if there exists some (α, β)-reconfiguration sequence, then there exists one in which the color of each vertex $v \in V(G)$ is changed in one go. ⌟

Algorithm 3 summarizes our algorithm for k-Recoloring Under $\overrightarrow{R}$-Recolorability when $\overrightarrow{R}$ is a multitree satisfying condition (S).

Algorithm 3 Algorithm for k-Recoloring Under $\overrightarrow{R}$-Recolorability with $\overrightarrow{R}$ a multitree satisfying (S).

Input : A graph G and two colorings $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$.
Output : "Yes" if there exists an (α, β)-reconfiguration sequence, and "no" otherwise.

Step 1. Check if (PR) holds for G, α, and β. Output "no" if (PR) does not hold.
Step 2. Compute $\overrightarrow{G}_{\alpha,\beta}$.
Step 3. Output "no" if $\overrightarrow{G}_{\alpha,\beta}$ contains a directed cycle; output "yes" otherwise.

▶ **Theorem 15.** *For any multitree $\overrightarrow{R}$ satisfying condition (S), k-Recoloring Under $\overrightarrow{R}$-Recolorability is solvable in* $\mathrm{O}(|V(G)| + |E(G)|)$ *time.*

Proof. As mentioned before Theorem 12, k-Recoloring Under $\overrightarrow{R}$-Recolorability can be solved by first checking whether (PR) holds for the input data (G, α, β) in $\mathrm{O}(|V(G)|)$ time, constructing $\overrightarrow{G}_{\alpha,\beta}$ in $\mathrm{O}(|E(G)|)$ time, and checking whether $\overrightarrow{G}_{\alpha,\beta}$ is a DAG or not in $\mathrm{O}(|V(G)| + |E(G)|)$ time (e.g., by using the depth first search). The correctness of this procedure follows from Theorem 12. ◀

References

1 Marthe Bonamy, Matthew Johnson, Ioannis Lignos, Viresh Patel, and Daniël Paulusma. Reconfiguration graphs for vertex colourings of chordal and chordal bipartite graphs. *Journal of Combinatorial Optimization*, 27(1):132–143, 2014. doi:10.1007/s10878-012-9490-y.

2 John Adrian Bondy and Uppaluri Siva Ramachandra Murty. *Graph Theory*, volume 244 of *Graduate Texts in Mathematics*. Springer, 2008.

3 Paul Bonsma and Luis Cereceda. Finding paths between graph colourings: Pspace-completeness and superpolynomial distances. *Theoretical Computer Science*, 410(50):5215–5226, 2009. doi:10.1016/j.tcs.2009.08.023.

4 Luis Cereceda, Jan van den Heuvel, and Matthew Johnson. Finding paths between 3-colorings. *Journal of Graph Theory*, 67(1):69–82, 2011. doi:10.1002/jgt.20514.

5 Erik D. Demaine, Isaac Grosof, Jayson Lynch, and Mikhail Rudoy. Computational complexity of motion planning of a robot through simple gadgets. In *Proceedings of the 9th International Conference on Fun with Algorithms (FUN 2018)*, volume 100 of *LIPIcs*, pages 18:1–18:21. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018. doi:10.4230/LIPIcs.FUN.2018.18.

6 Erik D. Demaine, Dylan H. Hendrickson, and Jayson Lynch. Toward a general complexity theory of motion planning: Characterizing which gadgets make games hard. In *Proceedings of the 11th Innovations in Theoretical Computer Science Conference (ITCS 2020)*, volume 151 of *LIPIcs*, pages 62:1–62:42. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.ITCS.2020.62.

7 Dorit Dor and Uri Zwick. Sokoban and other motion planning problems. *Computational Geometry*, 13(1):215–228, 1999. doi:10.1016/S0925-7721(99)00017-6.

8 Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* Freeman, San Francisco, CA, 1979.

9 Tatsuhiko Hatanaka, Takehiro Ito, and Xiao Zhou. The list coloring reconfiguration problem for bounded pathwidth graphs. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, 98-A(6):1168–1178, 2015. doi:10.1587/transfun.E98.A.1168.

10 Jan van den Heuvel. The complexity of change. *Surveys in Combinatorics 2013*, 409:127–160, 2013. doi:10.1017/CBO9781139506748.005.

11 Takehiro Ito, Erik D. Demaine, Nicholas J. A. Harvey, Christos H. Papadimitriou, Martha Sideri, Ryuhei Uehara, and Yushi Uno. On the complexity of reconfiguration problems. In *Proceedings of the 19th Annual International Symposium on Algorithms and Computation (ISAAC 2008)*, volume 5369 of *Lecture Notes in Computer Science*, pages 28–39, 2008. doi:10.1007/978-3-540-92182-0_6.

12 Takehiro Ito, Erik D. Demaine, Nicholas J. A. Harvey, Christos H. Papadimitriou, Martha Sideri, Ryuhei Uehara, and Yushi Uno. On the complexity of reconfiguration problems. *Theoretical Computer Science*, 412:1054–1065, 2011. doi:10.1016/j.tcs.2010.12.005.

13 Takehiro Ito, Yuni Iwamasa, Yasuaki Kobayashi, Yu Nakahata, Yota Otachi, Masahiro Takahashi, and Kunihiro Wasa. Independent set reconfiguration on directed graphs. In *Proceedings of the 47th International Symposium on Mathematical Foundations of Computer Science (MFCS 2022)*, to appear.

14 Matthew Johnson, Dieter Kratsch, Stefan Kratsch, Viresh Patel, and Daniël Paulusma. Finding shortest paths between graph colourings. *Algorithmica*, 75(2):295–321, 2016. doi:10.1007/s00453-015-0009-7.

15 Bernhard Korte and Jens Vygen. *Combinatorial Optimization.* Springer, sixth edition, 2018. doi:10.1007/978-3-662-56039-6.

16 Kazuo Murota. *Discrete Convex Analysis.* Monographs on Discrete Mathematics and Applications. SIAM, 2003. doi:10.1137/1.9780898718508.

17 Naomi Nishimura. Introduction to reconfiguration. *Algorithms*, 11(4):52, 2018. doi:10.3390/a11040052.

18 Hiroki Osawa. *Coloring Reconfiguration Problems and Their Generalizations.* PhD thesis, Tohoku University, 2020.

19 Hiroki Osawa, Akira Suzuki, Takehiro Ito, and Xiao Zhou. Complexity of Coloring Reconfiguration under Recolorability Constraints. In *Proceedings of the 28th International Symposium on Algorithms and Computation (ISAAC 2017)*, volume 92 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 62:1–62:12. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2017. doi:10.4230/LIPIcs.ISAAC.2017.62.

20 Hiroki Osawa, Akira Suzuki, Takehiro Ito, and Xiao Zhou. Algorithms for coloring reconfiguration under recolorability constraints. In *Proceedings of the 29th International Symposium on Algorithms and Computation (ISAAC 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018. doi:10.4230/LIPIcs.ISAAC.2018.37.

21 Alexander Schrijver. *Theory of Linear and Integer Programming.* John Wiley & Sons, 1998.

22 Roni Stern, Nathan R. Sturtevant, Ariel Felner, Sven Koenig, Hang Ma, Thayne T. Walker, Jiaoyang Li, Dor Atzmon, Liron Cohen, T. K. Satish Kumar, Roman Barták, and Eli Boyarski. Multi-agent pathfinding: Definitions, variants, and benchmarks. In *Proceedings of the 12th International Symposium on Combinatorial Search (SOCS 2019)*, pages 151–159. AAAI Press, 2019.

23 Marcin Wrochna. Reconfiguration in bounded bandwidth and tree-depth. *Journal of Computer and System Sciences*, 93:1–10, 2018. doi:10.1016/j.jcss.2017.11.003.

24 Marcin Wrochna. Homomorphism reconfiguration via homotopy. *SIAM Journal on Discrete Mathematics*, 34(1):328–350, 2020. doi:10.1137/17M1122578.

A Proofs for Section 4 (A class of multitrees)

▶ **Theorem 12** ($\star$)**.** *Let G be a graph and $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ be colorings. There exists an (α,β)-reconfiguration sequence if and only if (PR) and the following condition hold:*
(DAG) *the digraph $\overrightarrow{G}_{\alpha,\beta}$ does not contain a directed cycle.*

To prove the "only if" part of Theorem 12, we use the following lemma.

▶ **Lemma 16.** *Let G be a graph and $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ be colorings satisfying (PR) and*
($\not\subseteq$) *the digraph $\overrightarrow{G}_{\alpha,\beta}$ does not contain a directed cycle of length 2; in other words, for each $v, w \in V(G)$ with $vw \in E(G)$, we have $[\alpha(v), \beta(v)] \not\subseteq [\alpha(w), \beta(w)]$.*
Let $v, w \in V(G)$. If there exists a directed (v,w)-walk in $\overrightarrow{G}_{\alpha,\beta}$, then $\alpha(v) \le \beta(w)$ holds.

Proof. Let $(v = u_0, u_1, \dots, u_\ell = w)$ be a directed (v,w)-walk in $\overrightarrow{G}_{\alpha,\beta}$, and let $\overrightarrow{R}'$ be the subgraph $\bigcup_{k=0}^{\ell}[\alpha(u_k), \beta(u_k)]$ of $\overrightarrow{R}$, where we regard each directed path $[\alpha(u_k), \beta(u_k)]$ as the subgraph of $\overrightarrow{R}$ induced by the set $[\alpha(u_k), \beta(u_k)]$ of vertices. We shall see that the digraph $\overrightarrow{R}'$ has the following property:

(†) If $c \in V(\overrightarrow{R}')$ has indegree 0 in $\overrightarrow{R}'$, then c has outdegree ≤ 1 in $\overrightarrow{R}'$. If c has outdegree 0 in $\overrightarrow{R}'$, then c has indegree ≤ 1 in $\overrightarrow{R}'$.

To prove (†), we proceed as follows. We shall show below the statement of Lemma 16, *assuming* (†). Then, using this result, we can show (†) as follows. Suppose to the contrary that (†) does not hold. For each $\ell' \in \{0, 1, \dots, \ell\}$, let $\overrightarrow{R}_{\ell'}$ be the subgraph $\bigcup_{k=0}^{\ell'}[\alpha(u_k), \beta(u_k)]$ of $\overrightarrow{R}$. Let $r \in \{0, 1, \dots, \ell-1\}$ be the largest number such that $\overrightarrow{R}_r$ satisfies (†). Assume without loss of generality that $\alpha(u_{r+1})$ has indegree 0 in $\overrightarrow{R}_{r+1}$, but has outdegree > 1 in $\overrightarrow{R}_{r+1}$. Take $s \in \{1, 2, \dots, r\}$ such that $\alpha(u_s) = \alpha(u_{r+1})$. Since α is a coloring, we cannot have $(u_r, u_{r+1}) \in A(\overrightarrow{G}^1_{\alpha,\beta})$. Hence we have $(u_r, u_{r+1}) \in A(\overrightarrow{G}^2_{\alpha,\beta})$, i.e., $\beta(u_r) \in [\alpha(u_{r+1}), \beta(u_{r+1})]$. Using Lemma 16 for $\overrightarrow{R}_r$ (which satisfies (†)), we have $\alpha(u_s) \le \beta(u_r)$ in $\overrightarrow{R}_r$. On the other hand, $[\alpha(u_{r+1}), \beta(u_{r+1})]$ is not contained in $\overrightarrow{R}_r$. Therefore, since $\alpha(u_{r+1}) \ne \beta(u_r)$, $[\alpha(u_{r+1}), \beta(u_r)]$ is not contained in $\overrightarrow{R}_r$ either. Since $\alpha(u_s) = \alpha(u_{r+1})$, it follows that there exist two directed $(\alpha(u_s), \beta(u_r))$-paths in $\overrightarrow{R}_{r+1}$ and hence in $\overrightarrow{R}$, contradicting the assumption that $\overrightarrow{R}$ is a multitree.

Now we show the statement of Lemma 16 assuming (†). Since $\overrightarrow{R}'$ trivially satisfies (S), it follows from (†) that $\overrightarrow{R}'$ satisfies the stronger condition (S') mentioned above. (Indeed, to verify (S') for $\overrightarrow{R}'$, it suffices to show that (S') holds for all *maximal* directed paths in $\overrightarrow{R}'$. A directed path $(c_0, c_1, \dots, c_\ell)$ in $\overrightarrow{R}$ is maximal if and only if c_0 has indegree 0 and c_ℓ has outdegree 0 in $\overrightarrow{R}'$. Hence (†) implies that c_0 has outdegree ≤ 1 and c_ℓ has indegree ≤ 1 in $\overrightarrow{R}'$. Thus, among the three possible conclusions $i \leq j$, $i = \ell$ and $j = 0$ of (S), only the first can hold.) Since $c \leq d$ in $\overrightarrow{R}'$ implies $c \leq d$ in $\overrightarrow{R}$, it suffices to show $\alpha(v) \leq \beta(w)$ in $\overrightarrow{R}'$. We shall henceforth work within $\overrightarrow{R}'$; in particular, the terms "indegree" and "outdegree", as well as the relation $\leq$, are understood to be with respect to $\overrightarrow{R}'$.

Before proving the main statement, we show the following.

▷ Claim. If each $c \in V(\overrightarrow{R}')$ with $\alpha(v) \leq c$ has indegree ≤ 1, then $\alpha(v) \leq \alpha(w)$ holds.

Proof of Claim. First observe that if each $c \in V(\overrightarrow{R}')$ with $\alpha(v) \leq c$ has indegree ≤ 1, then we have:

$(*)$ for each $c, d, e \in V(\overrightarrow{R}')$ with $\alpha(v) \leq c \leq e$ and $d \leq e$, either $c \leq d$ or $d \leq c$ holds.

Indeed, if $\alpha(v) \leq e$, then any $d \in V(\overrightarrow{R}')$ with $d \leq e$ satisfies either $[\alpha(v), e] \subseteq [d, e]$ or $[d, e] \subseteq [\alpha(v), e]$, i.e., $d \leq \alpha(v)$ or $\alpha(v) \leq d$. In the former case we have $d \leq c$, and in the latter case we have either $c \leq d$ or $d \leq c$ since $c, d \in [\alpha(v), e]$.

We prove $\alpha(v) \leq \alpha(u_k)$ for all $k \in \{0, 1, \ldots, \ell\}$ by induction on k. Assume that we have shown $\alpha(v) \leq \alpha(u_k)$ for some $k \in \{0, 1, \ldots, \ell - 1\}$. We have $(u_k, u_{k+1}) \in A(\overrightarrow{G}_{\alpha,\beta})$. If $(u_k, u_{k+1}) \in A(\overrightarrow{G}^1_{\alpha,\beta})$, i.e., $\alpha(u_{k+1}) \in [\alpha(u_k), \beta(u_k)]$, then $\alpha(v) \leq \alpha(u_k) \leq \alpha(u_{k+1})$. Assume $(u_k, u_{k+1}) \in A(\overrightarrow{G}^2_{\alpha,\beta})$, i.e., $\beta(u_k) \in [\alpha(u_{k+1}), \beta(u_{k+1})]$. Since $[\alpha(u_k), \beta(u_k)] \not\subseteq [\alpha(u_{k+1}), \beta(u_{k+1})]$ by ($\not\subseteq$), we have $\alpha(u_{k+1}) \not\leq \alpha(u_k)$. But since $\alpha(v) \leq \alpha(u_k) \leq \beta(u_k)$ and $\alpha(u_{k+1}) \leq \beta(u_k)$ hold, we have $\alpha(u_k) \leq \alpha(u_{k+1})$ by $(*)$. Thus $\alpha(v) \leq \alpha(u_{k+1})$ as required. ◁

We now return to the proof of Lemma 16 (under the assumption (S')). We prove $\alpha(v) \leq \beta(u_k)$ for all $k \in \{0, 1, \ldots, \ell\}$ by induction on k. Assume that we have shown $\alpha(v) \leq \beta(u_k)$ for some $k \in \{0, 1, \ldots, \ell - 1\}$. We have $(u_k, u_{k+1}) \in A(\overrightarrow{G}_{\alpha,\beta})$. If $(u_k, u_{k+1}) \in A(\overrightarrow{G}^2_{\alpha,\beta})$, i.e., $\beta(u_k) \in [\alpha(u_{k+1}), \beta(u_{k+1})]$, then $\alpha(v) \leq \beta(u_k) \leq \beta(u_{k+1})$. Assume $(u_k, u_{k+1}) \in A(\overrightarrow{G}^1_{\alpha,\beta})$, i.e., $\alpha(u_{k+1}) \in [\alpha(u_k), \beta(u_k)]$. Let $[\alpha(u_k), \beta(u_k)] = (\alpha(u_k) = c_0, c_1, \ldots, c_p = \beta(u_k))$. Define

$$i = \min\{\, q \in \{0, 1, \ldots, p\} \mid \alpha(v) \leq c_q \,\} \quad \text{and} \quad j = \max\{\, q \in \{0, 1, \ldots, p\} \mid c_q \leq \beta(u_{k+1}) \,\}.$$

Note that i is well-defined since $\alpha(v) \leq \beta(v_k) = c_p$, and j is well-defined since $c_0 = \alpha(u_k) \leq \alpha(u_{k+1}) \leq \beta(u_{k+1})$. If $i \leq j$, then we have $\alpha(v) \leq c_i \leq c_j \leq \beta(u_{k+1})$ as required.

We finish the proof by showing that $i \leq j$ holds. Suppose to the contrary that $i > j$. First observe that c_j has outdegree > 1. Indeed, if c_j has outdegree ≤ 1, then (since $j < i \leq p$) $\beta(u_{k+1}) = c_j \in [\alpha(u_k), \beta(u_k)]$ and thus we have $[\alpha(u_{k+1}), \beta(u_{k+1})] \subseteq [\alpha(u_k), \beta(u_k)]$, contradicting ($\not\subseteq$). Hence, c_i has indegree ≤ 1 by (S'). Since $0 \leq j < i$, $\alpha(v) = c_i$. It follows from (S') that the assumption of Claim is satisfied. Thus, we have $\alpha(v) \leq \alpha(u_{k+1}) \leq \beta(u_{k+1})$, contradicting $\beta(u_{k+1}) = c_j < c_i = \alpha(v)$. ◀

Proof of Theorem 12. By Lemma 13, it suffices to show the "only if" part. We assume that the colorings $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ admit an (α, β)-reconfiguration sequence and show that conditions (PR) and (DAG) hold. Condition (PR) is clearly necessary for the existence of an (α, β)-reconfiguration sequence. To prove (DAG), assume to the contrary that there exists a directed cycle $D = (v_0, v_1, \ldots, v_\ell = v_0)$ in $\overrightarrow{G}_{\alpha,\beta}$. It suffices to show that for each coloring α' such that $(\alpha, \alpha') \in A(\mathcal{C}_{\overrightarrow{R}}(G))$ and that there exists an (α', β)-reconfiguration sequence, the directed cycle D persists in $\overrightarrow{G}_{\alpha',\beta}$; this would contradict the fact that $\overrightarrow{G}_{\beta,\beta}$ has no arcs.

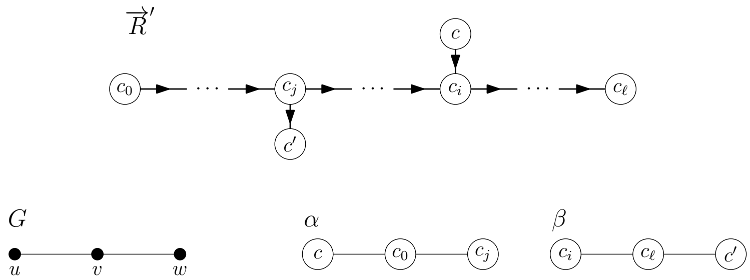

Figure 6 An example of a digraph $\overrightarrow{R}'$ that does not satisfy (S). We assume that $0 < j < i < \ell$.

We only have to treat the case where we recolor a vertex in D, say v_0. So we assume that $\alpha(v_0) \neq \beta(v_0)$ and, writing $[\alpha(v_0), \beta(v_0)] = (c_0, c_1, \ldots, c_p)$, the coloring $\alpha' \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ is given by

$$\alpha'(v) = \begin{cases} c_1 & \text{if } v = v_0; \\ \alpha(v) & \text{otherwise.} \end{cases}$$

Note that the possible differences between $\overrightarrow{G}_{\alpha,\beta}$ and $\overrightarrow{G}_{\alpha',\beta}$ are only of the following form:

- some arcs of the form (v, v_0) in $\overrightarrow{G}_{\alpha,\beta}$ where $v \in V(G) \setminus \{v_0\}$ might disappear in $\overrightarrow{G}_{\alpha',\beta}$;
- some arcs of the form (v, v_0) where $v \in V(G) \setminus \{v_0\}$ which is not in $\overrightarrow{G}_{\alpha,\beta}$ might appear in $\overrightarrow{G}_{\alpha',\beta}$.

Thus we still have a path $(v_0, v_1, \ldots, v_{\ell-1})$ in $\overrightarrow{G}_{\alpha',\beta}$. Since there exists an (α', β)-reconfiguration sequence, the pair (α', β) satisfies (PR) and ($\not\subseteq$). Therefore by Lemma 16 we have $\alpha'(v_0) \leq \beta(v_{\ell-1})$. If $(v_{\ell-1}, v_0) \in A(\overrightarrow{G}^1_{\alpha,\beta})$, i.e., $\alpha(v_0) \in [\alpha(v_{\ell-1}), \beta(v_{\ell-1})]$, then we have $\alpha'(v_0) \in [\alpha(v_{\ell-1}), \beta(v_{\ell-1})]$ and hence $(v_{\ell-1}, v_0) \in A(\overrightarrow{G}^1_{\alpha',\beta})$. If $(v_{\ell-1}, v_0) \in A(\overrightarrow{G}^2_{\alpha,\beta})$, i.e., $\beta(v_{\ell-1}) \in [\alpha(v_0), \beta(v_0)]$, then we have $\beta(v_{\ell-1}) \in [\alpha'(v_0), \beta(v_0)]$ and hence $(v_{\ell-1}, v_0) \in A(\overrightarrow{G}^2_{\alpha',\beta})$. So we still have the directed cycle D in $\overrightarrow{G}_{\alpha',\beta}$. ◀

▶ Remark 17. Let $\overrightarrow{R}$ be a multitree, not necessarily satisfying (S). We can still define the auxiliary digraph $\overrightarrow{G}_{\alpha,\beta}$ for each graph G and colorings $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}}(G))$ satisfying (PR). We remark that (S) is *almost* a necessary and sufficient condition on $\overrightarrow{R}$ for Lemma 16 to hold. More precisely, $\overrightarrow{R}$ satisfies (S) if and only if all subgraphs of $\overrightarrow{R}$ satisfy Lemma 16. To show this, it suffices to exhibit a counterexample to Lemma 16 for a suitable subgraph of any multitree $\overrightarrow{R}$ not satisfying (S). If $\overrightarrow{R}$ does not satisfy (S), then it must contain a subgraph $\overrightarrow{R}'$ as in Figure 6. Now consider the graph G and colorings $\alpha, \beta \in V(\mathcal{C}_{\overrightarrow{R}'}(G))$ in Figure 6. It turns out that $(u, v) \in A(\overrightarrow{G}^2_{\alpha,\beta})$ and $(v, w) \in A(\overrightarrow{G}^1_{\alpha,\beta})$, and hence (u, v, w) is a directed path in $\overrightarrow{G}_{\alpha,\beta}$. However, we have $\alpha(u) = c \not\leq c' = \beta(w)$ in $\overrightarrow{R}'$. ⌟