

Computing Homomorphisms in Hereditary Graph Classes: The Peculiar Case of the 5-Wheel and Graphs with No Long Claws

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Abstract

For graphs G and H , an H -coloring of G is an edge-preserving mapping from $V(G)$ to $V(H)$. In the H -COLORING problem the graph H is fixed and we ask whether an instance graph G admits an H -coloring. A generalization of this problem is H -COLORINGEXT, where some vertices of G are already mapped to vertices of H and we ask if this partial mapping can be extended to an H -coloring.

We study the complexity of variants of H -COLORING in F -free graphs, i.e., graphs excluding a fixed graph F as an induced subgraph. For integers $a, b, c \geq 1$, by $S_{a,b,c}$ we denote the graph obtained by identifying one endvertex of three paths on $a + 1$, $b + 1$, and $c + 1$ vertices, respectively. For odd $k \geq 5$, by W_k we denote the graph obtained from the k -cycle by adding a universal vertex.

As our main algorithmic result we show that W_5 -COLORINGEXT is polynomial-time solvable in $S_{2,1,1}$ -free graphs. This result exhibits an interesting non-monotonicity of H -COLORINGEXT with respect to taking induced subgraphs of H . Indeed, W_5 contains a triangle, and K_3 -COLORING, i.e., classical 3-coloring, is NP-hard already in claw-free (i.e., $S_{1,1,1}$ -free) graphs. Our algorithm is based on two main observations:

1. W_5 -COLORINGEXT in $S_{2,1,1}$ -free graphs can be in polynomial time reduced to a variant of the problem of finding an independent set intersecting all triangles, and
2. the latter problem can be solved in polynomial time in $S_{2,1,1}$ -free graphs.

We complement this algorithmic result with several negative ones. In particular, we show that W_5 -COLORING is NP-hard in P_t -free graphs for some constant t and W_5 -COLORINGEXT is NP-hard in $S_{3,3,3}$ -free graphs of bounded degree. This is again uncommon, as usually problems that are NP-hard in $S_{a,b,c}$ -free graphs for some constant a, b, c are already hard in claw-free graphs

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1 Introduction

Many computationally hard problems become tractable when restricted to instances with some special properties. In case of graph problems, typical families of such special instances come from forbidding certain substructures. For example, for a family \mathcal{F} of graphs, a graph G is \mathcal{F} -free if it does not contain any graph from \mathcal{F} as an induced subgraph. If \mathcal{F} consists of a single graph F , then we write F -free instead of $\{F\}$ -free. Classes defined by forbidden induced subgraphs are *hereditary*, i.e., closed under vertex deletion. Conversely, every hereditary class of graphs can be uniquely characterized by a (possibly infinite) set of minimal forbidden induced subgraphs.

Let us define two families of graphs that play a special role as forbidden induced subgraphs. For $t \geq 1$, by P_t we denote the path in t vertices. For $a, b, c \geq 1$, by $S_{a,b,c}$ we mean the graph consisting of three paths $P_{a+1}, P_{b+1}, P_{c+1}$ with one endvertex identified. Each graph $S_{a,b,c}$ is called a *subdivided claw*. The smallest subdivided claw, i.e., $S_{1,1,1}$ is the *claw*, and the graph $S_{2,1,1}$ is sometimes called the *fork* or the *chair*. Finally, let \mathcal{S} denote the class of graphs in which every connected component is a path or a subdivided claw.

MIS and k -Coloring in F -free graphs. Two problems, whose complexity in hereditary graph classes attracts significant attention, are MAX (WEIGHTED) INDEPENDENT SET (denoted by MIS) and k -COLORING. Let us briefly survey known results, focusing on F -free graphs for connected F . By an observation of Alekseev [2], MIS is NP-hard in F -free graphs, unless $F \in \mathcal{S}$. On the positive side, polynomial-time algorithms are known for some small graphs $F \in \mathcal{S}$. If $F = P_t$, then a polynomial-time algorithm for $t \leq 5$ was provided by Lokshтанov et al. [37], which was later extended to $t \leq 6$ by Grzesik et al. [28]. The case of $t = 7$ remains open and the general belief is that for every t the problem is polynomial-time solvable. Some evidence is given by the existence of quasipolynomial-time algorithms [25, 45].

The polynomial-time algorithm for MIS in claw-free graphs [41, 46] can be obtained by an extension of the augmenting path approach used for finding largest matchings [17]; note that a maximum matching is precisely a largest independent set in the line graph, and line graphs are in particular claw-free. There are also more modern approaches, based on certain decompositions of claw-free graphs [19, 42]. A polynomial-time algorithm for MIS in $S_{2,1,1}$ -free graphs was first obtained by Alekseev [3] (only for the unweighted case), and later an arguably simpler algorithm was provided by Lozin and Milanič [39] (also for the weighted case). Again, the complexity of MIS in F -free graphs for larger subdivided claws F remains open, but all these cases are believed to be tractable. This belief is supported by the existence of a subexponential-time algorithm [12, 40], a QPTAS [12, 13, 40], or a polynomial-time algorithm in the bounded-degree case [1].

If it comes to k -COLORING, then it follows from known results that if F is not a forest of paths, then for every $k \geq 3$ the problem is NP-hard in F -free graphs [18, 26, 33, 36]. The complexity of k -COLORING in P_t -free graphs is quite well understood. For $t = 5$, the problem is polynomial-time solvable for every constant k [31]. If $k \geq 5$, then the problem is NP-hard already in P_6 -free graphs [34]. The case of $k = 4$ is polynomial-time solvable for $t \leq 6$ [47] and NP-hard for $t \geq 7$ [34]. The case of $k = 3$ is much more elusive. We know a polynomial-time algorithm for P_7 -free graphs [5] and the cases for all $t \geq 8$ are open. The general belief that they should be tractable is again supported by the existence of a quasipolynomial-time algorithm [45].

Let us mention two generalizations of k -COLORING. In the k -COLORINGEXT problem we are given a graph G with a subset of its vertices colored using k colors, and we ask whether this partial coloring can be extended to a proper k -coloring of G . In the even more

general LIST k -COLORING problem, each vertex of the instance graph G is equipped with a subset of $\{1, \dots, k\}$ called *list*, and we ask whether there exists a proper k -coloring of G respecting all lists. Clearly any tractability result for a more general problem implies the same result for a less general one, and any hardness result for a less general problem implies the same hardness for more general variants. In almost all mentioned cases the algorithms for k -COLORING generalize to LIST k -COLORING. The only exception is the case $k = 4$ and $t = 6$: the polynomial time algorithm for 4-COLORING in P_6 -free graphs can be generalized to 4-COLORINGEXT [47], but LIST 4-COLORING in this class is NP-hard [27]

Minimal obstructions. One of the ways of designing polynomial-time algorithms for k -COLORING is to check if the instance graph contains some (hopefully small) subgraph that is not k -colorable. This approach is formalized by the notion of *critical graphs*. A graph G is $(k + 1)$ -*vertex critical* if it is not k -colorable, but each of its induced subgraphs is. Such graphs can be thought of *minimal obstructions* to k -coloring: a graph G is k -colorable if and only if it does not contain any $(k + 1)$ -vertex-critical graph as an induced subgraph. Thus if for some hereditary class \mathcal{G} of graphs, the number of $(k + 1)$ -vertex critical graphs is finite, we immediately obtain a polynomial-time algorithm for k -COLORING in graphs from \mathcal{G} . Indeed, it is sufficient to check if the instance graph contains any $(k + 1)$ -vertex-critical induced subgraph, which can be done in polynomial time by brute force. Such an algorithm, in addition to solving the instance, provides a *certificate* in case of a negative answer – a constant-size subgraph which does not admit a proper k -coloring. Thus the question whether for some class \mathcal{G} , the number of $(k + 1)$ -vertex critical graphs is finite, can be seen as a refined analysis of the polynomial cases of k -COLORING.

The finiteness of the families of $(k + 1)$ -vertex critical graphs in F -free graphs is fully understood. Recall that the only interesting (i.e., not known to be NP-hard) cases are for F being a forest of paths. Again focusing on connected F , i.e., k -COLORING of P_t -free graphs, we know that the families of minimal obstructions are finite for $t \leq 6$ and $k = 3$ [9], and for $t \leq 4$ and any k . The latter result follows from the fact that P_4 -free graphs are perfect and thus the only obstruction for k -coloring is K_{k+1} . In all other cases there are constructions of infinite families of minimal obstructions [9, 32].

Graph homomorphisms in F -free graphs. A homomorphism from a graph G to a graph H is a mapping from $V(G)$ to $V(H)$ that preserves edges, i.e., the image of every edge of G is an edge of H . Note that if H is the complete graph on k vertices, then homomorphisms to $H = K_k$ are exactly proper k -colorings. For this reason homomorphisms to H are called *H -colorings*, and we will also refer to vertices of H as *colors*. In the H -COLORING problem the graph H is fixed and we need to decide whether an instance graph G admits a homomorphism to H . By the analogy to coloring, we also define more general variants, i.e., H -COLORINGEXT and LIST H -COLORING. In the former problem we ask whether a given partial mapping from vertices of an instance graph G to $V(H)$ can be extended to a homomorphism, and in the latter one each vertex of G is equipped with a list which is a subset of $V(H)$ and we look for a homomorphism respecting all lists.

The complexity of H -COLORING was settled by Hell and Nešetřil [30]: the problem is polynomial-time solvable if H is bipartite or has a vertex with a loop, and NP-hard otherwise. The dichotomy is also known for LIST H -COLORING [22–24]: this time the tractable cases are the so-called *bi-arc graphs*. The case of H -COLORINGEXT is more tricky. The classification follows from the celebrated proof of the CSP complexity dichotomy [7, 49], but a graph-theoretic description of polynomial cases is unknown.

We are very far from understanding the complexity of variants of H -COLORING in F -free graphs. Chudnovsky et al. [10] showed that LIST C_k -COLORING for $k \in \{5, 7\} \cup [9, \infty)$ is polynomial-time solvable in P_9 -free graphs. On the negative side, they showed that for every $k \geq 5$ the problem is NP-hard in F -free graphs, unless $F \in \mathcal{S}$. This negative result was later extended by Piecyk and Rzażewski [44] who showed that if H is not a bi-arc graph, then LIST H -COLORING is NP-hard and cannot be solved in subexponential time (assuming the ETH) in F -free graphs, unless $F \in \mathcal{S}$.

The case of forbidden path or subdivided claw was later investigated by Okrasa and Rzażewski [43]. They defined a class of *predacious graphs* and showed that if H is not predacious, then for every t , the LIST H -COLORING problem can be solved in quasipolynomial time in P_t -free graphs. On the other hand, for every predacious H there exists t for which LIST H -COLORING cannot be solved in subexponential time in P_t -free graphs unless the ETH fails. They also provided some partial results for the case of forbidden subdivided claws. Finally, Chudnovsky et al. [11] considered a generalization of LIST H -COLORING in P_5 -free graphs and related classes.

The notion of vertex-critical graphs can be naturally translated to H -colorings. A graph G is a *minimal H -obstruction* if it is not H -colorable, but its every induced subgraph is. Minimal H -obstructions in restricted graph classes were studied by some authors, but the results are rather scattered [4, 8, 35].

Our motivation. Let us point a substantial difference between working with H -COLORING and working with LIST H -COLORING (with H -COLORINGEXT being somewhere between, but closer to H -COLORING). The LIST H -COLORING problem enjoys certain monotonicity: if H' is an induced subgraph of H , then every instance of LIST H' -COLORING can be seen as an instance of LIST H -COLORING, where no vertex from $V(H) - V(H')$ appears in any list. Thus any tractability result for LIST H -COLORING implies the analogous result for LIST H' -COLORING, while any hardness result for LIST H' -COLORING applies also to LIST H -COLORING. In particular, all hardness proofs for LIST H -COLORING follow the same pattern: first we identify a (possibly infinite) family \mathcal{H} of “minimal hard cases” and then show hardness of LIST H' -COLORING for every $H' \in \mathcal{H}$. This implies that LIST H -COLORING is hard unless H is \mathcal{H} -free. The lists are also useful in the design of algorithms: for example if for some reason we decide that some vertex $v \in V(G)$ must be mapped to $x \in V(H)$, we can remove from the lists of neighbors of v all non-neighbors of x , and then delete v from the instance graph. This combines well with e.g. branching algorithms or divide-&-conquer algorithms based on the existence of separators.

In contrast, when coping with H -COLORING we need to think about the global structure of H . This makes working with this variant of the problem much more complicated. In particular, hardness proofs often employ certain algebraic tools [6, 30], which in turn do not combine well with the world of F -free graphs. However, note that the complexity dichotomy for H -COLORING is *still monotone* with respect to taking induced subgraphs of H : the minimal NP-hard cases are odd cycles.

An interesting example of non-monotonicity of H -COLORING was provided by Feder and Hell in an unpublished manuscript [21]. For an odd integer $k \geq 5$, let W_k denote the *k -wheel*, i.e., the graph obtained from the k -cycle by adding a universal vertex. Feder and Hell proved that W_5 -COLORING is polynomial-time solvable in line graphs. This is quite surprising as W_5 contains a triangle and K_3 -COLORING, i.e., 3-COLORING, is NP-hard in line graphs [33]. (Note that this implies that LIST W_5 -COLORING is NP-hard in line graphs.) Feder and Hell also proved that for any odd $k \geq 7$, the W_k -COLORING problem is NP-hard in line graphs [21].

Our contribution. In this paper we study to which extent the result of Feder and Hell [21] can be generalized. We provide an algorithm and a number of lower bounds, each of a different kind. The main algorithmic contribution of our paper is the following theorem.

► **Theorem 1.** W_5 -COLORINGEXT can be solved in polynomial time in $S_{2,1,1}$ -free graphs.

Let us sketch the outline of the proof. Surprisingly, despite the fact that graph homomorphisms generalize colorings, our approach is much closer to the algorithms for MIS.

Consider any homomorphism φ from G to W_5 and let X be the set of vertices of G mapped by φ to the universal vertex of W_5 . We notice that X is independent, and $G - X$ admits a homomorphism to C_5 . This is exactly how we look at the problem: we aim to find an independent set X whose removal makes the graph C_5 -colorable (and make sure that the precoloring of vertices is respected).

So let us focus on describing the structure of $G - X$, i.e., recognizing C_5 -colorable graphs. Note that here we need to use the fact that our graph is $S_{2,1,1}$ -free, as C_5 -COLORING is NP-hard in general graphs. As a warm-up let us assume that our instance is claw-free and forget about precolored vertices. We observe that every C_5 -colorable graph must be triangle-free, as there is no homomorphism from K_3 to C_5 . But since $G - X$ is $\{S_{1,1,1}, K_3\}$ -free, it must be of maximum degree at most 2, i.e., every component of $G - X$ is a path or a cycle with at least 4 vertices. It is straightforward to verify that such graphs always admit a homomorphism to C_5 . Therefore in claw-free graphs, solving W_5 -COLORING boils down to finding an independent set intersecting all triangles.

We extend this simple observation in two ways. First, we show that the same phenomenon occurs in $S_{2,1,1}$ -free graphs: the only $S_{2,1,1}$ -free minimal C_5 -obstruction is the triangle. Second, we show that in the same way we can handle precolored vertices: the only no-instances of C_5 -COLORINGEXT in $\{S_{2,1,1}, K_3\}$ -free graphs can be easily recognized. Based on this result we show that W_5 -COLORINGEXT in $S_{2,1,1}$ -free graphs can be in polynomial time reduced to the INDEPENDENT TRIANGLE TRANSVERSAL EXTENSION (ITTE) problem on an induced subgraph of the original instance. Here, the “extension” means that some vertices of our instance can be prescribed to be in X or outside X . Thus from now on we focus on solving ITTE in $S_{2,1,1}$ -free graphs. Note that we still need to use the fact that our instances are $S_{2,1,1}$ -free, as ITTE is NP-hard in general graphs [20].

We start with the case that our instance graph G is claw-free. We use the result of Chudnovsky and Seymour [14] who show that each claw-free graph admits certain decomposition called a *strip structure*. Roughly speaking, this means that G “resembles” the line graph of some graph D : the vertices of G can be partitioned into sets $\eta(e)$ assigned to the edges e of D , such that (i) for each $e \in E(D)$ the set $\eta(e)$ induces a subgraph of G with a simple structure and (ii) the interactions between sets $\eta(e)$ and $\eta(f)$ for distinct edges e, f are well-defined. Due to property (i), for each edge e of D we can solve the problem in the subgraph of G induced by $\eta(e)$ in polynomial time. Then we can use property (ii) to combine these partial solutions into the final one by finding an appropriate matching in an auxiliary graph derived from D .

As the final step, we lift our algorithm for claw-free graphs to the class of $S_{2,1,1}$ -free graphs. We use an observation of Lozin and Milanič [39]: they show that if G is $S_{2,1,1}$ -free and *prime*, then every *prime* induced subgraph of the graph obtained from G by removing any vertex and its neighbors is claw-free (the exact definition of prime graphs can be found in Section 3.1). Using this observation we can reduce ITTE in $S_{2,1,1}$ -free graphs to the same problem in claw-free graphs, which we already know how to solve in polynomial time.

Combining the reduction from W_5 -COLORINGEXT in $S_{2,1,1}$ -free graphs to ITTE in $S_{2,1,1}$ -free graphs and the polynomial-time algorithm for ITTE in $S_{2,1,1}$ -free graphs we obtain Theorem 1. Let us point out that the frontier of the complexity of W_5 -COLORINGEXT in $S_{a,b,c}$ -free graphs is the same as for MIS: the minimal open cases are $S_{3,1,1}$ and $S_{2,2,1}$.

In the remainder of the paper we investigate several possible generalizations of Theorem 1 and show a number of negative results.

First, one can ask whether a simpler algorithm, at least for W_5 -COLORING, can be obtained by showing that the family of minimal $S_{2,1,1}$ -free W_5 -obstructions is finite. In the following theorem we show that this is not the case.¹

► **Theorem 2** (☆). *For all odd $k \geq 5$, there are infinitely many claw-free minimal W_k -obstructions.*

We remark that for every k , the LIST W_k -COLORING problem can be solved in polynomial time in P_5 -free graphs [11]. We show that this result cannot be extended to P_t -free graphs for every fixed t : there exists t such that W_5 -COLORING in P_t -free graphs is NP-hard and cannot be solved in subexponential time, unless the ETH fails. Note that this result in particular implies hardness in $S_{a,a,a}$ -free graphs for $a = \lceil (t-1)/2 \rceil$. For W_5 -COLORINGEXT we show that it is NP-hard and, assuming the ETH, there is no subexponential-time algorithm already in $S_{3,3,3}$ -free graphs of maximum degree 5. This should be contrasted with the fact that for any a, b, c , the MIS problem in $S_{a,b,c}$ -free graphs can be solved in subexponential time [12, 40], and even in polynomial time, if the instance is of bounded maximum degree [1]. Consequently, the complexity of W_5 -COLORING in $S_{a,b,c}$ -free graphs for large a, b, c differs from the complexity of MIS. Furthermore, we find our hardness results quite surprising, as typically problems that are hard in $S_{a,b,c}$ -free graphs for some fixed a, b, c are already hard in claw-free graphs.

Finally, we consider the complexity of variants of W_k -COLORING in F -free graphs for other pairs (k, F) . Our results are summarized in the following theorem.

► **Theorem 3** (☆). *Let F be a connected graph.*

1. *There exist a, b, c, t with the following property. If F is neither an induced subgraph of $S_{a,b,c}$ nor of P_t , then the W_5 -COLORING problem is NP-hard in F -free graphs and cannot be solved in subexponential time, unless the ETH fails.*
2. *For every odd $k \geq 7$ there exists t with the following property. If F is not an induced subgraph of P_t , then the W_k -COLORING problem is NP-hard in F -free graphs and cannot be solved in subexponential time, unless the ETH fails.*

► **Remark.** The curious reader might wonder why we only consider k -wheels for odd $k \geq 5$. The 3-wheel is exactly K_4 , and homomorphisms to K_4 are exactly proper 4-colorings. As mentioned above, both 4-COLORING and 4-COLORINGEXT are well studied in hereditary graph classes and behave substantially differently than W_k -COLORING for odd $k \geq 5$. On the other hand, if k is even, then the W_k -COLORING problem is equivalent to the 3-COLORING problem: a graph admits a homomorphism to W_k if and only if it is 3-colorable (this follows from the fact that K_3 is the core of W_k [29, Section 1.4]). Thus it only makes sense to consider variants of W_k -COLORING in F -free graphs, where F is a path or a forest of paths. However, as W_k is non-predacious, it follows from the result of Okrasa and Rzażewski [43] that then even LIST W_k -COLORING is quasipolynomial-time solvable in F -free graphs.

¹ The full proofs of the statements marked with (☆) can be found in the full version of the paper [16].

2 Preliminaries

For an integer k , by $[k]$ we denote the set $\{1, 2, \dots, k\}$.

Let G be a graph, v be a vertex and X be a set of vertices. By $N_G(v)$ we denote the set of neighbors of v , and by $N_G[v]$ we denote the set $N_G(v) \cup \{v\}$. If G is clear from the context, we omit the subscript, and write, respectively, $N(v)$, and $N[v]$. By $G[X]$ we denote the subgraph induced by X , and by $G - X$ we denote $G[V(G) - X]$. By \overline{G} we denote the complement of G .

We write $\varphi : G \rightarrow H$ to indicate that φ is a homomorphism from G to H . We also write $G \rightarrow H$ to indicate that some homomorphism from G to H exists.

For a fixed graph H , an instance of H -COLORINGEXT is a triple (G, U, φ) , where G is a graph, U is a subset of $V(G)$, and φ is a function that maps vertices from U to elements of $V(H)$. We ask whether there exists a homomorphism $\psi : G \rightarrow H$ such that $\psi|_U = \varphi$.

For a k -wheel W_k , we will always denote the consecutive vertices of the induced k -cycle in W_k by $1, 2, 3, \dots, k$, and the universal vertex by 0 . The following observation is straightforward and will be used implicitly throughout the paper.

► **Observation 4.** *Let $k \geq 5$ be an odd integer. Let φ be a homomorphism from a graph G to W_k . Let X be the set of vertices of G mapped by φ to 0 . Then the following properties are met: (i) G is 4-colorable, (ii) G is K_4 -free, (iii) X is an independent set, and (iv) $G - X$ has a homomorphism to C_k , in particular it is 3-colorable and triangle-free.*

For a graph G , a function $\mathfrak{w} : E(G) \rightarrow \mathbb{N} \cup \{0\}$, and a set $E' \subseteq E(G)$, we define $\mathfrak{w}(E') := \sum_{e \in E'} \mathfrak{w}(e)$.

Consider a certain variant of the MAXIMUM WEIGHT MATCHING problem. An instance of MAXIMUM WEIGHT MATCHING* (MWM*) is a tuple (G, U, \mathfrak{w}, k) , where G is a graph, U is a subset of its vertices, $\mathfrak{w} : E(G) \rightarrow \mathbb{N} \cup \{0\}$ is an edge weight function, and k is an integer. We ask whether G has a matching M such that $\mathfrak{w}(M) \geq k$ and M covers all vertices from U . By a simple reduction to the MAXIMUM WEIGHT MATCHING problem we show that MWM* can be solved in polynomial time.

► **Lemma 5** (\star). *The MWM* problem can be solved in polynomial time.*

3 ITTE in $S_{2,1,1}$ -free graphs

In this section we show that in $S_{2,1,1}$ -free graphs the W_5 -COLORINGEXT problem can be reduced to a variant of the problem of finding an independent set intersecting all triangles.

We denote the consecutive vertices of C_5 by $1, 2, 3, 4, 5$. We will refer to the vertices of C_5 as colors. We will say that two colors are *neighbors* if they are neighbors on the cycle C_5 .

Let G be a graph, let $W \subseteq V(G)$, and let $\varphi : W \rightarrow V(C_5)$ be a coloring of vertices of W . We say that a pair of vertices $\{u, v\} \subseteq V(G)$ is *conflicted* if $u, v \in W$ and there is a u - v path P of length at most 3 such that $W \cap V(P) = \{u, v\}$ and $\varphi|_{\{u, v\}}$ cannot be extended to a homomorphism from P to C_5 . Equivalently, a pair $\{u, v\} \subseteq W$ of vertices of G is conflicted in a coloring φ of G if and only if

- (i) $v \in N_G(u)$, and $\varphi(v)$ is non-adjacent to $\varphi(u)$ in C_5 , or
- (ii) there exists a path u, w, v in G with $w \notin W$, and $\varphi(v)$ is adjacent to $\varphi(u)$, or
- (iii) there exists a path u, w_1, w_2, v in G with $w_1, w_2 \notin W$, and $\varphi(u) = \varphi(v)$.

We say that φ is *conflict-free*, if there is no pair of conflicted vertices in G .

Clearly, being triangle-free and conflict-free are necessary conditions to be a yes-instance of C_5 -COLORINGEXT. It turns out that in $S_{2,1,1}$ -free graphs they are also sufficient.

► **Lemma 6** (\star). *Let G be an $\{S_{2,1,1}, K_3\}$ -free graph, let $W \subseteq V(G)$, and let $\varphi : W \rightarrow V(C_5)$. If φ is conflict-free, then φ can be extended to a homomorphism $\psi : G \rightarrow C_5$.*

The following auxiliary problem plays a crucial role in our algorithm.

INDEPENDENT TRIANGLE TRANSVERSAL EXTENSION (ITTE)

Instance: A graph G , sets $X', Y' \subseteq V(G)$, and $E' \subseteq E(G)$

Question: Is there an independent set $X \subseteq V(G)$, such that (i) $X' \subseteq X$, (ii) for every $e \in E'$ it holds that $e \cap X \neq \emptyset$, (iii) $Y' \cap X = \emptyset$, (iv) $G - X$ is triangle-free?

We observe that if G contains a K_4 , then at most one vertex of such a clique can belong to an independent set, and the graph induced by the remaining part is not triangle-free. Thus every yes-instance of ITTE must be K_4 -free. We use this observation implicitly throughout the paper.

As the final result of this section we show that W_5 -COLORINGEXT in $S_{2,1,1}$ -free graphs can be reduced in polynomial time to ITTE in $S_{2,1,1}$ -free graphs.

► **Theorem 7.** *The W_5 -COLORINGEXT in $S_{2,1,1}$ -free graphs can be reduced in polynomial time to the ITTE problem in $S_{2,1,1}$ -free graphs.*

Proof. Let (G, U, φ) be an instance of W_5 -COLORINGEXT. Note that we can assume that there are no vertices $u, v \in U$ such that $uv \in E(G)$ and $\varphi(u)\varphi(v) \notin E(W_5)$, otherwise we immediately report a no-instance (formally, we can return a trivial no-instance of ITTE, e.g., $(K_4, \emptyset, \emptyset, \emptyset)$). This can be verified in polynomial time.

We define the instance (G, X', Y', E') of ITTE as follows. We initialize X', Y' and E' as empty sets. We add to X' every vertex precolored with 0 and we add every vertex precolored with a vertex other than 0 to Y' . To construct E' , consider each pair $u, v \in U$ of conflicted vertices with respect to $\varphi : Y' \rightarrow [5]$, and let P be a witness of u and v . By our first assumption, $|P| \geq 2$. If the consecutive vertices of P are u, w, v , then we add w to X' . If the consecutive vertices of P are u, w_1, w_2, v , then we add the edge w_1w_2 to E' . This completes the construction of the instance (G, X', Y', E') of ITTE. Clearly the reduction is done in polynomial time.

Let us verify that the instance (G, X', Y', E') of ITTE is equivalent to the instance (G, U, φ) of W_5 -COLORINGEXT. First assume that there is a set $X \subseteq V(G)$ that is a solution to the instance (G, X', Y', E') of ITTE. Then $G - X$ is triangle-free. Suppose that there is a conflicted pair of vertices u, v in $G - X$ and let P be the path such that the precoloring of u, v cannot be extended to P . Observe that by the construction of E' at least one vertex of P must be in X , a contradiction. Hence, by calling Lemma 6 on $G - X$, we conclude that the precoloring of $G - X$ can be extended to all vertices of $G - X$ using only colors 1, 2, 3, 4, 5, and then extended to the whole graph G by coloring every vertex of X with 0.

So now suppose that (G, U, φ) is a yes-instance of W_5 -COLORINGEXT. Then there exists a W_5 -coloring ψ of G that extends φ . Define $X := \psi^{-1}(0)$. Let us verify that X satisfies the desired properties. It follows from the definition that X is an independent set. Suppose that $G - X$ contains a triangle. Then $G - X \not\rightarrow C_5$, and thus the vertices of $G - X$ cannot be colored using only colors 1, 2, 3, 4, 5, a contradiction. Consider a vertex $x \in X'$. Then either $\varphi(x) = 0$, or there is a path with consecutive vertices u, x, v with $u, v \in U$, such that φ cannot be extended to x using only colors 1, 2, 3, 4, 5. Therefore in both cases $\psi(x) = 0$, so $x \in X$. Now consider $y \in Y'$. Then $\varphi(y) \neq 0$, so $y \notin \psi^{-1}(0) = X$. Finally, consider an edge $xy \in E'$. Then there is a path with consecutive vertices u, x, y, v with $u, v \in U$, so that φ cannot be extended to x, y using only colors 1, 2, 3, 4, 5. We conclude that one of x, y is mapped by ψ to 0, and thus one of x, y is in X . That completes the proof. ◀

3.1 ITTE: basic toolbox

In this section we prove that ITTE behaves well under standard graph decompositions: modular decomposition, clique-cutset decomposition, and tree decomposition.

Modular decomposition and prime graphs. Let G be a graph and let $M \subsetneq V(G)$. We say that M is a *module* of G if for every vertex $v \in V(G) - M$ either v is adjacent to any vertex of M or is non-adjacent to every vertex of M . We say that a module M is *non-trivial* if $|M| > 1$, otherwise M is *trivial*. We say that G is *prime* if every module of G is trivial.

In the next lemma we show that ITTE combines well with modular decompositions.

► **Lemma 8** (☆). *Let \mathcal{X} be a hereditary class of graphs and let \mathcal{X}^* be the class of all induced subgraphs of the graphs in \mathcal{X} that are either prime or cliques. If ITTE can be solved in polynomial time on \mathcal{X}^* , then it can be solved in polynomial time on \mathcal{X} .*

Clique cutsets and atoms. A (possibly empty) set $C \subseteq V(G)$ is a *cutset* in G , if $V(G) - C$ is disconnected. We say that C is a *clique cutset* if C is a cutset and a clique. We say that G is an *atom* if it does not contain clique cutsets. Note that, in particular, every atom is connected. A triple (A, C, B) is a *clique cutset partition* if it is a partition of $V(G)$ such that C is a clique cutset, there is no edge between A and B , and $G[A \cup C]$ is an atom.

We show that we can reduce ITTE to instances (G, X', Y', E') such that G is an atom. The algorithm is a standard recursion that exploits the existence of clique cutsets [48].

► **Lemma 9** (☆). *Let \mathcal{X} be a family of graphs, and let \mathcal{X}^a be the family of all atoms in \mathcal{X} . If ITTE can be solved in polynomial time in \mathcal{X}^a , then ITTE can be solved in polynomial time in \mathcal{X} .*

Bounded-treewidth graphs. *Monadic Second-Order Logic* (MSO_2) over graphs consists of formulas with vertex variables, edge variables, vertex set variables, and edge set variables, quantifiers, and standard logic operators. We also have a predicate $\text{inc}(v, e)$, indicating that the vertex v belongs to the edge e .

For a graph G , let $\text{tw}(G)$ denote the treewidth of G . The classic result of Courcelle [15] asserts that problems that can be expressed in MSO_2 can be efficiently solved on graphs of bounded treewidth. As ITTE can be expressed in such a way, we obtain the following.

► **Corollary 10** (☆). *ITTE can be solved in polynomial time on bounded-treewidth graphs.*

4 Solving ITTE in claw-free graphs

Let G be a connected graph. A *strip structure* of G is a pair (D, η) that consists of a simple graph D , a set $\eta(xy) \subseteq V(G)$ for every $xy \in E(D)$, and its non-empty subsets $\eta(xy, x), \eta(xy, y) \subseteq \eta(xy)$, satisfying the following conditions:

- (S1) $|E(D)| \geq 3$, and there are no vertices of degree 2 in D ,
- (S2) the set $\{\eta(e) : e \in E(D)\}$ forms a partition of $V(G)$,
- (S3) if $u \in \eta(e), v \in \eta(f)$, for some $e, f \in E(D)$, then $uv \in E(G)$ if and only if there exists $x \in V(D)$ such that e and f are incident to x , $u \in \eta(e, x)$ and $w \in \eta(f, x)$,
- (S4) for every $x \in V(D)$, the set $\bigcup_{y \in N(x)} \eta(xy, x)$ induces a clique in G .

Chudnovsky and Seymour [14] proved that every claw-free graph G is either very simple or admits a strip structure where the subgraphs induced by vertices assigned to a single edge have “simple” structure. If G is additionally of bounded maximum degree, then this result becomes particularly useful. The following corollary of the result of Chudnovsky and Seymour comes from the paper of Abrishami et al. [1, Corollary 3.5].

► **Theorem 11** (Chudnovsky, Seymour [14]). *If G is a connected claw-free graph with maximum degree at most Δ , then either $\text{tw}(G) \leq 4\Delta + 3$, or G admits a strip structure (D, η) such that for every $e \in E(D)$, we have $\text{tw}(G[\eta(e)]) \leq 4\Delta + 4$. Moreover, (D, η) can be found in time polynomial in $|V(G)|$.*

The main result of this section is the following theorem.

► **Theorem 12.** *ITTE can be solved in polynomial time in claw-free graphs.*

Proof. Let (G, X', Y', E') be an instance of ITTE such that G is claw-free. By Lemma 9, we can assume that G is an atom, and, in particular, G is connected.

Recall that we can safely assume that G is K_4 -free, as otherwise we deal with a no-instance. This implies that $\Delta(G) \leq 5$. Indeed, if G has a vertex v of degree at least six, then v either has three pairwise non-adjacent neighbors (and hence a claw), or three pairwise adjacent neighbors (and hence a K_4).

Since G is claw-free and $\Delta(G) \leq 5$, by Theorem 11, either (i) $\text{tw}(G) \leq 23$ or (ii) there exists a strip structure (D, η) such that for every $e \in E(D)$ we have $\text{tw}(G[\eta(e)]) \leq 24$. By Corollary 10, if (i) holds, it can be checked in polynomial time if G is a yes-instance. Therefore, we will assume that (ii) holds. By Theorem 11 the strip structure (D, η) can be computed in polynomial time. As G is connected, so is D .

Recall that by the definition of a strip structure, if y_1, \dots, y_d are neighbors of some $x \in V(D)$, the set $\bigcup_{i \in [d]} \eta(xy_i, x)$ induces a clique in G . Hence, as G is K_4 -free, we have $\Delta(D) \leq 3$, which implies that the vertices of D are either of degree 1 or 3. Moreover, if x is of degree 3, let $N(x) = \{y_1, y_2, y_3\}$ and note that since G is K_4 -free, $|\eta(xy_1, x)| = |\eta(xy_2, x)| = |\eta(xy_3, x)| = 1$. In this case, we denote the single member of $\eta(xy, x)$ by $v_{x,y}$.

On the other hand, if D contains a vertex y of degree 1, two things can happen: either D is isomorphic to K_2 , or y is adjacent to a vertex x that is of degree 3 in D . In the first case, if we denote by e the edge of D , we have $\text{tw}(G) = \text{tw}(G[\eta(e)]) \leq 24$, so again, we can solve the problem in polynomial time by Corollary 10. In the second case, let y_1, y_2 be the other (than y) neighbors of x in D . We observe that $\{v_{x,y_1}, v_{x,y_2}\}$ or $\{v_{x,y}\}$ is a clique cutset in G , a contradiction with G being an atom. Hence, we can assume that D is 3-regular.

For $x \in V(D)$ and $y_1, y_2, y_3 \in N_D(x)$, let $E(x) := \{v_{x,y_1}v_{x,y_2}, v_{x,y_2}v_{x,y_3}, v_{x,y_3}v_{x,y_1}\}$. Recall that by the definition of a strip structure, $G[E(x)]$ is a triangle. The following reduction rule is straightforward.

► **Reduction Rule 12.1.** *If $E(x) \subseteq E'$, then (G, X', Y', E') is a no-instance. If $v_{x,y_1}v_{x,y_2}, v_{x,y_2}v_{x,y_3} \in E'$, but $v_{x,y_3}v_{x,y_1} \notin E'$, we can safely add v_{x,y_2} to X' and remove $v_{x,y_1}v_{x,y_2}, v_{x,y_2}v_{x,y_3}$ from E' . Hence, for every $x \in V(D)$ it holds that $|E(x) \cap E'| \leq 1$.*

Consider an edge xy of D . Let $G_{xy} := G[\eta(xy)]$, and let $I_{xy} = \{v_{x,y}, v_{y,x}\} - Y'$. Let $\mathcal{A}(xy)$ be the (possibly empty) set of these sets A that satisfy $X' \cap I_{xy} \subseteq A \subseteq I_{xy}$ and

$$\mathcal{I}_A = (G_{xy}, (X' \cap V(G_{xy})) \cup A, (Y' \cap V(G_{xy})) \cup (I_{xy} - A), E' \cap E(G_{xy}))$$

is a yes-instance of ITTE. Observe that if there exists a solution X of ITTE for (G, X', Y', E') , then $X \cap \eta(xy)$ is a solution to $\mathcal{I}_{A'}$ for $A' = \{v_{x,y}, v_{y,x}\} \cap X$, and in this case $A' \in \mathcal{A}(xy)$. We obtain the following.

► **Reduction Rule 12.2.** *If $\mathcal{A}(xy) = \emptyset$ for some $xy \in E(D)$, then (G, X', Y', E') is a no-instance.*

Our aim is to reduce the ITTE problem to the MWM* problem. Let $\mathcal{E} := \{xy \in E(D) : \emptyset \notin \mathcal{A}(xy)\}$. We construct an instance $(D', U, \mathfrak{w}, |\mathcal{E}|)$ of MWM* as follows. To obtain D' , we take a copy of the set $V(D)$, and for every $xy_1 \in E(D)$ we introduce an additional vertex t_{xy} . Then, for every $xy \in E(D)$, we do the following (below we always use the notation $N_D(x) = \{y, y', y''\}$).

- (1) if $\{v_{x,y}, v_{y,x}\} \in \mathcal{A}(xy)$, we add xy to $E(D')$,
- (2) if $\{v_{x,y}\} \in \mathcal{A}(xy)$, we add xt_{xy} to $E(D')$,
- (3) if $\{v_{y,x}\} \in \mathcal{A}(xy)$, we add yt_{xy} to $E(D')$.
- (4) if $\{v_{x,y'}v_{x,y''}\} = E' \cap E(x)$, we remove xy and xt_{xy} from $E(D')$ (if they were introduced).

Let $U = V(D) \subseteq V(D')$, i.e., the set of original vertices of D . For $xy \in E(D)$, we define $T(xy) := \{xy, xt_{xy}, yt_{xy}\} \cap E(D')$. We set $\mathfrak{w} : E(D') \rightarrow \{0, 1\}$ to be $\mathfrak{w}(uv) = 1$ if $uv \in T(xy)$ for some $xy \in \mathcal{E}$, and $\mathfrak{w}(uv) = 0$ otherwise. This concludes the construction of $(D', U, \mathfrak{w}, |\mathcal{E}|)$.

By the definition of the strip structure, each edge $uv \in V(G)$ is either contained in $E(G_{xy})$, for some $xy \in E(D)$, or $u = v_{x,y'}$ and $v = v_{x,y''}$ for some $x \in U$. In the first case, we say that uv is of *type I* for xy , in the latter – that uv is of *type II* for $\{xy', xy''\}$. Similarly, each triangle $uvt \in V(G)$ is either contained in $E(G_{xy})$, for some $xy \in E(D)$ (so it is of *type I* for xy), or $u = v_{x,y}$, $v = v_{x,y'}$ and $t = v_{x,y''}$ for some $x \in U$ (so it is of *type II* for x).

▷ Claim 13 (☆). (G, X', Y', E') is a yes-instance of ITTE if and only if $(D', U, \mathfrak{w}, |\mathcal{E}|)$ is a yes-instance of MWM*.

Sketch of proof of Claim. There is a correspondence between a matching M which is a solution to $(D', U, \mathfrak{w}, |\mathcal{E}|)$ and a set X which is a solution to (G, X', Y', E') . Let us present the intuition.

Consider $xy \in E(D)$ and note that $X \cap \eta(xy)$ is a solution of the instance induced by $\eta(xy)$. Note that the only way how such a partial solution interacts with the rest of the graph is by including (or not) vertices $v_{x,y}$ and $v_{y,x}$ to X . Each of the four possibilities is reflected by the way how M intersects $T(xy)$ (see the definition of $E(D')$). The requirement that M is of weight at least $|\mathcal{E}|$ means that for each $xy \in E(D)$ such that $\emptyset \notin \mathcal{A}(xy)$ we have to choose some edge from $T(x, y)$ to M . This is crucial as for these edges there are no solutions of ITTE restricted to $\eta(xy)$ containing neither $v_{x,y}$ nor $v_{y,x}$.

Finally, the set U is used to ensure that the partial solutions chosen for distinct edges of D are compatible with each other. It enforces that for each $x \in V(D)$, the set X intersects the triangle $\bigcup_{xy \in E(D)} \eta(xy, x)$. ◀

Therefore, for an instance $(G, X', Y', |\mathcal{E}|)$ of ITTE, we create an equivalent (by Claim 13) instance $(D', U, \mathfrak{w}, |\mathcal{E}|)$ of MWM*. Then we solve $(D', U, \mathfrak{w}, |\mathcal{E}|)$.

It remains to estimate the running time. We check in polynomial time whether G contains a K_4 . As for each $xy \in E(D)$, the set $\eta(xy)$ is non-empty, $|E(D)| \leq n$. For every $xy \in E(D)$ we compute $\mathcal{A}(xy)$, which requires solving at most four instances $\mathcal{I}_\emptyset, \mathcal{I}_{\{v_{x,y}\}}, \mathcal{I}_{\{v_{y,x}\}}, \mathcal{I}_{\{v_{x,y}, v_{y,x}\}}$. Each of them consists of a graph G_{xy} with $\text{tw}(G_{xy}) \leq 24$ (by Theorem 11), hence this can also be done in polynomial time. Finally, the instance $(D', U, \mathfrak{w}, |\mathcal{E}_N|)$ of MWM* can be constructed and solved in polynomial time by Lemma 5. ◀

5 Solving ITTE in $S_{2,1,1}$ -free graphs

In this section we generalize the algorithm from Theorem 12 to the class of $S_{2,1,1}$ -free graphs. The main combinatorial tool is the following theorem used to solve MIS in this class [38].

► **Theorem 14** (Lozin, Milanič [38, Theorem 4.1]). *Let G be an $S_{2,1,1}$ -free prime graph and let $v \in V(G)$. Let G' be an induced prime subgraph of $G - N[v]$. Then G' is claw-free.*

Equipped with Theorem 14, we can present our algorithm.

► **Theorem 15.** *ITTE can be solved in polynomial time in $S_{2,1,1}$ -free graphs.*

Proof. Let (G, X', Y', E') be an instance of ITTE such that G is $S_{2,1,1}$ -free. By calling Lemma 8 for the class of $S_{2,1,1}$ -free graphs, it is enough to consider the case that G is either a prime $S_{2,1,1}$ -free graph or a clique. If $Y' = V(G)$, then we can verify in polynomial time if $X = \emptyset$ is a solution. So since now we assume that Y' is a proper subset of $V(G)$. Observe that if (G, X', Y', E') is a yes-instance, then there exists a solution X that is non-empty. Indeed, for a solution $X = \emptyset$, we can safely add a vertex $v \in V(G) - Y'$.

For every $v \notin Y'$ we define the instance $\mathcal{I}_v := (G, X' \cup \{v\}, Y', E')$. Now (G, X', Y', E') is a yes-instance if and only if there exists $v \notin Y'$ for which \mathcal{I}_v is a yes-instance. Consider one fixed $v \notin Y'$. Clearly, for any solution X , we have that $X \cap N(v) = \emptyset$, so if there is a vertex in $N(v) \cap X'$ or there is an edge in E' with both endpoints in $N(v)$, we report a no-instance.

Define $G_v := G - N[v]$. We initialize $X'' := X'$ and $Y'' := Y' - N[v]$, and let E'' contain these edges from E' that have both endpoints in $V(G_v)$. Consider a triangle xyz . If $x \in N(v)$ and $y, z \in V(G_v)$ we add yz to E'' . If $x, y \in N(v)$ and $z \in V(G_v)$ we add z to X'' . Finally, for every edge $uw \in E'$ with $u \in N(v)$ and $w \in V(G_v)$, we add w to X'' . Hence, it is enough to focus on the instance (G_v, X'', Y'', E'') , as it is clearly equivalent to the instance \mathcal{I}_v . Note that this reduction can be performed in polynomial time.

We call again Lemma 8, now for the class $\mathcal{Y} := \{G_w \mid G \in \mathcal{X}, w \in V(G)\}$, so it is enough to solve the problem for the class \mathcal{Y}^* of all induced subgraphs of the graphs in \mathcal{Y} that are either prime or cliques. Clearly, every clique is claw-free. Together with Theorem 14, it implies that every graph in \mathcal{Y}^* is claw-free. By Theorem 12, ITTE can be solved in polynomial time on \mathcal{Y}^* , and therefore it can be solved in polynomial time on \mathcal{Y} . Since $G_v \in \mathcal{Y}$, the instance (G_v, X'', Y'', E'') can be solved in polynomial time. ◀

Combining Theorem 7 with Theorem 15 we obtain our main algorithmic result.

► **Theorem 1.** *W_5 -COLORINGEXT can be solved in polynomial time in $S_{2,1,1}$ -free graphs.*

6 W_5 -ColoringExt in $S_{3,3,3}$ -free graphs is hard

In this section we prove the following hardness result.

► **Theorem 16** (\star). *The W_5 -COLORINGEXT problem is NP-hard in $S_{3,3,3}$ -free graphs of maximum degree at most 5. Furthermore, it cannot be solved in time $2^{o(n)}$, where n is the number of vertices of the input graph, unless the ETH fails.*

Sketch of proof. Let G be an instance of 3-COLORING such that G is a claw-free graph of maximum degree at most 4 [33]. Note that we can assume that the minimum degree of G is at least 3. Since G is claw-free, this means that every vertex of G belongs to a triangle.

We construct an instance (\tilde{G}, U, φ) of W_5 -COLORINGEXT. We initialize $V(\tilde{G}) := V(G)$, $E(\tilde{G}) := E(G)$, and $U := \emptyset$. For each $v \in V(G)$ we proceed as follows. We add to $V(\tilde{G})$ vertices x_v, y_v, z_v and to $E(\tilde{G})$ we add $x_v y_v, y_v z_v$, and $y_v v$. Moreover we add x_v, z_v to U and set $\varphi(x_v) := 0$ and $\varphi(z_v) := 4$.

Note that the gadget attached to every vertex v of G simulates imposing the list $\{0, 1, 2, 4\}$. However, recall that every vertex of G belongs to a triangle, so it will never receive color 4. Thus (\tilde{G}, U, φ) is a yes-instance of W_5 -COLORINGEXT if and only if G is 3-colorable.

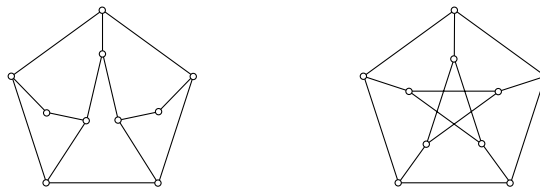
It is straightforward to verify that \tilde{G} is $S_{3,3,3}$ -free: the central vertex of a hypothetical $S_{3,3,3}$ must be a vertex of G , and at least one leg of $S_{3,3,3}$ must be fully contained in the gadget attached to v . Thus this path cannot have three vertices. ◀

7 Conclusion

Let us conclude the paper with pointing out some open questions and directions for future research.

Variants of W_5 -Coloring in $S_{a,b,c}$ -free graphs. Recall that W_5 -COLORINGEXT is polynomial-time solvable in $S_{2,1,1}$ -free graphs but NP-hard in $S_{3,3,3}$ -free graphs. We point out that this leaves an infinite family of open cases, and the minimal ones are $S_{2,2,1}$ and $S_{3,3,3}$.

Initial research shows that Lemma 6 can probably be extended to $S_{2,2,1}$ -free graphs, at least without precolored vertices. Thus there is hope to solve W_5 -COLORING by a reduction to ITTE. However, an analogue of Lemma 6 does not hold for $S_{2,2,2}$ -free and for $S_{3,1,1}$ -free graphs; see Figure 1.



■ **Figure 1** An $\{S_{2,2,2}, K_3\}$ -free (left) and an $\{S_{3,1,1}, K_3\}$ -free (right) graph that are not C_5 -colorable.

Of course this does not mean that some other approach cannot work for W_5 -COLORING. However, as we show in Theorem 3, the problem becomes hard if we exclude some long subdivided claw. This leads to a natural question about the boundary between easy and hard cases.

► **Question 1.** For which a, b, c are W_5 -COLORING and W_5 -COLORINGEXT polynomial-time solvable in $S_{a,b,c}$ -free graphs?

Minimal obstructions. Recall that by Theorem 2 and Theorem 3, the only connected graphs F for which we can hope for a finite family of F -free minimal W_k -obstructions are paths. This leads to the following question.

► **Question 2.** For which k and t is the family of P_t -free minimal W_k -obstructions finite?

Let us point out that for all k , the number of P_4 -free minimal W_k -obstructions is finite. Indeed, P_4 -free graphs are perfect. Thus if G is P_4 -free and does not contain K_4 , then G is 3-colorable (and thus it admits a homomorphism to W_k). On the other hand, K_4 is a minimal graph that does not admit a homomorphism to W_k . Concluding, K_4 is the only P_4 -free (even P_3 -free) W_k -obstruction.

On the negative side, recall that there exists an infinite family of P_7 -free 4-vertex-critical graphs [9]. An inspection of this family shows that these graphs are minimal W_k -obstructions for every odd $k \geq 5$. Note that for each such graph G , the fact that an induced subgraph G' of G has a proper 3-coloring implies that G' has a homomorphism to W_k . However, it still needs to be verified that G itself does not admit a homomorphism to W_k .

Thus the open cases in Question 2 are $t = 5$ and $t = 6$.

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