Graph Product Structure for h-Framed Graphs

Michael A. Bekos
University of Ioannina, Greece

Giordano Da Lozzo
Roma Tre University, Rome, Italy

Petr Hliněný
Masaryk University, Brno, Czech Republic

Michael Kaufmann
Universität Tübingen, Germany

Abstract

Graph product structure theory expresses certain graphs as subgraphs of the strong product of much simpler graphs. In particular, an elegant formulation for the corresponding structural theorems involves the strong product of a path and of a bounded treewidth graph, and allows to lift combinatorial results for bounded treewidth graphs to graph classes for which the product structure holds, such as to planar graphs [Dujmović et al., J. ACM, 67(4), 22:1-38, 2020].

In this paper, we join the search for extensions of this powerful tool beyond planarity by considering the h-framed graphs, a graph class that includes 1-planar, optimal 2-planar, and k-map graphs (for appropriate values of h). We establish a graph product structure theorem for h-framed graphs stating that the graphs in this class are subgraphs of the strong product of a path, of a planar graph of treewidth at most 3, and of a clique of size $3\lfloor\frac{h}{2}\rfloor + \lfloor\frac{h}{3}\rfloor - 1$. This allows us to improve over the previous structural theorems for 1-planar and k-map graphs. Our results constitute significant progress over the previous bounds on the queue number, non-repetitive chromatic number, and p-centered chromatic number of these graph classes, e.g., we lower the currently best upper bound on the queue number of 1-planar graphs and k-map graphs from 115 to 82 and from $\lfloor\frac{33}{2}(k+3\lfloor\frac{k}{2}\rfloor-3)\rfloor$ to $\lfloor\frac{33}{2}(3\lfloor\frac{k}{2}\rfloor+\lfloor\frac{k}{3}\rfloor-1)\rfloor$, respectively. We also employ the product structure machinery to improve the current upper bounds on the twin-width of 1-planar graphs from $O(1)$ to 80. All our structural results are constructive and yield efficient algorithms to obtain the corresponding decompositions.

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1 Introduction

Graph product structure theory [15] was recently introduced and is receiving considerable attention, as it gives deep insights that allow a host of mathematical and algorithmic tools to be applied. Despite being a relatively new development, it is having significant impact [25]. Initially, it was introduced to settle a long-standing conjecture by Heath, Leighton and Rosenberg [21] related to the queue number of planar graphs [15]. Recently, it has been further exploited to solve several other combinatorial problems that were open for years, e.g., it was used to prove that planar graphs have bounded non-repetitive chromatic number [15], to improve the best known bounds for p-centered colorings of planar graphs and graphs excluding any fixed graph as a subdivision [11], to find shorter adjacency labelings of planar graphs [7], and to find asymptotically optimal adjacency labelings of planar graphs [13].

In its simplest form, the product structure theorem states that every planar graph is a subgraph of the strong product of a path and of a planar graph of treewidth at most 6 [15, 27]. The bound on the treewidth can be improved by allowing more than two graphs in the strong product, as it is known that every planar graph is a subgraph of the strong product of a path, of a 3-cycle and of a planar graph of treewidth at most 3 [15]. These theorems are attractive, since they describe planar graphs in terms of graphs of bounded treewidth, which are considered much simpler than the planar ones. Furthermore, they enable combinatorial results that hold for graphs of bounded treewidth to be generalised for planar graphs and, more generally, for graphs where similar structural theorems can be obtained. On the algorithmic side, it has been recently shown that the graphs involved in the product structure theorem can be computed in linear time [10], improving upon [24].

Analogous results are known for graphs of bounded Euler genus [15], apex-minor-free graphs [15], graphs with bounded degree in proper minor-closed classes [14], and graphs in non-minor closed classes [17]; see [19] for a survey. Related to our work are the structural theorems for k-planar and k-map graphs (the former ones are the graphs that can be drawn with at most k crossings per edge, whereas the latter ones are the contact-graphs of regions homeomorphic to closed disks such that at most k regions may share the same point). In particular, it is known that every k-planar graph is a subgraph of the strong product of a path, of a graph of treewidth at most $\frac{1}{6}(k+4)(k+3)(k+2)-1$, and of a clique on $18k^2+48k+30$ vertices, while every k-map graph is a subgraph of the strong product of a path, of a planar graph of treewidth at most 3, and of a clique on $k+3\lfloor\frac{k}{2}\rfloor-3$ vertices [17].

Our contribution. In this work, our focus is on the class of h-framed graphs, which were recently introduced as a notable subclass of k-planar and a superclass of k-map graphs (for appropriate values of k) [5]; a graph is *h-framed*, if it admits a drawing on the Euclidean plane whose uncrossed edges induce a biconnected spanning plane graph with faces of size at most h. Since any h-framed graph is $O(h^2)$-planar, it follows from the aforementioned product structure theorem for k-planar graphs that every h-framed graph is a subgraph of a path, of a graph with treewidth $O(h^6)$ and of a clique of size $O(h^4)$. Our main contribution is to show the following structural result (which guarantees that the graph of bounded treewidth is planar and of improved treewidth, from $O(h^6)$ to $O(1)$, and lowers the size of the involved clique to $O(h)$): Every h-framed graph is a subgraph of the strong product of a path, of a planar graph of treewidth at most 3, and of a clique on $3\lfloor\frac{h}{2}\rfloor+\lfloor\frac{h}{3}\rfloor-1$ vertices[1];

[1] Note that, almost concurrently with our result, a slightly weaker version of our product structure theorem for h-framed graphs appeared in [17], where the size of the involved clique is larger, namely, $h+3\lfloor\frac{h}{2}\rfloor-3$.

Table 1 Previous [17] and new bounds on the queue number, non-repetitive and p-centered chromatic number, and twin-width for h-framed, 1-planar, optimal 2-planar, and k-map graphs. We denote by $\chi_p(H)$ the p-centered chromatic number of planar 3-trees.

		h-framed/h-map	**1-planar**	**opt 2-planar**
queue num	old	$\lfloor\frac{33}{2}(h+3\lfloor\frac{h}{2}\rfloor-3)\rfloor$	115	132
	new	$\lfloor\frac{33}{2}(3\lfloor\frac{h}{2}\rfloor+\lfloor\frac{h}{3}\rfloor-1)\rfloor$	82	82
non-repetitive chr. num	old	$4^4\cdot(h+3\lfloor\frac{h}{2}\rfloor-3)$	1792	2048
	new	$4^4\cdot(3\lfloor\frac{h}{2}\rfloor+\lfloor\frac{h}{3}\rfloor-1)$	1536	1536
p-centered chr. num	old	$(h+3\lfloor\frac{h}{2}\rfloor-3)(p+1)\chi_p(H)$	$7(p+1)\chi_p(H)$	$8(p+1)\chi_p(H)$
	new	$(3\lfloor\frac{h}{2}\rfloor+\lfloor\frac{h}{3}\rfloor-1)(p+1)\chi_p(H)$	$6(p+1)\chi_p(H)$	$6(p+1)\chi_p(H)$
twin-width	old	–	$O(1)$	$O(1)$
	new	$17h+13$	80	80

see Theorem 3.1. Note that, since any planar graph is a subgraph of some triangulation (and thus of a 3-framed graph), we have that, for $h = 3$, Theorem 3.1 coincides with the product structure theorem for planar graph proved in [15]. This is an indication that our theorem may be tight for the graphs in this family. Furthermore, we provide an alternative formulation, where the role played by a path is instead played by the $\lfloor h/2\rfloor$-th power of a path, which allows us to further reduce the size of the clique involved in the product to $\max(3, h-2)$; see Theorem 3.5. All our structural results are constructive and yield efficient algorithms to obtain the corresponding decompositions. Our techniques provide improved upper bounds on the queue number, on the non-repetitive chromatic number, and on the p-centered chromatic number of h-framed graphs that are linear in h; see Theorem 4.2, Corollary 4.7, and Lemma 4.8, respectively. Finally, by extending the product structure machinery, we are able to give an efficient construction to obtain an explicit, linear in h, upper bound on the twin-width of h-framed graphs, while the currently best explicit upper bound derives from the one for k-planar graphs and it is hence exponential in $O(h^2)$ [8, 9]; see Theorem 5.1.

Consequences on related graph classes. Since 1-planar and optimal 2-planar graphs are subgraphs of 4- and 5-framed graphs, respectively [2, 4], and since k-map graphs are subgraphs of k-framed graphs [3, 5], the product structure theorems mentioned above imply significant improvements on the currently best bounds for the following parameters (refer to Table 1). For definitions, see Section 4.

- **Queue number:** Using Theorem 3.1, we improve the best known upper bound on the queue number of k-map graphs from $\lfloor\frac{33}{2}(k+3\lfloor\frac{k}{2}\rfloor-3)\rfloor$ [17] to $\lfloor\frac{33}{2}(3\lfloor\frac{k}{2}\rfloor+\lfloor\frac{k}{3}\rfloor-1)\rfloor$ (Corollary 4.3), whereas, using Theorem 3.5, we lower the best known upper bounds on the queue number of 1-planar and optimal 2-planar graphs from 115 and 132 [17], respectively, both to 82 (Theorem 4.5).
- **Non-repetitive chromatic number:** Theorem 3.1 allows us to improve the best known upper bound on the non-repetitive chromatic number of k-map graphs from $4^4\cdot(k+3\lfloor\frac{k}{2}\rfloor-3)$ [17] to $4^4\cdot(3\lfloor\frac{k}{2}\rfloor+\lfloor\frac{k}{3}\rfloor-1)$. In particular, for the class of 1-planar graphs our improvement is from 1792 to 1536 (Corollary 4.7). The latter is a bound that notably holds also for optimal 2-planar graphs, which forms an improvement over the one of 2048 that was previously known [17].

For the sake of completeness, even though the results in [17] are still unpublished, we will compare the bounds stemming from our product structure theorem with those stemming from the one in [17].

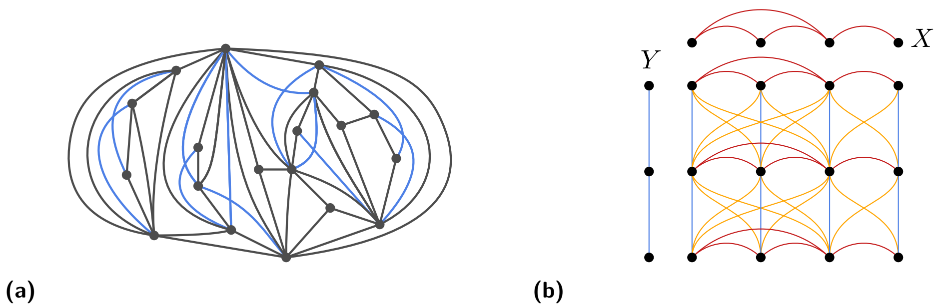


Figure 1 Illustration of: (a) a 4-framed topological graph whose skeleton edges (crossing edges) are black (blue), and (b) the strong product $X \boxtimes Y$ of a planar graph X (red) and a path Y (blue).

- ***p*-centered chromatic number:** Theorem 3.1 allows us to improve the best known upper bound on the non-repetitive chromatic number of k-map graphs from $(k+3\lfloor\frac{k}{2}\rfloor-3)(p+1)\chi_p(H)$ [17] to $(3\lfloor\frac{k}{2}\rfloor+\lfloor\frac{k}{3}\rfloor-1)(p+1)\chi_p(H)$, where $\chi_p(H) \leq (p+1)(p\lceil\log(p+1)+2p+1)$ denotes the p-centered chromatic number of planar 3-trees [11]. In particular, we lower the best known upper bounds on the p-centered chromatic number of 1-planar and optimal 2-planar graphs from $7(p+1)\chi_p(H)$ and $8(p+1)\chi_p(H)$ [17], respectively, both to $6(p+1)\chi_p(H)$ (Corollary 4.9).
- **Twin-width:** Theorem 5.1 improves the currently best upper bound on the twin-width of 1-planar and optimal 2-planar graphs, from $O(1)$ [8] to 80, whereas our improvement for k-map graphs is limited to certain value of k, as these graphs have bounded twin-width independently of k [8].

2 Preliminaries

For standard graph-theoretic terminology and notation we refer the reader, e.g., to [12].

Graphs. A graph is *simple* if it contains neither loops nor multi-edges. For a general graph G (not-necessarily simple), let $\mathrm{si}(G)$ denote the *simplification* of G, i.e., the simple graph obtained from G by removing all loops and replacing each bunch of parallel edges with a single edge. For any $i \geq 1$, the *i-th power* G^i of a graph G is the graph with the same vertex set as G, in which two vertices are adjacent if and only if they are at distance at most i in G. Clearly, $G \subseteq G^i$. A graph H is a *minor* of a graph G, if H can be obtained from a subgraph of G by contracting edges.

Topological graphs. A *topological graph* is a graph drawn on the plane such that any two edges cross in at most one point and no edge crosses itself. In this paper, we will solely consider topological graphs in which no two adjacent edges cross and no three edges cross in the same point (in the literature, such drawings are commonly referred to as "simple"). A *plane graph* is a topological graph with no crossing edges. A graph is *k-planar* if it is isomorphic to a topological graph in which each edge crosses at most k other edges. Furthermore, a k-planar graph with the maximum number of edges is called *optimal*. A *k-map graph* is one that admits a *k-map*, i.e., a representations where each vertex is a region homeomorphic to a closed disk, such that regions have pairwise disjoint interiors, no more than k regions share the same boundary point, and two regions touch if and only if the corresponding vertices are adjacent.

Given a topological graph G, the subgraph $\mathrm{sk}(G)$ of G consisting of all its vertices and uncrossed edges is the *skeleton* of G; refer to Fig. 1a. A topological graph G whose skeleton $\mathrm{sk}(G)$ is biconnected is called *h-framed* [5], if all the faces of $\mathrm{sk}(G)$ have size at most h, and *internally h-framed*, if all the faces of $\mathrm{sk}(G)$, except for possibly one, have size at most h. The importance of this class lies in the following connections with k-planar and k-map graphs [3, 5]. Optimal 1-planar and optimal 2-planar graphs are 4- and 5-framed, respectively, while general 1-planar graphs can be augmented to 8-framed graphs, if multi-edges are forbidden, or to 4-framed graphs, if multi-edges are allowed. Finally, note that any k-map graph is a subgraph of a k-framed (multi-)graph and of a $2k$-framed simple graph [3, 5].

Treewidth. Let $(\mathcal{X}, T)$ be a pair such that $\mathcal{X} = \{X_1, X_2, \dots, X_\ell\}$ is a collection of subsets of vertices of a graph G, called *bags*, and T is a tree whose nodes are in one-to-one correspondence with the elements of $\mathcal{X}$. The pair $(\mathcal{X}, T)$ is a *tree-decomposition* of G if it satisfies the following two conditions: (i) for every edge (u, v) of G, there exists a bag $X_i \in \mathcal{X}$ that contains both u and v, and (ii) for every vertex v of G, the set of nodes of T whose bags contain v induces a non-empty subtree of T. The *width* of a tree-decomposition $(\mathcal{X}, T)$ of G is $\max_{i=1}^{\ell} |X_i| - 1$, while the *treewidth* $\mathrm{tw}(G)$ of G is the minimum width over all tree-decomposition of G.

Quotient graph. For a graph G and a partition $\mathcal{P}$ of $V(G)$, the *quotient of G by $\mathcal{P}$*, denoted by $G/\mathcal{P}$, is a graph containing a vertex v_P for each part P in $\mathcal{P}$ (we say that *v_P stems from P*) and an edge $(v_{P'}, v_{P''})$ if and only if there exists a vertex in P' adjacent to a vertex in P'' in G. Note that, $G/\mathcal{P}$ is a minor of G, if every part in $\mathcal{P}$ induces a connected subgraph of G.

Strong product. The *strong product* of two graphs X and Y, denoted by $X \boxtimes Y$, is the graph whose vertex set $V(X \boxtimes Y)$ is the Cartesian product $V(X) \times V(Y)$, such that there exists an edge in $E(X \boxtimes Y)$ between the vertices $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in V(X \boxtimes Y)$ if and only if one of the following occurs: (a) $x_1 = x_2$ and $(y_1, y_2) \in E(Y)$, (b) $y_1 = y_2$ and $(x_1, x_2) \in E(X)$, or (c) $(x_1, x_2) \in E(X)$ and $(y_1, y_2) \in E(Y)$; see Fig. 1b. Dujmović et al. [15, 17] and Ueckerdt, Wood, and Yi [27] showed the following main graph product structure results.

▶ **Theorem 2.1** (Dujmović et al. [15, 17], Ueckerdt et al. [27]). *For a graph G, the next hold:*

(a) *If G is planar, then $G \subseteq P \boxtimes H$, for a path P and a planar graph H with $\mathrm{tw}(H) \leq 6$.*
(b) *If G is planar, then $G \subseteq P \boxtimes H \boxtimes K_3$, for a path P and a planar graph H with $\mathrm{tw}(H) \leq 3$.*
(c) *If G is 1-planar, then $G \subseteq P \boxtimes H \boxtimes K_7$, for a path P and a planar graph H with $\mathrm{tw}(H) \leq 3$.*
(d) *If G is k-planar with $k > 1$, then $G \subseteq P \boxtimes H \boxtimes K_{18k^2+48k+30}$, for a path P and a graph H with $\mathrm{tw}(H) \leq \frac{1}{6}(k+4)(k+3)(k+2) - 1$.*
(e) *If G is a k-map graph, then $G \subseteq P \boxtimes H \boxtimes K_{21k(k-3)}$, for a path P and a graph H with $\mathrm{tw}(H) \leq 9$.*

Layering. Consider a graph G. A *layering* of G is an ordered partition $(V_0, V_1, \dots)$ of $V(G)$ such that, for every edge (v, w) of G with $v \in V_i$ and $w \in V_j$, it holds $|i - j| \leq 1$. If $i = j$, then (v, w) is an *intra-level edge*; otherwise, (v, w) is an *inter-level edge*. Each part V_i is called a *layer*. Let T be a BFS tree of G rooted at a vertex r. The *BFS layering* of G *determined by* r is the layering $(V_0, V_1, \dots)$ of G such that V_i contains all vertices of G at distance i from r. Given a partition $\mathcal{P}$ of $V(G)$ and a layering $\mathcal{L}$ of G, the *layered width of $\mathcal{P}$ with respect to $\mathcal{L}$* is the size of the largest set obtained by intersecting a part in $\mathcal{P}$ and a layer in $\mathcal{L}$. The *layered width* of $\mathcal{P}$ is the minimum layered width of $\mathcal{P}$ over all layerings of G.

3 Computing the Product Structure

This section is devoted to the proof of a product structure theorem for h-framed graphs, summarized in the next theorem; several applications of this result are presented in Section 4.

▶ **Theorem 3.1** (Product Structure Theorem for h-Framed Graphs)**.** *Let G be a not-necessarily simple h-framed graph with $h \geq 3$. Then, $\mathrm{si}(G)$ is a subgraph of the strong product $H \boxtimes P \boxtimes K_{3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1}$, where H is a planar graph with $\mathrm{tw}(H) \leq 3$ and P is a path.*

The algorithm supporting Theorem 3.1 is going to recursively decompose the graph G into parts with special properties, such that the resulting quotient graph will be H, and the additional properties of the constructed partition will imply the claimed product structure. We start with a technical setup followed by the core recursion in Lemma 3.2.

Layering G. Let T be a BFS tree of $\mathrm{sk}(G)$ rooted at an arbitrary vertex r incident to the unbounded face of $\mathrm{sk}(G)$. For an arbitrary H and its implicitly fixed BFS tree $T' \subseteq H$ (such as $H = \mathrm{sk}(G)$ and $T' = T$ in our case), we call a path $P \subseteq \mathrm{sk}(G)$ *vertical* if P is a subpath of some root-to-leaf path of T'. Let $\mathcal{L} = (V_0, V_1, \dots, V_b)$ be the BFS layering of $\mathrm{sk}(G)$ determined by r. Observe that, if P is a vertical path in $\mathrm{sk}(G)$, then P intersects every part of $\mathcal{L}$ in at most one vertex. Given $\mathcal{L}$, we define a new ordered partition $\mathcal{W} = (W_0, W_1, \dots, W_\ell)$ of the vertex set of G with $\ell = \lceil b / \lfloor \frac{h}{2} \rfloor \rceil - 1$, by merging consecutive $\lfloor \frac{h}{2} \rfloor$-tuples of layers of $\mathcal{L}$. This is done as follows. For $i = 0, 1, \dots, \ell$, we let $W_i := \bigcup_{j=0}^{\lfloor h/2 \rfloor - 1} V_{i \lfloor h/2 \rfloor + j}$ (assuming $V_x = \emptyset$ if $x > b$). Then, $\mathcal{W} := (W_0, W_1, \dots, W_\ell)$ is a layering of G; see [6].

Partitioning G. The core of our algorithm is a construction of a special partition $\mathcal{R}$ of $V(G)$ such that $H = G/\mathcal{R}$ is a planar graph with $\mathrm{tw}(H) \leq 3$, and the layered width of $\mathcal{R}$ with respect to $\mathcal{W}$ is not large. Our recursive decomposition of G is analogous to the one in [15] (as applied to planar graphs); however several non-trivial changes are needed to exploit the existence of the underlying (plane) skeleton of G. The algorithm starts from the unbounded face and recursively "dives" into gradually-shrinking areas of G.

Central in our approach is the following notion. For a cycle $C \subseteq \mathrm{sk}(G)$, the *subgraph of G bounded by C*, denoted by G_C, is the subgraph of G formed by the vertices and edges of C and the vertices and edges of G drawn inside C. Consider a subset $U \subseteq V(G)$. For the partition $\mathcal{L}$ (resp., the partition $\mathcal{W}$), the *width of U with respect to $\mathcal{L}$* (resp. *to* $\mathcal{W}$), denoted by $\lambda_{\mathcal{L}}(U)$ (resp. by $\lambda_{\mathcal{W}}(U)$), is the largest size of a set obtained by intersecting U and a part of $\mathcal{L}$ (resp. of $\mathcal{W}$). We are now ready to present our main technical lemma.

▶ **Lemma 3.2.** *Let G be an h-framed graph with $h \geq 3$ and let $\mathcal{L}$ be a BFS layering of G. Also, let C be a cycle in $\mathrm{sk}(G)$, and let G_C be the subgraph of G bounded by C. Further, for some $k \in \{1, 2, 3\}$, let $P_1, \dots, P_k$ be paths belonging to C such that $\mathcal{R}^0 = \{X_i : X_i := V(P_i), 1 \leq i \leq k\}$ is a partition of $V(C)$. Then, it is possible to construct in quadratic time a* good *partition $\mathcal{R}'$ of $V(G_C)$, i.e., one that satisfies the following properties:*

1. *$\mathcal{R}' \supseteq \mathcal{R}^0$, and for every part $X \in \mathcal{R}' \setminus \mathcal{R}^0$, there exist $q \in \{1, 2, 3\}$ and $X' \subseteq X$ such that*[2]
   - *$X \setminus X'$ is a union of the vertex sets of at most q vertical paths of $\mathrm{sk}(G)$, and so, in particular, $\lambda_{\mathcal{L}}(X \setminus X') \leq q$, and*
   - *$|X'| \leq h - 3$ if $q = 1$, $|X'| \leq \lfloor (h-1)/2 \rfloor - 1$ if $q = 2$, and $|X'| \leq \lfloor h/3 \rfloor - 1$ if $q = 3$.*

[2] The somehow technical Property 1 of Lemma 3.2 will imply that $\lambda_{\mathcal{W}}(X) \leq 3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1$ in the proof of Theorem 3.1, but we will also make use of the stated more detailed treatment.

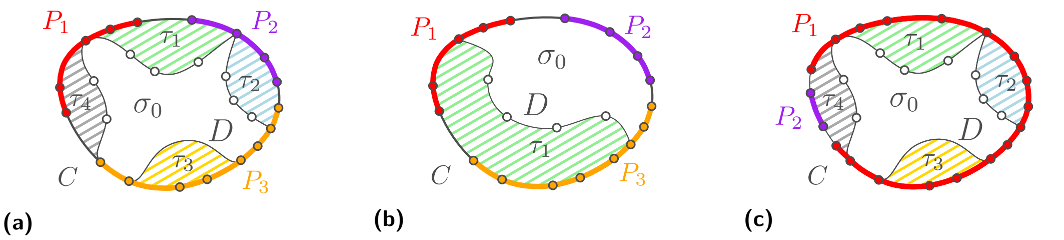


Figure 2 Illustrations of graph $C \cup D$ for the separable case of G_C (Definition 3.3). The number a of τ-faces is 4 in (a) and 1 in (b). (c) A case when Property 2 of Definition 3.3 is not met (by τ_4).

2. *the quotient graph $H' = G_C/\mathcal{R}'$ is a planar graph with* $\mathrm{tw}(H') \leq 3$*, and*
3. *the vertices of H' that stem from X_i, with $1 \leq i \leq k$, are incident to the same face of H' and induce a clique (i.e., either a vertex, or an edge, or a triangle).*

Proof. We prove Lemma 3.2 by providing a recursive procedure that we describe in the following. The base case of the recursion occurs when $V(G_C) = V(C)$ (i.e., there are no vertices in the interior of C and the edges in $E(G_C) \setminus E(C)$ are chords of C). In this case, the algorithm returns the partition $\mathcal{R}' = \mathcal{R}^0$, which is clearly good since the graph H' is a plane clique of size k whose vertices stem from the parts of $\mathcal{R}^0$. Note that, if $|E(G_C) \setminus E(C)| = 0$, then $V(G_C) = V(C)$, since G_C cannot have isolated vertices.

In the recursive step of the algorithm, we assume that there exist vertices and edges of G_C that lie in the interior of C. Our aim is to recurse on instances that contain fewer edges in the interior, but not on the boundary, of the cycle delimiting their unbounded face. We first need to handle a possible degenerate case[3] of G_C. Recall that, since G is h-framed, all bounded faces of $G_C \cap \mathrm{sk}(G)$ have length at most h.

▶ **Definition 3.3.** *We say that G_C is* separable *if the following conditions hold (see Fig. 2):*
1. *The plane graph $G_C \cap \mathrm{sk}(G)$ contains a bounded face σ_0 that intersects C in $a \geq 1$ disjoint maximal subpaths (some of the paths may consist of single vertices). Denoting by D the facial cycle of σ_0, let $\tau_1, \ldots, \tau_a$ be the bounded faces of the plane graph $C \cup D$ other than σ_0.*
2. *For each τ_j, with $j = 1, \ldots, a$, the boundary of τ_j is a cycle C_j of $\mathrm{sk}(G)$, at most two parts of $\mathcal{R}^0$ intersect C_j, and every part of $\mathcal{R}^0$ intersecting C_j does so in a single subpath.*[4]

Separable case. Suppose that G_C is separable and let G_j, $j = 1, \ldots, a$, denote the subgraph of G_C bounded by the facial cycle C_j of τ_j[5]. By definition, we have that $|E(G_j) \setminus E(C_j)| \subseteq |E(G_C) \setminus E(C)|$ (even when $a = 1$). Since $E(C_j) \setminus E(C) \neq \emptyset$, the latter implies $|E(G_j) \setminus E(C_j)| < |E(G_C) \setminus E(C)|$. Also, let Y denote the vertices of D that do not belong to C, i.e., $Y = V(D) \setminus V(C)$; refer to the hollow vertices of Fig. 2. By the previous, we have $|Y| \leq |D| - 2 \leq h - 2$.

For $j = 1, \ldots, a$, let $\mathcal{R}_j^0$ be the partition of $V(C_j)$ consisting of the set $Y_j = Y \cap V(C_j)$, if it is not empty, and of the sets $X_i^j = X_i \cap V(C_j)$, $i = 1, \ldots, k$, if X_i^j is not empty; by Property 2 of Definition 3.3, R_j^0 consists of at most three parts. Therefore, by a recursive application of our algorithm, each graph G_j, $j = 1, \ldots, a$, admits a good partition $\mathcal{R}_j \supseteq \mathcal{R}_j^0$ of $V(G_j)$.

[3] Such a case does not explicitly occur in the planar proof of [15], but the implicit case of a so-called [15] “tripod” with degenerate legs is analogous to what we are defining here.
[4] A part of $\mathcal{R}^0$ indeed may intersect the boundary C_j of τ_j in two subpaths (see Fig. 2c). This, however, can happen only if $k \leq 2$.
[5] Note that, if $a = 1$, we have $|V(D) \cap V(C)| \geq 2$, since otherwise the face τ_1 would not be bounded by a cycle as required by Property 2 of Definition 3.3.

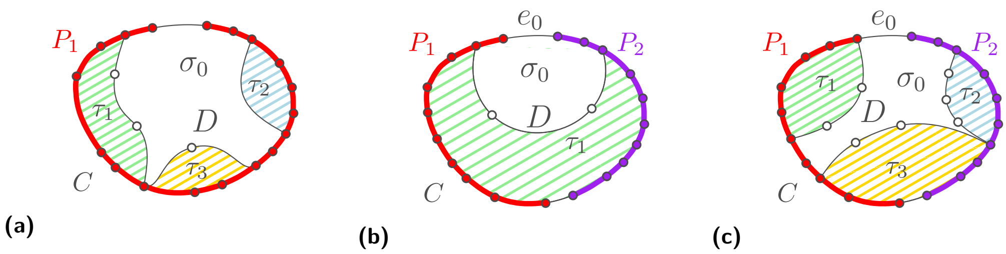


Figure 3 Illustrations of graph $C \cup D$ for the general case of G_C, for $k = 1$ (a) and $k = 2$ (b)(c).

We construct a partition $\mathcal{R}'$ of $V(G_C)$ by putting into $\mathcal{R}'$ the parts of $\mathcal{R}_0$, the set Y (if non-empty), and the recursively obtained parts of each G_j that do not touch C_j; formally, $\mathcal{R}' = \mathcal{R}_0 \cup \{Y\} \cup \bigcup_{j=1,\dots,a}(\mathcal{R}_j \setminus \mathcal{R}_j^0)$, or $\mathcal{R}' = \mathcal{R}_0 \cup \bigcup_{j=1,\dots,a}(\mathcal{R}_j \setminus \mathcal{R}_j^0)$ if $Y = \emptyset$. Note that $\mathcal{R}'$ is indeed a partition of $V(G_C)$, since each vertex of G_C that lies in the interior of C must belong either to Y or to a part $X \in \mathcal{R}_j \setminus \mathcal{R}_j^0$, for some $j \in \{1, \dots, a\}$. We show that the constructed partition $\mathcal{R}'$ is good in [6].

General case. Now we move to the general (i.e., not-necessarily separable) case of G_C in Lemma 3.2. If $\mathbf{k} = \mathbf{1}$, we pick the bounded face σ_0 of $G_C \cap \mathrm{sk}(G)$ incident to the single edge of $E(C) \setminus E(P_1)$; refer to Fig. 3a. The face σ_0 then witnesses the separable case for G_C, by Definition 3.3, which is solved as above. If $\mathbf{k} = \mathbf{2}$, then we pick $e_0 \in E(C)$ as one of the edges joining P_1 and P_2 on C, and σ_0 as the bounded face of $G_C \cap \mathrm{sk}(G)$ incident to e_0; see Figures 3b and 3c. Then we are back to the separable case for G_C with σ_0, by Definition 3.3.

In the remainder, we assume $\mathbf{k} = \mathbf{3}$. First, we color every vertex v of G_C by the color $i \in \{1, 2, 3\}$ if the (unique) path in the BFS tree T from v to the root r hits $V(P_i)$ before possibly hitting other parts of $\mathcal{R}^0$. In particular, the vertices of P_i are colored i. Our aim is to find, in the plane graph $F := G_C \cap \mathrm{sk}(G)$, a bounded face σ_1 containing vertices of all the three colors on its boundary. There our arguments divert from those used in [15] – since F is generally not a near-triangulation, and we additionally need that the face σ_1 intersects the boundary cycle C at most once (which requires additional care). We exploit the following.

▷ Claim 3.4. **(*)** In the setting above, there exists a cycle R bounding a bounded face σ_1 of F, such that $V(R)$ contains all three of our colors, and R intersects C in at most one connected piece. Furthermore, the colors on R appear in three consecutive sections.

Next, consider the set $V(R) \cap V(C)$. If this set contains all three colors, then all three colors occur on the path $R_0 := C \cap R$, and one of them, say color 1, occurs only on internal vertices of R_0 (and nowhere else on C). In this case, the face σ_1 again witnesses the case of separable G_C with $a = 1$ which is solved as above. If, instead, the set $V(R) \cap V(C)$ does not contain all three colors, then we choose on R representatives – vertices $t_i \in V(R)$ of color i where $i = 1, 2, 3$, as in one of the following three possible cases of $V(R) \cap V(C)$ (refer to Fig. 4):

C.1 If $V(R) \cap V(C)$ contains two colors, say 1 and 2, we choose $t_1, t_2 \in V(R) \cap V(C)$ as neighbors on C and $t_3 \in V(R) \setminus V(C)$ arbitrarily; refer to Fig. 4a.

C.2 If $V(R) \cap V(C)$ contains one color, say 1, we choose $t_1 \in V(R) \cap V(C)$ arbitrarily and pick $t_2, t_3 \in V(R) \setminus V(C)$ such that $t_2t_3 \in E(R)$ (this is unique). Furthermore, up to symmetry between the colors 2 and 3, we may assume that the distance on R between t_2 and $V(C)$ is not smaller than the distance on R between t_3 and $V(C)$; refer to Fig. 4b.

C.3 If $V(R) \cap V(C) = \emptyset$, then, up to symmetry between the colors, we may assume that the color 3 occurs in $V(R)$ no more times than each of the colors 1 and 2. We then choose $t_3 \in V(R)$ arbitrarily (of color 3), and set t_1 and t_2 to the two (unique) vertices colored 1 and 2 on R that are neighbors of vertices of color 3 on R; refer to Fig. 4c.

For $i = 1, 2, 3$, let R_i denote the unique vertical path in T from t_i to $V(P_i)$; see Fig. 4. Note that some vertices t_i may lie on C, and then $R_i = t_i$ is a single-vertex path. Let Q be the subpath of R with the ends t_1 and t_2 and avoiding t_3. We define $R' \subseteq R$ as the subpath or cycle (in the case $R' = R$) obtained from R by deleting all internal vertices of Q. Finally, we set $R^+ := R' \cup R_1 \cup R_2 \cup R_3$ which is a connected subgraph of F (R^+ will play the same role here as the so-called tripods in [15]).

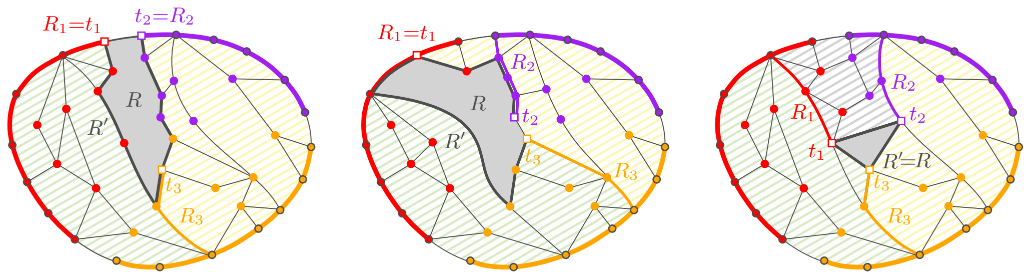
Observe that $C \cup R^+$ is a 2-connected plane graph (in each of the three cases above). Moreover, it contains $a \in \{2, 3\}$ bounded faces $\tau_1, \dots, \tau_a$, plus the bounded face σ_1 in the case of $R' = R$; for the latter see Fig. 4c. We denote by C_j, $j \in \{1, \dots, a\}$, the facial cycle of τ_j. It is now important to notice that each cycle C_j intersects at most two parts of $\mathcal{R}^0$, which follows from our "multi-colored" choice of t_1, t_2, t_3 and R_1, R_2, R_3 in all three cases. Furthermore, every two parts of $\mathcal{R}^0$ are together intersected by at most one of C_j.

We next proceed similarly as in the separable case above. Let $G_j \subsetneq G_C$, $j \in \{1, \dots, a\}$, be the strict subgraph of G_C bounded by C_j, and let $\mathcal{R}^0_j$ be the partition of $V(C_j)$ consisting of $V(C_j) \setminus V(C)$ and of the non-empty parts $X \cap V(C_j)$ over $X \in \mathcal{R}^0$. So, $|\mathcal{R}^0_j| \leq 3$. Therefore, by a recursive application of our algorithm, we may assume that each graph G_j admits a good partition $\mathcal{R}_j \supseteq \mathcal{R}^0_j$ of $V(G_j)$, with $j = 1, \dots, a$.

We construct a partition $\mathcal{R}' \supseteq \mathcal{R}^0$ of $V(G_C)$ similarly as before; besides $\mathcal{R}^0$ we add the set $Z := V(R^+) \setminus V(C) \neq \emptyset$ as whole, and the recursively obtained parts of each G_j that do not touch C_j. Formally, $\mathcal{R}' = \mathcal{R}_0 \cup \{Z\} \cup \bigcup_{j=1,\dots,a} (\mathcal{R}_j \setminus \mathcal{R}^0_j)$. Note that $\mathcal{R}'$ is a partition of $V(G_C)$ – in particular, each vertex of G_C which is not on C must belong either to Z or to a part $X \in \mathcal{R}_j \setminus \mathcal{R}^0_j$, for some $j \in \{1, \dots, a\}$, by induction. We show that the constructed partition $\mathcal{R}'$ is good in [6].

We conclude the proof of Lemma 3.2 by discussing the time complexity of our algorithm, which follows the same ideas as the ones by Dujmovic et al. [15] to compute the decomposition deriving from their product structure theorem for n-vertex planar graphs[6]. To show that a

[6] Note that subsequent improvements have brought the running time of this procedure first to $O(n \log n)$ [24] and finally to $O(n)$ [10].



(a) Case 1. **(b)** Case 2. **(c)** Case 3.

Figure 4 Illustrations for the representatives t_1, t_2, and t_3, and the vertical paths R_1, R_2, and R_3. The vertices of R bound the gray shaded region. R' is depicted with black thick edges.

good partition of G_C can be obtained in $O(|V(G_C)|^2)$ time, it suffices to observe that the non recursive work needed to compute the graphs on which the recursive calls are applied can be easily implemented to run in $O(|V(G_C)|)$ time, by performing a visit of the planar skeleton of the input h-framed graph and of its BFS-tree (provided that G_C is a topological h-framed graph). Since the total number of recursive calls is at most linear in $|V(G_C)|$, the total running time is thus quadratic in $|V(G_C)|$. ◀

Proof of Theorem 3.1. Let C denote the cycle bounding the unbounded face of $\mathrm{sk}(G)$, which, by a possible homeomorphism of the sphere, may be assumed to satisfy $|V(C)| \geq 3$. Based on the BFS tree T of $\mathrm{sk}(G)$ rooted in a vertex $r \in V(C)$, we define the following partition $\mathcal{R}^0$ of C: We split C into a path P_1 only consisting of the vertex r, and two paths P_2 and P_3 of lengths at most $\lfloor \frac{h-1}{2} \rfloor$ and $\lfloor \frac{h}{2} \rfloor$, respectively. Then, we set $\mathcal{R}^0 = \{V(P_1), V(P_2), V(P_3)\}$ and apply the algorithm given in the proof of Lemma 3.2. This way we obtain a good partition $\mathcal{R}'$ of $V(G_C) = V(G)$ and graph $H' := G_C/\mathcal{R}'$ in $O(|V(G)|^2)$ time.

Note that, in general, $G_C \neq G$ as G may have edges drawn in the unbounded face (bounded by C) of $\mathrm{sk}(G)$. However, by setting $H = H'$, we guarantee all edges of G in the unbounded face of $\mathrm{sk}(G)$ are "captured", since the quotient graph H' anyway contains a triangle on the vertices that stem from $\mathcal{R}_0$. In fact, we have just obtained the graph H with the desired properties, i.e., H is planar and of $\mathrm{tw}(H) \leq 3$.

What remains to prove is that G indeed is a subgraph of the strong product $H \boxtimes P \boxtimes K_{3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1}$ for some path P. Recall that the number of layers of the layering $\mathcal{W}$ is $\ell + 1$, and that $\mathcal{W}$ was obtained by merging consecutive $\lfloor \frac{h}{2} \rfloor$-tuples of layers of $\mathcal{L}$. We set P to be the path on $\ell + 1$ vertices denoted in order by $p_0, p_1, \ldots, p_\ell$. To a vertex $v \in V(G)$, we assign the pair (t, p_i) where $t \in V(H)$ if t stems from the part of $\mathcal{R}'$ that v belongs to, and $v \in W_i \in \mathcal{W}$. This assignment is sound and unique.

If $vv' \in E(G)$ is any edge of G, and v and v' are assigned the pairs (t, p_i) and (t', p_j) as above, then $tt' \in E(H)$ or $t = t'$ since $H = G/\mathcal{R}'$ is the quotient graph, and $p_i p_j \in E(P)$ or $i = j$ since $\mathcal{W}$ is a layering of G. Using Property 1 of Lemma 3.2, we furthermore estimate, for every part $X \in \mathcal{R}'$ and its $X' \subseteq X$ (cf. Property 1),

$$\begin{aligned}\lambda_{\mathcal{W}}(X) &\leq |X'| + \lambda_{\mathcal{L}}(X \setminus X') \cdot \lfloor h/2 \rfloor \\ &\leq \max\big(h - 3 + \lfloor h/2 \rfloor,\ \lfloor h/2 \rfloor - 1 + 2\lfloor h/2 \rfloor,\ \lfloor h/3 \rfloor - 1 + 3\lfloor h/2 \rfloor\big) \\ &\leq 3\lfloor h/2 \rfloor + \lfloor h/3 \rfloor - 1\,,\end{aligned}$$

and hence at most $3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1$ vertices of G are assigned to the same pair (t, p_i). This concludes the proof that $\mathrm{si}(G) \subseteq H \boxtimes P \boxtimes K_{3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1}$. ◀

We next present a variant of Theorem 3.1, which reduces the size of the clique in the product by replacing the path with a power of it. The variant does not immediately follow from the statement of Theorem 3.1. However, it can easily be derived by adopting in the proof of Theorem 3.1 the layering $\mathcal{L}$ instead of the layering $\mathcal{W}$.

▶ **Theorem 3.5.** *Let G be an h-framed graph (where G is not necessarily simple). Then $\mathrm{si}(G)$ is a subgraph of the strong product of three graphs $H \boxtimes P^{\lfloor h/2 \rfloor} \boxtimes K_{\max(3,h-2)}$, where H is a planar graph with $\mathrm{tw}(H) \leq 3$ and P is a path.*

Proof. Recall that $\mathcal{L} = (V_0, V_1, \ldots, V_b)$ is a BFS layering of the skeleton $\mathrm{sk}(G)$, and thus every edge of G has ends in parts $V_i, V_j \in \mathcal{L}$ such that $|i - j| \leq \lfloor h/2 \rfloor$. Hence, we may choose P as the path on $b+1$ vertices $(p_0, p_1, \ldots, p_b)$, use $P^{\lfloor h/2 \rfloor}$, and assign each vertex $v \in V(G)$ to the pair (t, p_i) where $t \in V(H)$ if t stems from the part of $\mathcal{R}'$ that v belongs to, and $v \in V_i \in \mathcal{L}$. Now, the number of vertices of G assigned to the same pair (t, p_i) (where t stems from a part X) is at most $\lambda_{\mathcal{L}}(X) \leq |X'| + \lambda_{\mathcal{L}}(X \setminus X') \leq \max(h-3+1, \lfloor h/2 \rfloor -1+2, \lfloor h/3 \rfloor -1+3) = \max(3, h-2)$. This concludes that $\mathrm{si}(G) \subseteq H \boxtimes P^{\lfloor h/2 \rfloor} \boxtimes K_{\max(3,h-2)}$. ◀

4 Consequences of the Product Structure

As mentioned in the introduction, Dujmović et al. [17] have derived upper bounds on the queue number, on the non-repetitive chromatic number, and on the p-centered chromatic number of k-planar and k-map graphs exploiting Theorem 2.1. In the following, we present our improvements to each of these problems.

Queue number. A *queue layout* of a graph G is a linear order σ of the vertices of G together with an assignment of its edges to sets, called *queues*, such that no two edges in the same set nest. The *queue number* $\mathrm{qn}(G)$ of a graph G is the minimum number of queues over all queue layouts of G. In [15], Dujmović et al. have proved the following useful lemma concerning the queue number of graphs that can be expressed as subgraphs of the strong product of a path P, a graph H with queue number $\mathrm{qn}(H)$, and a clique K_ℓ on ℓ vertices.

▶ **Lemma 4.1** (Dujmović et al. [15]). *If $G \subseteq P \boxtimes H \boxtimes K_\ell$ then $\mathrm{qn}(G) \leq 3\ell\,\mathrm{qn}(H) + \lfloor \frac{3}{2}\ell \rfloor$.*

Combining Lemma 4.1 and Theorem 2.1(d), together with the fact that the queue number of planar 3-trees is at most 5 [1], Dujmović, Morin, and Wood showed the first constant upper bound on the queue number of k-planar graphs [17], thus resolving a long-standing open question. Analogously, by combining Lemma 4.1 and Theorem 3.1, we obtain the following.

▶ **Theorem 4.2.** *The queue number of h-framed graphs is at most*

$$\left\lfloor \frac{33}{2}\left(3\lfloor\frac{h}{2}\rfloor + \lfloor\frac{h}{3}\rfloor - 1\right)\right\rfloor.$$

Dujmović et al. [15] first showed the queue number of k-map graphs is at most $2(98(k+1))^3$. Later, by combining Theorem 2.1(e) and Lemma 4.1, Dujmović et al. [16] improved this bound to $32225k(k-3)$. More recently, Dujmović et al. [17] have also observed that k-map graphs are k-framed and have exploited this observation to further improve this bound to $\lfloor \frac{33}{2}(k+3\lfloor\frac{k}{2}\rfloor-3)\rfloor$. By Theorem 4.2, we can further improve these bounds by also leveraging the fact that these graphs are subgraphs of k-framed graphs [3, 5].

▶ **Corollary 4.3.** *The queue number of k-map graphs is at most*

$$\left\lfloor \frac{33}{2}\left(3\lfloor\frac{k}{2}\rfloor + \lfloor\frac{k}{3}\rfloor - 1\right)\right\rfloor.$$

For $h \in 4, 5$, Theorem 4.2 gives us an upper bound of 95. Since any 1-planar graph can be augmented to a (not-necessarily simple) 4-framed graph [2], Theorem 4.2 improves the currently best upper bound of 1-planar graphs from 115 [17] to 95. Since any optimal 2-planar graph is 5-framed, Theorem 4.2 improves the currently best upper bound on their queue number from 132 [17] to 95. Next, we show a generalization of Lemma 4.1 that allows further improvements.

▶ **Lemma 4.4.** (*) *If $G \subseteq H \boxtimes P^i \boxtimes K_\ell$ then $\mathrm{qn}(G) \leq i\ell + (2i+1)\ell\mathrm{qn}(H) + \lfloor \frac{\ell}{2} \rfloor$.*

Lemma 4.4 in conjunction with Theorem 3.5 yields a quadratic (in h) upper bound on the queue number of h-framed graphs. However, for $h \leq 5$, it implies an improved bound on the queue number of 1-planar and optimal 2-planar graph, which we summarize in the following.

▶ **Theorem 4.5.** *The queue number of 1-planar and optimal 2-planar graphs is at most 82.*

Non-repetitive chromatic number. An r*-coloring* of a graph G is a function ϕ :V(G)$\rightarrow [r]$. A path $p = (v_1, v_2, \dots, v_{2\tau})$ is *repetitively colored* by ϕ if $\phi(v_i) = \phi(v_{i+\tau})$ for $i = 1, 2, \dots, \tau$. A coloring ϕ of G is *non-repetitive* if no path of G is repetitively colored by ϕ. Clearly, non-repetitive colorings are *proper*, i.e., $\phi(u) \neq \phi(v)$ if u and v are adjacent in G. The *non-repetitive chromatic number* $\pi(G)$ of G is the minimum integer r such that G admits a non-repetitive r-coloring. In [18], Dujmović et al. developed the following lemma to upper-bound the non-repetitive chromatic number of graphs that can be expressed as subgraphs of the strong product of a path P, a graph H with tw(H), and a clique K_ℓ on ℓ vertices.

▶ **Lemma 4.6** (Dujmović et al. [18]). *If $G \subseteq P \boxtimes H \boxtimes K_\ell$, then $\pi(G) \leq 4^{tw(H)+1} \cdot \ell$.*

Using Lemma 4.6 and Theorem 2.1(c), Dujmović et al. [17] provide an upper bound of 1792 and of 2048 on the non-repetitive chromatic number of 1-planar and optimal 2-planar graphs, respectively. Since 1-planar and optimal 2-planar graphs are 4-framed and 5-framed, respectively, we improve both bounds to 1536. By Lemma 4.6 and the product structure theorem for h-framed graph in [17], one can obtain an upper bound of $4^4 \cdot (h + 3\lfloor \frac{h}{2} \rfloor - 3)$ for the non-repetitive chromatic number of the graphs in this family. By Lemma 4.6 and Theorem 3.1, we can further improve this upper bound to $4^4 \cdot (3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1)$. Also, since k-map graphs are subgraphs of k-framed graphs [3, 5], their non-repetitive chromatic number is also improved from to $4^4 \cdot (k + 3\lfloor \frac{k}{2} \rfloor - 3)$ to $4^4 \cdot (3\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor - 1)$. Specifically, we get the following.

▶ **Corollary 4.7.** *For a graph G, it holds:*

- $\pi(G) \leq 4^4 \cdot 6$, *if G is 1-planar,*
- $\pi(G) \leq 4^4 \cdot (3\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor - 1)$, *if G is k-map, and*
- $\pi(G) \leq 4^4 \cdot (3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1)$, *if G is h-framed.*

***p*-centered chromatic number.** For any $c, p \in \mathbb{N}$ with $c \geq p$, a c-coloring of a graph G is p*-centered* if, for every connected component X of G, at least one of the following holds: (i) the vertices of X are colored with more than p colors, or (ii) there exists a vertex v of X that is assigned a color different from the ones of the remaining vertices of X. For any graph G, the p*-centered chromatic number* $\chi_p(G)$ of G is the minimum integer c such that G admits a p-centered c-coloring. The following lemma is implied by combining results by Pilipczuk and Siebertz [26], Debski et al. [11] and Dujmović et al. [17].

▶ **Lemma 4.8** ([11, 17, 26]). *If $G \subseteq P \boxtimes H \boxtimes K_\ell$, where H is a planar graph with* tw$(H) \leq 3$, *it holds that $\chi_p(G) \leq \ell(p+1)^2(p\lceil\log(p+1) + 2p + 1)$.*

By Lemma 4.8 and Theorem 2.1, Dujmović et al. [17] showed the following upper bounds: $\chi_p(G) \leq 7(p+1)^2(p\lceil\log(p+1) + 2p + 1)$ if G is a 1-planar graph, $\chi_p(G) \leq 8(p+1)^2(p\lceil\log(p+1) + 2p + 1)$ if G is an optimal 2-planar graph, $\chi_p(G) \leq (k + 3\lfloor \frac{k}{2} \rfloor - 3)(p+1)^2(p\lceil\log(p+1) + 2p + 1)$ if G is a k-map graph, and $\chi_p(G) \leq (h + 3\lfloor \frac{h}{2} \rfloor - 3)(p+1)^2(p\lceil\log(p+1) + 2p + 1)$ if G is an h-framed graph. By exploiting Theorem 3.1 and Lemma 4.8, we get the next.

▶ **Corollary 4.9.** *For a graph G, it holds:*

- $\chi_p(G) \leq 6(p+1)^2(p\lceil\log(p+1) + 2p + 1)$, *if G is 1-planar or optimal 2-planar,*
- $\chi_p(G) \leq (3\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor - 1)(p+1)^2(p\lceil\log(p+1) + 2p + 1)$, *if G is k-map, and*
- $\chi_p(G) \leq (3\lfloor \frac{h}{2} \rfloor + \lfloor \frac{h}{3} \rfloor - 1)(p+1)^2(p\lceil\log(p+1) + 2p + 1)$, *if G is h-framed.*

5 Bounding Twin-width

Besides the direct consequences of the product structure theorem(s) surveyed in Section 4, the construction presented in Section 3 has another strong implication described next.

Consider only simple graphs for the coming definition.[7] A *trigraph* is a simple graph G in which some edges are marked as *red*, and we then naturally speak about *red neighbors* and *red degree* in G. We denote the set of red neighbors of a vertex v by $N_r(v)$. For a pair of (possibly not adjacent) vertices $x_1, x_2 \in V(G)$, we define a *contraction* of the pair x_1, x_2 as the operation creating a trigraph G' which is the same as G except that x_1, x_2 are replaced with a new vertex x_0 whose full neighborhood is the union of neighborhoods of x_1 and x_2 in G, that is, $N(x_0) = (N(x_1) \cup N(x_2)) \setminus \{x_1, x_2\}$, and the red neighbors $N_r(x_0)$ of x_0 inherit all red neighbors of x_1 and of x_2 and those in $N(x_1) \oplus N(x_2)$, that is, $N_r(x_0) = \big((N_r(x_1) \cup N_r(x_2)) \setminus \{x_1, x_2\}\big) \cup \big(N(x_1) \oplus N(x_2)\big)$, where $\oplus$ denotes the symmetric difference.

A *contraction sequence* of a trigraph G is a sequence of successive contractions turning G into a single vertex, and its *width* is the maximum red degree of any vertex in any trigraph of the sequence. The *twin-width* is the minimum width over all possible contraction sequences (where for an ordinary graph, we start with the same trigraph having no red edges). As noted already in the pioneering paper on this concept [8], the twin-width of k-planar graphs is bounded for any fixed k by means of FO transductions (which, therefore, gives a not-even-asymptotically expressible bound). Explicit asymptotic bounds for the twin-width of k-planar graphs (albeit with $O(k)$ in the exponent, and so not giving an explicit number for e.g. $k = 1$) are in [9] (with a generalization to higher surfaces), and specially for planar graphs, the current upper bounds on twin-width are 583 in [9], improved to 183 in [23], to 37 in [6] and to 9 in [22], which is currently the best known one. It is worth to mention that both [9, 23], more or less explicitly, use the product structure machinery of planar graphs. Here, we give a new explicit (non-asymptotic) bounds on the twin-width of h-framed and 1-planar graphs.

▶ **Theorem 5.1.** **(*)** *Let G be a simple spanning subgraph of an h-framed graph with $h \geq 4$. Then the twin-width of G is at most $33\lfloor h/2\rfloor + \lfloor h/3\rfloor + 13 \leq 17h + 13$.*

Proof sketch. The rough idea (inspired in part by [23, Section 4]) is to recursively decompose G into subgraphs bounded by cycles of the plane skeleton of an h-framed supergraph of G – the same decomposition that has been used to obtain the product structure in Lemma 3.2. Then, on the "way back" from the recursion, we contract vertices inside these cycles which are at the same time in the same BFS layer in a controlled way, that is, not creating too-high red degrees in general, and completely avoiding red edges to suitably selected vertices on the bounding cycles (however, unlike [23], we do not exploit planarity for the latter task). ◀

▶ **Corollary 5.2.** *The twin-width of simple 1-planar and optimal 2-planar graphs is at most 80.*

We point out that Theorem 5.1 implies an improvement on the twin-width of k-map graphs only up to a certain k, as these graphs have bounded twin-width independently of k [8].

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[7] In general, the concept of twin-width is defined for binary relational structures of a finite signature, and so one may either define the twin-width of a multigraph as the twin-width of its simplification, or allow only bounded multiplicities of edges and use the more general matrix definition of twin-width.

6 Conclusions

In this paper we have provided a product structure theorem for h-framed graphs. Our approach is constructive and can easily be implemented to run in quadratic time to obtain the corresponding decomposition, provided that the input graph is a topological h-framed graph.

A major open question is to obtain a speed up in the construction; the recent algorithmic advances in [10, 24] have the potential to lead to improvements in the running time. Another important open problem is whether each k-planar graph is a subgraph of the strong product of a path, a (planar) graph of constant treewidth, and a clique whose size is a function of k; our results suggest that such a structure might be possible.

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